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Combinatorial designs with two singular values—I: uniform multiplicative designs

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Abstract

In this and a sequel paper (Combinatorial designs with two singular values. II. Partial geometric designs, preprint) we study combinatorial designs whose incidence matrix has two distinct singular values. These generalize 2-(v,k,λ) designs, and include partial geometric designs and uniform multiplicative designs. Here we study the latter, which are precisely the nonsingular designs. We classify all such designs with smallest singular value at most $\sqrt{2}$, generalize the Bruck–Ryser–Chowla conditions, and enumerate, partly by computer, all uniform multiplicative designs on at most 30 points.

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1. Introduction

Combinatorial designs (a set of points, a set of blocks, and an incidence relation between those) are usually defined in terms of nice combinatorial properties, such as "each block has the same size", "every pair of points occurs in the same number of

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blocks", etc. Many combinatorial designs defined in this way have the property that their (0,1)-incidence matrix has nice algebraic properties. These algebraic properties are in turn relevant to the statistical properties of the designs.

Here we start from the point of view of such an algebraic property, i.e., the property that the incidence matrix N has two distinct singular values (the positive square roots of the (nonzero) eigenvalues of NN^T). Designs with zero or one singular value are trivial: they are empty or complete, respectively. Designs with two singular values include 2-(v, k, λ) designs and certain group divisible designs, but also some less familiar designs such as partial geometric designs and uniform multiplicative designs. The latter are precisely the nonsingular designs, and these form the subject of this paper. In a sequel paper [10] we will study the partial geometric designs, that is, the singular and non-square 1-designs with constant block size and two singular values.

Multiplicative designs were introduced by Ryser [13], and have been studied by Bridges [1], Bridges and Mena [2–4] and Host [11,12]. Here we shall collect some of the known results on uniform multiplicative designs, give some new examples, and classify, partly by computer, all designs on at most 30 points. Some of these designs have four distinct block sizes, while up to now only designs with at most three distinct block sizes were known. We also classify all uniform multiplicative designs with smallest singular value at most $\sqrt{2}$, and give a generalization of the Bruck–Ryser–Chowla conditions.

There is an important connection to algebraic graph theory in the sense that the incidence graphs of the studied designs are precisely the bipartite graphs with four eigenvalues. Graphs with few distinct eigenvalues have been studied before by the authors, cf. [6–9], but so far not much attention has been paid to bipartite graphs. As a consequence of our results all bipartite graphs with four eigenvalues up to 60 vertices have now been classified.

In order to eliminate some trivialities, we assume that the studied designs (and their bipartite incidence graphs) are connected, i.e., that there is no (nontrivial) subset of points and subset of blocks such that all incidences are between those subsets, or between their complements. Consequently, the Perron–Frobenius theory (cf. [5, p. 80]) can be applied, and it follows that the largest singular value has multiplicity one and a positive eigenvector.

2. Uniform multiplicative designs

If the incidence graph of a design with two singular values $\sigma_0 > \sigma_1$ has four distinct eigenvalues $(\pm\sigma_0, \pm\sigma_1)$ then the design (i.e., its incidence matrix) must be square and nonsingular. It is clear then that $NN^T - \sigma_1^2 I$ is a rank one matrix. It follows that $NN^T = \sigma_1^2 I + \alpha\alpha^T$, where α is the positive eigenvector of NN^T with eigenvalue σ_0^2 such that $\alpha^T\alpha = \sigma_0^2 - \sigma_1^2$. Such designs are called (square) uniform multiplicative designs by Ryser [13]. We note that the dual design of such a design is also uniform multiplicative, since there must similarly be a (positive) vector β such that $N^TN = \sigma_1^2 I + \beta\beta^T$; in fact, this vector is $\beta = \frac{1}{\sigma_0}N^T\alpha$. If the incidence matrix can be rearranged such that $NN^T = N^TN$ $(\alpha = \beta)$, then the design is called normal. In this case the design and its dual have the same intersection pattern. Most known examples of multiplicative designs are indeed normal, such as symmetric 2-(v, k, λ) designs.

2.1. Parameter restrictions

From the equation $NN^T = \sigma_1^2 I + \alpha\alpha^T$, we derive that

$$\begin{cases} r_p = \sigma_1^2 + \alpha_p^2, \\ \lambda_{pq} = \alpha_p\alpha_q, \end{cases}$$

where r_p equals the replication of point p, i.e. the number of blocks incident with p (also row sum p in N); and λ_{pq} is the number of blocks containing the pair of points p, q.

From this it follows that if the design has constant replication r, then α is a constant vector, and thus $\lambda = \lambda_{pq}$ is constant. Hence $NN^T = (r - \lambda)I + \lambda J$ and $NJ = rJ$. From this it follows that $N^{-1} = \frac{1}{r-\lambda}(N^T - \frac{\lambda}{r}J)$, and consequently that $\frac{\lambda}{r}JN = N^TN - (r-\lambda)I$, which is symmetric with rank one. Now we may conclude that $JN = rJ$, i.e., the design has constant block size r, and thus is a symmetric design. Thus we have the following.

Proposition 1. *A uniform multiplicative design is a symmetric design if and only if it has constant replication or constant block size.*

Since symmetric designs are well-studied objects, we will focus on nonsymmetric designs, that is, we will assume in the remainder of this paper that the designs do not have constant replication, and do not have constant block size. To be absolutely clear, we remark that a non-symmetric design can have a symmetric incidence matrix. Indeed, we shall see some of such examples.

Let us first make some observations about the form of the singular values (of the integer $v \times v$ matrix N). The characteristic polynomial $(x - \sigma_0^2)(x - \sigma_1^2)^{v-1}$ of NN^T is a monic polynomial with integer coefficients. The minimal polynomial $(x - \sigma_0^2)(x - \sigma_1^2)$ is monic with rational coefficients (since it can be obtained by Gaussian elimination from a system of v^2 equations with integer coefficients), and since it divides the characteristic polynomial, it has integer coefficients. The quotient $(x - \sigma_1^2)^{v-2}$ of the two polynomials therefore also has integer coefficients, hence σ_1^2, and consequently also σ_0^2, is an integer, unless maybe when $v = 2$. Indeed, for $v = 2$ there is one design with two singular values: its incidence matrix is

$$N = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

which has singular values $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$. Now let us assume in the remainder of this section that $v \geqslant 3$. As derived we know then that the singular values are square roots of integers. Furthermore, since $\sigma_0^2\sigma_1^{2(v-1)} = \det(NN^T)$ is a square integer, we have the following.

Proposition 2. *Let $v \geqslant 3$ be the number of points of a uniform multiplicative design with singular values $\sigma_0 > \sigma_1$. If v is odd, then σ_0 is an integer, and if v is even, then $\sigma_0\sigma_1$ is an integer.*

From the equations $r_p = \sigma_1^2 + \alpha_p^2$ and $\lambda_{pq} = \alpha_p\alpha_q$, it now follows that $\alpha = w\sqrt{\delta}$, where δ is a square-free integer and w is a positive integer vector. Dually, we have that $\beta = u\sqrt{\varepsilon}$, where ε is a square-free integer, and u a positive integer vector. Since $N^T\alpha = \sigma_0\beta$, we have that $\sigma_0\sqrt{\delta\varepsilon}$ is rational, and hence an integer. If the design is normal (then $\delta = \varepsilon$), then σ_0 is an integer. We thus have the following.

Proposition 3. *For a uniform multiplicative design with singular values $\sigma_0 > \sigma_1$ on $v \geqslant 3$ points, with vectors $\alpha = w\sqrt{\delta}$ and $\beta = u\sqrt{\varepsilon}$ as above, we have that $\sigma_0\sqrt{\delta\varepsilon}$ is an integer. If moreover the design is normal, then σ_0 is an integer.*

Some examples we shall see have two distinct replications and the same block sizes, and moreover they are normal. Such multiplicative designs have been studied by Bridges and Mena [3]. Here we shall use the following.

Proposition 4. *A uniform multiplicative design with two distinct replications r_1 and r_2, which also has block sizes r_1 and r_2, is normal and its singular values are both integers. Moreover, each point with replication r_i is in r_{ij} blocks of size r_j, where r_{ij} is uniquely determined by the equations $r_{i1} + r_{i2} = r_i$ and $r_{i1}w_1 + r_{i2}w_2 = \sigma_0 w_i$.*

Proof. Consider such a design. Let v_i be the number of points with replication r_i $(i = 1, 2)$. Then $v_1 + v_2 = v$, and $v_1r_1 + v_2r_2 = \sigma_0^2 + (v-1)\sigma_1^2$ (which follows from the trace of NN^T). Thus v_1 and v_2 are uniquely determined. Similarly, this hold for the blocks: the number of blocks of size r_i is also v_i $(i = 1, 2)$. It follows now that the incidence matrix can be rearranged such that $NN^T = N^TN$, hence N is normal. By the previous proposition, we now have that σ_0 is an integer.

Consider now a point with replication r_i $(i = 1, 2)$, and suppose that it is contained in r_{ij} blocks of size r_j $(j = 1, 2)$. Then $r_{i1} + r_{i2} = r_i$ and $r_{i1}w_1 + r_{i2}w_2 = \sigma_0 w_i$. It follows that r_{ij} $(i, j = 1, 2)$ is uniquely determined, i.e., it only depends on i and j. Thus N has a regular partition with quotient matrix

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

which has eigenvalues σ_0 and $r_{11} + r_{22} - \sigma_0$. The latter eigenvalue must be $\pm\sigma_1$, from which it follows that also σ_1 is an integer. $\square$

More generally, we have the following on the numbers r_{ij}.

Proposition 5. *In a uniform multiplicative design, let p be a point with replication* $r_p = \sigma_1^2 + \alpha_p^2$. *If* r_{pj} *is the number of blocks of size* $k_j = \sigma_1^2 + \beta_j^2$ *containing p, then*

$$\sum_j r_{pj} = r_p, \quad \sum_j r_{pj}\beta_j = \sigma_0\alpha_p, \quad \text{and} \quad \sum_j r_{pj}k_j = \sigma_1^2 + (\alpha^T\mathbf{j})\alpha_p. \tag{1}$$

If the design has three distinct block sizes then the numbers r_{pj} *are uniquely determined by the replication* r_p.

Proof. The first equation is clear, while the second follows from the equation $N\beta = \sigma_0\alpha$. The third follows from the fact that $N(N^T\mathbf{j}) = \sigma_1^2\mathbf{j} + (\alpha^T\mathbf{j})\alpha$, and by observing that $N^T\mathbf{j}$ is a vector containing the block sizes k_j. It is easy to show that if there are only three block sizes, then the obtained system (three equations with three unknowns, for each p) is nonsingular, hence has a unique solution. $\square$

Host [11] derived rational congruence conditions for uniform multiplicative designs by using the Hasse–Minkowski theorem. These conditions seem to be rather complicated though. Here we derive the following elementary generalization of the well-known Bruck–Ryser–Chowla conditions for symmetric designs, by adjusting Ryser's proof (cf. [14]) for these conditions.

Proposition 6. *Let v be odd. If a uniform multiplicative design on v points exists, with singular values* $\sigma_0 > \sigma_1$, *and eigenvector* $\alpha = w\sqrt{\delta}$ *as before, then the equation* $x^2 = \sigma_1^2 y^2 + (-1)^{(v-1)/2}\delta z^2$ *has a nontrivial integer solution* (x, y, z).

Proof. Let $\beta = u\sqrt{\varepsilon}$ be as before, then $N\beta = \sigma_0\alpha$. Let M be the rational matrix given by

$$M = \begin{bmatrix} N & w \\ \delta\varepsilon u^T & \sigma_0\sqrt{\delta\varepsilon} \end{bmatrix}.$$

Then $M(I \oplus [-\delta])M^T = \sigma_1^2 I \oplus [-\delta^2\varepsilon\sigma_1^2]$, hence $I \oplus [-\delta]$ is rationally congruent to $\sigma_1^2 I \oplus [-\delta^2\varepsilon\sigma_1^2]$. By using Lagrange's four squares theorem, this implies that for $v \equiv 1 \pmod 4$, we have that $[1] \oplus [-\delta]$ is rationally congruent to $[\sigma_1^2] \oplus [-\delta^2\varepsilon\sigma_1^2]$. This implies that $x^2 = \sigma_1^2 y^2 + \delta z^2$ has a nontrivial integer solution. For $v \equiv 3 \pmod 4$, we have that $I_3 \oplus [-\delta]$ is rationally congruent to $[\sigma_1^2 I_3] \oplus [-\delta^2\varepsilon\sigma_1^2]$, and hence that $[\sigma_1^2] \oplus I_3 \oplus [-\delta] \cong \sigma_1^2 I_4 \oplus [-\delta^2\varepsilon\sigma_1^2] \cong I_4 \oplus [-\delta^2\varepsilon\sigma_1^2]$, which implies that $[\sigma_1^2] \oplus [-\delta]$ is rationally congruent to $[1] \oplus [-\delta^2\varepsilon\sigma_1^2]$. This implies that $x^2 = \sigma_1^2 y^2 - \delta z^2$ has a nontrivial integer solution. $\square$

As an application we mention a parameter set which is ruled out by this rational congruence condition. This parameter set has $v = 31, \sigma_0 = 19, \sigma_1 = \sqrt{6}, \delta = 1$, and it satisfies all other known conditions. A design with these parameters is normal with 10 points with replication 10, 3 points with replication 15, and 18 points with replication 22. Proposition 5 implies that if the points and blocks are partitioned according to replications and block sizes, then the incidence matrix has a corresponding regular quotient matrix $[1\ 0\ 9; 0\ 3\ 12; 5\ 2\ 15]$. The Bruck–Ryser–Chowla condition is however not satisfied, so such a design cannot exist.

2.2. Reducible designs

A design is called reducible if there exist a set of t blocks (called the reducing set of blocks) such that the union of these blocks is a set of t points, called the reducing set of points. In [3], Bridges and Mena classified the reducible multiplicative designs. We specialize to obtain the following on the uniform ones.

Proposition 7. *If a uniform multiplicative design is reducible, then the reducing blocks form a symmetric design on the reducing points, the remaining blocks contain all reducing points, and with these points deleted they form a symmetric design on the remaining points. The parameters* (v_1, k_1, λ_1) *and* (v_2, k_2, λ_2) *of these two symmetric designs are related by the equations* $k_1 - \lambda_1 = k_2 - \lambda_2 = \lambda_1\lambda_2 = \sigma_1^2$.

Proof. Let N be the incidence matrix of a reducible design, say

$$N = \begin{bmatrix} N_1 & M \\ O & N_2 \end{bmatrix},$$

where N_1 has size $t \times t$ (the reducing "design"). It follows by inspection of NN^T and N^TN that both designs N_1 and N_2 are, like N, uniform multiplicative (and thus nonsingular). Moreover, MN_2^T has rank 1, so M must have rank 1. Since each entry of NN^T is positive, M has no zero rows or columns, so $M = J$. It now follows that if i is a reducing point, and j is not a reducing point, then $\alpha_i\alpha_j = r_j$ (notation is as usual). This implies that α is constant over the reducing points, and thus that N_1 is a symmetric design. Similarly (dually) N_2 is also a symmetric design. The parameter restrictions easily follow from working out NN^T. We remark further that these restrictions are also sufficient. $\square$

2.3. The designs with small second singular value

Propositions 4 and 7 are useful in the following classifications of the designs with $\sigma_1 \leqslant \sqrt{2}$.

Proposition 8. *There are two non-symmetric uniform multiplicative designs with singular values* $\sigma_0 > \sigma_1 = 1$. *They are described by the incidence matrices*

$$\begin{bmatrix} J_3 - I_3 & J_3 \\ O_3 & J_3 - I_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & j^T \\ j & I_4 \end{bmatrix}.$$

Proof. Let N be the incidence matrix of such a design, such that $NN^T = I + \alpha\alpha^T$, with $\alpha = w\sqrt{\delta}$, with δ a square-free integer, and w an integer vector. Consider two points p and q with distinct replications $r_p > r_q$. Then $w_p \geqslant w_q + 1$, and hence $r_q \geqslant \lambda_{pq} = \delta w_p w_q \geqslant \delta w_q^2 + \delta w_q \geqslant \delta w_q^2 + 1 = r_q$. Thus we have equality in the entire chain of inequalities, and hence $\delta = 1, w_p = 2, w_q = 1$. The only possible replications are therefore 2 and 5. For the dual the same holds, hence the design is normal.

Since $\lambda_{pq} = 2$, the two blocks containing p contain also q, and moreover, all points with replication 5. It thus follows that N can be rearranged such that

$$N = \begin{bmatrix} N_1 & J \\ O & N_2 \end{bmatrix},$$

where N_1 is on the points with replication 5 and N_2 is on the points with replication 2. Say these designs have v_1 and v_2 points, and b_1 and b_2 blocks, respectively.

If $b_1 = 0$, then $v = b_2 = 5$ and $v_1 = 1$ (since two points with replications 5 meet in 4 blocks). Since the design is normal, it follows that there is also one block containing all points, and we obtain the second design in the proposition.

Finally assume that $b_1 > 0$. It follows from inspecting NN^T that $N_1 N_1^T = I + 4J - b_2 J$ and $N_2 N_2^T = I + J$. By considering ranks we find that $b_1 \geqslant v_1$ and $b_2 \geqslant v_2$ (note that $b_2 < 4$ since $NN^T > 0$). But the total number of blocks $b_1 + b_2$ equals the total number of points $v_1 + v_2$, hence N_1 and N_2 are square, and hence they are symmetric designs by Proposition 7. It also follows that N_1 and N_2 are both 2-(3,2,1) designs. $\square$

Proposition 9. *There are two non-symmetric uniform multiplicative designs with singular values* $\sigma_0 > \sigma_1 = \sqrt{2}$ *(up to duality). They are described by the incidence matrices*

$$\begin{bmatrix} N_1 & J_7 \\ O_7 & N_2 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 1 & j^T & j^T \\ j & I_5 & I_5 \\ j & I_5 & J_5 - I_5 \end{bmatrix},$$

where N_1 *and* N_2 *are the incidence matrices of symmetric* 2-(7,3,1) *and* 2-(7,4,2) *designs, respectively.*

Proof. Let N be the incidence matrix of such a design, such that $NN^T = 2I + \alpha\alpha^T$, with $\alpha = w\sqrt{\delta}$ (with δ a square-free integer, and w an integer vector, as before). A similar argument as in the classification of designs with $\sigma_1 = 1$ shows that $\delta \leqslant 2$, and moreover the replications can be either 4 and 10 ($\delta = 2$) or 3, 6, and 11 ($\delta = 1$).

Let us first consider the case $\delta = 2$. A point with replication 4 ($w_i = 1$) and a point with replication 10 ($w_j = 2$) meet in 4 blocks, hence N can be rearranged such that

$$N = \begin{bmatrix} N_1 & J \\ O & N_2 \end{bmatrix},$$

where N_1 is on the points with replication 10 and N_2 is on the points with replication 4. Say these designs have v_1 and v_2 points, and b_1 and b_2 blocks, respectively.

If $b_1 = 0$, then $v = 10$ and $v_1 = 1$ (since two points with replications 10 meet in 8 blocks). From the trace of NN^T we find that $\sigma_0^2 = -\sigma_1^2(v-1) + 10 + 4(v-1) = 28$, which contradicts Proposition 2.

Hence we may assume that $b_1 > 0$. As before, we find that $N_1 N_1^T = 2I + 8J - b_2 J$ and $N_2 N_2^T = 2I + 2J$, from which it follows that $b_1 \geqslant v_1$ and $b_2 \geqslant v_2$ (note that $NN^T > 0$, hence $b_2 \leqslant 7$). But the total numbers of points and blocks are equal, so N_1 and N_2 are square, and hence they are symmetric designs. From the parameters it follows now that N_1 is a 2-(7,3,1) design, and N_2 is a 2-(7,4,2) design.

Secondly, consider the case $\delta = 1$. Without loss of generality, we may also assume that the dual design has $\delta = 1$. Proposition 4 implies that the design or its dual must then have a point with replication 11. We assume the design itself has one. From the parameters it follows that also here we can write the incidence matrix as

$$N = \begin{bmatrix} N_1 & J \\ O & N_2 \end{bmatrix},$$

where N_1 is on the points with replications 11 and N_2 on the points with replications 3 or 6.

Like before it follows now that if $b_1 > 0$, then N_1 and N_2 are square, hence N_1 and N_2 are symmetric designs by Proposition 7. It follows that N_2 is a 2-(7,3,1) design (a 2-$(v_2, 6, 4)$ design does not exist), and N_1 is a 2-(7,4,2) design, which is the dual of the example found above (so dually $\delta = 2$ after all).

If $b_1 = 0$ however, then $v = 11$, and $v_1 = 1$. If v_3 is the number of points with replication 3, then $\sigma_0^2 = -\sigma_1^2(v-1) + 11 + 3v_3 + 6(v-1-v_3) = 51 - 3v_3$. By Proposition 2 this number must be square, which implies that $v_3 = 5$. If b_3, b_6, b_{11} are the numbers of blocks of sizes 3, 6, and 11, respectively, then it follows (from the trace of $N^T N$) that $3b_3 + 6b_6 + 11b_{11} = 56$. Since there can be at most one block of size 11, this implies that $b_{11} = 1$, and $b_3 = b_6 = 5$. From these and the other parameters it now follows easily that this gives the second design in the proposition. □

We remark that the first example in Proposition 9 is interesting in view of Proposition 4, since it has only two replications (4 and 10) and two block sizes (3 and 11), but σ_1 is not an integer.

2.4. Enumeration of small designs

In this section we will enumerate all nonsymmetric uniform multiplicative designs on at most 30 points (all symmetric designs on at most 30 points have already been enumerated). We found already 5 designs in the above having $\sigma_1 < \sqrt{3}$. To determine the other ones, we may assume $v \geqslant 3$ and $\sigma_1 \geqslant \sqrt{3}$.

Since $v \leqslant 30$, the integer eigenvector w has entries at most 5. We will show first that $w_i = 5$ cannot occur however (we remark that $w_i = 5$ implies that $r_i = \sigma_1^2 + \delta w_i^2 \geqslant 28$).

The case $w_i = 5, \sigma_1 = \sqrt{3}, \delta = 1$ is easily excluded by similar arguments as used in Propositions 8 and 9.

A design with $v = 29, w_i = 5, \sigma_1 = 2, \delta = 1$ has one point p with replication 29 and the other points can only have replications 5 and 20 (otherwise the number of blocks where a point and p meet is too large). If v_1 and v_2 are the numbers of points with replications 5 and 20, respectively, then $v_1 + v_2 = 28$ and $5v_1 + 20v_2 + 29 = \sigma_0^2 + 28\sigma_1^2$. This implies that $\sigma_0^2 = 57 + 15v_2$ which is however never a square (for the relevant v_2), a contradiction. A design with $v = 30, w_i = 5, \sigma_1 = 2, \delta = 1$ also has one point with replication 29. From the intersection pattern it follows however that the block not containing this point must be empty, a contradiction. A design with $v = 30, w_i = 5, \sigma_1 = \sqrt{5}, \delta = 1$ has one point with replication 30, but also here the intersection numbers and the replications do not match.

Hence we may assume that $w_i \leqslant 4$, and consequently at most four distinct replications and four distinct block sizes can occur.

By computer we generated all parameter sets $(v, \sigma_0, \sigma_1, v_1, \ldots, v_4, r_1, \ldots, r_4, b_1, \ldots, b_4, k_1, \ldots, k_4)$ for designs on $v \leqslant 30$ points with singular values $\sigma_0 > \sigma_1 \geqslant \sqrt{3}$, satisfying Propositions 2 and 3, with v_i points with replication r_i and b_i blocks of size k_i, satisfying the equations $\sum_i v_i = \sum_i b_i = v$, $\sum_i v_i r_i = \sum_i b_i k_i = \sigma_0^2 + (v-1)\sigma_1^2$, and $\sum_i b_i k_i^2 = \sigma_1^2 v + (\sum_i v_i \alpha_i)^2$ (and the dual equation). The last equation follows from summing all entries in the equation $NN^T = \sigma_1^2 I + \alpha\alpha^T$, i.e., by working out the equation $\mathbf{j}^T NN^T \mathbf{j} = \mathbf{j}^T(\sigma_1^2 I + \alpha\alpha^T)\mathbf{j}$. We also checked that the parameters are such that for any two points p, q we have that $\lambda_{pq} \leqslant r_p$ (and the same for the dual). We obtained 26 parameter sets, as displayed in Table 1 (together with the five parameter sets with $\sigma_1 < \sqrt{3}$). The column "#" gives the number of designs for each parameter set. Comments on these parameter sets now follow:

- $v = 17$: This must be a symmetric 2-$(16, 6, 2)$ design extended by a point and block in the obvious way. Since there are 3 such symmetric designs, there are 3 nonsingular designs with two singular values on 17 points.
- $v = 18$: The two possible parameter sets are related. Both are normal with two block sizes. If the incidence matrix of the first one is partitioned (regularly) according to block sizes and replications as

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix},$$

which (by using Proposition 4) has quotient matrix

$$\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix},$$

then the design with incidence matrix

$$\begin{bmatrix} J - N_{21} & J - N_{22} \\ N_{11} & N_{12} \end{bmatrix}$$

Table 1
Nonsymmetric uniform multiplicative designs with $v \leqslant 30$

v	σ_0, σ_1	$(v_1, \dots, v_4)$	$(r_1, \dots, r_4)$	$(b_1, \dots, b_4)$	$(k_1, \dots, k_4)$	#	Remarks
2	$\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$	(1,1)	(1,2)	(1,1)	(1,2)	1	
5	3,1	(4,1)	(2,5)	(4,1)	(2,5)	1	Proposition 8
6	4,1	(3,3)	(2,5)	(3,3)	(2,5)	1	Proposition 8
11	$6, \sqrt{2}$	(5,5,1)	(3,6,11)	(5,5,1)	(3,6,11)	1	Proposition 9
14	$\sqrt{72}, \sqrt{2}$	(7,7)	(3,11)	(7,7)	(4,10)	1	Proposition 9
17	8,2	(16,1)	(7,16)	(16,1)	(7,16)	3	From (16,6,2)
18	11,2	(9,9)	(8,13)	(9,9)	(8,13)	3	Computer
18	7,2	(9,9)	(5,8)	(9,9)	(5,8)	3	Computer
20	6,2	(16,4)	(5,8)	(16,4)	(5,8)	1	From PG(2,4)
20	10,2	(16,4)	(7,16)	(16,4)	(7,16)	1	From PG(2,4)
21	11,2	(15,6)	(7,16)	(15,6)	(7,16)	1	From hyperoval in PG(2,4)
22	10,2	(8,9,4,1)	(5,8,13,20)	(8,9,4,1)	(5,8,13,20)	0	
22	10,2	(15,6,1)	(6,12,22)	(15,6,1)	(6,12,22)	1	From hyperoval in PG(2,4)
22	8,2	(21,1)	(6,22)	(21,1)	(6,22)	1	From PG(2,4)
22	14,2	(8,14)	(7,16)	(8,14)	(7,16)	4	Computer; from (8,4,3)
22	13,2	(11,11)	(7,16)	(11,11)	(7,16)	1	From (11,6,3)
23	13,2	(7,14,2)	(5,13,20)	(7,14,2)	(5,13,20)	0	$v_4 = 2$
24	$\sqrt{192}, \sqrt{3}$	(13,11)	(4,19)	(11,13)	(6,15)	1	$(11, 6, 3) + (13, 4, 1)$
24	$\sqrt{147}, \sqrt{3}$	(16,8)	(4,19)	(16,8)	(6,15)	0	
25	$12, \sqrt{3}$	(14,7,4)	(4,12,19)	(14,7,4)	(4,12,19)	0	
25	$13, \sqrt{5}$	(6,4,14,1)	(6,9,14,21)	(6,4,14,1)	(6,9,14,21)	0	
25	$12, \sqrt{5}$	(6,9,9,1)	(6,9,14,21)	(6,9,9,1)	(6,9,14,21)	5	Computer
25	$10, \sqrt{5}$	(10,10,5)	(6,9,14)	(10,10,5)	(6,9,14)	5	Computer
27	12,2	(12,7,4,4)	(5,8,13,20)	(12,7,4,4)	(5,8,13,20)	0	
29	$12, \sqrt{5}$	(11,11,4,3)	(6,9,14,21)	(11,11,4,3)	(6,9,14,21)	0	
29	$9, \sqrt{5}$	(20,5,4)	(6,9,14)	(20,5,4)	(6,9,14)	1	From PG(2,5)
29	$13, \sqrt{6}$	(3,19,4,3)	(7,10,15,22)	(3,19,4,3)	(7,10,15,22)	0	
29	$15, \sqrt{7}$	(1,7,21)	(8,11,16)	(1,7,21)	(8,11,16)	137,541	Computer
30	14,2	(12,9,9)	(5,8,20)	(12,9,9)	(5,8,20)	0	
30	20,2	(9,21)	(6,22)	(9,21)	(6,22)	0	
30	12,2	(25,5)	(6,22)	(25,5)	(6,22)	0	

has the other parameter set with 18 points, and the other way around (see also [3]). By computer we enumerated all (three) designs with these parameter sets. All these designs are self-dual. One design (for each parameter set) was already known by Bridges and Mena [3].

- $v = 20$: A design with the first parameter set can be constructed from the unique symmetric 2-(21,5,1) design of PG(2,4) by deleting an incident point-block pair (p, B), and adding all remaining four points of B to all blocks incident with p (cf. [3]). The obtained design has a regular partition as desired with quotient matrix $[4\ 1; 4\ 4]$. Moreover, it follows that each design with this parameter set must be constructed in this way, and hence is unique.

Similarly the design with the second parameter set is obtained from the complementary 2-(21,16,12) design of PG(2,4).

- $v = 21$: A design with this parameter set can be regularly partitioned with quotient matrix $[3\ 4; 10\ 6]$. It is straightforward to check that such a design must be obtained from the unique hyperoval in PG(2,4) by complementing all incidences except between the points not in the hyperoval and the blocks not in the dual hyperoval (cf. [3]; a hyperoval consist of 6 points, no three on a line; there is a dual hyperoval consisting of the 6 blocks not intersecting the hyperoval), and hence is unique.
- $v = 22$: Similarly, the second parameter set with $v = 22$ is realized uniquely by considering a hyperoval in PG(2,4), by complementing the incidences between the hyperoval and the dual hyperoval, and by extending the obtained design by a point and block in the obvious way. We remark that the design is normal, and by Proposition 5 the incidence matrix can be partitioned regularly with quotient matrix $[3\ 2\ 1; 5\ 6\ 1; 15\ 6\ 1]$.

  The first parameter set with $v = 22$ is excluded by the following argument. If such a design would exist, then the unique point p with replication 20 and a point with replication 5, 8, or 13 meet in 4, 8, or 12 blocks, respectively. This implies that the points with replications 5, 8, or 13 are contained in 1, 0, or 1 of the two blocks not containing p. Thus the sum of the block sizes of these two blocks is $v_1 + v_3 = 12$, which gives a contradiction.

  The third parameter set with $v = 22$ is realized uniquely by extending PG(2,4) by a point and block in the obvious way.

  A design with the fourth parameter set with $v = 22$ can be regularly partitioned with quotient matrix $[0\ 7; 4\ 12]$. It follows that the incidences between the 8 points with replications 7 and the 14 blocks of sizes 16 form a 2-(8,4,3) design. Dually, the same holds. Previously, one example was known, where the 2 designs are the unique resolvable 2-(8,4,3) design. The incidence matrix of this example can even be rearranged such that it is symmetric with zero diagonal, and hence can be seen as the adjacency matrix of a graph. This graph has three distinct eigenvalues (14, 2, and -2), cf. [3,8]. By computer we determined that there are 3 more designs. Also in these designs the corresponding 2-(8,4,3) designs are resolvable, and all designs are self-dual.

  A design with the final parameter set with $v = 22$ can be regularly partitioned with quotient matrix $[1\ 6; 6\ 10]$. It follows that the incidence matrix can be rearranged as
  $$\begin{bmatrix} I & N_{12} \\ N_{21} & J - I \end{bmatrix}.$$
  It then follows that N_{12} and N_{21} are symmetric 2-(11,6,3) designs, and that $N_{21} = N_{12}^T$. Hence there is a unique design with this parameter set.
- $v = 23$: This parameter set cannot be realized since there should be two points with replication 20, and these should meet in 16 blocks, a contradiction.
- $v = 24$: There is a unique design with the first parameter set. From the parameters it follows that it is reducible (and not normal), and consequently that it

must be obtained from the unique 2-(13,4,1) and 2-(11,6,3) designs, see Proposition 7.

Also a design with the other parameter set would be reducible, but the required symmetric designs (on 16 and 8 points) do not exist, since the parameters are not right. Hence such a design on 24 points does not exist.

- $v = 25$: A design with the first parameter set does not exist since each point with replication 19 must be in 5 blocks of size 19, according to Proposition 5, while there are only 4 such blocks.

  A design with the second parameter set does not exist either. Each point with replication 14 in such a design would be contained in the block of size 21, since there is a unique (nonnegative integral) solution to system (1) in Proposition 5 with variables r_{3j}, satisfying $r_{34} \leqslant 1$. This solution has $r_{34} = 1$. But then $r_{43} = 14$, which gives a contradiction with the system with variables r_{4j}.

  For a design with the third parameter set the system of equations (1) for r_{4j} has one (nonnegative integral) solution with $r_{4j} \leqslant b_j$. This solution is given by $r_{41} = 3, r_{42} = 9, r_{43} = 9, r_{44} = 0$. This implies that $r_{24} = r_{34} = 1$. Now the systems of equations for the r_{2j} and r_{3j} have unique solutions $r_{21} = 1, r_{22} = 2, r_{23} = 5$ and $r_{31} = 1, r_{32} = 5, r_{33} = 7$. Since $r_{41} = 3$, r_{14} equals 1 for three points, and 0 for the remaining three points with replication 6. If $r_{14} = 1$, then $r_{11} = 2, r_{12} = 3, r_{13} = 0$; if $r_{14} = 0$, then $r_{11} = 3, r_{12} = 0, r_{13} = 3$. Dually the same holds. If we partition the incidence matrix according to replications and block sizes, and further partition the points with replication 6 into the ones occurring in the block of size 21 (type A) and the others, and the blocks similarly (dually), then it follows by counting the blocks containing a given pair of points with replications 6 and 21, that each point with replication 6 occurs in 1 block of type A. Consequently, the (finer) partition is regular with quotient matrix

$$\begin{bmatrix} 1 & 1 & 3 & 0 & 1 \\ 1 & 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 5 & 1 \\ 0 & 1 & 5 & 7 & 1 \\ 3 & 0 & 9 & 9 & 0 \end{bmatrix}.$$

  By computer we determined that there are 5 such designs, one of which is given in the appendix. We remark that these designs are all self-dual, and they are the first known uniform multiplicative designs with four distinct block sizes!

  A design with the last parameter with $v = 25$ is normal with three distinct replications. The quotient matrix obtained from Proposition 5 is $[3\ 2\ 1; 2\ 3\ 4; 2\ 8\ 4]$. This determines already part of the structure of the design. By computer we enumerated all (five) such designs. All these designs are self-dual. One of them is given in the appendix.
- $v = 27$: A design with this parameter set does not exist. The system of equations (1) on r_{4j} does not have a (nonnegative integral) solution with $r_{43} \leqslant 4$ and $r_{44} \leqslant 4$.

- $v = 29$: A design with the first parameter set does not exist. The system of equations (1) on r_{4j} has a unique (nonnegative integral) solution with $r_{43} \leqslant 4$ and $r_{44} \leqslant 3$. This solution is given by $r_{41} = 4, r_{42} = 10, r_{43} = 4, r_{44} = 3$. This implies that the pairs of points with replication 21 occur in all 7 blocks of sizes 14 and 21, and in 9 blocks of size 9. Hence such a pair cannot occur in a block of size 6. Since $r_{41} = 4$, $b_1 = 11$, and $v_4 = 3$, this gives a contradiction.

  According to Proposition 5, a design with the second parameter set with $v = 29$ can be regularly partitioned with quotient matrix $[4\ 1\ 1; 4\ 1\ 4; 5\ 5\ 4]$. It is straightforward to show that such a design must be constructed in the following way, and hence is unique. Consider the unique 2-(31,6,1) design of PG(2,5). Fix an incident point-block pair (p, B), and another point p' on B, and another block B' through p. Remove p, p', B, B', include the remaining four points p'' of B in all blocks through p or p', and include the remaining five points p''' through B' in all blocks through p.

  The third parameter set with $v = 29$ cannot be realized. The system of equations (1) on r_{1j} has a unique (nonnegative integral) solution $r_{11} = 3, r_{12} = 2, r_{13} = 2, r_{14} = 0$. But $b_1 = 3$ then implies that a pair of points with replications 7 occurs in at least 3 blocks, a contradiction.

  According to Proposition 5, a design with the last parameter set with $v = 29$ can be regularly partitioned with quotient matrix $[1\ 7\ 0; 1\ 1\ 9; 0\ 3\ 13]$. This implies among others that the incidences between the points with replication 11 and the blocks of size 16 form a 2-(7,3,3) design, and the same holds for the dual design. By computer we enumerated all possible designs, and we found 137,541 pairwise non-isomorphic designs. One of these is given in the appendix. Up to duality there are 69,460 designs.
- $v = 30$: There exist no designs with these parameter sets. According to Proposition 5, a design with the first parameter set can be regularly partitioned with quotient matrix $[2\ 0\ 3; 0\ 2\ 6; 4\ 6\ 10]$. But there are only 9 blocks of size 20, a contradiction. Similarly the other two parameter sets are excluded after using Proposition 4 (the first of these has $r_{11} = -1$; the second has $r_{33} = 7 > b_3$).

2.5. *Final remarks*

Not many infinite families of uniform multiplicative designs are known. Ryser [13] already mentioned a family of reducible examples (which can easily be rediscovered using Proposition 7), and a family of "borderings" of symmetric designs. These are symmetric designs on v points extended by a point and block of size v or $v + 1$. We saw examples of these with 17 points and 22 points.

Besides these, Bridges and Mena [3] mentioned sporadic examples on 39 points, constructed from a 2-(40,13,4) design in the same way as the examples on 20 points constructed from PG(2,4), examples of "borderings" on 46 and 97 points with three distinct replications, an example on 45 points constructed from a 2-(45,12,3) design with a 9×9 empty sub-design, and an example on 52 points with a cyclic structure.

All these examples are normal and have two distinct replications, unless mentioned otherwise.

In this paper we found examples for 5 new parameter sets: on 22 points we constructed one from the hyperoval in PG(2,4); on 29 points we constructed one from PG(2,5); and for three parameter sets (two with $v = 25$, one with $v = 29$) we constructed examples by computer. The 5 designs for one of the parameter sets with $v = 25$ have 4 distinct replications and 4 distinct block sizes. Such designs were not known before.

We found no counterexamples to the conjecture (cf. [2]) that a uniform multiplicative design is normal or reducible. It is however interesting to note that one candidate parameter set for such a counterexample suggested that a (nonexisting) projective plane of order 6 with a hyperoval would give a counterexample by adding all (8) points of the hyperoval to all (15) lines not intersecting the hyperoval. We challenge the interested reader to come up with the first "real" counterexample. Another candidate parameter set for such a counterexample has $v = 47, \sigma_0 = 22, \sigma_1 = 3$, with 12, 28, and 7 points with replications 13, 18, and 34, respectively, and 28, 7, and 12 blocks of sizes 13, 18, and 34, respectively. According to Proposition 5, a design with this parameter set can be regularly partitioned with quotient matrix $[7\ 0\ 6; 6\ 3\ 9; 16\ 6\ 12]$. The dual quotient matrix for this partition is $[3\ 6\ 4; 0\ 12\ 6; 6\ 21\ 7]$. It follows that the incidence matrix N can be written as

$$N = \begin{bmatrix} N_{22} & O & N_{25} \\ N_{32} & N_{33} & N_{35} \\ N_{52} & J - I & J \end{bmatrix},$$

where N_{52} is the incidence matrix of a 2-(7,4,8) design, and N_{33}^T is the incidence matrix of a 2-(7,3,4) design.

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Appendix

In this appendix we give three designs found by computer with parameter sets for which no designs were previously known. The left one has $v = 25, \sigma_0 = 12, \sigma_1 = \sqrt{5}$ (with 4 distinct replications), the middle one has $v = 25, \sigma_0 = 10, \sigma_1 = \sqrt{5}$, and the right one has $v = 29, \sigma_0 = 15, \sigma_1 = \sqrt{7}$ (with a symmetric incidence matrix, and the corresponding 2-(7,3,3) designs consisting of three copies of the Fano plane).

$$
\begin{bmatrix}
1111111110000000000000000\\
1111000001111100000000000\\
1111000000000011111000000\\
1100110001100011000100000\\
1100110000011000110010000\\
1010101001010010100001000\\
1010010101010001001010000\\
1001100100010101100100000\\
1001010011001001100001000\\
1000001000001000001100100\\
1000000100100000010001010\\
1000000010000110000010001\\
0100000101000000100000101\\
0010100000000100100000011\\
0001010000001001000000110\\
1111110001000100101111010\\
1111100011010001010111100\\
1111011000110001100111001\\
1110110010110101101001100\\
1110011011010101110100010\\
1101111001010101011001001\\
1101101011110001101010010\\
1011111001100101110010100\\
1011110011110000111100001\\
0111111111111111111111000
\end{bmatrix}
\begin{bmatrix}
1110000000110000000010000\\
1001100000001100000001000\\
1000011000000011000000100\\
0101010000000000110000010\\
0100100100000010001000001\\
0011000010000000001100100\\
0010001001001000100000001\\
0000100011000001010010000\\
0000010101100000000101000\\
0000001110010100000000010\\
1000000010000010100111011\\
1000000001010000011001111\\
0100001000000100010111101\\
0100000010101001000001111\\
0010000100001010010011110\\
0010010000000101001011011\\
0001000001100110000010111\\
0000101000100000101011110\\
0001000100010001100011101\\
0000110000011000000110111\\
1000000100101101111110111\\
0100000001011111101111110\\
0010100000110111110101111\\
0001001000111011011111011\\
0000010010111110111011101
\end{bmatrix}
\begin{bmatrix}
11111111000000000000000000000\\
11000000111111111000000000000\\
10100000111000000111111000000\\
10010000111000000000000111111\\
10001000000111000111000111000\\
10000100000111000000111000111\\
10000010000000111111000000111\\
10000001000000111000111111000\\
01110000100110110110110110110\\
01110000001101101101101101101\\
01110000010011011011011011011\\
01001100110100110101011101011\\
01001100101001101011110011110\\
01001100011010011110101110101\\
01000011110110100011101011101\\
01000011101101001110011110011\\
01000011011011010101110101110\\
00101010110101011100110011101\\
00101010101011110001101110011\\
00101010011110101010011101110\\
00100101110011101110100101011\\
00100101101110011101001011110\\
00100101011101110011010110101\\
00011001110101011011101100110\\
00011001101011110110011001101\\
00011001011110101101110010011\\
00010110110011101101011110100\\
00010110101110011011110101001\\
00010110011101110110101011010
\end{bmatrix}
$$

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