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Strong time-consistency in the cartel-versus-fringe model

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Abstract

Due to developments on the oil market in the 1970s, the theory of exhaustible resources was extended with the cartel-versus-fringe model to characterize markets with one big coherent cartel and a large number of small suppliers called the fringe. Because cartel and fringe are leader and follower, the von Stackelberg solution concept is appropriate for the supply side of this market. The solution for the cartel-versus-fringe model, presented in the previous literature, proved to be time-inconsistent for a large plausible range of values for extraction costs and initial reserves. This paper provides a (strongly) time-consistent solution for the cartel-versus-fringe model.

JEL classification: 720

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1. Introduction

The oil price shocks in the 1970s increased the interest for the theory of the supply of raw materials from natural exhaustible resources which dates back to the seminal work of Hotelling (1931). Salant (1976) suggested to characterize the supply side of the oil market by one big coherent cartel and a large number of small suppliers, called the fringe. This market structure was analyzed in a number of papers, e.g. Pindyck (1978), Salant (1982), Lewis and Schmalensee (1980a, 1980b), and Ulph and Folie (1980). All these studies use the Nash equilibrium concept. Gilbert (1978) put forward that, for this market structure, the von Stackelberg solution concept might be more

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appropriate. The cartel is the leader and determines the extraction path first, whereas the suppliers in the fringe are followers and just react to that. However, the von Stackelberg solution that was derived for this model had an undesirable property. For plausible values of the extraction costs and initial reserves, the strategy of the cartel proved to be time-inconsistent: the ex ante optimal extraction path ceases to be optimal ex post in case the strategy is reconsidered at a future point in time. This means that, in the absence of commitments, the cartel has an incentive to renege on the announced extraction path (Ulph and Folie, 1981; Newbery, 1981; Ulph, 1982; Groot et al., 1992). This problem attracted quite some attention (Karp and Newbery, 1991, 1992, 1993) but up to now the literature on exhaustible resources does not provide a time-consistent von Stackelberg solution for the cartel-versus-fringe model.

It is important to note that in the models employed so far it is assumed that the suppliers choose extraction paths as functions of the initial stocks of the resource and time. In the theory of differential games the resulting solution is called the *open-loop* solution. In order to check time-consistency, strategies are reconsidered at some point in time, so that it is implicitly assumed that the extraction paths are functions of the resource stocks at that point in time. This is somewhat confusing but might be clarified by using a more precise notion of time-consistency introduced by Başar (1989). An open-loop solution is called weakly time-consistent if the solution does not change when reconsidered at any point in time, starting from resource stocks that result from applying the solution up to that point in time. The conclusion of the previous literature on the cartel-versus-fringe model is that the open-loop von Stackelberg solution is weakly time-consistent for some values of the extraction costs and initial reserves but time-inconsistent for the other values. Time-inconsistency is a property of the solution and can only be solved by putting constraints on the solution (see e.g. Fischer, 1980) or by using another solution concept. The most common approach is to derive the solution with dynamic programming, because then time-consistency is achieved by construction, but this implies that extractions are assumed to be functions of the current resource stocks. The information structure is changed and in the terminology of differential games the *feedback* von Stackelberg solution results. The combination of requiring time-consistency and conditioning on the current state is also referred to as Markov perfectness. Note that in this case time-consistency holds not only when starting from resource stocks on the solution trajectory but also when starting from resource stocks off the solution trajectory. Therefore Başar (1989) refers to *strong* time-consistency. Note also that as a consequence feedback solutions are robust against unexpected shocks to the resource stocks such as unexpected new oil discoveries: the solutions in terms of functions are the same but the resulting extractions change with the changes in the stocks, of course.

In this paper the feedback von Stackelberg solution is derived for the cartel-versus-fringe model. This means that the set of Hamilton–Jacobi–Bellman equations for dynamic programming has to be solved sequentially, starting with the fringe (see e.g. Başar and Olsder, 1982; Dockner et al., 2000). It can be shown that for a large number of small identical fringe members, the reaction of the total fringe can be modelled as one producer who supplies the rest of the market in case the resulting price equals a Hotelling competitive price. It follows that the reaction of the total fringe is similar to the one in the open-loop von Stackelberg solution. The hard part, however, is to solve the Hamilton–Jacobi–Bellman (HJB) equation for the cartel, given the reaction of the fringe. In a linear-quadratic framework, the usual procedure is to postulate a quadratic value function, but in this case one has no clue what the value function will look like. Therefore, we start from a class of plausible extraction schedules and check the optimality conditions and the dynamics of the HJB equation. In this way the optimal extraction schedule is found and also the value function (which proves to be very complicated, as is to be expected).

More specifically, it is first shown that for values of the extraction costs and initial reserves, for which the open-loop von Stackelberg solution is weakly time-consistent, the solution is also strongly time-consistent. The main contribution of the paper, however, is to present strongly time-consistent extraction schedules for values of the extraction costs and initial reserves, for which the open-loop von Stackelberg solution is time-inconsistent. It will be interesting to note that a conjecture by Karp and Newbery (1993), that the open-loop Nash equilibrium may be an answer in certain cases, is confirmed. It will also be interesting to note that the strongly time-consistent extraction schedules for the cartel-versus-fringe model are not more complicated but, on the contrary, more transparent and more intuitive than the open-loop von Stackelberg extraction schedules.

Section 2 presents the cartel-versus-fringe model with linear demand and constant marginal extraction costs as was introduced by Ulph and Folie (1981) and Newbery (1981).[1] In Section 3 the approach is described and Section 4 presents the strongly time-consistent solution. Section 5 is a short conclusion.

2. The cartel-versus-fringe model

The supply side of some markets for exhaustible natural resources, such as the oil market, can be characterized by one big coherent cartel and a large number of small suppliers called the fringe. The cartel has initial resource stock S_0^{c} and extracts $E^{\mathrm{c}}(t)$ at time t with constant marginal extraction costs k^{c}. Similarly, each fringe member has the same initial resource stock S_{0i}^{f} and extracts $E_i^{\mathrm{f}}(t)$ at time t with constant and equal marginal extraction costs k^{f}, $i=1,\dots,N$, where N denotes the number of fringe members. The demand function is assumed to be linear. A so-called "choke" price $\bar{p}$ indicates that above this price the demand for the resource will be zero (for example because a backstop technology becomes available). It follows that in market equilibrium the price is a linear function of total supply:

$$p(t) = \bar{p} - E^{\mathrm{c}}(t) - E^{\mathrm{f}}(t), \quad E^{\mathrm{f}}(t) = \sum_{i=1}^{N} E_i^{\mathrm{f}}(t). \tag{1}$$

The marginal extraction costs k^{c} and k^{f} are assumed to be smaller than the choke price $\bar{p}$ in order to make sure that both the cartel and the fringe want to exploit the resource.

[1] Karp and Tahvonen (1995) solve the time-consistency issue but change the model. They consider the case of extraction costs that linearly depend on remaining stocks.

The objectives of the cartel and each fringe member i are to maximize discounted profits, given by

$$\int_0^\infty \mathrm{e}^{-rt}[\bar{p} - E^{\mathrm{c}}(t) - E^{\mathrm{f}}(t) - k^{\mathrm{c}}]E^{\mathrm{c}}(t)\,\mathrm{d}t, \tag{2}$$

$$\int_0^\infty \mathrm{e}^{-rt}[\bar{p} - E^{\mathrm{c}}(t) - E^{\mathrm{f}}(t) - k^{\mathrm{f}}]E_i^{\mathrm{f}}(t)\,\mathrm{d}t, \quad i = 1,\ldots,N, \tag{3}$$

respectively, where r is the constant discount rate, subject to the dynamics of the extraction of the resource stocks, given by

$$\dot{S}^{\mathrm{c}}(t) = -E^{\mathrm{c}}(t), \quad S^{\mathrm{c}}(0) = S_0^{\mathrm{c}}, \tag{4}$$

$$\dot{S}_i^{\mathrm{f}}(t) = -E_i^{\mathrm{f}}(t), \quad S_i^{\mathrm{f}}(0) = S_{0i}^{\mathrm{f}}, \quad i = 1,\ldots,N, \tag{5}$$

where the extraction rates and the stocks (S^{c} and S_i^{f}) have to be non-negative at each point in time.

The cartel is a von Stackelberg leader and moves first but in choosing the supply, the cartel has to take the reaction of the fringe into account. When this problem is solved by sequentially applying Pontryagin's maximum principle (first to the problem of the fringe in order to get the optimal reaction and then to the problem of the cartel), the open-loop von Stackelberg solution results. A summary can be found in Section 3. However, this solution has the undesirable property that the optimal extraction path of the cartel is time-inconsistent for plausible values of the extraction costs and initial reserves (Ulph and Folie, 1981; Newbery, 1981; Ulph, 1982; Groot et al., 1992). Therefore, the next sections consider the feedback von Stackelberg solution.

3. The approach

By construction, time-inconsistency is removed if one requires that the extractions are optimal at each point in time for each value of the resource stocks at that point in time. Başar (1989) calls this strong time-consistency in order to distinguish it from weak time-consistency, that holds only on the solution path. To put it differently, the feedback (or Markov perfect) von Stackelberg solution has to be derived. As in all dynamic programming problems, the solution is characterized by a set of HJB equations (see e.g. Başar and Olsder, 1982; Dockner et al., 2000).

The leader–follower structure implies that at each point in time for each value of the resource stocks, the cartel determines the extraction first. The fringe members act simultaneously given the extraction of the cartel. The solution is found by first determining the reaction of the fringe members to the action of the cartel. The cartel-versus-fringe model assumes a large number of small fringe members where each individual fringe member has an insignificant influence on the game that is played (see e.g. Jovanovic and Rosenthal, 1988). This can be modelled by starting from N individual fringe members and letting N go to infinity. It will be shown that a reaction of the aggregate fringe results that has to be taken into account in the optimization problem for the cartel. This reaction is similar to what was found in the open-loop von Stackelberg solution.

First, consider the HJB equations for the fringe, given by (omitting the time argument when no confusion can occur)

$$\frac{\partial V_i^{\mathrm{f}}}{\partial t} + \max_{E_i^{\mathrm{f}}} \left[\mathrm{e}^{-rt}(p - k^{\mathrm{f}})E_i^{\mathrm{f}} - \frac{\partial V_i^{\mathrm{f}}}{\partial S^{\mathrm{c}}} E^{\mathrm{c}} - \sum_{j=1}^{N} \frac{\partial V_i^{\mathrm{f}}}{\partial S_j^{\mathrm{f}}} E_j^{\mathrm{f}} \right] = 0, \quad i = 1, \ldots, N, \tag{6}$$

where V_i^{f} denote the value functions and where the price p is given by (1).

The maximization yields $E_i^{\mathrm{f}} = 0$ or the first-order conditions

$$\bar{p} - E^{\mathrm{c}} - E^{\mathrm{f}} - k^{\mathrm{f}} - \mathrm{e}^{rt} \frac{\partial V_i^{\mathrm{f}}}{\partial S_i^{\mathrm{f}}} - E_i^{\mathrm{f}} = 0, \quad i = 1, \ldots, N. \tag{7}$$

The fringe members are identical. Summing up yields $E^{\mathrm{f}} = E_i^{\mathrm{f}} = 0$ or

$$E^{\mathrm{f}} = \frac{N}{N+1} \left(\bar{p} - E^{\mathrm{c}} - k^{\mathrm{f}} - \mathrm{e}^{rt} \frac{\partial V_i^{\mathrm{f}}}{\partial S_i^{\mathrm{f}}} \right) > 0, \tag{8}$$

$$E_i^{\mathrm{f}} = \frac{1}{N+1} \left(\bar{p} - E^{\mathrm{c}} - k^{\mathrm{f}} - \mathrm{e}^{rt} \frac{\partial V_i^{\mathrm{f}}}{\partial S_i^{\mathrm{f}}} \right) > 0, \quad i = 1, \ldots, N. \tag{9}$$

It follows that in the limit for $N \to \infty$

$$E^{\mathrm{f}} \downarrow \left(\bar{p} - E^{\mathrm{c}} - k^{\mathrm{f}} - \mathrm{e}^{rt} \frac{\partial V_i^{\mathrm{f}}}{\partial S_i^{\mathrm{f}}} \right), \quad E_i^{\mathrm{f}} \downarrow 0, \quad i = 1, \ldots, N. \tag{10}$$

With value functions of the form $V_i^{\mathrm{f}} = \lambda^{\mathrm{f}} S_i^{\mathrm{f}}$, $i = 1, \ldots, N$, it is easy to see that in the limit for $N \to \infty$, the HJB equations (6) are satisfied and that

$$E^{\mathrm{f}} = \max\{0, \bar{p} - E^{\mathrm{c}} - k^{\mathrm{f}} - \mathrm{e}^{rt}\lambda^{\mathrm{f}}\}. \tag{11}$$

This result can be interpreted as follows. From Hotelling (1931) we know that in a competitive equilibrium, where all suppliers are price-takers with marginal extraction costs k^{f}, a price results of the type

$$p(t) = P^{\mathrm{f}}(t) := k^{\mathrm{f}} + \mathrm{e}^{rt}\lambda^{\mathrm{f}}. \tag{12}$$

This is referred to as the Hotelling rule or the competitive price path. The reaction of the aggregate fringe, given by Eq. (11), can therefore be characterized as price-taking, where the fringe supplies only at that competitive price and supplies the market in as far the cartel is not supplying it. Note, however, that λ^{f} is a parameter that still has to be determined. In most previous studies it was assumed from the outset that the fringe as a whole can be treated as a price-taker but here it follows naturally from the HJB equations (see for a similar idea Lewis and Schmalensee (1980a) and Groot et al. (1990), for the open-loop case). Since the fringe can be treated as an aggregate, one of the constraints, Eq. (5), can be replaced by

$$\dot{S}^{\mathrm{f}}(t) = -E^{\mathrm{f}}(t), \quad S^{\mathrm{f}}(0) = S_0^{\mathrm{f}} = \sum_{i=1}^{N} S_{0i}^{\mathrm{f}}. \tag{13}$$

The HJB equation for the cartel becomes

$$\frac{\partial V}{\partial t} + \max_{E^{\mathrm{c}}} \left[\mathrm{e}^{-rt}(p - k^{\mathrm{c}})E^{\mathrm{c}} - \frac{\partial V}{\partial S^{\mathrm{c}}} E^{\mathrm{c}} - \frac{\partial V}{\partial S^{\mathrm{f}}} E^{\mathrm{f}} \right] = 0, \tag{14}$$

where V denotes the value function and where the price p is given by (1).

The normal procedure would be to postulate a (parametrized) functional form for the value function and then to solve (14), but in this case one has no clue what the value function will look like. Therefore, another route is followed that will eventually yield the value function.

Following the usual outcomes of these type of models, we will consider extraction schedules with combinations of four possible regimes, denoted by F, C, S and C^{m}. In the first three regimes the competitive price, given by Eq. (12), results. In F the fringe produces alone, in C the cartel produces alone, and in S cartel and fringe produce simultaneously. In the last regime C^{m}, the cartel produces alone as a monopolist. Note that in this regime the resulting price (called the monopoly price) cannot be above the competitive price, because otherwise the cartel cannot keep the fringe from producing.

A closer look at the maximization in the HJB equation (14), subject to the price equation (1) and the reaction of the fringe given by (11), reveals the following.

If $E^{\mathrm{c}} > \bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rt}\lambda^{\mathrm{f}}$, the fringe does not produce and the objective in the HJB equation is concave with the maximum at $E^{\mathrm{c}} = \frac{1}{2}(\bar{p} - k^{\mathrm{c}} - \mathrm{e}^{rt}\frac{\partial V}{\partial S^{\mathrm{c}}})$. The price that results is given by

$$p(t) = \frac{1}{2}\left(\bar{p} + k^{\mathrm{c}} + \mathrm{e}^{rt}\frac{\partial V}{\partial S^{\mathrm{c}}}\right) \tag{15}$$

and this will be called the monopoly price, that will be denoted henceforth by P^{m}.

If $E^{\mathrm{c}} \leqslant \bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rt}\lambda^{\mathrm{f}}$, it follows from Eq. (11) that cartel and fringe supply the market at the competitive price, given by (12), and the objective in the HJB equation becomes linear with slope $k^{\mathrm{f}} + \mathrm{e}^{rt}\lambda^{\mathrm{f}} - k^{\mathrm{c}} - \mathrm{e}^{rt}\frac{\partial V}{\partial S^{\mathrm{c}}} + \mathrm{e}^{rt}\frac{\partial V}{\partial S^{\mathrm{f}}}$.

In order to get the regime C, it must be optimal for the cartel to produce alone at the competitive price, which implies that the slope of the linear part of the HJB equation must be positive and the maximum location of the concave part cannot be to the right of $\bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rt}\lambda^{\mathrm{f}}$.

In order to get the regime C^{m}, it must be optimal for the cartel to produce alone at the monopoly price, which is guaranteed if the slope of the linear part of the HJB equation is non-negative and if the maximum location of the concave part is to the right of $\bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rt}\lambda^{\mathrm{f}}$.

In order to get the regime F, it must be optimal for the cartel to have the fringe produce alone at the competitive price, which is guaranteed if the slope of the linear part of the HJB equation is negative and if the maximum location of the concave part is not to the right of $\bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rt}\lambda^{\mathrm{f}}$.

Finally, in order to get the regime S, it must be optimal for the cartel to have simultaneous production at the competitive price, which implies that the slope of the linear part of the HJB equation must be zero and the maximum location of the concave part again cannot be to the right of $\bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rt}\lambda^{\mathrm{f}}$.

The idea is to start from plausible extraction schedules and to verify that these optimality conditions are satisfied and that the resulting HJB differential equation holds. In the next section the feedback von Stackelberg solution is derived in this way.

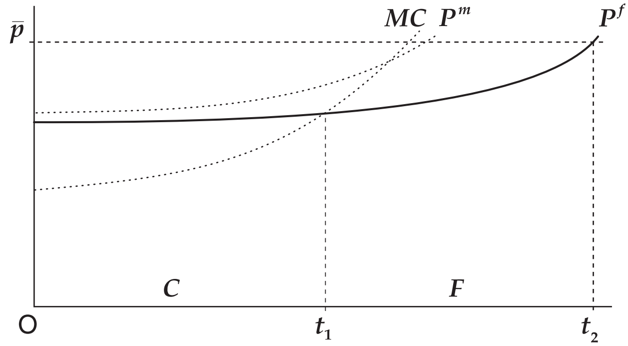

Fig. 1.

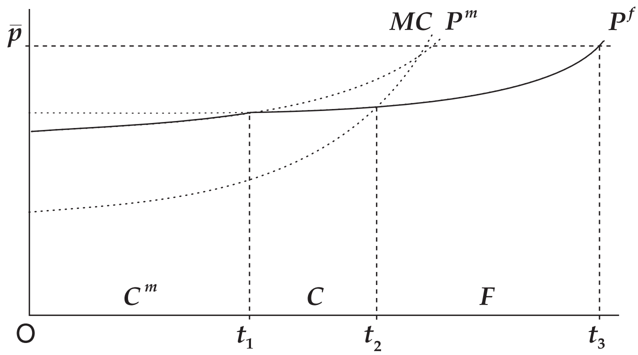

Fig. 2.

4. The feedback von Stackelberg solution

In characterizing the solutions of the cartel-versus-fringe model it is convenient to fix k^{c}, the marginal extraction cost of the cartel, and to vary k^{f}, the marginal extraction cost of the fringe. First we recollect the open-loop von Stackelberg solution. In the figures above and below, which are derived in detail by Groot et al. (1990), P^{f} denotes the competitive price, P^{m} denotes the monopoly price, and MC denotes the full marginal cost of the cartel when producing at the competitive price. The full marginal cost consists of the marginal extraction cost (k^{c}) plus the shadow price of one unit of its own stock plus the shadow price of leaving one unit in the stock of the fringe (denoted later by μ^{c}).

If $k^{\mathrm{f}} > \frac{1}{2}(\bar{p} + k^{\mathrm{c}})$ and the initial resource stock of the cartel is small relative to the initial resource stock of the fringe, the result is the extraction schedule $C \rightarrow F$ in Fig. 1. If, on the other hand, the initial resource stock of the cartel is relatively large, the result is the extraction schedule $C^{\mathrm{m}} \rightarrow C \rightarrow F$ in Fig. 2. Both outcomes are weakly time-consistent.

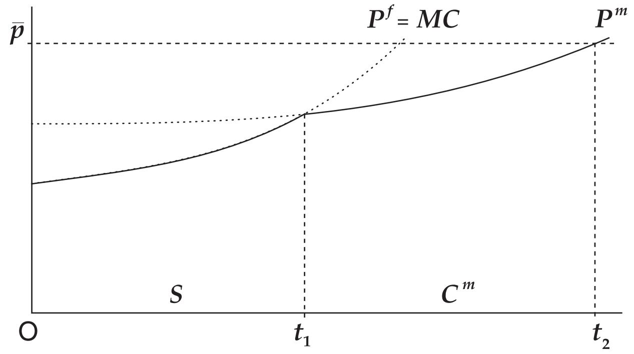

Fig. 3.

If $k^c < k^f \leqslant \frac{1}{2}(\bar{p}+k^c)$ and the initial resource stock of the cartel is relatively small, Fig. 1 with $C \rightarrow F$ applies again, but if the initial resource stock of the cartel is relatively large, time-inconsistent extraction schedules result.

If $k^f = k^c$, infinitely many solutions occur. One is the open-loop Nash equilibrium, given by the extraction schedule $S \rightarrow C^m$ in Fig. 3, which is weakly time-consistent.

Finally if $k^f < k^c$, time-inconsistent extraction schedules result.

The main part of the paper is to show that the weakly time-consistent extraction schedules $C \rightarrow F$, $C^m \rightarrow C \rightarrow F$ and $S \rightarrow C^m$ are also strongly time-consistent, using the approach presented in Section 3. It will be shown that these extraction schedules satisfy the HJB equations for a wider range of parameter values than for which they are the open-loop von Stackelberg solution. More precisely, the extraction schedule $C^m \rightarrow C \rightarrow F$ is strongly time-consistent, for the parameter values for which it also is the open-loop von Stackelberg solution but the extraction schedules $C \rightarrow F$ and $S \rightarrow C^m$ satisfy the HJB equations for a wider range of parameter values. These ranges are even overlapping, so that in first instance these schedules can be seen as candidate feedback von Stackelberg solutions. Later the overlapping parameter space will be divided in an area where $C \rightarrow F$ is the solution and one where $S \rightarrow C^m$ is, simply using the criterium of largest profits for the cartel, since the cartel is the leader in this game. The extraction schedule $F \rightarrow S \rightarrow C^m$ is also a candidate feedback von Stackelberg solution but a proof will be omitted, because it is similar to the other proofs. When the final result is presented, it will be shown for which parameter values the last extraction schedule is the solution.

First, consider the extraction schedule $C \rightarrow F$. In Proposition 1 it is shown that this is a candidate solution for $k^f > k^c$ up to a certain level of k^f. This borderline marks the point where the solution becomes the extraction schedule $C^m \rightarrow C \rightarrow F$, as will be seen later. Proposition 1 assumes given initial reserves and then yields an upper level for k^f. One can also fix $k^f > \frac{1}{2}(\bar{p}+k^c)$ and state that for relatively large initial reserves for the fringe, the solution is $C \rightarrow F$ and for relatively small initial reserves for the fringe, the solution is $C^m \rightarrow C \rightarrow F$. The borderline is in fact the same as what was found in the open-loop von Stackelberg solution. Generally, the results are

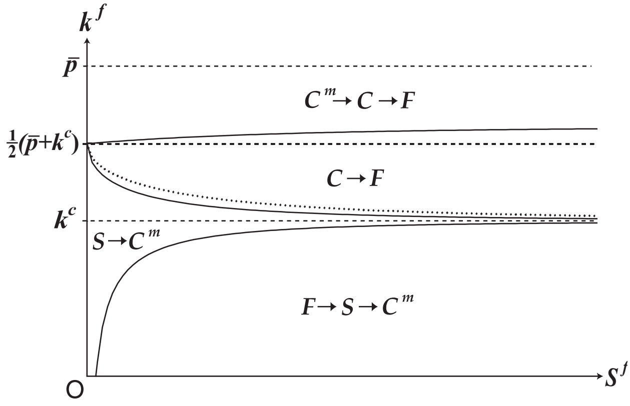

Fig. 4.

most clearly presented in a graph where k^c and S_0^c are fixed and k^f is varied on one axis and S_0^f on the other axis: see Fig. 4.

Proposition 1. *A* $\hat{k}$ *with* $\frac{1}{2}(\bar{p}+k^c) < \hat{k} < \bar{p}$ *exists such that the following holds. If* $k^c < k^f < \hat{k}$, *the extraction schedule* $C \to F$ *in Fig.* 1 (*where the cartel produces first at the competitive price, followed by the fringe*) *is a candidate feedback von Stackelberg solution.*

Proof. We have to show that at each time t between 0 and t_1 (i.e. in the C-regime), for each set of resource stocks S^c and S^f, it is optimal for the cartel to fully supply the market at the competitive price. In Section 3 we derived the optimality conditions for the C-regime

$$e^{-rt}(k^f + e^{rt}\lambda^f - k^c) > \frac{\partial V}{\partial S^c} - \frac{\partial V}{\partial S^f}, \tag{16}$$

$$\frac{1}{2}\left(\bar{p} + k^c + e^{rt}\frac{\partial V}{\partial S^c}\right) > k^f + e^{rt}\lambda^f \tag{17}$$

and the HJB partial differential equation becomes

$$\frac{\partial V}{\partial t} + e^{-rt}\left(k^f + e^{rt}\lambda^f - k^c - e^{rt}\frac{\partial V}{\partial S^c}\right)(\bar{p} - k^f - e^{rt}\lambda^f) = 0. \tag{18}$$

Condition (17) implies that the cartel cannot act as a monopolist, because the resulting price would be above the competitive price and the fringe would start producing. Condition (16) implies that it is better for the cartel to fully supply the market at the competitive price instead of letting the fringe produce or producing simultaneously (the slope of the linear part in the objective is positive: see Section 3). Finally, condition (18) yields strong time-consistency.

Appendix A provides the rest of the formal proof but a sketch of that proof is as follows. Because the competitive price P^f given by Eq. (12) equals the choke price

$\bar{p}$ at t_2, the variable λ^{f} is a function of t_2 (and the parameters $\bar{p}$, k^{f} and r). Since we start from a given extraction schedule, the profits-to-go (the value function) of the cartel at time t can be expressed in t, t_1 and t_2 (and the parameters $\bar{p}$, k^{f}, k^{c} and r). Furthermore, t_1 and t_2 are implicitly given as functions of t, $S^{\mathrm{c}}(t)$ and $S^{\mathrm{f}}(t)$, using the conditions that both the cartel and the fringe exhaust their resource stocks. It follows that the value function is implicitly given and that the partial derivatives with respect to t, S^{c} and S^{f} can be determined. Tedious but straightforward calculations then show that condition (16) is satisfied for $k^{\mathrm{f}} > k^{\mathrm{c}}$, condition (17) is satisfied for $k^{\mathrm{f}} < \hat{k}$ for some $\hat{k}$ with $\frac{1}{2}(\bar{p}+k^{\mathrm{c}}) < \hat{k} < \bar{p}$. Condition (18) is satisfied everywhere. □

In the range $\frac{1}{2}(\bar{p}+k^{\mathrm{c}}) \leqslant k^{\mathrm{f}}$, Proposition 1 states that for small k^{f} the extraction schedule $C \to F$ is a candidate solution but for large k^{f} it is not. As was already mentioned above, Proposition 1 determines a dividing curve in this range in the $k^{\mathrm{f}} - S_0^{\mathrm{f}}$ plane (see Fig. 4), below which the extraction schedule $C \to F$ is both the open-loop and the feedback von Stackelberg solution: $C \to F$ is weakly time-consistent but also strongly time-consistent in this area. Above this dividing curve, the extraction schedule $C^{\mathrm{m}} \to C \to F$ is the open-loop von Stackelberg solution, which is weakly time-consistent. The next step is to prove that this extraction schedule is also strongly time-consistent. First we have to show that it holds for the last two stages $C \to F$, but this is immediately clear because when entering these last two stages, the stock of the cartel has decreased and we are back in that area of the $k^{\mathrm{f}} - S_0^{\mathrm{f}}$ plane where $C \to F$ is shown to be the strongly time-consistent solution.[2] Thus the proof of the next proposition only has to deal with the first stage C^{m}.

Proposition 2. *The weakly time-consistent open-loop von Stackelberg solution $C^{\mathrm{m}} \to C \to F$ in Fig.* 2 (*where the cartel produces first at the monopoly price and then at the competitive price*, *followed by the fringe*) *is strongly time-consistent*.

Proof. We have to show that at each time t between 0 and t_1, for each set of resource stocks S^{c} and S^{f}, it is optimal for the cartel to fully supply the market at the monopoly price. In Section 3 we derived the optimality conditions

$$\mathrm{e}^{-rt}(k^{\mathrm{f}} + \mathrm{e}^{rt}\lambda^{\mathrm{f}} - k^{\mathrm{c}}) \geqslant \frac{\partial V}{\partial S^{\mathrm{c}}} - \frac{\partial V}{\partial S^{\mathrm{f}}}, \tag{19}$$

$$\frac{1}{2}\left(\bar{p} + k^{\mathrm{c}} + \mathrm{e}^{rt}\frac{\partial V}{\partial S^{\mathrm{c}}}\right) < k^{\mathrm{f}} + \mathrm{e}^{rt}\lambda^{\mathrm{f}} \tag{20}$$

and the HBJ equation reduces to

$$\frac{\partial V}{\partial t} + \frac{1}{4}\,\mathrm{e}^{-rt}\left(\bar{p} - k^{\mathrm{c}} - \mathrm{e}^{rt}\frac{\partial V}{\partial S^{\mathrm{c}}}\right)^2 = 0. \tag{21}$$

[2] A formal proof can be given and is available from the authors upon request.

Condition (20) implies that the cartel can act as a monopolist and condition (19) assures that it cannot be better for the cartel to let the fringe produce alone at the competitive price (the slope of the linear part in the objective is non-negative: see Section 3). As before, condition (21) yields strong time-consistency.

The rest of the proof is similar in structure to the proof of Proposition 1 and only a sketch of the proof is given here.[3] Consider Fig. 2. The curve P^{f} intersects $\bar{p}$ at t_3, the curve MC at t_2 and the curve P^{m} at t_1, so that the parameters of these curves are functions of t_1, t_2 and t_3 (and the parameters $\bar{p}$, k^{f}, k^{c} and r). Since we start from a given extraction schedule, the profits-to-go (the value function) of the cartel at time t can be expressed in t, t_1, t_2 and t_3 (and the parameters $\bar{p}$, k^{f}, k^{c} and r). Exhaustion of the two resource stocks plus one of the conditions for the open-loop von Stackelberg solution (see Groot et al., 1990) yield t_1, t_2 and t_3 as implicit functions of $t, S^{\mathrm{c}}(t)$ and $S^{\mathrm{f}}(t)$. It follows that the value function of the cartel is implicitly given and that the partial derivatives with respect to t, S^{c} and S^{f} can be determined. The last two partial derivatives prove to be equal to the parameters (λ^{c} and μ^{c}) of the curves P^{m} and MC. Conditions (20) and (19) then read that the curve P^{f} should be above the curve P^{m} and should not be below the curve MC. This can easily be checked from the positions of the three curves in the stage C^{m}. Finally, straightforward calculations show that Condition (21) is also satisfied. □

We now move to the lower part of Fig. 4. As was recollected above, the extraction schedule $S \to C^{\mathrm{m}}$ is one of the open-loop von Stackelberg solutions in the case $k^{\mathrm{f}} = k^{\mathrm{c}}$ but it is also the open-loop Nash equilibrium. Karp and Newbery (1993) already conjectured that the open-loop Nash equilibrium may be the strongly time-consistent solution for certain parameter values. The next proposition indeed shows that $S \to C^{\mathrm{m}}$ is a candidate solution for a wide range of parameter values.

Proposition 3. *If $\frac{1}{2}(\bar{p} + k^{\mathrm{c}}) > k^{\mathrm{f}}$ and the initial resource stock of the cartel is large relative to the initial resource stock of the fringe, the extraction schedule $S \to C^{\mathrm{m}}$ in Fig.* 3 (*where the cartel and the fringe produce first simultaneously at the competitive price, followed by the cartel as a monopolist*) *is a candidate feedback von Stackelberg solution.*

Proof. We have to show that at each time t between 0 and t_1 (i.e., in the S-regime), for each set of resource stocks S^{c} and S^{f}, it is optimal for the cartel to have simultaneous production at the competitive price. In Section 3 we derived the optimality conditions

$$\mathrm{e}^{-rt}(k^{\mathrm{f}} + \mathrm{e}^{rt}\lambda^{\mathrm{f}} - k^{\mathrm{c}}) = \frac{\partial V}{\partial S^{\mathrm{c}}} - \frac{\partial V}{\partial S^{\mathrm{f}}}, \tag{22}$$

$$\frac{1}{2}\left(\bar{p} + k^{\mathrm{c}} + \mathrm{e}^{rt}\frac{\partial V}{\partial S^{\mathrm{c}}}\right) > k^{\mathrm{f}} + \mathrm{e}^{rt}\lambda^{\mathrm{f}}. \tag{23}$$

[3] The full proof is available from the authors upon request.

Using Eq. (22) we get the HJB equation

$$\frac{\partial V}{\partial t} - (\bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rt}\lambda^{\mathrm{f}})\frac{\partial V}{\partial S^{\mathrm{f}}} = 0. \tag{24}$$

Condition (22) states that the cartel is indifferent whether to produce alone or to let the fringe produce alone or to produce simultaneously at the competitive price (the slope of the linear part in the objective is zero: see Section 3).

The rest of the proof is much more difficult than for Propositions 1 and 2. Appendix B provides a full proof but the main steps are summarized in what follows. "Ex post" the extraction schedule $S \to C^{\mathrm{m}}$ in Fig. 3 can be characterized by variables λ^{f} and λ^{c} of the competitive and monopoly price paths, respectively, so that we can write the extractions as functions of time s, λ^{f} and λ^{c}. The idea is to perform a transformation of these coordinates which finally leads to a partial differential equation in the value function that can be solved. The transformation to the set of coordinates z, x and y is

$$s = t - \frac{\ln z}{r}, \quad \lambda^{\mathrm{f}} = \tfrac{1}{2}\,\mathrm{e}^{-rt}\left\{(\bar{p} + k^{\mathrm{c}} - 2k^{\mathrm{f}})x + (\bar{p} - k^{\mathrm{c}})y\right\},$$

$$\lambda^{\mathrm{c}} = \mathrm{e}^{-rt}(\bar{p} - k^{\mathrm{c}})y.$$

Note that t denotes the point in time at which we want to verify the HJB equation and s denotes the running time. This transformation may look a bit strange at first sight but the reverse transformation

$$x = \mathrm{e}^{r(t-t_1)}, \quad y = \mathrm{e}^{r(t-t_2)}, \quad z = \mathrm{e}^{r(t-s)}$$

probably makes it clear. The resource stocks will be exhausted, which yields the restrictions

$$\int_t^{t_1} E^{\mathrm{f}}(s)\,\mathrm{d}s = S^{\mathrm{f}}, \quad \int_t^{t_2} (E^{\mathrm{c}}(s) + E^{\mathrm{f}}(s))\,\mathrm{d}s = S^{\mathrm{c}} + S^{\mathrm{f}}.$$

This leads to

$$\int_x^1 z^{-1}E^{\mathrm{f}}(x, y, z)\,\mathrm{d}z = rS^{\mathrm{f}},$$

$$(\bar{p} + k^{\mathrm{c}} - 2k^{\mathrm{f}})(x - 1 - \ln x) + (\bar{p} - k^{\mathrm{c}})(y - 1 - \ln y) = 2r(S^{\mathrm{f}} + S^{\mathrm{c}}).$$

These restrictions define x and y as implicit functions of S^{c} and S^{f}, from which the partial derivatives of x and y with respect to S^{c} and S^{f} can be derived. The profits-to-go of the cartel can be written as a function of t, x and y. It is now possible, but rather complex, to derive a linear partial differential equation in the profits-to-go or the value function of the cartel (see Appendix B). In this derivation optimality condition (22) is used as well as the property that E^{f} as a function of x, y and z is homogeneous of degree zero. This partial differential equation can be solved and we get an expression for the value function of the cartel. It is now straightforward, although again rather tedious, to derive the optimal extraction schedule of the cartel in the simultaneous phase. The optimality condition (22) and the HJB equation (24) are always satisfied. The optimality condition (23) is satisfied for $\frac{1}{2}(\bar{p} + k^{\mathrm{c}}) > k^{\mathrm{f}}$. Finally, the extraction rates found in Appendix B have to be non-negative. From this it can be shown that, in the range $\frac{1}{2}(\bar{p}+k^{\mathrm{c}})>k^{\mathrm{f}}>k^{\mathrm{c}}$, the extraction schedule $S\rightarrow C^{\mathrm{m}}$ is only a solution, if the initial resource stock of the cartel is relatively large. In the range $k^{\mathrm{f}}<k^{\mathrm{c}}$, we do not have an analytical proof, but numerical calculations show that it is better for the cartel to let the fringe produce alone for a while before producing simultaneously, if the initial resource stock of the cartel is relatively small. It follows that in that case, $S\rightarrow C^{\mathrm{m}}$ is not the solution but the extraction schedule $F\rightarrow S\rightarrow C^{\mathrm{m}}$, provided of course that $F\rightarrow S\rightarrow C^{\mathrm{m}}$ also satisfies the HJB equation. This is indeed the case. The proof is similar to the earlier proofs and will be omitted here. □

We are now ready to put the pieces together and provide a strongly time-consistent solution for the cartel-versus-fringe model.

Proposition 4. *If $\frac{1}{2}(\bar{p}+k^{\mathrm{c}})\leqslant k^{\mathrm{f}}$, the feedback von Stackelberg solution is either represented by the extraction schedule $C\rightarrow F$ or by $C^{\mathrm{m}}\rightarrow C\rightarrow F$. The last schedule occurs if k^{f} is relatively high or, to put it differently, if the initial resource stock of the fringe S_0^{f} is relatively small. Fixing all the other parameters, a dividing curve can be drawn in the k^{f}–S_0^{f} plane above which $C^{\mathrm{m}}\rightarrow C\rightarrow F$ is the solution and below which $C\rightarrow F$ is the solution.*

If $k^{\mathrm{c}}<k^{\mathrm{f}}<\frac{1}{2}(\bar{p}+k^{\mathrm{c}})$, a dividing curve in the $k^{\mathrm{f}}-S_0^{\mathrm{f}}$ plane exists above which the extraction schedule $C\rightarrow F$ represents the feedback von Stackelberg solution and below which $S\rightarrow C^{\mathrm{m}}$ is the solution.

If $k^{\mathrm{f}}=k^{\mathrm{c}}$, the extraction schedule $S\rightarrow C^{\mathrm{m}}$ is the solution.

If $k^{\mathrm{f}}<k^{\mathrm{c}}$, the feedback von Stackelberg solution is either represented by the extraction schedule $S\rightarrow C^{\mathrm{m}}$ or by $F\rightarrow S\rightarrow C^{\mathrm{m}}$. The last schedule occurs if k^{f} is relatively low or, to put it differently, if the initial resource stock of the fringe S_0^{f} is relatively high. Fixing all the other parameters, a dividing curve can be drawn in the $k^{\mathrm{f}}-S_0^{\mathrm{f}}$ plane above which $S\rightarrow C^{\mathrm{m}}$ is the solution and below which $F\rightarrow S\rightarrow C^{\mathrm{m}}$ is the solution.

Proof. These results follow directly from Propositions 1–3, except for the result in the range $k^{\mathrm{c}}<k^{\mathrm{f}}<\frac{1}{2}(\bar{p}+k^{\mathrm{c}})$. In Propositions 1 and 3 it was found that in this range the extraction schedule $C\rightarrow F$ is a candidate solution but also the extraction schedule $S\rightarrow C^{\mathrm{m}}$ (that is below a dividing curve in the k^{f}–S_0^{f} plane where it exists). Which one will occur simply depends on which one gives the highest profits to the cartel, since the cartel is the leader in the game. This yields another dividing curve in the k^{f}–S_0^{f} plane above which $C\rightarrow F$ is the solution and below which $S\rightarrow C^{\mathrm{m}}$ is the solution. □

Fig. 4 gives an example of Proposition 4 where the initial resource stock of the cartel S_0^{c} is fixed at 200, the marginal extraction cost of the cartel k^{c} at 50, the choke price $\bar{p}$ at 100 and the interest rate r at 10%. The dashed curve denotes the dividing curve below which the extraction schedule $S\rightarrow C^{\mathrm{m}}$ as a candidate solution exists.

It can be argued that the story is not complete because we have not checked whether other candidate solutions may exist that yield higher profits for the cartel. However, the following arguments complete the story. We make a distinction between two cases: $k^f < k^c$ and $k^f > k^c$.

If $k^f < k^c$, it cannot be a solution to have C in the final stage: at the moment of exhaustion of the fringe, the cartel will start exploiting its monopoly power. It cannot be a solution either to have S at the end. The formal proof is given in Appendix B, in which we show that the sequence $S \to C^m$ is a solution where C^m occurs on a non-degenerate interval of time. Let us now consider F as a final stage. A stage C^m cannot precede this phase because the price trajectories P^f and P^m can intersect only once. Also S is not eligible as a predecessor of F. It is better for the cartel to have an interval with simultaneous supply split into two subintervals: a first interval in which the fringe supplies alone and a second one in which the cartel supplies alone. The same type of argument applies to show that F cannot be preceded by C. It follows that in the case at hand the final stage must be a monopoly stage. It can be shown that $F \to C^m$ does not satisfy the HJB equation. The candidate solutions are therefore $S \to C^m$, $F \to S \to C^m$, and $C \to S \to C^m$. The latter can be excluded using a continuity argument with regard to the value function.[4]

If $k^f > k^c$, then clearly also no C-stage can occur at the end. A regime S at the end is excluded because it is better for the cartel to supply first at the competitive price. In the case at hand this is a strongly time-consistent extraction schedule. Let us consider F as a final stage. The sequence $C^m \to F$ can be excluded based on the fact that $C \to F$ is strongly time-consistent. Clearly the sequence $S \to F$ cannot be optimal as a final sequence. We conclude that the only regime preceding F can be the one where the cartel produces alone at the competitive price. It is clear from the arguments used above that we cannot have S or F preceding $C \to F$. Hence, a monopoly phase is the only candidate for preceding $C \to F$. We now consider C^m as the final stage. Obviously C^m can not be preceded by C. Moreover, it cannot be preceded by F. Such a sequence violates the HJB conditions. Hence the only possibility before the monopoly phase is simultaneous supply: $S \to C^m$. Another candidate solution is therefore $C \to S \to C^m$. As before, this one can be excluded using a continuity argument with regard to the value function.

Proposition 4 and Fig. 4 present the strongly time-consistent solution for the cartel-versus-fringe model. As compared to the solutions presented in the previous literature, it does not suffer from time-inconsistency but it is also more transparent and intuitive. While the solutions presented in the previous literature are often complicated sequences of extraction schedules, here two simple mechanisms are at work. If the marginal extraction costs of the fringe k^f are relatively high, the cartel wants the fringe to produce last. Furthermore, if the initial reserve of the cartel is relatively high, it can start to produce as a monopolist. On the other hand, if the marginal extraction costs of the fringe k^f are relatively low, the cartel wants to produce last (as a monopolist) and simultaneously with the fringe before that. Furthermore, if the initial reserve of the fringe is relatively high, the cartel lets the fringe start producing. Time-inconsistency is resolved but the intuition for the solution is simpler as well.

[4] The proofs are available upon request.

5. Conclusion

This paper provides a strongly time-consistent solution for the cartel-versus-fringe model for the supply side of markets for exhaustible natural resources. For a long time it has been an open problem to solve time-inconsistency, which occurred for the solution concept used in the previous literature, but the result is actually more transparent and more intuitive than the previous results. The last two stages of the extraction schedules are either $C \to F$, in case the marginal extraction costs of the fringe are relatively high, or $S \to C^{\mathrm{m}}$, in case the marginal extraction costs of the cartel are relatively high. The stages $C \to F$ can be preceded by a stage C^{m}, in case the initial resource stock of the cartel is relatively high, and the stages $S \to C^{\mathrm{m}}$ can be preceded by a stage F, in case the initial resource stock of the fringe is relatively high.

In our view, this completes the theory of the cartel-versus-fringe model with constant marginal extraction costs, linear demand and fixed initial resource stocks. It may be interesting for further research to consider models with marginal extraction costs that depend on the stocks, with other demand schedules, or with stocks that change due to exploration activities, induced by higher prices.

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Appendix A.

Consider Fig. 1, with extraction schedule $C \to F$. The corresponding extraction rates are given as follows. There exist a positive constant λ^{f} and points in time t_1 and t_2 with $t_2 > t_1 > 0$ such that

$$\begin{aligned} &E^{\mathrm{c}}(s) = \bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rs}\lambda^{\mathrm{f}} \\ &E^{\mathrm{f}}(s) = 0 \qquad (0 \leqslant s < t_1), \end{aligned} \tag{A.1}$$

$$\begin{aligned} &E^{\mathrm{c}}(s) = 0 \\ &E^{\mathrm{f}}(s) = \bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rs}\lambda^{\mathrm{f}} \qquad (t_1 \leqslant s \leqslant t_2). \end{aligned} \tag{A.2}$$

Fix some t with $0 \leqslant t < t_1$. At t_2 the price must reach the choke price $\bar{p}$, so that

$$k^{\mathrm{f}} + \lambda^{\mathrm{f}}\mathrm{e}^{rt_2} = \bar{p}. \tag{A.3}$$

We also have $\int_t^{t_2} E^{\mathrm{f}}(s)\,\mathrm{d}s = S^{\mathrm{f}}(t)$. Hence, using (A.3),

$$(\bar{p} - k^{\mathrm{f}})[\mathrm{e}^{rt_1 - rt_2} - 1 + rt_2 - rt_1] = rS^{\mathrm{f}}(t). \tag{A.4}$$

Similarly, $\int_t^{t_2} E^c(s)\,\mathrm{d}s = S^c(t)$. Hence, using (A.3),

$$(\bar{p} - k^f)[\mathrm{e}^{rt-rt_2} - \mathrm{e}^{rt_1-rt_2} + rt_1 - rt] = rS^c(t). \tag{A.5}$$

The profits of the cartel, from t onwards with $0 \leqslant t < t_1$, are

$$\begin{aligned}\Pi(t,t_1,t_2) &= \int_t^{t_2} \mathrm{e}^{-rs}[p(s) - k^c]E^c(s)\,\mathrm{d}s \\ &= \frac{\mathrm{e}^{-rt}}{r}(\bar{p} - k^f)[(k^f - k^c)(1 - \mathrm{e}^{rt-rt_1}) + (\bar{p} + k^c - 2k^f)\mathrm{e}^{rt-rt_2}(rt_1 - rt) \\ &\quad - (\bar{p} - k^f)(\mathrm{e}^{rt_1-rt_2} - \mathrm{e}^{rt-rt_2})\mathrm{e}^{rt-rt_2}].\end{aligned} \tag{A.6}$$

Using the implicit function theorem we can find t_1 and t_2 as functions of t, S^c and S^f from (A.4) and (A.5), where S^c and S^f denote the stocks at time t. The value function of the cartel is then implicitly defined by

$$V(t,S^c,S^f) = \Pi(t,t_1(t,S^c,S^f),t_2(t,S^c,S^f)).$$

The partial derivatives needed in the HJB equation are found as

$$\frac{\partial V}{\partial t} = \frac{\partial \Pi}{\partial t} + \frac{\partial \Pi}{\partial t_1}\frac{\partial t_1}{\partial t} + \frac{\partial \Pi}{\partial t_2}\frac{\partial t_2}{\partial t} = -r\Pi(t,t_1,t_2),$$

$$\begin{aligned}\frac{\partial V}{\partial S^c} &= \frac{\partial \Pi}{\partial t_1}\frac{\partial t_1}{\partial S^c} + \frac{\partial \Pi}{\partial t_2}\frac{\partial t_2}{\partial S^c} \\ &= \frac{(\bar{p}-k^f)(\mathrm{e}^{rt_1-rt_2}+1-2\mathrm{e}^{rt-rt_2})-(k^f-k^c)(1-\mathrm{e}^{rt_2-rt_1})-(\bar{p}+k^c-2k^f)(rt_1-rt)}{(\mathrm{e}^{rt_2}-\mathrm{e}^{rt})},\end{aligned}$$

$$\begin{aligned}\frac{\partial V}{\partial S^f} &= \frac{\partial \Pi}{\partial t_1}\frac{\partial t_1}{\partial S^f} + \frac{\partial \Pi}{\partial t_2}\frac{\partial t_2}{\partial S^f} \\ &= \frac{(\bar{p}-k^f)(\mathrm{e}^{rt_1-rt_2}-\mathrm{e}^{rt-rt_2})-(k^f-k^c)(1-\mathrm{e}^{rt-rt_1})-(\bar{p}+k^c-2k^f)(rt_1-rt)}{(\mathrm{e}^{rt_2}-\mathrm{e}^{rt})}.\end{aligned}$$

We can now verify Eqs. (16)–(18). It is easy to check that Eq. (18) is satisfied. Eq. (16) reduces to $(k^f - k^c)(1 - \mathrm{e}^{rt-rt_1}) > 0$ which is satisfied for $k^f > k^c$. Eq. (17) can be seen as a linear expression in k^f which has to be positive. It can be shown that this expression is decreasing, positive for $k^f = \frac{1}{2}(\bar{p} + k^c)$ and negative for $k^f = \bar{p}$. It follows that Eq. (17) is satisfied for $k^f < \hat{k}$ with $\frac{1}{2}(\bar{p} + k^c) < \hat{k} < \bar{p}$.

Appendix B.

Consider Fig. 3, with extraction schedule $S \to C^m$. There exist positive constants λ^f and λ^c and points in time t_1 and t_2 with $t_2 > t_1 > 0$ such that

$$E^c(s) + E^f(s) = \bar{p} - k^f - \mathrm{e}^{rs}\lambda^f \quad (0 \leqslant s < t_1), \tag{B.1}$$

$$E^{\mathrm{c}}(s) = \tfrac{1}{2}(\bar{p} - k^{\mathrm{c}}) - \tfrac{1}{2}\mathrm{e}^{rs}\lambda^{\mathrm{c}}$$
$$E^{\mathrm{f}}(s) = 0 \qquad (t_1 \leqslant s \leqslant t_2). \tag{B.2}$$

Fix some t with $0 \leqslant t < t_1$. At t_2 the price must reach the choke price $\bar{p}$, so that

$$\tfrac{1}{2}(\bar{p} + k^{\mathrm{c}}) + \tfrac{1}{2}\mathrm{e}^{rt_2}\lambda^{\mathrm{c}} = p. \tag{B.3}$$

The price path is continuous at t_1, so that

$$k^{\mathrm{f}} + \mathrm{e}^{rt_1}\lambda^{\mathrm{f}} = \tfrac{1}{2}(\bar{p} + k^{\mathrm{c}}) + \tfrac{1}{2}\mathrm{e}^{rt_1}\lambda^{\mathrm{c}}. \tag{B.4}$$

Given λ^{f} and λ^{c} it can easily be calculated how much the cartel and the fringe will supply in total over the interval $[0, t_1)$. Central in our approach is the claim that, given λ^{f} and λ^{c}, there is also only one possible division of supply along this first interval if $k^{\mathrm{f}} \neq k^{\mathrm{c}}$. This follows from the fact that the cartel will simply choose the best one for itself if there exist multiple candidates. Note that at any point in time between 0 and t_1 it is optimal to have the same λ's. We write

$$E^{\mathrm{c}}(s; \lambda^{\mathrm{f}}, \lambda^{\mathrm{c}}) = \hat{E}^{\mathrm{c}}(\mathrm{e}^{rs}\lambda^{\mathrm{f}}, \mathrm{e}^{rs}\lambda^{\mathrm{c}}) \quad (0 \leqslant s < t_1), \tag{B.5}$$

$$E^{\mathrm{f}}(s; \lambda^{\mathrm{f}}, \lambda^{\mathrm{c}}) = \bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rs}\lambda^{\mathrm{f}} - E^{\mathrm{c}}(s; \lambda^{\mathrm{f}}, \lambda^{\mathrm{c}}) = \hat{E}^{\mathrm{f}}(\mathrm{e}^{rs}\lambda^{\mathrm{f}}, \mathrm{e}^{rs}\lambda^{\mathrm{c}}) \quad (0 \leqslant s < t_1). \tag{B.6}$$

In view of the transformation to x, y and z we have

$$\hat{E}^{\mathrm{f}}(\mathrm{e}^{rs}\lambda^{\mathrm{f}}, \mathrm{e}^{rs}\lambda^{\mathrm{c}}) = \hat{E}^{\mathrm{f}}\left(\frac{ax + by}{2z}, \frac{by}{z}\right) =: \bar{E}^{\mathrm{f}}(x, y, z).$$

The restrictions with respect to the stocks are

$$\int_t^{t_1} \hat{E}^{\mathrm{f}}(\mathrm{e}^{rs}\lambda^{\mathrm{f}}, \mathrm{e}^{rs}\lambda^{\mathrm{c}})\,\mathrm{d}s = S^{\mathrm{f}}(t),$$
$$\int_t^{t_2} [\hat{E}^{\mathrm{c}}(\mathrm{e}^{rs}\lambda^{\mathrm{f}}, \mathrm{e}^{rs}\lambda^{\mathrm{c}}) + \hat{E}^{\mathrm{f}}(\mathrm{e}^{rs}\lambda^{\mathrm{f}}, \mathrm{e}^{rs}\lambda^{\mathrm{c}})]\,\mathrm{d}s = S^{\mathrm{c}}(t) + S^{\mathrm{f}}(t),$$

which imply (omitting the time argument in S^{c} and S^{f})

$$F(x, y) := \int_x^1 \frac{1}{z}\bar{E}^{\mathrm{f}}(x, y, z)\,\mathrm{d}z = rS^{\mathrm{f}}, \tag{B.7}$$

$$a(x - 1 - \ln x) + b(y - 1 - \ln y) = 2r(S^{\mathrm{c}} + S^{\mathrm{f}}), \tag{B.8}$$

where $a = \bar{p} + k^{\mathrm{c}} - 2k^{\mathrm{f}}$ and $b = \bar{p} - k^{\mathrm{c}}$. The profits of the cartel, from time t onwards with $0 \leqslant t < t_1$, are

$$\Pi(t, x, y) = \int_t^{t_2} \mathrm{e}^{-rs}[\bar{p} - E^{\mathrm{f}}(s) - E^{\mathrm{c}}(s) - k^{\mathrm{c}}]E^{\mathrm{c}}(s)\,\mathrm{d}s$$
$$= \int_t^{t_1} \mathrm{e}^{-rs}[k^{\mathrm{f}} + \mathrm{e}^{rs}\lambda^{\mathrm{f}} - k^{\mathrm{c}}][\bar{p} - k^{\mathrm{f}} - \mathrm{e}^{rs}\lambda^{\mathrm{f}} - \hat{E}^{\mathrm{f}}(\mathrm{e}^{rs}\lambda^{\mathrm{f}}, \mathrm{e}^{rs}\lambda^{\mathrm{c}})]\,\mathrm{d}s$$
$$+ \int_{t_1}^{t_2} \mathrm{e}^{-rs}\frac{1}{2}[\bar{p} - k^{\mathrm{c}} + \mathrm{e}^{rs}\lambda^{\mathrm{c}}]\frac{1}{2}[\bar{p} - k^{\mathrm{c}} - \mathrm{e}^{rs}\lambda^{\mathrm{c}}]\,\mathrm{d}s$$

$$= \frac{e^{-rt}}{4r}\,[b^2(1-y)^2 - a^2(x-1)^2 + 2(ax+by)a(x-1-\ln x)$$
$$- 2(b-a)G(x,y) - 2(ax+by)F(x,y)], \tag{B.9}$$

where

$$G(x,y) := \int_x^1 \bar{E}^{\mathrm{f}}(x,y,z)\,\mathrm{d}z. \tag{B.10}$$

Using the implicit function theorem we can find x and y as functions of S^{c} and S^{f} from (B.7) and (B.8). Hence, the value function at time t is

$$V(t,S^{\mathrm{c}},S^{\mathrm{f}}) = \Pi(t,x(S^{\mathrm{c}},S^{\mathrm{f}}),y(S^{\mathrm{c}},S^{\mathrm{f}})).$$

A necessary condition for the proposed exploitation patterns to be a strongly time-consistent solution is (22). In terms of x and y the equation boils down to

$$\frac{1}{2}(b-a) + \frac{1}{2}(ax+by) = \bar{\Pi}_x(t,x,y)\left(\frac{\partial x}{\partial S^{\mathrm{c}}} - \frac{\partial x}{\partial S^{\mathrm{f}}}\right)$$
$$+ \bar{\Pi}_y(t,x,y)\left(\frac{\partial y}{\partial S^{\mathrm{c}}} - \frac{\partial y}{\partial S^{\mathrm{f}}}\right), \tag{B.11}$$

where $\bar{\Pi} = e^{rt}\Pi$. First we claim

$$G(x,y) = xG_x(x,y) + yG_y(x,y) - xF_x(x,y) - yF_y(x,y). \tag{B.12}$$

This can be seen as follows. Definitions (B.7) and (B.10) imply

$$F_x(x,y) = -\frac{1}{x}\bar{E}^{\mathrm{f}}(x,y,x) + \int_x^1 \frac{1}{z}\bar{E}^{\mathrm{f}}_x(x,y,z)\,\mathrm{d}z; \quad F_y(x,y) = \int_x^1 \frac{1}{z}\bar{E}^{\mathrm{f}}_y(x,y,z)\,\mathrm{d}z,$$
$$G_x(x,y) = -\bar{E}^{\mathrm{f}}(x,y,x) + \int_x^1 \bar{E}^{\mathrm{f}}_x(x,y,z)\,\mathrm{d}z; \quad G_y(x,y) = \int_x^1 \bar{E}^{\mathrm{f}}_y(x,y,z)\,\mathrm{d}z.$$

Moreover, $\bar{E}^{\mathrm{f}}$ is homogeneous of degree zero which is immediate from the definition of $\bar{E}^{\mathrm{f}}$, so that

$$\int_x^1 \left(\frac{z-1}{z}\right)[x\bar{E}^{\mathrm{f}}_x(x,y,z) + y\bar{E}^{\mathrm{f}}_y(x,y,z) + z\bar{E}^{\mathrm{f}}_z(x,y,z)]\,\mathrm{d}z = 0.$$

It is easily seen that these equations yield (B.12).

The next step is to solve G from (B.9), to calculate G_x and G_y and to insert all three of them into (B.12). This yields

$$xF_x + yF_y = \tfrac{1}{2}(a(x-1) + b(y-1))$$
$$+2r(\bar{\Pi} - x\bar{\Pi}_x - y\bar{\Pi}_y)/(a(x-1) + b(y+1)). \tag{B.13}$$

It follows from (B.7) and (B.8) that

$$\frac{\partial x}{\partial S^{\mathrm{c}}} - \frac{\partial x}{\partial S^{\mathrm{f}}} = -brx(y-1)/(b(y-1)xF_x - a(x-1)yF_y),$$

$$\frac{\partial y}{\partial S^{\mathrm{c}}} - \frac{\partial y}{\partial S^{\mathrm{f}}} = ary(x-1)/(b(y-1)xF_x - a(x-1)yF_y).$$

Hence, we find a second equation in xF_x and yF_y by inserting these results in (B.11). We can then solve for xF_x and yF_y:

$$xF_x = \frac{1}{2}a(x-1) + \frac{2ra(x-1)\bar{\Pi} - (a(x-1)+b(y-1))2rx\bar{\Pi}_x}{(a(x-1)+b(y+1))(a(x-1)+b(y-1))}, \tag{B.14}$$

$$yF_y = \frac{1}{2}b(y-1) + \frac{2rb(y-1)\bar{\Pi} - (a(x-1)+b(y-1))2ry\bar{\Pi}_y}{(a(x-1)+b(y+1))(a(x-1)+b(y-1))}. \tag{B.15}$$

When (B.14) is differentiated with respect to y and (B.15) with respect to x, we find the following partial differential equation for Π, because $F_{xy} = F_{yx}$:

$$2ab(x-y)(a(x-1)+by)\Pi + b(a(x-1)+b(y-1))(b(y-1)-a(x-1))x\Pi_x$$
$$+a(a(x-1)+b(y-1))(b(y+1)+a(x-1)-2bx)y\Pi_y = 0. \tag{B.16}$$

When solving the partial differential equation it has to be taken into account that if $x = 1$, then $t = t_1$ and $F(x,y) = G(x,y) = 0$, so that

$$\Pi(t,1,y) = \frac{e^{-rt}}{4r} b^2 (1-y)^2, \quad 0 < y \leqslant 1.$$

We then find as the value function (the proof is available from the authors upon request)

$$V(t,S^c,S^f) = -\frac{e^{-rt}}{4r}[a(x-1)+b(y-1)] \times\sqrt{[a(x-1)+b(y+1)]^2 - 4b^2x^{a/b}y}, \tag{B.17}$$

where it is to be understood that x and y are functions of S^c and S^f.

The last step is to find the functions $\bar{E}^f$ and $\bar{E}^c$. Since we know the function Π we can solve F from (B.14) and (B.15). Then G follows from (B.9). From the definitions of F and G this yields $E^f(t)$ and subsequently $E^c(t)$ through (B.1):

$$E^c(t) = \frac{(\bar{p}-k^f)(k^f-k^c) + (e^{rt}\lambda^f)^2 - (\bar{p}-k^f)\lambda^c e^{rt}\left[\dfrac{e^{rt}(2\lambda^f-\lambda^c)}{\bar{p}+k^c-2k^f}\right]^{\frac{\bar{p}+k^c-2k^f}{\bar{p}-k^c}}}{\sqrt{(k^f-k^c+e^{rt}\lambda^f)^2 - (\bar{p}-k^c)e^{rt}\lambda^c\left[\dfrac{e^{rt}(2\lambda^f-\lambda^c)}{\bar{p}+k^c-2k^f}\right]^{\frac{\bar{p}+k^c-2k^f}{\bar{p}-k^c}}}}.$$

We can now verify Eqs. (22)–(24). It is easy to check that Eqs. (22) and (24) are satisfied. Eq. (23) reduces to $SQR < 2(\bar{p} - k^f - e^{rt}\lambda^f)$, where SQR is the square root in the value function V given in (B.17). It can be shown that this is precisely satisfied for $k^f < \frac{1}{2}(\bar{p} + k^c)$.

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