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THE POWER OF WEIGHTED AND ORDINARY LEAST SQUARES WITH ESTIMATED UNEQUAL VARIANCES IN EXPERIMENTAL DESIGN

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ABSTRACT

Response variances, $var(y_i)$, are estimated using replications for each experimental condition. The resulting estimated variances s_i^2 can be used to derive the correct variances of the Ordinary Least Squares (OLS) estimators $\hat{\beta}$. The estimates s_i^2 can also be used to compute the Estimated Weighted Least Squares (EWLS) estimators $\hat{\beta}^*$. The asymptotic covariance formula for EWLS might be utilized to test the EWLS estimators. The type I and type II errors of this test procedure are compared to the corresponding errors for the OLS estimators $\hat{\beta}$.

1. INTRODUCTION

This paper is a continuation of Kleijnen, Brent and Brouwers (1981) and Nozari (1984); also see Deaton, Reynolds and Myers

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(1983). The problem we tackle is as follows. In the classical general linear model

$$\underset{\sim}{y} = \underset{\sim}{X} \cdot \underset{\sim}{\beta} + \underset{\sim}{e} \tag{1.1}$$

the errors $\underset{\sim}{e}$ may show strong <u>heterogeneity of variance</u>. We have variance estimators s_i^2 based on replicating the experimental conditions i, say, m_i times:

$$s_i^2 = \sum_{j=1}^{m_i} (y_{ij} - \bar{y}_i)^2/(m_i-1) \qquad (i = 1,\ldots,n) \tag{1.2}$$

We examine the following questions:

(1) Can we continue to use the <u>classical</u> Ordinary Least Squares (OLS) <u>formulas</u>? So we compute

$$\hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}' \cdot \underset{\sim}{X})^{-1} \cdot \underset{\sim}{X}' \cdot \underset{\sim}{y} \tag{1.3}$$

$$\underset{\sim}{\Omega}_{\hat{\beta}} = (\underset{\sim}{X}' \cdot \underset{\sim}{X})^{-1} \cdot \sigma^2 \tag{1.4}$$

$$\hat{\sigma}^2 = \sum_{i=1}^{n} \sum_{j=1}^{m_i} (y_{ij} - \hat{y}_{ij})^2/(N-q) \tag{1.5}$$

where $N = \sum_{i=1}^{n} m_i$ and $\hat{y}_{ij} = \hat{y}_i$ and q denotes the number of parameters. The classical t statistic with v = N-q degrees of freedom is:

$$t_v = \frac{\hat{\beta}_j - \beta_j}{\{\widehat{\text{var}}\,(\hat{\beta}_j)\}^{\frac{1}{2}}} \qquad (j = 1,\ldots,q) \tag{1.6}$$

(2) Can we use the OLS estimator $\hat{\underset{\sim}{\beta}}$ of eq. (1.3) combined with the <u>correct</u> expression for the <u>covariance matrix</u> $\underset{\sim}{\Omega}_{\hat{\beta}}$ in case of unequal variances? Obviously this expression is:

$$\Omega_{\hat{\beta}} = (\underset{\sim}{X}' \cdot \underset{\sim}{X})^{-1} \cdot \underset{\sim}{X}' \cdot \underset{\sim}{\Omega}_y \cdot \underset{\sim}{X} \cdot (\underset{\sim}{X}' \cdot \underset{\sim}{X})^{-1} \tag{1.7}$$

We can estimate $\underset{\sim}{\Omega}_y$ in eq. (1.7), using s_i^2 of eq. (1.2). But how many degrees of freedom has the t statistic corresponding to eq. (1.6)? It is easy to derive that eq. (1.3) reduces to eq. (1.8), where $\bar{x}_{ij}$ is the (i,j)th element of the n × q matrix $\underset{\sim}{\bar{X}}$ formed by the n <u>different</u> rows of the N × q matrix $\underset{\sim}{X}$ (remember: $N = \Sigma m_i$; m_i replicates), and we restrict this study to orthogonal experimental designs, that is, $\underset{\sim}{\bar{X}}'.\underset{\sim}{\bar{X}} = n.\underset{\sim}{I}$.

$$\hat{\beta}_j = \sum_{i=n}^{n} \bar{x}_{ij}.\bar{y}_i/n \qquad (j = 1,\ldots,q) \tag{1.8}$$

Because $(\bar{x}_{ij})^2$ equals plus one and the observations are independent, we obtain:

$$\text{var}\ (\hat{\beta}_j) = \sum_{i=1}^{n} \text{var}\ (\bar{y}_i)/n^2 \tag{1.9}$$

Using the estimator s_i^2 of eq. (1.2) we get:

$$\hat{\text{var}}\ (\hat{\beta}_j) = \sum_{i=1}^{n} (s_i^2/m_i)/n^2 \tag{1.10}$$

We further restrict our study to an equal number of replications ($m_i = m$); if the variances were homogeneous, then $\hat{\text{var}}\ (\hat{\beta}_j)$ would reduce to a sum of χ^2 variates; because of the additivity of χ^2 variates, the t statistic of eq. (1.6) would have degrees of freedom $v = n(m-1)$.

<u>Note</u>: If $m_i \neq m$ then we might take a pilot sample to estimate $\text{var}(y_i)$; next we make m_i equal to $c\hat{\text{var}}\ (y_i)$ so that $\text{var}\ (\bar{y}_i)$ is approximately constant (c denotes a constant).

(3) Can we use the variance estimators s_i^2 to compute the <u>Estimated Weighted Least Squares</u> (EWLS) estimators $\hat{\beta}^*$? In other words, we have

$$\hat{\beta}^* = (\bar{X}' \cdot \hat{\Omega}_{\bar{y}}^{-1} \cdot \bar{X})^{-1} \cdot \bar{X}' \cdot \hat{\Omega}_{\bar{y}}^{-1} \cdot \bar{y} \tag{1.11}$$

Its asymptotic covariance matrix is:

$$\hat{\Omega}_{\hat{\beta}^*} = (\bar{X}' \cdot \hat{\Omega}_{\bar{y}}^{-1} \cdot \bar{X})^{-1} \tag{1.12}$$

Eqs. (1.11) and (1.12) result in the analogue of the t statistic of eq. (1.6). However, we do not know the correct degrees of freedom v^*. We might investigate:

(i) $v^* = N-q = n.m-q$; see the classical OLS formulas.

(ii) $v^* = \sum_1^n (m_i-1) = n.(m-1) = n.m-n$; see eq. (1.10).

(iii) $v^* = \min_i (m_i-1) = m-1$; see Scheffé (1964).

(iv) $v^* = \infty$ or $t_v = z$ with $z \sim N(0,1)$; the asymptotic case.

Actually we do not investigate approach (i). One reason is that approach (i) assumes a correctly specified regression model, whereas the other approaches use the unbiased estimators s_i^2. The difference between (i) and (ii) is negligible if $q \approx n$ (with $q \leq n$).

2. MONTE CARLO INPUT PARAMETERS

Appendix 1 gives details on the parameters of our Monte Carlo experiment. All $n \times q$ matrices $\bar{X}$ satisfy the condition $\bar{X}'\bar{X} = nI$. One $\bar{X}$ is a 16×13 matrix taken from a simulation study of the Rotterdam harbor (with design generators $1 = 5.6$ and $3 = 4.5$). The other matrices $\bar{X}$ are based on a 2^3 and a 2^2 experimental design. We combine each of these three cases with several degrees of heterogeneity, measured by

$$H = (\sigma^2_{max} - \sigma^2_{min})/\sigma^2_{min} \tag{2.1}$$

where σ^2_{max} (and σ^2_{min}) is the maximum (and minimum) element of Ω_y. H varies between zero (constant variances) and 1,455.69 (taken from the harbor case-study). The variances are estimated from m replications; we vary m between two (a technical minimum) and twenty-five. We repeat each Monte Carlo experiment (specified by $\bar{\underset{\sim}{X}}$, $\underset{\sim}{\beta}$, $\underset{\sim}{\Omega}_y$, and m) 150 times to reduce chance effects.

3. MONTE CARLO RESULTS

Appendix 2 gives details on an experiment that substantiates the experimental results of Kleijnen et al. (1981), i.e., we repeat their experiment with different random numbers and find the following results:

(i) Bias: Both OLS and EWLS give unbiased estimators of $\underset{\sim}{\beta}$, as we knew.

(ii) Standard errors: The asymptotic covariance formula of eq. (1.12) holds, provided we estimate var(y) from twenty-five replications (m = 25). For m = 9 our results deviate from Kleijnen et al. (1981), i.e., the asymptotic formula may underestimate the variance.

(iii) Relative efficiency: In case of strong heterogeneity EWLS give smaller variances for the β estimators, provided we have more than two replications (m > 2).

Next we try to answer a new set of questions, namely can we use the Student t statistic t_v when we estimate the unknown variances $var(\bar{y}_i)$ and apply OLS and EWLS respectively, where the degrees of freedom v may equal n(m-1) for OLS and (m-1), n(m-1) or ∞ for EWLS. We estimate the true distribution from 150 realizations, and apply three popular goodness-of-fit tests, namely the χ^2, the Kolmogorov-Smirnov, and the Anderson-Darling tests. We apply each goodness-of-fit test to each of the q parameters β_j, using a 1% significance level. We do not present the resulting mass of data but report only preliminary results (which are further investigated below): EWLS based on only two replications result in distributions not well approximated by any Student distribution. If we have more replications ($m > 2$), then we may use the Student t statistic with the (conservative) degrees of freedom equal to m-1. If m is as high as 25, then we may use the normal approximation. OLS with the corrected variance formula accounting for unequal variances (eq. 1.7) result in a t distribution with degrees of freedom equal to $n(m-1)$, provided $n(m-1) > 15$ (as n increases the variance of $\hat{var}(\hat{\beta})$ decreases). We shall give more detailed results for the following more specialized question.

Because we use the t distribution only to select the critical constant $t_v^{\alpha/2}$, we test the hypothesis:

$$H_0 : P\{\frac{|\hat{\beta}_j - \beta_j|}{\{\hat{var}\ (\hat{\beta}_j)\}^{\frac{1}{2}}} > t_v^{\alpha/2}\} = \alpha \qquad (j = 1,\ldots,q) \tag{3.1}$$

If e denotes the event within the outer brackets of eq. (3.1) then the alternative hypothesis H_1 is $P\{e\} \neq \alpha$ or H_1 is the one-sided and conservative alternative hypothesis: $P\{e\} > \alpha$. To test H_0 we use the binomial test as follows. We estimate $P\{e\}$ from 150 independent replications and compute a confidence interval. For example, for the one-sided H_1 the lower limit of the $1-\gamma_0$ confidence interval is given by the following expression where $z \sim N(0,1)$ and $P(z > z^{\gamma_0}) = \gamma_0$:

$$\hat{p}-z^{\gamma_0}.\{\hat{p}.(1-\hat{p})/150\}^{\frac{1}{2}} \tag{3.2}$$

so that we reject H_0 if α is smaller than this limit. Actually we reject H_0 if <u>any</u> of the q parameters β_j exceeds the critical level: Applying the Bonferroni inequality, we reject H_0 of eq. (3.1) if

$$\max_{1\leq j\leq q} [\hat{p}_j-z^{\gamma/q}.\{\hat{p}_j.(1-\hat{p}_j)/150\}^{\frac{1}{2}}] > \alpha \tag{3.3}$$

For γ in eq. (3.3) we select the value 5%. We apply the procedure

TABLE I

Testing the tail of the t_v distribution; one-sided test

	α = 1%				α = 5%				α = 10%			
	OLS	EWLS			OLS	EWLS			OLS	EWLS		
	$t_{n(m-1)}$	t_{m-1}	$t_{n(m-1)}$	t_∞	$t_{n(m-1)}$	t_{m-1}	$t_{n(m-1)}$	t_∞	$t_{n(m-1)}$	t_{m-1}	$t_{n(m-1)}$	t_∞
C1H11m2[1]			*	*			*	*			*	*
C1H1455m2			*	*	*		*	*	*		*	*
C2H10m2			*	*			*	*			*	*
C2H1455m2			*	*	*		*	*	*	*	*	*
C3H10m4			*	*			*	*			*	*
C3H1289m4			*	*			*	*			*	*
C3H10m5				*				*				*
C3H1289m5			*	*			*	*			*	*
C1H0m9							*	*			*	*
C1H11m9												
C1H1455m9												
C2H0m9												
C2H10m9			*	*			*	*			*	*
C2H1455m9				*			*	*			*	*
C3H10m9												
C3H1289m9			*	*			*	*	*	*	*	*
C1H0m25												
C1H11m25												
C1H1455m25												
C2H0m25												
C2H10m25												
C2H1455m25												
C3H10m25												
C3H1289m25												

1) C1H11m2 means: Case 1 (n = 16, q = 13), Heterogeneity factor H = 11, replications m = 2. Etc.

TABLE II

Testing the tail of the t_v distribution; two-sided test

	α = 1%				α = 5%				α = 10%			
	OLS	EWLS			OLS	EWLS			OLS	EWLS		
	$t_{n(m-1)}$	t_{m-1}	$t_{n(m-1)}$	t_∞	$t_{n(m-1)}$	t_{m-1}	$t_{n(m-1)}$	t_∞	$t_{n(m-1)}$	t_{m-1}	$t_{n(m-1)}$	t_∞
C1H11m2	*	*	*	*		*	*	*		*	*	*
C1H1455m2		*	*	*	*	*	*	*		*	*	*
C2H10m2		*	*	*		*	*	*			*	*
C2H1455m2		*	*	*	*	*	*	*	*		*	*
C3H10m4		*	*	*			*	*			*	*
C3H1289m4				*			*	*			*	*
C3H10m5		*						*				
C3H1289m5		*		*		*	*	*			*	*
C1H0m9	*	*						*		*	*	*
C1H11m9	*	*	*			*						
C1H1455m9	*	*								*		
C2H0m9	*	*										
C2H10m9												*
C2H1455m9		*					*	*			*	*
C3H10m9	*	*				*						
C3H1289m9				*			*	*	*		*	*
C1H0m25	*	*	*	*		*				*		
C1H11m25	*	*	*	*		*	*	*	*	*	*	*
C1H1455m25	*	*	*	*								
C2H0m25	*	*	*	*								
C2H10m25	*	*	*	*								
C2H1455m25	*	*	*									
C3H10m25												
C3H1289m25	*	*	*	*		*						

of eq. (3.3) for three classical α values in eq. (3.1), namely 1%, 5% and 10%. This approach results in Tables I and II where the symbol * means that we reject H_0. These tables suggest the following conclusions: If the n responses $\bar{y}$ have different variances and we can estimate these variances from more than two replications ($m > 2$), then the OLS estimators $\hat{\beta}$ can be tested using a Student t statistic with degrees of freedom equal to $v = n(m-1)$, provided we test β with an α exceeding 1%. Testing the EWLS estimators $\hat{\beta}^*$ requires more replications, say $m = 25$ (and $\alpha > 0.01$). This conclusion agrees with Nozari (1984)'s conclusion.

If and only if both OLS and EWLS pass the test of eq. (3.3), then it makes sense to compare their power functions. We estimate the power function in eight to ten points, using different random numbers per point (Kleijnen (1983) presents a more efficient procedure). For each point we generate 150 replicates. The result is that in all experiments EWLS dominate OLS (as we might expect because in previous experiments we found that $\mathrm{var}(\hat{\beta}^*) < \mathrm{var}(\hat{\beta})$); we test this dominance using the sign test. Appendix 3 gives some details.

4. CONCLUSIONS

We limited our study to experimental designs with $\bar{X}'\bar{X} = n\,I$. If we suspect heterogeneity of variances, then we should try to estimate the n different variances, obtaining more than two replications ($m > 2$). We can use these estimated variances to derive the correct variances of the OLS estimators $\hat{\beta}$ and to test their significance, through the Student t statistic with $n(m-1)$ degrees of freedom. If we have firm estimators of the response variances - say 25 replications - then it is better to use the EWLS estimators $\hat{\beta}^*$ with the t distribution with degrees of freedom equal to $n(m-1)$. We should test OLS and EWLS estimators using an α higher than 1%.

BIBLIOGRAPHY

Deaton, M., M.R. Reynolds and R.M. Myers, (1983). Estimation and hypothesis testing in regression in the presence of nonhomogeneous error variances. Communications in Statistics, Simulation and Computation, 12, no. 1, pp. 45-66.

Kleijnen, J.P.C., (1983). Efficient estimation of power functions in simulation experiments. Tilburg University, Tilburg (Neth.), (Submitted for publication.)

Kleijnen, J.P.C., R. Brent and R. Brouwers, (1981). Small-sample behavior of weighted least squares in experimental design applications. Communications in Statistics, Simulation and Computation, B10, no. 3, pp. 303-313.

Kleijnen, J.P.C., A.J. van de Burg and R.T. van der Ham, (1979). Generalization of simulation results: practicality of statistical methods. European Journal of Operational Research, 3, pp. 50-64.

Nozaro, A., (1984). Generalized and ordinary least squares with estimated and unequal variances. Communications in Statistics, Simulation and Computation, 13, no. 4, pp. 521-537.

Scheffe, H., (1964). The Analysis of Variance. John Wiley & Sons, Inc., New York, 4th printing, 1964.

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APPENDIX 1: DETAILS OF MONTE CARLO INPUT

Case 1 is inspired by a case study, namely a simulation of a harbor in Rotterdam, reported in Kleijnen, Van den Burg and Van der Ham (1979). There are sixteen experimental conditions and thirteen parameters β, i.e., the matrix $\bar{X}$ is as follows (the number one is not displayed):

+	+	+	+	+	+	+	+	+	+	+	+	+
+	-	+	+	+	+	+	-	+	-	-	-	-
+	+	-	+	+	+	-	+	-	-	-	+	+
+	-	-	+	+	+	-	-	-	+	+	-	-
+	+	+	-	+	-	-	-	+	-	+	-	+
+	-	+	-	+	-	-	+	+	+	-	+	-
+	+	-	-	+	-	+	-	-	+	-	-	+
+	-	-	-	+	-	+	+	-	-	+	+	-
+	+	+	+	-	-	+	-	-	+	+	+	-
+	-	+	+	-	-	+	+	-	-	-	-	+
+	+	-	+	-	-	-	-	+	-	-	+	-
+	-	-	+	-	-	-	+	+	+	+	-	+
+	+	+	-	-	+	-	+	-	-	+	-	-
+	-	+	-	-	+	-	-	-	+	-	+	+
+	+	-	-	-	+	+	+	+	+	-	-	-
+	-	-	-	-	+	+	-	+	-	+	+	+

We use the estimates of the case study as the (true) population parameters, i.e., $\underset{\sim}{\beta}'$ = (-1.42 -0.769 13.4 -11.508 3.5 -1.375 140.918 15.391 0.046 281.098, 21.25 11.875 -49.483). The three degrees of heterogeneity are: If H = 0 (constant variances) then $var(\bar{y}_i) = 1$ for $i = 1,\ldots,16$. If H = 11.84 (intermediate heterogeneity) then $\underset{\sim}{var}(\bar{y})$ = (1 2 3 4 4.5 5 6 7 7.5 8 9 9.5 10 11 12 12.84). If H = 1455 (extreme heterogeneity, found in case study) then $\underset{\sim}{var}(\bar{y})$ = (93228.38 821.78 2809 69.44 2567.11 177.78 15129 576 27115.11 560.11 4181.77 64 12693.77 529 20640.11 608.44).

Case 2 is a 2^3 design (see the submatrix of Case 1 formed by the first four columns and the first eight rows). $\underset{\sim}{\beta}'$ = (-1.42 -0.769 13.44 -11.508). If H = 0 then $var(\bar{y}_i) = 1$. If H = 10.83 then $\underset{\sim}{var}(\bar{y})$ = (1 2 4 5 6 7 9 11.83). If H = 1455 then $\underset{\sim}{var}(\bar{y})$ = (93228.38 821.78 2809 64 2567.11 177.78 15129 576).

Case 3 is a 2^2 design (see Case 1, first three columns and first four rows) with $\underset{\sim}{\beta}'$ = (1 1 1). If H = 0 then $var(\bar{y}_i) = 1$.

If $H = 10.38$ then $\underset{\sim}{\text{var}}(\bar{y}) = (1 \quad 4 \quad 8 \quad 11.38)$. If $H = 1289$ then $\underset{\sim}{\text{var}}(\bar{y}) = (1 \quad 200 \quad 600 \quad 1290.15)$.

We use a multiplicative random number generator with multiplier 13^{13} and modulus 2^{59}. This generator was developed by NAG (Numerical Algorithms Group) and it is standard on our ICL 2960 computer.

APPENDIX 2: REPEATING THE EXPERIMENT OF KLEIJNEN ET AL. (1981)

We first verify the correctness of our (Monte Carlo) computer program as follows. We know that the OLS estimator $\hat{\beta}$ of eq. (1.3) or (1.8) is unbiased and that its covariance matrix is given by eq. (1.7) or (1.9) where $\underset{\sim}{\Omega}_y$ or var $(\bar{y})$ is known in the Monte Carlo experiment. So we estimate the expected values $E(\hat{\beta}_j)$ and the variances $\text{var}(\hat{\beta}_j)$ from the 150 Monte Carlo repetitions, and test these values using the standard normal statistic z and the χ^2 statistic with 149 degrees of freedom. Next we examine the quality of the various β estimators in several steps:

<u>(i) Bias of β estimator</u>

We know that OLS always give unbiased estimators $\hat{\beta}$, and that (under mild technical assumptions) EWLS also give unbiased estimators $\hat{\beta}^*$ (under normality $\bar{y}$ and s^2 are independent so that EWLS give unbiased estimators). In the preceding paragraph we verified the lack of bias in OLS, using the standard normal statistic z. For EWLS we compute the (approximate) Student t statistic:

$$t_{149}^{(j)} \approx \frac{\left(\sum_{g=1}^{150} \hat{\beta}^*_{jg}/150\right) - \beta_j}{\left\{\sum_{g=1}^{150} \left(\hat{\beta}^*_{jg} - \sum_{g=1}^{150} \hat{\beta}^*_{jg}/150\right)^2/(149 \times 150)\right\}^{\frac{1}{2}}} \quad (j = 1,\ldots,q) \quad (A.1)$$

Note: We do not use the equality sign in eq. (A.1) because the EWLS estimator $\hat{\beta}^*$ is not a linear transformation of $\bar{\underset{\sim}{y}}$; $\hat{\beta}^*$ also uses the random vector with the elements s_i^2. However, the t statistic is supposed to be robust, especially with as many observations as 150.

We obtain 160 realizations of t_{149} (the number 160 follows from Table III later on: eight combinations of H and m, and q parameters, i.e., $160 = 8 \times 13 + 8 \times 4 + 8 \times 3$). We use a 5% significance level per realization, so that we expect eight false significances. We find zero significances for OLS and six for EWLS. We conclude that OLS and EWLS indeed give unbiased estimators of β, which agrees with Kleijnen et al. (1981).

(ii) Standard error of β estimator

The standard errors of the OLS estimators $\hat{\beta}$ follow from eq. (1.7) or eq. (1.10). For EWLS we have the asymptotic formula of eq. (1.12). We compute the χ^2 approximation:

$$\chi^{2(j)}_{149} \approx \frac{\sum_{g=1}^{150} (\hat{\beta}^*_{jg} - \sum_{g=1}^{150} \hat{\beta}^*_{jg}/150)^2/149}{(\bar{X}'.\hat{\Omega}^{-1}_{\bar{y}}.\bar{X})^{-1}_{jj}} \quad (j = 1,\ldots,q) \qquad \text{(A.2)}$$

where $(\ )_{jj}$ means the j^{th} element on the main diagonal of ( ). Table III displays the maximum and the minimum of the q realizations. We compare the maximum and minimum using a two-sided χ^2_{149} test with 1% significance, resulting in the critical values 0.73 and 1.32. Table III suggests the following conclusions. If we have only two replications (m) to estimate var(y), then we underestimate the true variance of $\hat{\beta}^*$. With m = 25 the asymptotic formula gives unbiased estimators of var($\hat{\beta}^*$). With m = 9 it is very well possible that we underestimate the variance; our results for m = 9 conflict with Kleijnen et al. (1981) who reported unbiased estimators.

Note: We use the χ^2 statistic even though $\hat{\beta}^*$ may be nonnormal and we know that the χ^2 statistic is not robust. We do not apply a

TABLE III

Adequacy of asymptotic variance formula

Case 1: n = 16, q = 13

Heterogeneity H

	0		11.84			1,455.69		
m	9	25	2	9	25	2	9	25
max χ^2	1.399*	1.207	1.643*	1.215	1.238	1.792*	1.186	1.224
min χ^2	0.834	0.823	1.013	0.914	0.674*	1.236	0.923	1.005

Case 2: n = 8, q = 4

Heterogeneity H

	0		10.83			1,455.69		
m	9	25	2	9	25	2	9	25
max χ^2	1.195	1.211	2.427*	1.348*	1.038	3.635*	1.183	0.918
min χ^2	0.948	1.047	1.680	1.074	0.899	2.642	0.947	0.873

Case 3: n = 4, q = 3

Heterogeneity H

	10.38				1,289.15			
m	4	5	9	25	4	5	9	25
max χ^2	1.251	1.100	1.092	1.120	1.315	1.184	1.399*	1.049
min χ^2	1.169	0.903	0.945	0.962	1.012	0.978	0.993	0.792

TABLE IV

Efficiency of OLS versus EWLS

Case 1: n = 16, q = 13

Heterogeneity H

	0		11.84			1,455.69		
m	9	25	2	9	25	2	9	25
max χ^2	1.399*	1.207	1.453*	1.195	1.217	1.436*	0.949	0.941
min χ^2	0.834	0.823	0.993	0.864	0.673*	0.097*	0.077*	0.075*

Case 2: n = 8, q = 4

Heterogeneity H

	0		10.83			1,455.69		
m	9	25	2	9	25	2	9	25
max χ^2	1.195	1.211	1.827*	1.203	0.989	0.560	0.190	0.162
min χ^2	0.948	1.047	1.389	0.732	0.635*	0.150*	0.052*	0.046*

Case 3: n = 4, q = 3

Heterogeneity H

	10.38				1,289.15			
m	4	5	9	25	4	5	9	25
max χ^2	1.237	1.100	1.017	1.096	1.214	1.119	1.242	0.915
min χ^2	0.763	0.589*	0.684*	0.730	0.352*	0.395*	0.345*	0.275*

distribution-free procedure, because we have 149 degrees of freedom and ultimately we are not interested in the standard errors themselves but only in their role in the t statistic of eq. (1.6); see Section 3.

(iii) Efficiency of OLS versus EWLS

We quantify the efficiency through the variance. Therefore we compare the estimated variance of the EWLS estimator (the numerator of eq. (A.2)) to the known variance of the OLS estimator (see eq. (1.9)) and this results in a χ^2_{149} statistic analogous to eq. (A.2). Table IV suggests the following conclusions (which agree with Kleijnen et al. (1981)):

(i) If we knew that the variances $var(\bar{y})$ are constant (H = 0), then we should not estimate them, i.e., we should not use EWLS.

(ii) In case of strong heterogeneity EWLS is more efficient provided we can estimate $var(\bar{y})$ from more than two observations.

APPENDIX 3: POWER FUNCTIONS

An example is as follows. We consider Case 3 where we have three parameters (β_1, β_2, β_3) besides the general mean (β_0). We test the null hypothesis:

$$H_0: \beta_3 = 1$$

We may estimate β_3 using OLS and test H_0 applying the t test with $\alpha = 0.05$ and degrees of freedom $v = n(m-1)$. Suppose that actually β_3 has the value 1.15. Then we sample the responses y from the linear model of eq. (1.1) with the value $\beta_3 = 1.15$. Repeating this sampling 150 times, we obtain an estimate of the power at the value $\beta_3 = 1.15$:

$$P(H_0: \beta_3 = 1 \text{ rejected} | \beta_3 = 1.15)$$

TABLE V

Estimated power: $\hat{P}(H_0: \beta_3 = 1|\beta_3)$

β_3	1	1.15	1.25	1.37	1.50	1.62	1.75	2.05	2.45
OLS, v=n(m-1)	.047	.073	.213	.313	.527	.687	.867	.980	1
EWLS, v = m-1	.047	.080	.167	.267	.507	.687	.893	.993	1
v=n(m-1)	.053	.087	.220	.307	.533	.707	.907	.993	1
v = ∞	.053	.093	.227	.533	.720	.920	.993	1	1

We repeat this process for eight values different from 1.15, including the value $\beta_3 = 1$ (where the power coincides with the α error). We estimate the power function, not only for OLS but also for EWLS, using three different values for v (see the end of section 1) and we obtain Table V.

Obviously the power increases as the true β deviates from the hypothesized value, and the power increases as the degrees of freedom used for the t statistic increase. For the other two parameters (β_1 and β_2) of Case 3 (with H = 10, m = 25) we obtain similar tables. The results for the three parameters are independent.

We further study several other m values, degrees of heterogeneity (H), α values, and X matrices, provided OLS and EWLS pass the test of eq. (3.3).

We never use common random numbers, i.e., outputs are independent (obviously outputs within a particular column of tables like Table V are dependent). Further we know that a two-sided test results in a symmetric power function, and therefore we estimate the power function only for $\beta_3 > 1$ in Table V. We compare the power of OLS and EWLS through the sign test, using a 5% significance level and eight to ten observations (the power is estimated in so many points).

BIBLIOGRAPHY

Deaton, M., M.R. Reynolds and R.M. Myers, (1983). Estimation and hypothesis testing in regression in the presence of nonhomogeneous error variances. Communications in Statistics, Simulation and Computation, 12, no. 1, pp. 45-66.

Kleijnen, J.P.C., (1983). Efficient estimation of power functions in simulation experiments. Tilburg University, Tilburg (Neth.), (Submitted for publication.)

Kleijnen, J.P.C., R. Brent and R. Brouwers, (1981). Small-sample behavior of weighted least squares in experimental design applications. Communications in Statistics, Simulation and Computation, B10, no. 3, pp. 303-313.

Kleijnen, J.P.C., A.J. van de Burg and R.T. van der Ham, (1979). Generalization of simulation results: practicality of statistical methods. European Journal of Operational Research, 3, pp. 50-64.

Nozaro, A., (1984). Generalized and ordinary least squares with estimated and unequal variances. Communications in Statistics, Simulation and Computation, 13, no. 4, pp. 521-537.

Scheffe, H., (1964). The Analysis of Variance. John Wiley & Sons, Inc., New York, 4th printing, 1964.

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