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Binary Codes of Strongly Regular Graphs

WILLEM H. HAEMERS
Department of Econometrics, Tilburg University, 5000 LE Tilburg, The Netherlands

RENÉ PEETERS
Department of Econometrics, Tilburg University, Brabant 5000 LE Tilburg, The Netherlands

JEROEN M. VAN RIJCKEVORSEL
Department of Econometrics, Tilburg University, Brabant 5000 LE Tilburg, The Netherlands

Dedicated to the memory of E. F. Assmus

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Abstract. For strongly regular graphs with adjacency matrix A, we look at the binary codes generated by A and $A + I$. We determine these codes for some families of graphs, we pay attention to the relation between the codes of switching equivalent graphs and, with the exception of two parameter sets, we generate by computer the codes of all known strongly regular graphs on fewer than 45 vertices.

Keywords: binary codes, strongly regular graphs, regular two-graphs

1. Introduction

Codes generated by the incidence matrix of combinatorial designs and related structures have been studied rather extensively. This interplay between codes and designs has provided several useful and interesting results. The best reference for this is the book by Assmus and Key [3] (see also the update [4]). Codes generated by the adjacency matrix of a graph did get much less attention. Especially for strongly regular graphs (for short: SRG's) there is much analogy with designs and therefore similar results may be expected. Concerning the dimension of these codes, that is, the p-rank of SRG's, several results are known: see [5], [17]. In addition, it has turned out that some special SRG's generate nice codes, see [12] and [22]. The present paper is a first step to a more structural approach. We restrict to binary codes, not only because it is the obvious case to start with, but also since for the binary case there is a relation with regular two-graphs and Seidel switching that has already proved to be useful: see [12] and [7].

The paper is organised as follows. First some conclusions about the code are presented that are based upon the parameters of the SRG only. Often, but not always, the dimension follows from the parameters, but only in a few obvious cases the code itself is determined by the parameters. Next we pay attention to codes of some famous SRG's. This includes the Paley graphs, the graphs related to a symplectic form over $\mathbf{F}_2$, the Latin square graphs and the block graphs of Steiner 2-designs. Several good codes appear. For example the quadratic residue and simplex codes. In Section 5 we study the relation between Seidel switching in a graph and the generated code. Here we introduce the concept of a two-graph code, which unifies in a sense the codes of SRG's that are equivalent under Seidel switching. This section contains a useful result (Theorem 5.1), that relates the presence of the all-one vector to the dimension of the code. It implies for example that the code of an SRG with parameters (36,14,4,6) contains the all-one vector. In Section 6 we present the computer results. Here we give the weight enumeraters of the codes (and two-graph codes) of the known SRG's (and regular two-graphs) on fewer than 45 vertices. For two parameter sets ((35,16,6,8) and (36,15,6,6)) there are too many SRG's known to present all weight enumerators, so we restrict to the ones from Steiner triple systems and Latin squares. In some cases we find interesting codes. For example there is a Latin square of order 6 that generates an optimal code of length 36, dimension 13 with minimum weight 12 (see Section 6.3). We also find out that for several parameter sets the non-isomorphic graphs produce non-isomorphic codes. But there are interesting exceptions. For example the 180 known SRG's with parameters (36,14,4,6) together generate only 76 non-isomorphic codes.

The reader is assumed to be familiar with the basis concepts of codes and designs, which can for example be found in [3] or [10].

2. Preliminaries

For an integral $n \times v$ matrix A we define the binary code $\mathcal{C}_A$ of A to be the subspace of $\mathbf{V} = \mathbf{F}_2^v$ generated by the rows of A (mod 2). We start with some known lemmas for symmetric integral matrices (see [5], [6] or [17]).

LEMMA 2.1 *If A is a symmetric integral matrix with zero diagonal, then* 2-rank(A) *(i.e. the dimension of $\mathcal{C}_A$) is even.*

Proof. Let A' be a non-singular principal submatrix of A with the same 2-rank as A. Over $\mathbf{Z}$, any skew symmetric matrix of odd order has determinant 0 (since $\det(A) = -\det(A^\top)$). Reduction mod 2 shows that A' has even order. ∎

LEMMA 2.2 *If A is a symmetric binary matrix, then* $\mathrm{diag}(A) \in \mathcal{C}_A$.

Proof. Suppose $x \in \mathcal{C}_A^\perp$. Then $\sum_i (A)_{ii} x_i = \sum_{i,j} (A)_{ij} x_i x_j = x^\top A x = 0$ (mod 2), so $x \perp \mathrm{diag}(A)$. Hence $\mathrm{diag}(A) \perp \mathcal{C}_A^\perp$. ∎

With these lemmas we easily find a relation between the codes $\mathcal{C}_A$ and $\mathcal{C}_{A+J}$ (J is the all-one matrix, $\mathbf{1}$ is the all-one vector).

PROPOSITION 2.1 *Suppose A is the adjacency matrix of a graph then $\mathcal{C}_A \subseteq \mathcal{C}_{A+J}$ and the following are equivalent:*

$i.\ \mathcal{C}_A = \mathcal{C}_{A+J}$, $\quad ii.\ \mathbf{1} \in \mathcal{C}_A$, $\quad iii.\ \dim(\mathcal{C}_{A+J})$ *is even.*

Proof. By Lemma 2.2, $\mathrm{diag}(A+J) = \mathbf{1} \in \mathcal{C}_{A+J}$, so $\mathcal{C}_{A+J} = \mathcal{C}_A + \langle \mathbf{1} \rangle$ and the equivalence of i and ii follows. By Lemma 2.1 we have 2-rank(A) is even and so $i \Leftrightarrow iii$. ■

The next proposition gives a trivial but useful relation between $\mathcal{C}_A$ and $\mathcal{C}_{A+I}$.

PROPOSITION 2.2 *If A is a symmetric integral matrix, then $\mathcal{C}_A^{\perp} \subseteq \mathcal{C}_{A+I}$ with equality if and only if $A(A+I) = 0 \pmod 2$.*

Proof. Suppose $x \in \mathcal{C}_A^{\perp}$. Then $Ax = 0 \pmod 2$, so $(A+I)x = x$ and hence $x \in \mathcal{C}_{A+I}$. Clearly $A(A+I) = 0 \pmod 2$ reflects that $\mathcal{C}_{A+I} \subseteq \mathcal{C}_A^{\perp}$, which completes the proof. ■

A *strongly regular graph* with parameters (v, k, λ, μ), for short $SRG(v, k, \lambda, \mu)$, is a regular graph Γ with v vertices, valency k $(1 \le k \le v-2)$, and the number of common neighbours of two distinct vertices x and y is equal to λ or μ respectively, depending on whether x and y are adjacent or not. Let Γ be a $SRG(v, k, \lambda, \mu)$. Clearly $\overline{\Gamma}$, the complement of Γ, is a $SRG(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$ and moreover, the adjacency matrix A of Γ satisfies

$$A^2 = kI + \lambda A + \mu \overline{A}, \tag{1}$$

where $\overline{A} = J - I - A$ is the adjacency matrix of $\overline{\Gamma}$. The valency k is an eigenvalue of A with eigenvector $\mathbf{1}$. It follows that A has just two more eigenvalues, say r and s $(r > s)$, with multiplicities f and g respectively, which satisfy:

$$-rs = k - \mu,\ r + s = \lambda - \mu,\ (k-r)(k-s) = \mu v,\ v = f + g + 1$$
$$\text{and } 0 = k + fr + gs.$$

Moreover, either r and s are integers or $r = -1-s = (-1+\sqrt{v})/2$, $f = g = k = (v-1)/2$ and $\mu = \lambda+1 = k/2$. If Γ or $\overline{\Gamma}$ is a disjoint union of complete graphs, Γ is called *imprimitive* and the codes $\mathcal{C}_A$ and $\mathcal{C}_{A+I}$ are trivial. Throughout, we will assume that Γ is primitive (not imprimitive).

If $\lambda = \mu$, Equation (1) becomes $A^2 = (k-\lambda)I + \lambda J$, that is, A is the incidence matrix of symmetric 2-(v, k, λ) design and Γ is called a (v, k, λ) graph. Similarly, if $\lambda+2 = \mu$, $A+I$ represents a 2-$(v, k+1, \lambda+2)$ design. It may happen, however, that two non-isomorphic (v, k, λ) graphs, Γ_1 and Γ_2 with adjacency matrices A_1 and A_2 say, give isomorphic designs. If this is the case then $A_1 = PA_2$ for some permutation matrix P and it is obvious that $\mathcal{C}_{A_1} = \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1+J} = \mathcal{C}_{A_2+J}$. Similarly, if A_1 and $A_2 + I$ represent the same design, then $A_1 = P(A_2 + I)$ and then $\mathcal{C}_{A_1} = \mathcal{C}_{A_2+I}$. The standard example is given by the two $SRG(16, 6, 2, 2)$'s (the lattice graph $L(4)$ and the Shrikhande graph) and the unique $SRG(16, 5, 0, 2)$ (the Clebsch graph). The three graphs produce the same symmetric 2-(16, 6, 2) design, and therefore the same code (take $\mathcal{C}_{A+I}$ for the Clebsch graph). We refer to [10] for these and other results on strongly regular graphs and designs.

3. Facts from the Parameters

Here we present some properties of the binary codes of a strongly regular graph Γ, using only the parameters of Γ.

PROPOSITION 3.1 *Suppose Γ has non-integral eigenvalues.*

i. If μ is odd (i.e. $v = 5 \bmod 8$) then $\mathcal{C}_A = \mathbf{1}^\perp$ and $\mathcal{C}_{A+I} = \mathbf{V}$.

ii. If μ is even ($v = 1 \bmod 8$) then $\mathcal{C}_A^\perp = \mathcal{C}_{A+I}$ and $\dim(\mathcal{C}_A) = \dim(\mathcal{C}_{A+I}) - 1 = 2\mu$ ($= f = g = k = (v-1)/2$).

Proof. If μ is odd, Equation (1) becomes $A^2 = A + I + J \pmod 2$, so $(A+J)(A+I) = I \pmod 2$, hence $\mathcal{C}_{A+J} = \mathcal{C}_{A+I} = \mathbf{V}$ and $\mathcal{C}_A = \mathbf{1}^\perp$. Suppose μ is even. Then $A^2 = A \pmod 2$ so $\mathcal{C}_{A+I} = \mathcal{C}_A^\perp$. The characteristic polynomial of A is given by:

$$\det(xI - A) = (x+k)(x^2 + x + \mu)^f = x^{f+1}(x+1)^f \pmod 2.$$

Therefore 2-rank$(A+I) \geq v - f$ and 2-rank$(A) \geq v - (f+1) = f$. We know (Proposition 2.2) 2-rank(A) + 2-rank$(A+I) = v$ and the result follows. ■

PROPOSITION 3.2 Suppose the eigenvalues r and s of Γ are integers.

i. If $k = r = s = 1 \pmod 2$ then $\mathcal{C}_A = \mathbf{V}$, $\mathcal{C}_{A+I}$ is self-orthogonal and $\dim(\mathcal{C}_{A+I}) \leq \min\{f+1, g+1\}$.

ii. If $r = s = 1 \pmod 2$ and k is even, then $\mathcal{C}_A = \mathbf{1}^\perp$, $\mathcal{C}_{A+I}$ is orthogonal to $\mathcal{C}_{\overline{A}}$ and $\dim(\mathcal{C}_{A+I}) \leq \min\{f+1, g+1\}$.

iii. If $r \neq s \pmod 2$ and k is even, then $\mathcal{C}_{A+I} = \mathcal{C}_A^\perp$, $\dim(\mathcal{C}_A) = f'$ and $\dim(\mathcal{C}_{A+I}) = v - f'$, where f' is the multiplicity of the odd eigenvalue.

iv. If $r \neq s \pmod 2$ and k is odd, then $\mathcal{C}_{\overline{A}} = \mathcal{C}_A^\perp$, $\dim(\mathcal{C}_A) = f'+1$ and $\dim(\mathcal{C}_{A+I}) = v - f'$.

v. If $r = s = 0 \pmod 2$ then k is even, $\mathcal{C}_{A+I} = \mathbf{V}$, $\mathcal{C}_A$ is self-orthogonal and $\dim(\mathcal{C}_A) \leq \min\{f+1, g+1\}$ and even.

Proof. i. Equation (1) gives $A^2 = I$ and $(A+I)^2 = 0 \pmod 2$. Over the real numbers, rank$(A - rI) = v - f = g + 1$, hence 2-rank$(A+I) \leq g+1$ and similarly, 2-rank$(A+I) \leq f+1$.

ii. Now $A\mathbf{1} = 0$, $A^2 = I + J$, and $(A+I)^2 = J \pmod 2$, proving the first two claims. For the dimension bound see case i.

iii. Now (1) becomes $A(A+I) = 0 \pmod 2$, so $\mathcal{C}_{A+I} = \mathcal{C}_A^\perp$ by Proposition 2.2. The characteristic polynomial of $A \pmod 2$ reads $x^{v-f'}(x+1)^{f'}$, so $\dim(\mathcal{C}_{A+I}) \geq v - f'$ and $\dim(\mathcal{C}_A) \geq f'$ and, since they add up to v the result follows.

iv. Here $A\overline{A} = 0 \pmod 2$. Similar to case iii we get $\dim(\mathcal{C}_{A+I}) \geq v - f' - 1$ and $\dim(\mathcal{C}_A) \geq f'+1$. Now the dimensions add up to $v+1$, but f' is odd (from trace(A)) and v is even (since k is odd), so by Proposition 2.1 we find $\dim(\mathcal{C}_A) = f' + 1$, $\dim(\mathcal{C}_{A+I}) = v - f'$ and $\dim(\mathcal{C}_{\overline{A}}) = v - f' - 1$.

v. Now $A^2 = kJ$ and $(A + I)^2 = kJ + I \pmod 2$. From $k + fr + gs = 0$ it follows that k is even. By Lemma 2.1 $\dim(\mathcal{C}_A)$ is even. The rest follows by similar arguments as above. ■

Thus, unless r and s are both even, the dimension of $\mathcal{C}_A$ (i.e. 2-rank(A)) follows from the parameters of Γ and similarly, $\dim(\mathcal{C}_{A+I})$ follows, unless r and s are both odd. This result is due to Brouwer and Van Eijl [5]. It also follows from a more general lemma of Peeters [17] (p. 15) on the p-ranks of symmetric integral matrices. From the two propositions above we also see that if rs $(= \mu - k)$ is odd $\mathcal{C}_A$ and $\mathcal{C}_{A+J}$ $(= \mathcal{C}_{\overline{A+I}})$ are determined by the parameters of Γ. Similarly, $\mathcal{C}_{A+I}$ and $\mathcal{C}_{\overline{A}}$ are determined if $(r + 1)(s + 1)$ is odd. So in these cases non-isomorphic strongly regular graphs with the same parameters (of which there are many examples) generate the same (trivial) codes.

4. Some Families and Their Codes

4.1. Triangular Graphs

The triangular graph $T(n)$ is the line graph of the complete graph K_n. It follows that $T(n)$ is a strongly regular graph with $v = n(n-1)/2$, $k = 2(n-2)$, $\lambda = n-2$, $\mu = 4$, $r = n-4$ and $s = -2$. $T(n)$ is known to be determined by these parameters if $n \neq 8$. If N is the vertex-edge incidence matrix of K_n, then $A = N^\top N \pmod 2$ is the adjacency matrix of $T(n)$. The words of $\mathcal{C}_N$, $\mathcal{C}_A$ and $\mathcal{C}_{A+I}$ are characteristic vectors of subsets of the edge set of K_n, so can be interpreted as graphs on a fixed vertex set of size n. It is easily seen that $\mathcal{C}_N$ is the $n - 1$ dimensional binary code consisting of all complete bipartite graphs and that $\mathcal{C}_N^\perp$ consists of disjoint unions of Euler graphs. Note that $\mathbf{1} \notin \mathcal{C}_N$.

THEOREM 4.1 *Let Γ be the triangular graph $T(n)$.*
If n is even then $\mathcal{C}_A = \mathcal{C}_N \cap \mathbf{1}^\perp$ (the Eulerian complete bipartite graphs), $\mathcal{C}_{A+I} = \mathbf{V}$, $\mathcal{C}_{\overline{A}} = \mathbf{V}$ if $n = 0 \pmod 4$ and $\mathcal{C}_{\overline{A}} = \mathbf{1}^\perp$ if $n = 2 \pmod 4$.
If n is odd then $\mathcal{C}_A = \mathcal{C}_N$, $\mathcal{C}_{A+I} = \mathcal{C}_N^\perp$, $\mathcal{C}_{\overline{A}} = \mathcal{C}_N^\perp$ if $n = 1 \pmod 4$ and $\mathcal{C}_{\overline{A}} = \mathcal{C}_N^\perp \cap \mathbf{1}^\perp$ (the unions of Euler graphs with an even total number of edges) if $n = 3 \pmod 4$.

Proof. Since $N^\top N = A \pmod 2$, we have $\mathcal{C}_A \subset \mathcal{C}_N$. First suppose n is odd. By iii of Proposition 3.2, $\dim(\mathcal{C}_A) = f = n - 1$, hence $\mathcal{C}_A = \mathcal{C}_N$ and $\mathcal{C}_{A+I} = \mathcal{C}_N^\perp$. Proposition 2.1 gives $\mathcal{C}_{\overline{A}} = \mathcal{C}_{A+I}$ whenever $(n - 1)(n - 2)/2 = \dim(C_{A+I})$ is even, that is $n = 1 \bmod 4$. If $n = 3 \bmod 4$, $\mathcal{C}_{\overline{A}}$ has dimension one less and is orthogonal to $\mathcal{C}_A$ and to $\mathbf{1}$. Since $\mathbf{1} \notin \mathcal{C}_A$, this proves the last claim. Next take n even. By i and ii of Proposition 3.2 we find $\mathcal{C}_{A+I}$ and $\mathcal{C}_{\overline{A}}$. Since $\dim(\mathrm{kernel}(N^\top)) = 1 \pmod 2$, $\dim(\mathcal{C}_A) \geq \dim(\mathcal{C}_N) - 1 = n-2$. Clearly $\mathbf{1} \in \mathcal{C}_A^\perp$ but (since n is even), $\mathbf{1} \notin \mathcal{C}_N^\perp$. Therefore $\mathcal{C}_A^\perp = \mathcal{C}_N^\perp + \langle \mathbf{1} \rangle$ and so $\mathcal{C}_A = \mathcal{C}_N \cap \mathbf{1}^\perp$. ■

From Theorem 4.1 it follows that the codes $\mathcal{C}_N$ and $\mathcal{C}_A$ only have weights $w_i = i(n-i)$ $(0 \le i \le \frac{n}{2})$. In n is odd, the number of codewords of weight w_i equals $\binom{n}{i}$ (for both $\mathcal{C}_N$ and $\mathcal{C}_A$). If n is even, $\mathcal{C}_N$ has $\binom{n}{i}$ codewords of weight w_i for $0 \le i < \frac{n}{2}$ and $\frac{1}{2}\binom{n}{n/2}$ codewords of weight $w_{n/2}$. The code $\mathcal{C}_A$ consists of the codewords from $\mathcal{C}_N$ with even weight.

4.2. Lattice Graphs

The lattice graph $L(m)$ is the line graph of the complete bipartite graph $K_{m,m}$. It is strongly regular with parameters $v = m^2$, $k = 2(m-1)$, $\lambda = m-2$, $\mu = 2$, $r = n-2$ and $s = -2$. If $m \ne 4$, $L(m)$ is determined by these parameters. Similar to above the adjacency matrix $A = M^\top M \pmod 2$ if M is the vertex-edge incidence matrix of $K_{m,m}$. The code $\mathcal{C}_M^\perp$ consist of the edge sets of $K_{m,m}$ that form a union of Euler graphs. The code $\mathcal{C}_M$ has dimension $2m-1$ and consists of disjoint unions of two bipartite graphs, one on $m_1 + m_2$ and one on $(m-m_1)+(m-m_2)$ vertices. Each choice of m_1, m_2 $(0 \le m_1 \le m,\ 0 \le m_2 \le m/2)$ gives codewords of weight $m_1m_2 + (m-m_1)(m-m_2)$. The number of these codewords equals $\binom{m}{m_1}\binom{m}{m_2}$ if $m_2 < m/2$ and $\frac{1}{2}\binom{m}{m_1}\binom{m}{m/2}$ if $m_2 = m/2$ (but note that different choices for m_1, m_2 can lead to the same weight). The weight enumerators of the codes $\mathcal{C}_A$ now follow easily from the next result.

THEOREM 4.2 *Let Γ be the lattice graph $L(m)$. If m is even then $\mathcal{C}_A$ consists of the graphs from $\mathcal{C}_M$ with $m_1 + m_2$ odd, and moreover, $\mathcal{C}_A + \langle \mathbf{1} \rangle = \mathcal{C}_M$ and $\mathcal{C}_{A+I} = \mathcal{C}_{\overline{A}} = \mathbf{V}$.*
If m is odd then $\mathcal{C}_A$ consists of the graphs from $\mathcal{C}_M$ with $m_1 + m_2$ even, and moreover, $\mathcal{C}_A = \mathcal{C}_M \cap \mathbf{1}^\perp$, $\mathcal{C}_{A+I} = \mathcal{C}_A^\perp$ and $\mathcal{C}_{\overline{A}} = \mathcal{C}_{A+I} \cap \mathbf{1}^\perp$.

Proof. From $M^\top M = A \pmod 2$, we deduce $\mathcal{C}_A \subseteq \mathcal{C}_M$ and $\dim(\mathcal{C}_A) \ge \dim(\mathcal{C}_M) - 1 = 2m-2$. Let $\chi \in \mathbf{F}_2^v$ represent a subgraph of $K_{m,m}$ with all vertex degrees odd (if m is odd, we may choose $\chi = \mathbf{1}$). Then $\chi \in \mathcal{C}_A^\perp$, but $\chi \notin \mathcal{C}_M^\perp$, hence $\mathcal{C}_A = \mathcal{C}_M \cap \chi^\perp$. Now all statements follow straightforwardly. ■

4.3. Paley Graphs

Suppose $v = 1 \pmod 4$ is a prime power. The *Paley graph* has $\mathbf{F}_v$ as vertex set and two vertices are adjacent if the difference is a non-zero square in $\mathbf{F}_v$. The Paley graph is a $SRG(v, (v-1)/2, (v-1)/4 - 1, (v-1)/4)$ and isomorphic to its complement. By Propositions 3.1 and 3.2, the code $\mathcal{C}_A$ of a Paley graph is only non-trivial if $v = 1 \pmod 8$. Then $\mathcal{C}_A$ and $\mathcal{C}_{A+I}$ are well known as the (binary) quadratic residue codes, see for example [10] or [14] (which are usually only defined for primes v). For $v = 5$, 9, 13 and 17, the Paley graph is the only one with the given parameters. If $v \ge 25$, other graphs with the same parameters exist. If $v = 5 \pmod 8$ all these graphs give isomorphic (trivial) codes. If $v = 25$ or 41 (see Section 6), the known non-isomorphic graphs give non-isomorphic codes and amongst them, the codes of the Paley graphs have the largest minimum distance. We conjecture that the second part of this statement is true in general.

4.4. Symplectic Graphs

Let V be a vector space of dimension $2n$ over $\mathbf{F}_2$ provided with a nondegenerate symplectic form $B\colon V \times V \to \mathbf{F}_2$. Now the symplectic graph $\mathcal{S}p(2n, 2)$ is the graph of the perpendicular relation induced on the non-zero vectors of V. So by definition its complementary graph $\overline{\mathcal{S}p(2n, 2)}$ has adjacency matrix

$$A = [B(u, v)]_{u,v \in V \setminus \{0\}}$$

of 2-rank $2n$. There are essentially two quadratic forms $Q\colon V \to \mathbf{F}_2$ which have B as their associated symplectic form; one, Q^+ say, with $2^{2n-1} + 2^{n-1}$ zeros and one, Q^- say, with $2^{2n-1} - 2^{n-1}$ zeros. For each of these there is a partition of $\mathcal{S}p(2n, 2)$ into two subgraphs $\mathcal{N}^{\epsilon}_{2n}$ and $\mathcal{S}^{\epsilon}_{2n}$ of vectors achieving value 1 or 0 under Q^{ϵ} respectively. So for the adjacency matrix of $\overline{\mathcal{N}^{\epsilon}_{2n}}$ we have

$$A = [B(u, v)]_{u,v \in V \setminus \{0\}, Q^{\epsilon}(u)=Q^{\epsilon}(v)=1}$$

and for the adjacency matrix of $\overline{\mathcal{S}^{\epsilon}_{2n}}$:

$$A = [B(u, v)]_{u,v \in V \setminus \{0\}, Q^{\epsilon}(u)=Q^{\epsilon}(v)=0}$$

so both have 2-rank equal to $2n$. The graphs $\mathcal{S}p(2n, 2)$, $\mathcal{N}^{\epsilon}_{2n}$ and $\mathcal{S}^{\epsilon}_{2n}$ are strongly regular with parameters as displayed in Table 1.

A graph G is said to possess the *cotriangle property* if for every pair $\{x, y\}$ of non-adjacent vertices in G there exists a third vertex z forming a subgraph $T = \{x, y, z\}$ isomorphic to a cotriangle (i.e. $\overline{K_3}$) having the property that any vertex u of G not lying in the cotriangle T is adjacent to exactly one or all of the vertices of T. Similarly, a graph G is said to have the *triangle property* if, whenever $\{x, y\}$ is an edge in G, a third vertex z adjacent to both x and y can be found such that any vertex of G not lying in the triangle $\{x, y, z\} = T$ is adjacent to one or all members of T. In terms of the adjacency matrix of its complement, G has the cotriangle property if and only if the sum modulo 2 of two rows of the adjacency matrix of $\overline{G}$ corresponding to two adjacent (in $\overline{G}$) vertices is again a row of this adjacency matrix and G has the triangle property if and only if the sum modulo 2 of two rows of the adjacency matrix of $\overline{G}$ corresponding to two non-adjacent vertices is again a row of this matrix.

Using this it follows straightforwardly from the definitions that the graphs $\mathcal{S}p(2n, 2)$ and $\mathcal{N}^{\epsilon}_{2n}$ satisfy the cotriangle property and the graphs $\mathcal{S}p(2n, 2)$ and $\mathcal{S}^{\epsilon}_{2n}$ satisfy the triangle property.

Let A be the adjacency matrix of the complement of the symplectic graph $\mathcal{S}p(2n, 2)$, then

$$A = M \Delta_n M^T,$$

where M is the $(2^{2n} - 1) \times 2n$ matrix whose rows are precisely all $2^{2n} - 1$ nonzero binary vectors of length $2n$ and Δ_n is the block diagonal matrix with n diagonal blocks of the form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Since M^T is the parity check matrix of the Hamming code $\mathcal{H}_{2n}$, the code $\mathcal{C}_A$ is the dual of this Hamming code: the binary simplex code. The rows of A are precisely all nonzero codewords of $\mathcal{C}_A$.

Table 1. Parameters of the strongly regular graphs related to the symplectic graphs.

Name	v	k	λ r	μ s
$\mathcal{S}p(2n,2)$	$2^{2n}-1$	$2^{2n-1}-2$	$2^{2n-2}-3$ $2^{n-1}-1$	$2^{2n-2}-1$ $-2^{n-1}-1$
$\overline{\mathcal{S}p(2n,2)}$	$2^{2n}-1$	2^{2n-1}	2^{2n-2} 2^{n-1}	2^{2n-2} -2^{n-1}
$\mathcal{N}_{2n}^{+}$	$2^{2n-1}-2^{n-1}$	$2^{2n-2}-1$	$2^{2n-3}-2$ $2^{n-2}-1$	$2^{2n-3}+2^{n-2}$ $-2^{n-1}-1$
$\overline{\mathcal{N}_{2n}^{+}}$	$2^{2n-1}-2^{n-1}$	$2^{2n-2}-2^{n-1}$	$2^{2n-3}-2^{n-2}$ 2^{n-1}	$2^{2n-3}-2^{n-1}$ -2^{n-2}
$\mathcal{N}_{2n}^{-}$	$2^{2n-1}+2^{n-1}$	$2^{2n-2}-1$	$2^{2n-3}-2$ $2^{n-1}-1$	$2^{2n-3}-2^{n-2}$ $-2^{n-2}-1$
$\overline{\mathcal{N}_{2n}^{-}}$	$2^{2n-1}+2^{n-1}$	$2^{2n-2}+2^{n-1}$	$2^{2n-3}+2^{n-2}$ 2^{n-2}	$2^{2n-3}+2^{n-1}$ -2^{n-1}
$\mathcal{S}_{2n}^{+}$	$2^{2n-1}+2^{n-1}-1$	$2^{2n-2}+2^{n-1}-2$	$2^{2n-3}+2^{n-1}-3$ $2^{n-1}-1$	$2^{2n-3}+2^{n-2}-1$ $-2^{n-2}-1$
$\overline{\mathcal{S}_{2n}^{+}}$	$2^{2n-1}+2^{n-1}-1$	2^{2n-2}	$2^{2n-3}-2^{n-2}$ 2^{n-2}	2^{2n-3} -2^{n-1}
$\mathcal{S}_{2n}^{-}$	$2^{2n-1}-2^{n-1}-1$	$2^{2n-2}-2^{n-1}-2$	$2^{2n-3}-2^{n-1}-3$ $2^{n-2}-1$	$2^{2n-3}-2^{n-2}-1$ $-2^{n-1}-1$
$\overline{\mathcal{S}_{2n}^{-}}$	$2^{2n-1}-2^{n-1}-1$	2^{2n-2}	$2^{2n-3}+2^{n-2}$ 2^{n-1}	2^{2n-3} -2^{n-2}

It follows from the triangle property (see [17]) that for $\mathcal{S}_{2n}^{\epsilon}$ all codewords of $\mathcal{C}_{\overline{A}}$ are linear combinations of at most two rows from $\overline{A}$. Except for the zero codeword, all codewords have either weight 2^{2n-2} or weight $2^{2n-2}\pm 2^{n-1}$. The rows of $\overline{A}$ are precisely all $2^{2n-1}\pm 2^{n-1}-1$ codewords of weight 2^{2n-2}. So the weight enumerator of $\mathcal{C}_{\overline{A}}$ of length $2^{2n-1}\pm 2^{n-1}-1$ is

weight	0	2^{2n-2}	$2^{2n-2}\pm 2^{n-1}$
number	1	$2^{2n-1}\pm 2^{n-1}-1$	$2^{2n-1}\mp 2^{n-1}$

Similarly one can derive that for $\mathcal{N}_{2n}^{\epsilon}$ the code $\mathcal{C}_{\overline{A}}$ of length $2^{2n-1}\pm 2^{n-1}$ has weight enumerator

weight	0	2^{2n-2}	$2^{2n-2}\pm 2^{n-1}$
number	1	$2^{2n-1}\mp 2^{n-1}-1$	$2^{2n-1}\pm 2^{n-1}$

Again, the rows of $\overline{A}$ are precisely all codewords of weight $2^{2n-2}\pm 2^{n-1}$.

4.5. Graphs from Designs

Let D denote a 1-(n, κ, m) design where two distinct blocks have at most one point in common (i.e. D is a partial linear space). Then the *block graph* $\Gamma(D)$ has the blocks of D as vertices and two vertices are adjacent whenever the blocks intersect. Let N denote the point-block incidence matrix of D. Clearly $A = N^\top N - \kappa I$ is the adjacency matrix of $\Gamma(D)$ and so $\mathcal{C}_A = \mathcal{C}_{N^\top N} \subseteq \mathcal{C}_N$ if κ is even and $\mathcal{C}_{A+I} = \mathcal{C}_{N^\top N} \subseteq \mathcal{C}_N$ if κ is odd. It is well-known that if D is a 2-$(n, \kappa, 1)$ design, then $\Gamma(D)$ is a $SRG(m^2 - m(m-1)/\kappa, \kappa(m-1), \kappa^2 - 2\kappa + m - 1, \kappa^2)$, and if D is a transversal design $TD_1(\kappa, m)$ (or dually, a net of order m and degree κ), then $\Gamma(D)$ is a $SRG(m^2, \kappa(m-1), \kappa^2 - 3\kappa + m, \kappa(\kappa-1))$. If $\kappa = 2$, $\Gamma(D)$ is a triangular or a lattice graph and the related codes are given above. If $\kappa = 3$ D is a Steiner triple system $STS(n)$ or a Latin square ($LS(m)$). Then the situation is already much more difficult, but the dimensions of $\mathcal{C}_N$ and $\mathcal{C}_{A+I}$ are known in terms of the number of sub-triple systems and quotient Latin squares, see [11], [15] and [17]. In some cases the relation between $\mathcal{C}_N$ and $\mathcal{C}_{A+I}$ is easy.

PROPOSITION 4.1 *If D is an $STS(n)$ then*

i. if $n = 1 \pmod 4$ (i.e. m is even), then $\mathcal{C}_{A+I} = \mathcal{C}_N$ and $\dim(\mathcal{C}_{A+I}) = n$;

ii. if $n = 3 \pmod 4$ (i.e. m is odd), then $\dim(\mathcal{C}_{A+I}) = 2\dim(\mathcal{C}_N) - n$ (so $\mathcal{C}_{A+I} = \mathcal{C}_N$ if and only if $\dim(\mathcal{C}_N) = n$).

If D represents an $LS(m)$ then $\dim(\mathcal{C}_N) \le 3m - 2$ and

iii. if m is odd then $\mathcal{C}_{A+I} = \mathcal{C}_N$ and $\dim(\mathcal{C}_{A+I}) = 3m - 2$;

iv. if m is even then $\dim(\mathcal{C}_{A+I}) \le 3m - 4$ with equality if and only if $\dim(\mathcal{C}_N) = 3m - 2$; equality also implies that $\mathcal{C}_{A+I} = \mathcal{C}_N \cap \mathcal{C}_N^\perp$.

Proof. The cases i and iii follow from Proposition 3.2 and the results about dimensions in ii and iv can be found in Chapter 3 of [17]. So we are left with the last statement. We have $NN^\top = (J_3 + I_3) \otimes J_m \pmod 2$ and $\dim(\mathcal{C}_N \cap \mathcal{C}_N^\perp) \le \dim(\mathcal{C}_N) - 2\text{-rank}(NN^\top) = 3m - 4$. Moreover, $NN^\top N = 0$, so $\mathcal{C}_{A+I} \perp \mathcal{C}_N$ and hence $\mathcal{C}_{A+I} \subseteq \mathcal{C}_N \cap \mathcal{C}_N^\perp$ and the result follows. ∎

For Steiner triple systems the problem has been raised (see [22]) whether or not non-isomorphic designs always give non-isomorphic codes $\mathcal{C}_N$. This is true for $n \le 15$. If $\dim(\mathcal{C}_{A+I}) < n$ (the $STS(n)$ has subsystems) then $\mathcal{C}_{A+I} \ne \mathcal{C}_N$. We checked the 80 block graphs of the $STS(15)$'s ($n < 15$ is trivial) and found that also the codes $\mathcal{C}_{A+I}$ are mutually non-isomorphic (see Table 7). However, we did find some examples of non-isomorphic strongly regular graphs with the parameters of the block graph of an $STS(15)$, but with isomorphic codes $\mathcal{C}_{A+I}$ of dimension 15. Because of this, we believe that the question above has a negative answer.

The binary codes of Latin squares have also been studied by Assmus [2]. Similar to the situation for Steiner triple systems, he wonders if non-isomorphic Latin squares (regarded as nets of degree 3) give non-isomorphic codes C_N. This is true for $m \leq 7$. In particular if $m = 4$ the codes C_N of the two Latin squares even have different dimension. However the codes C_{A+I} of the graphs are isomorphic, because they correspond to the same 2-(16, 10, 6) design (see the end of Section 2).

5. Two-Graph Codes

We briefly explain Seidel switching. For details we refer to [10]. Let $\Gamma = (V, E)$ be a graph and let $\{V_1, V \setminus V_1\}$ be a partition of V, then we define the result of switching Γ with respect to this partition to be the graph $\Gamma' = (V, E')$ whose edges are those edges of Γ contained in V_1 or $V \setminus V_1$ together with the pairs $\{v_1, v_2\}$, with $v_1 \in V_1, v_2 \in V \setminus V_1$ for which $\{v_1, v_2\} \notin E$. The graphs Γ and Γ' are said to be switching equivalent. It is not hard to check that switching defines an equivalence relation on graphs. An equivalence class is called a two-graph. Note that, if we switch with respect to the set of neighbours Γ_x of a vertex x, then x becomes an isolated vertex in Γ'. If we order the vertices in a suitable way then, in terms of the adjacency matrices A and A', Seidel switching comes down to

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{12}^\top & A_2 \end{bmatrix}, A' = \begin{bmatrix} A_1 & A_{12} + J \\ A_{12}^\top + J & A_2 \end{bmatrix} \pmod 2.$$

Suppose we switch with respect to a subset V_1 of V with characteristic vector χ. Then we have

$$\mathcal{C}_A + \langle \mathbf{1} \rangle + \langle \chi \rangle = \mathcal{C}_{A'} + \langle \mathbf{1} \rangle + \langle \chi \rangle .$$

Let us not worry about $\mathbf{1}$ and look at the codes $\mathcal{C}_{A+J} = \mathcal{C}_A + \langle \mathbf{1} \rangle$ and $\mathcal{C}_{A'+J}$. It is clear that if $\chi \in \mathcal{C}_{A+J}$ then $\mathcal{C}_{A'+J} \subseteq \mathcal{C}_{A+J}$. Suppose Γ and Γ' both have an isolated vertex (not the same one) then χ is in $\mathcal{C}_{A+J}$ and $\mathcal{C}_{A'+J}$, hence $\mathcal{C}_{A+J} = \mathcal{C}_{A'+J}$. So this code is independent of the isolated vertex and we will call it the *two-graph code*. Note that $\mathbf{1} \notin \mathcal{C}_A$ (because of the isolated vertex), so $\dim(\mathcal{C}_{A+J}) = \dim(\mathcal{C}_A) + 1$ is odd.

Assume Γ is a $SRG(v, k, \lambda, \mu)$ with $k = 2\mu$ (or equivalently, $k = -2rs$). Extend Γ with an isolated vertex x to $\tilde{\Gamma}$ (i.e. $\tilde{\Gamma} \setminus \{x\} = \Gamma$). If we switch in $\tilde{\Gamma}$ to $\tilde{\Gamma}'$, such that another vertex y becomes isolated, then it follows that $\Gamma' = \tilde{\Gamma}' \setminus \{y\}$ is again a $SRG(v, k, \lambda, \mu)$, but not necessarely isomorphic to Γ. In this case the switching class of $\tilde{\Gamma}$ is called a *regular two-graph* and Γ (and Γ') is the *descendant* of $\tilde{\Gamma}$ with respect to x (and y). Clearly, the code $\mathcal{C}_A$ of a descendant is the shortened code of the corresponding two-graph code. Regular two-graphs can produce interesting two-graph codes. For example the Paley graph is the descendant of a regular two-graph and the corresponding two-graph code is the extended quadratic residue code. Also the symplectic graphs are the descendants of a regular two-graphs with two-graph codes the first order Reed-Muller codes. For other intersting two-graph codes, see [7] and [12]. If $\tilde{\Gamma}$ can be switched into a regular graph $\tilde{\Gamma}'$, then it follows that $\tilde{\Gamma}'$ is strongly regular with the same r and s as Γ, but with two possibilities for the valency: Either $k = -2rs - r$ or $k = -2rs - s$ (so r and s need to be integral). On the other hand, a strongly regular graph with degree $-2rs - r$ or $-2rs - s$ is in the switching class of a regular two-graph (so isolating a vertex yields a strongly regular graph with

$k = -2rs$). For example the Shrikhande graph, $L(4)$ and the complement of the Clebsch graph are switching equivalent We observed already that these three graphs generate the same (6-dimensional) code. By isolating a vertex we get $T(6)$ and the two-graph code is a 5-dimensional subspace of the $L(4)$ code. The shortened code (with respect to any vertex) is the 4-dimensional code of $T(6)$.

THEOREM 5.1 *Suppose Ω is a regular two-graph with eigenvalues r and s and two-graph code $\mathcal{C}$. Suppose Γ is a k-regular graph in Ω (so Γ is strongly regular) and let Δ be the graph in Ω with a given vertex x isolated (so switching in Γ with respect to the neighbors Γ_x of x gives Δ). Let A and B be the adjacency matrices of Γ and Δ respectively, and let χ denote the characteristic vector of the switching set Γ_x. Then either*

$$\left.\begin{array}{l} \dim \mathcal{C}_A = \dim \mathcal{C}_B = \dim \mathcal{C} - 1 \\ \mathbf{1} \notin \mathcal{C}_A \\ \chi \in \mathcal{C}_B \\ \mathcal{C}_{A+J} = \mathcal{C}_A + \langle \mathbf{1} \rangle = \mathcal{C}_B + \langle \mathbf{1} \rangle = \mathcal{C} \end{array}\right\} \text{ or } \left\{\begin{array}{l} \dim \mathcal{C}_A - 2 = \dim \mathcal{C}_B = \dim \mathcal{C} - 1 \\ \mathbf{1} \in \mathcal{C}_A \\ \chi \notin \mathcal{C}_B \\ \mathcal{C}_{A+J} = \mathcal{C}_A = \mathcal{C}_B + \langle \mathbf{1} \rangle + \langle \chi \rangle = \mathcal{C} + \langle \chi \rangle. \end{array}\right.$$

If k is even and $r + s$ is odd, we are in the first case.
If $k = 2 \bmod 4$ and $r + s$ is even, or k is odd, we are in the second case.

Proof. The results follow from the fact that

$$\mathcal{C}_A + \langle \mathbf{1} \rangle + \langle \chi \rangle = \mathcal{C}_B + \langle \mathbf{1} \rangle + \langle \chi \rangle,\ \mathcal{C}_A + \langle \mathbf{1} \rangle = \mathcal{C}_{A+J},\ \mathcal{C} = \mathcal{C}_B + \langle \mathbf{1} \rangle,$$

and that $\dim \mathcal{C}_A$ and $\dim \mathcal{C}_B$ are even. Clearly $\mathbf{1} \notin \mathcal{C}_B$, $\mathbf{1} \in \mathcal{C}_{A+J}$ and $\chi \in \mathcal{C}_A$.
If $\mathbf{1} \in \mathcal{C}_A$ then $\mathcal{C}_A = \mathcal{C}_{A+J}$ and $\mathcal{C}_A = \mathcal{C}_B + \langle \mathbf{1} \rangle + \langle \chi \rangle$, so $\mathcal{C}_B$ is a proper subspace of $\mathcal{C}_A$ and hence $\dim \mathcal{C}_A = \dim \mathcal{C}_B + 2$ and $\chi \notin \mathcal{C}_B$. On the other hand, if $\mathbf{1} \notin \mathcal{C}_A$, then χ must be a codeword of $\mathcal{C}_B$ and $\dim \mathcal{C}_A = \dim \mathcal{C}_B$. Furthermore, $\mathcal{C}_{A+J} = \mathcal{C}_A + \langle \mathbf{1} \rangle = \mathcal{C}_B + \langle \mathbf{1} \rangle = \mathcal{C}$.

If k is even and $r + s$ is odd, then $\mu = k + rs$ is even and $\lambda = \mu + r + s$ is odd. Now the rows of B corresponding to Γ_x add up to the characteristic vector χ of Γ_x. So $\chi \in \mathcal{C}_B$ and hence we are in the first case.

It is clear that $\mathbf{1} \in \mathcal{C}_A$ if k is odd. Suppose $k = 2 \bmod 4$ and $r + s$ is even. Then r and s are both even (since $-k = 2rs + s$ or $2rs + r$). Let B' be the adjacency matrix of the descendant $\Delta' = \Delta \setminus \{x\}$. Then $\mathcal{C}_{B'}$ is self-orthogonal by 3.2.v. Moreover, the degree of Δ' is $2rs$, which is divisible by 4, and hence all weights in $\mathcal{C}_{B'}$ and $\mathcal{C}_B$ are divisible by 4. Therefore $\chi \notin \mathcal{C}_B$, so we are in the second case. ■

For example, the last statement implies that $\mathbf{1} \in \mathcal{C}_A$ for an $SRG(36, 14, 4, 6)$. If k is even and $r + s$ is odd then $\mathcal{C} = \mathcal{C}_{A+J}$. So, in this case, non-isomorphic switching equivalent strongly regular graphs give isomorphic codes of the form $\mathcal{C}_{A+J}$. Examples are given by the switching equivalent $SRG(26, 10, 3, 4)$'s (see the next section).

It is clear that if two two-graph codes are isomorphic then so are the codes of corresponding descendants. And vice versa, two descendants Γ_1 and Γ_2 with isomorphic codes $\mathcal{C}_{A_1} = \mathcal{C}_{A_2}$ give isomorphic two-graph codes. Among the regular two-graphs on 36 vertices ($r = 2$, $s = -4$) we found several non-isomorphic ones with isomorphic two-graph codes, therefore we also have non-isomorphic $SRG(35, 16, 6, 8)$'s with isomorphic codes $\mathcal{C}_A$.

Table 2. Primitive strongly regular graphs on fewer than 45 vertices.

no.	(v, k, λ, μ)	a name	#graphs	#two-graphs	dim($\mathcal{C}_A$)	dim($\mathcal{C}_{\overline{A}}$)
1	(5,2,0,1)	pentagon (Paley)	1	1	4	4
2	(9,4,1,2)	$L(3)$	1	1	4	4
3	(10,3,0,1)	Petersen ($\overline{T(5)}$)	1		6	4
4	(13,6,2,3)	Paley	1	1	12	12
5	(15,6,1,3)	$\overline{T(6)}$	1	1	14	4
6	(16,5,0,2)	Clebsch	1		16	6
7	(16,6,2,2)	$L(4)$	2		6	16
8	(17,8,3,4)	Paley	1	1	8	8
9	(21,10,3,6)	$\overline{T(7)}$	1		14	6
10	(25,8,3,2)	$L(5)$	1		8	16
11	(25,12,5,6)	$LS(5)$	15	4	12	12
12	(26,10,3,4)	$\overline{STS(13)}$	10		12	14
13	(27,10,1,5)	Schläfli	1	1	26	6
14	(28,12,6,4)	$T(8)$	4		6, 8	28
15	(29,14,6,7)	Paley	41	6	28	28
16	(35,16,6,8)	$\overline{STS(15)}$	$\geq$ 3854	$\geq$ 227	6, . . . , 14	34
17	(36,10,4,2)	$L(6)$	1		10	36
18	(36,14,4,6)	HJsub	$\geq$ 180		8, . . . , 14	36
19	(36,14,7,4)	$T(9)$	1		8	27
20	(36,15,6,6)	$LS(6)$	$\geq$ 32548		36	6, . . . , 16
21	(37,18,8,9)	Paley	$\geq$ 82	$\geq$ 11	36	36
22	(40,12,2,4)	$GQ(3, 3)$	28		10, . . . , 16	40
23	(41,20,9,10)	Paley	$\geq$ 120	$\geq$ 18	20	20

6. Small Cases

In Table 2 we give the parameters of all primitive strongly regular graphs on fewer than 45 vertices (up to taking complements). We indicate how many non-isomorphic graphs there exist with the given parameters and, if $k = 2\mu$ we give the number of corresponding non-isomorphic regular two-graphs. In the previous sections we have obtained the codes of most of these graphs. For the parameters no. 11, 12, 13, 14, 16, 18, 20, 22 and 23 we have no complete answer yet. These cases have been investigated by computer.

6.1. Case No. 11 and 12

There are exactly four non-isomorphic regular two-graphs on 26 vertices with eigenvalues 2 and -3. Together they have fifteen $SRG(25, 12, 5, 6)$'s (two from $LS(5)'s$ one of which is the Paley graph) as a descendant and ten $SRG(26, 15, 8, 9)$'s (two from $STS(13)'s$) in the switching class, see [16] and [1]. The corresponding codes of the form $\mathcal{C}_A$ have been generated and the weight enumerators are given in Table 3 and Table 4 (keeping the names and order from [16]; the lines give the partition into the four switching-equivalence classes (two-graphs)). All codes turn out to be non-isomorphic. In most cases this follows from the weight enumerator, but in some cases more information is needed. To distinguish between

Table 3. Weight enumerators of the codes of the $SRG(25, 12, 5, 6)$'s.

name	dim	0	4	6	8	10	12	14	16	18	20	22
s1	12	1		50	225	880	1225	1050	550	100	15	
s2	12	1	10	37	279	712	1343	1140	432	124	15	3
s3	12	1	12	43	279	696	1331	1152	448	124	9	1
s4	12	1	4	54	213	868	1237	1062	546	96	15	
s5	12	1	4	66	225	832	1201	1098	582	84	3	
s6	12	1	3	51	213	876	1243	1056	538	96	18	1
s7	12	1		54	225	864	1225	1074	550	84	15	4
s8	12	1	6	32	291	728	1331	1122	436	132	15	2
s9	12	1	8	38	291	712	1319	1134	452	132	9	
s10	12	1	7	39	295	708	1313	1140	456	128	8	1
s11	12	1	5	41	303	700	1301	1152	464	120	6	3
s12	12	1	7	35	291	720	1325	1128	444	132	12	1
s13	12	1	6	36	295	716	1319	1134	448	128	11	2
s14	12	1	7	35	291	720	1325	1128	444	132	12	1
s15	12	1	6	44	303	692	1295	1158	472	120	3	2

s12 and s14 we look at the codewords of weight 4 and count the number of times a 1 occurs on every coordinate. We find the following multisets:

$$\text{s12}: \{0^4, 1^{14}, 2^7\}\ ; \quad \text{s14}: \{0^5, 1^{13}, 2^6, 3^1\}.$$

So the two codes are non-isomorphic. Similarly, we computed these multisets for st21, st23, st22 and st25 for weight 8 and found:

$$\text{st21: } \{27^1, 31^9, 35^7, 39^7, 43^2\}\ ; \qquad \text{st23: } \{27^4, 31^5, 35^5, 39^{11}, 43^1\};$$

$$\text{st22: } \{23^2, 27^2, 31^5, 35^7, 39^6, 43^3, 47^1\}\ ; \qquad \text{st25: } \{15^1, 31^9, 35^3, 39^{12}, 43^1\}.$$

It follows that also the four two-graph codes are non-isomorphic and by Theorem 5.1 we have that the ten graphs on 26 vertices give rise to just four non-isomorphic codes of the form $\mathcal{C}_{\overline{A}+J}$ $(= \mathcal{C}_A^{\perp} + \langle \mathbf{1} \rangle)$. In other words, by deleting the words of odd weight, the ten codes of length 26 collapse to the four two-graph codes.

6.2. Case No. 13 and 14

There exist exactly four non-isomorphic $SRG(28, 12, 6, 4)$'s, being $T(8)$ and the three Chang graphs. All these graphs are switching equivalent, so there is a unique regular two-graph. Theorem 4.1 gives the code $\mathcal{C}_A$ of $T(8)$, which has dimension 6. The three Chang graphs can be constructed by switching in $T(8)$ on the vertices corresponding in K_8 with four disjoint edges, an 8-cycle and with a combination of a 3-cycle and a 5-cycle respectively. The weight enumerators are given in Table 5. Note that Theorem 5.1 implies

Table 4. Weight enumerators of the codes of the $SRG(26, 15, 8, 9)$'s.

		0	4	5	6	7	8	9	10	11	12	13
name	dim	26	22	21	20	19	18	17	16	15	14	
ls11	14	1		10	65	190	325	740	1430	1826	2275	2660
st11	14	1	13		52	130	403	884	1144	1950	2483	2264
st12	14	1	13	24	52	130	403	788	1144	1950	2483	2408
ls21	14	1	4	14	69	190	309	724	1414	1826	2299	2684
ls22	14	1	4	10	69	190	309	740	1414	1826	2299	2660
st21	14	1	8	26	47	130	423	780	1164	1950	2453	2420
st22	14	1	8	22	47	130	423	796	1164	1950	2453	2396
st23	14	1	8	26	47	130	423	780	1164	1950	2453	2420
st24	14	1	8	10	47	130	423	844	1164	1950	2453	2324
st25	14	1	8	22	47	130	423	796	1164	1950	2453	2396

Table 5. Weight enumerators of the codes of the $SRG(28, 12, 6, 4)$'s.

name	dim	0	4	8	12	16	20	24	28
T(8)	6	1			28	35			
Chang 1	8	1		6	121	121	6		1
Chang 2	8	1		6	121	121	6		1
Chang 3	8	1	1	3	123	123	3	1	1

that $\mathbf{1} \in \mathcal{C}_A$ whenever the dimension of $\mathcal{C}_A$ equals 8, that is, for the Chang graphs. Chang 1 and Chang 2 have the same weight enumerator. For the invariant multisets of the codewords of weight 8 we find:

$$\text{Chang 1}: \{0^4, 2^{24}\}; \ \text{Chang 2}: \{1^{18}, 3^{10}\}.$$

Hence these codes are not isomorphic. The code $\mathcal{C}_{A+J}$ of $T(8)$ has dimension 7 (since $\mathbf{1} \notin \mathcal{C}_A$), and therefore $\mathcal{C}_{A+J}$ is the corresponding two-graph code with weight enumerator $0^1\ 12^{63}\ 16^{63}\ 28^1$. The descendant is the unique $SRG(27, 16, 10, 8)$, the complement of the Schläfli graph. The code of this decendant is the shortened two-graph code. It has weight enumerator $0^1\ 12^{36}\ 16^{27}$. Since the Schläfli graph is isomorphic with S_6^- (see Section 4.4), it has the triangle property, so the nonzero words are just the rows of A together with the mod 2 sums of pairs of rows that correspond to adjacent vertices.

6.3. Case No. 16, 18 and 20

There are 3854 non-isomorphic $SRG(35, 16, 6, 8)$ known, descending from 227 regular two-graphs, see Spence [19]. We computed the two-graph codes of these 227 regular two-graphs. The weight enumerators of the two-graph codes are in Table 6. (with the numeration

Table 6. Weight enumerators of the two-graph codes of the 227 known regular two-graphs on 36 points (weights 0 and 36 are omitted). The letters show which codes are isomorphic.

no.	dim	4 32	8 28	12 24	16 20	no.	dim	4 32	8 28	12 24	16 20	no.	dim	4 32	8 28	12 24	16 20
1	7				63	f 77	15	13	176	1944	14250	153	15	9	228	1812	14334
2	9	1	4	12	238	78	13	1	48	480	3566	y 154	15	21	232	1728	14402
3	11	3	18	78	924	k 79	15	9	164	2004	14206	z 155	15	9	228	1812	14334
4	11	5	16	72	930	l 80	15	7	174	1986	14216	z 156	15	9	228	1812	14334
5	11	2	19	81	921	k 81	15	9	164	2004	14206	z 157	15	9	228	1812	14334
6	11		21	87	915	82	13	1	52	468	3574	y 158	15	21	232	1728	14402
7	11	2	15	93	913	h 83	15	15	198	1866	14304	x 159	15	45	240	1560	14538
8	13	7	54	426	3608	n 84	15	5	152	2064	14162	160	15	4	101	2223	14055
9	13	3	58	438	3596	85	15	4	157	2055	14167	161	15	4	101	2223	14055
10	13	4	53	447	3591	86	13		40	510	3545	162	15	4	97	2235	14047
11	13	6	59	417	3613	j 87	15	6	147	2073	14157	163	15	4	105	2211	14063
a 12	13	6	59	417	3613	88	15	9	132	2100	14142	164	15	4	97	2235	14047
13	13	7	62	402	3624	m 89	15	7	142	2082	14152	165	15	5	112	2184	14082
14	13	11	58	390	3636	90	13		37	519	3539	166	15	2	91	2265	14025
15	11	3	18	78	924	91	13		34	528	3533	167	15	2	95	2253	14033
16	13	13	64	360	3658	92	15	3	114	2190	14076	168	15	2	87	2277	14017
17	11	2	19	81	921	o 93	15	3	114	2190	14076	169	15	1	88	2280	14014
b 18	13	7	62	402	3624	94	15	3	114	2190	14076	170	15	1	88	2280	14014
19	13	13	64	360	3658	o 95	15	3	114	2190	14076	171	15	1	92	2268	14022
b 20	13	7	62	402	3624	96	15	3	114	2190	14076	172	15	2	91	2265	14025
21	13	5	56	432	3602	97	13		31	537	3527	173	15	1	80	2304	13998
22	13	5	64	408	3618	p 98	15	3	114	2190	14076	174	15	1	80	2304	13998
a 23	13	6	59	417	3613	99	13		28	546	3521	175	15	1	68	2340	13974
24	13	2	63	429	3601	p 100	15	3	114	2190	14076	176	15	1	88	2280	14014
25	13	3	58	438	3596	q 101	15	13	208	1848	14314	177	15	1	88	2280	14014
26	11		19	93	911	q 102	15	13	208	1848	14314	178	15	1	84	2292	14006
c 27	13	4	61	423	3607	r 103	15	13	208	1848	14314	179	15	1	72	2328	13982
d 28	13	9	68	372	3646	s 104	15	15	198	1866	14304	180	15	1	68	2340	13974
d 29	13	9	68	372	3646	t 105	15	11	218	1830	14324	181	15	1	80	2304	13998
c 30	13	4	61	423	3607	r 106	15	13	208	1848	14314	182	15	1	92	2268	14022
31	13	3	50	462	3580	u 107	15	11	218	1830	14324	183	15	1	92	2268	14022
e 32	13	4	61	423	3607	r 108	15	13	208	1848	14314	184	13		9	603	3483
33	13	1	60	444	3590	s 109	15	15	198	1866	14304	185	13		9	603	3483
e 34	13	4	61	423	3607	q 110	15	13	208	1848	14314	186	13		5	615	3475
d 35	13	9	68	372	3646	q 111	15	13	208	1848	14314	187	15		77	2319	13987
36	13	5	56	432	3602	r 112	15	13	208	1848	14314	188	15		69	2343	13971
37	13	3	66	414	3612	s 113	15	15	198	1866	14304	189	15		77	2319	13987
38	13	2	55	453	3585	q 114	15	13	208	1848	14314	190	15		77	2319	13987
39	11		21	87	915	q 115	15	13	208	1848	14314	191	15		81	2307	13995
d 40	13	9	68	372	3646	v 116	15	23	222	1746	14392	192	15		81	2307	13995
c 41	13	4	61	423	3607	v 117	15	23	222	1746	14392	193	15		81	2307	13995
42	13	3	44	480	3568	v 118	15	23	222	1746	14392	194	15		81	2307	13995
43	15	13	176	1944	14250	119	15	15	198	1866	14304	195	15		73	2331	13979
f 44	15	13	176	1944	14250	120	15	13	208	1848	14314	196	15		81	2307	13995
g 45	15	13	176	1944	14250	v 121	15	23	222	1746	14392	197	15		81	2307	13995
46	13	3	44	480	3568	v 122	15	23	222	1746	14392	198	15		73	2331	13979
f 47	15	13	176	1944	14250	v 123	15	23	222	1746	14392	199	15		65	2355	13963
48	15	15	166	1962	14240	v 124	15	23	222	1746	14392	200	15		73	2331	13979
g 49	15	13	176	1944	14250	v 125	15	23	222	1746	14392	201	15		85	2295	14003
50	13	4	57	435	3599	t 126	15	11	218	1830	14324	202	15		69	2343	13971
51	15	25	212	1764	14382	u 127	15	11	218	1830	14324	203	15		73	2331	13979
52	13	3	50	462	3580	u 128	15	11	218	1830	14324	204	15		65	2355	13963
h 53	15	15	198	1866	14304	t 129	15	11	218	1830	14324	205	15		65	2355	13963
h 54	15	15	198	1866	14304	t 130	15	11	218	1830	14324	206	15		77	2319	13987
h 55	15	15	198	1866	14304	t 131	15	11	218	1830	14324	207	15		73	2331	13979
56	13	4	49	459	3583	v 132	15	23	222	1746	14392	208	15		69	2343	13971
57	13	3	37	501	3554	v 133	15	23	222	1746	14392	209	15		53	2391	13939
58	15	8	137	2091	14147	v 134	15	23	222	1746	14392	210	15		85	2295	14003
i 59	15	7	142	2082	14152	t 135	15	11	218	1830	14324	211	15		65	2355	13963
j 60	15	6	147	2073	14157	136	15	11	218	1830	14324	212	15		65	2355	13963
k 61	15	9	164	2004	14206	137	15	13	208	1848	14314	213	15		85	2295	14003
62	15	7	174	1986	14216	v 138	15	23	222	1746	14392	214	15		77	2319	13987
l 63	15	7	174	1986	14216	w 139	15	25	212	1764	14382	215	15		69	2343	13971
64	15	9	164	2004	14206	w 140	15	25	212	1764	14382	216	15		69	2343	13971
65	13	1	38	510	3546	v 141	15	23	222	1746	14392	217	15		69	2343	13971
66	15	7	142	2082	14152	v 142	15	23	222	1746	14392	218	13		7	609	3479
m 67	15	7	142	2082	14152	x 143	15	45	240	1560	14538	219	13		11	597	3487
68	15	7	142	2082	14152	x 144	15	45	240	1560	14538	220	13		4	618	3473
n 69	15	5	152	2064	14162	x 145	15	45	240	1560	14538	221	15		69	2343	13971
i 70	15	7	142	2082	14152	x 146	15	45	240	1560	14538	222	13		15	585	3495
71	13	7	70	378	3640	x 147	15	45	240	1560	14538	223	13		19	573	3503
72	15	15	198	1866	14304	y 148	15	21	232	1728	14402	224	13		27	549	3519
73	15	11	186	1926	14260	y 149	15	21	232	1728	14402	225	13		15	585	3495
f 74	15	13	176	1944	14250	x 150	15	45	240	1560	14538	226	15		45	2415	13923
75	13	1	48	480	3566	y 151	15	21	232	1728	14402	227	13			630	3465
76	15	13	176	1944	14250	x 152	15	45	240	1560	14538						

Table 7. Weight enumerators of the codes $\mathcal{C}_{A+I}$ of the 80 $SRG(35, 18, 9, 9)$'s, that are the block graphs of the $STS(15)$'s (weights 0 and 35 are omitted).

		4	7	8	11	12	15	16
no.	dim	31	28	27	24	23	20	19
1	7						28	35
2	9	1	1	3	1	11	115	123
3	11	2	5	10	16	77	444	469
4	11	3	9	9	3	75	457	467
5	11	2	8	11	5	76	458	463
6	11		4	15	17	76	448	463
7	11		6	15	9	78	460	455
8	13	3	13	24	138	363	1645	1909
9	13	3	21	23	105	375	1697	1871
10	13	3	24	26	96	366	1703	1877
11	13	3	20	24	110	370	1687	1881
12	13	1	12	26	136	374	1661	1885
13	13	4	32	25	67	368	1740	1859
14	13	4	25	24	94	365	1702	1881
15	13		7	27	153	375	1642	1891
16	13	1	19	29	111	369	1691	1875
17	13		7	24	150	387	1654	1873
18	13	1	20	28	106	374	1701	1865
19	13		13	27	129	381	1678	1867
20	13		7	21	147	399	1666	1855
21	13		10	27	141	378	1660	1879
22	13	7	49	21	7	371	1813	1827
23	13	1	22	30	100	368	1705	1869
24	15	5	60	52	642	1542	6345	7737
25	15	4	54	51	661	1550	6328	7735
26	15	4	53	48	662	1561	6334	7721
27	15	4	53	48	662	1561	6334	7721
28	15	4	51	46	668	1567	6330	7717
29	15	2	41	46	700	1577	6300	7717
30	15	2	42	49	699	1566	6294	7731
31	15	4	49	48	678	1557	6310	7737
32	15	1	41	51	701	1567	6295	7727
33	15	2	44	51	693	1560	6298	7735
34	15	1	39	49	707	1573	6291	7723
35	15	1	36	44	714	1590	6293	7705
36	15	1	37	47	713	1579	6287	7719
37	15	1	36	44	714	1590	6293	7705
38	15	1	39	49	707	1573	6291	7723
39	15	1	35	45	719	1585	6283	7715
40	15	1	41	51	701	1567	6295	7727
41	15	1	39	49	707	1573	6291	7723
42	15	1	39	49	707	1573	6291	7723
43	15	1	41	51	701	1567	6295	7727
44	15	2	42	49	699	1566	6294	7731
45	15	1	32	40	726	1602	6285	7697
46	15	1	29	39	737	1603	6271	7703
47	15	1	32	36	722	1618	6301	7673
48	15		27	42	744	1599	6262	7709
49	15		31	46	732	1587	6270	7717
50	15		35	50	720	1575	6278	7725
51	15		35	50	720	1575	6278	7725
52	15		30	43	733	1598	6276	7703
53	15		29	40	734	1609	6282	7689
54	15		26	39	745	1610	6268	7695
55	15		33	48	726	1581	6274	7721
56	15		32	45	727	1592	6280	7707
57	15		32	49	713	1576	6264	7731
58	15		33	48	726	1581	6274	7721
59	15		34	51	725	1570	6268	7735
60	15		33	48	726	1581	6274	7721
61	15		32	45	727	1592	6280	7707
62	15		33	48	726	1581	6274	7721
63	15		27	38	740	1615	6278	7685
64	15		34	47	721	1586	6284	7711
65	15		30	43	733	1598	6276	7703
66	15		26	39	745	1610	6268	7695
67	15		30	43	733	1598	6276	7703
68	15		32	45	727	1592	6280	7707
69	15		28	45	743	1588	6256	7723
70	15		30	47	737	1582	6260	7727
71	15		26	39	745	1610	6268	7695
72	15		28	45	743	1588	6256	7723
73	15		26	43	749	1594	6252	7719
74	15		28	41	739	1604	6272	7699
75	15		23	30	748	1643	6286	7653
76	15		26	39	745	1610	6268	7695
77	15		25	44	754	1589	6242	7729
78	15		25	44	754	1589	6242	7729
79	15		21	48	774	1569	6202	7769
80	15		15	30	780	1635	6238	7685

of Spence [19]). The letters indicate which two-graph codes are isomorphic. All together we found 158 non-isomorphic two-graph codes.

We did not compute the codes of all known $SRG(35, 16, 6, 8)$'s. We did check the code of the form $\mathcal{C}_{A+I}$ of the 80 block graphs of an $STS(15)$. The details are in Table 7. Table 8 gives the invariant multisets in case the weight enumerators are the same. It follows that all these codes are non-isomorphic. It is known that the corresponding two-graphs are non-isomorphic and hence the corresponding two-graph codes are non-isomorphic too.

Note that the possible dimensions of $\mathcal{C}_A$ are 8, 10, 12 and 14 for an $SRG(36, 14, 4, 6)$ and 6, 8, 10, 12 and 14 for an $SRG(35, 16, 6, 8)$. So by Theorem 5.1 we have the following result:

Table 8. Invariant multisets for the graphs of the $STS(15)$'s.

26: $\{7^6, 9^9, 11^4, 12^4, 13^{12}\}$
27: $\{6^2, 7^4, 8^8, 9^1, 10^2, 11^2, 13^8, 14^8\}$

30: $\{5^2, 6^6, 7^{10}, 10^{12}, 11^4, 14^1\}$
44: $\{4^1, 5^3, 6^2, 7^8, 8^4, 9^4, 10^7, 11^5, 14^1\}$

32: $\{4^2, 5^6, 6^4, 7^3, 8^5, 9^3, 10^1, 11^7, 12^3, 14^1\}$
40: $\{5^6, 6^3, 7^6, 8^6, 9^3, 10^3, 11^4, 12^4\}$
43: $\{5^3, 6^6, 7^6, 8^6, 9^7, 11^3, 12^3, 14^1\}$

34: $\{4^5, 5^5, 6^2, 7^3, 8^4, 9^3, 10^9, 11^3, 13^1\}$
38: $\{5^4, 7^{13}, 8^6, 9^6, 10^6\}$
41: $\{4^4, 5^2, 6^5, 7^6, 8^2, 9^8, 10^5, 11^2, 15^1\}$
42: $\{5^4, 6^6, 7^4, 8^6, 9^{10}, 10^4, 11^1\}$

35: $\{4^6, 5^3, 6^4, 7^3, 8^{10}, 9^5, 10^2, 11^1, 12^1\}$
37: $\{4^6, 5^2, 6^4, 7^5, 8^{10}, 9^4, 10^3, 13^1\}$

50: $\{4^5, 5^5, 6^5, 8^{10}, 9^{10}\}$
51: $\{3^2, 5^4, 6^8, 7^8, 8^8, 9^4, 15^1\}$

52: $\{3^2, 4^4, 5^6, 6^{10}, 7^7, 8^5, 9^1\}$
65: $\{3^2, 4^7, 5^7, 6^6, 7^7, 8^5, 9^1, 10^1\}$
67: $\{3^1, 4^6, 5^5, 6^{12}, 7^5, 8^4, 9^1, 10^1\}$

54: $\{3^4, 4^8, 5^9, 6^8, 7^3, 8^3\}$
66: $\{3^6, 4^6, 5^5, 6^{12}, 7^5, 8^1\}$
71: $\{3^5, 4^7, 5^6, 6^{13}, 7^1, 8^3\}$
76: $\{3^1, 4^9, 5^{13}, 6^6, 7^6\}$

55: $\{3^2, 4^5, 5^4, 6^5, 7^7, 8^4, 9^6, 10^2\}$
58: $\{3^2, 4^5, 5^8, 6^2, 7^3, 8^7, 9^5, 10^2, 11^1\}$
60: $\{3^1, 4^3, 5^6, 6^8, 7^6, 8^5, 9^5, 11^1\}$
62: $\{3^1, 4^4, 5^2, 6^8, 7^9, 8^8, 9^3\}$

56: $\{3^1, 4^3, 5^7, 6^9, 7^6, 8^5, 9^2, 10^2\}$
61: $\{3^3, 4^6, 5^3, 6^6, 7^5, 8^3, 9^9\}$
68: $\{3^1, 4^1, 5^8, 6^7, 7^{11}, 8^6, 10^1\}$

69: $\{3^4, 4^6, 5^5, 6^{11}, 7^6, 8^1, 9^1, 10^1\}$
72: $\{3^1, 4^4, 5^{12}, 6^{12}, 7^3, 8^3\}$

77: $\{4^{10}, 5^{15}, 6^{10}\}$
78: $\{4^{14}, 5^{12}, 6^6, 7^2, 9^1\}$

Table 9. Weight enumerators of the codes of the known $SRG(36, 14, 4, 6)$'s (weights 0 and 36 are omitted). The letters show which codes are isomorphic. Switching equivalent graphs are on consecutive places; "tgr" refers to the no. of the corresponding two-graph (see Table 6).

			4	6	8	10	12	14	16	18
tgr	no.	dim	32	30	28	26	24	22	20	
1	1	8						36	63	56
2	2	10	1		4	4	12	128	238	248
	3	10	1		4		12	144	238	224
3	4	12	3		18	16	78	512	924	992
4	5	12	5		16	16	72	512	930	992
	6	12	5		16		72	576	930	896
5	a 7	12	2		19	8	81	544	921	944
	a 8	12	2		19	8	81	544	921	944
	9	12	2	4	19	16	81	476	921	1056
	10	12	2		19	24	81	480	921	1040
6	11	12		4	21		87	540	915	960
	12	12			21	24	87	480	915	1040
7	b 13	12	2		15	8	93	544	913	944
	c 14	12	2		15	20	93	496	913	1016
	b 15	12	2		15	8	93	544	913	944
	c 16	12	2		15	20	93	496	913	1016
8	d 17	14	7		54	80	426	1984	3608	4064
	d 18	14	7		54	80	426	1984	3608	4064
9	e 19	14	3	4	58	88	438	1916	3596	4176
	e 20	14	3	4	58	88	438	1916	3596	4176
	e 21	14	3	4	58	88	438	1916	3596	4176
	e 22	14	3	4	58	88	438	1916	3596	4176
10	f 23	14	4	4	53	88	447	1916	3591	4176
	f 24	14	4	4	53	88	447	1916	3591	4176
	f 25	14	4	4	53	88	447	1916	3591	4176
	f 26	14	4	4	53	88	447	1916	3591	4176
11	g 27	14	6		59	80	417	1984	3613	4064
	g 28	14	6		59	80	417	1984	3613	4064
12	h 29	14	6		59	80	417	1984	3613	4064
	h 30	14	6		59	80	417	1984	3613	4064
13	i 31	14	7		62	64	402	2048	3624	3968
	i 32	14	7		62	64	402	2048	3624	3968
	i 33	14	7		62	64	402	2048	3624	3968
	i 34	14	7		62	64	402	2048	3624	3968
14	j 35	14	11		58	32	390	2176	3636	3776
	j 36	14	11		58	32	390	2176	3636	3776
15	37	12	3		18	24	78	480	924	1040
	38	12	3		18		78	576	924	896
16	39	14	13		64	64	360	2048	3658	3968
17	40	12	2		19	16	81	512	921	992
	41	12	2		19	24	81	480	921	1040
18	42	14	7		62	64	402	2048	3624	3968
19	43	14	13		64	64	360	2048	3658	3968
20	k 44	14	7	8	62	80	402	1912	3624	4192
	k 45	14	7	8	62	80	402	1912	3624	4192
21	l 46	14	5	8	56	48	432	2040	3602	4000
	l 47	14	5	8	56	48	432	2040	3602	4000
22	m 48	14	5		64	80	408	1984	3618	4064
	m 49	14	5		64	80	408	1984	3618	4064
23	h 50	14	6		59	80	417	1984	3613	4064
	h 51	14	6		59	80	417	1984	3613	4064
24	n 52	14	2	4	63	88	429	1916	3601	4176
	n 53	14	2	4	63	88	429	1916	3601	4176
	n 54	14	2	4	63	88	429	1916	3601	4176
25	o 55	14	3	4	58	88	438	1916	3596	4176
	o 56	14	3	4	58	88	438	1916	3596	4176
	o 57	14	3	4	58	88	438	1916	3596	4176
26	p 58	12			19	20	93	496	911	1016
	p 59	12			19	20	93	496	911	1016
27	q 60	14	4		61	96	423	1920	3607	4160
	q 61	14	4		61	96	423	1920	3607	4160
28	r 62	14	9		68	96	372	1920	3646	4160
29	r 63	14	9		68	96	372	1920	3646	4160
30	s 64	14	4	12	61	88	423	1844	3607	4304
	s 65	14	4	12	61	88	423	1844	3607	4304
	s 66	14	4	12	61	88	423	1844	3607	4304
31	t 67	14	3	8	50	80	462	1912	3580	4192
	t 68	14	3	8	50	80	462	1912	3580	4192
	t 69	14	3	8	50	80	462	1912	3580	4192
	t 70	14	3	8	50	80	462	1912	3580	4192
32	u 71	14	4	12	61	88	423	1844	3607	4304
	u 72	14	4	12	61	88	423	1844	3607	4304
33	v 73	14	1	8	60	80	444	1912	3590	4192
	v 74	14	1	8	60	80	444	1912	3590	4192
	v 75	14	1	8	60	80	444	1912	3590	4192

			4	6	8	10	12	14	16	18
tgr	no.	dim	32	30	28	26	24	22	20	
42	z 91	14	3		44	92	480	1936	3568	4136
	z 92	14	3		44	92	480	1936	3568	4136
	z 93	14	3		44	92	480	1936	3568	4136
46	A 94	14	3		44	92	480	1936	3568	4136
	A 95	14	3		44	92	480	1936	3568	4136
	A 96	14	3		44	92	480	1936	3568	4136
	A 97	14	3		44	92	480	1936	3568	4136
50	B 98	14	4		57	104	435	1888	3599	4208
	B 99	14	4		57	104	435	1888	3599	4208
52	100	14	3		50	96	462	1920	3580	4160
56	C 101	14	4		49	88	459	1952	3583	4112
	C 102	14	4		49	88	459	1952	3583	4112
57	D 103	14	3		37	74	501	2008	3554	4028
	D 104	14	3		37	74	501	2008	3554	4028
	D 105	14	3		37	74	501	2008	3554	4028
	D 106	14	3		37	74	501	2008	3554	4028
65	E 107	14	1		38	92	510	1936	3546	4136
	E 108	14	1		38	92	510	1936	3546	4136
	E 109	14	1		38	92	510	1936	3546	4136
	E 110	14	1		38	92	510	1936	3546	4136
71	111	14	7		70	112	378	1856	3640	4256
75	F 112	14	1		48	104	480	1888	3566	4208
	F 113	14	1		48	104	480	1888	3566	4208
78	G 114	14	1		48	104	480	1888	3566	4208
	G 115	14	1		48	104	480	1888	3566	4208
82	116	14	1		52	112	468	1856	3574	4256
86	H 117	14			40	98	510	1912	3545	4172
	H 118	14			40	98	510	1912	3545	4172
	H 119	14			40	98	510	1912	3545	4172
	H 120	14			40	98	510	1912	3545	4172
90	I 121	14			37	104	519	1888	3539	4208
	I 122	14			37	104	519	1888	3539	4208
91	J 123	14			34	86	528	1960	3533	4100
	J 124	14			34	86	528	1960	3533	4100
	J 125	14			34	86	528	1960	3533	4100
	J 126	14			34	86	528	1960	3533	4100
97	K 127	14			31	92	537	1936	3527	4136
	K 128	14			31	92	537	1936	3527	4136
	K 129	14			31	92	537	1936	3527	4136
	K 130	14			31	92	537	1936	3527	4136
99	L 131	14			28	98	546	1912	3521	4172
	L 132	14			28	98	546	1912	3521	4172
184	M 133	14		6	9	108	603	1818	3483	4328
	M 134	14		6	9	108	603	1818	3483	4328
	M 135	14		6	9	108	603	1818	3483	4328
	M 136	14		6	9	108	603	1818	3483	4328
185	N 137	14		6	9	108	603	1818	3483	4328
	N 138	14		6	9	108	603	1818	3483	4328
	O 139	14		6	9	108	603	1818	3483	4328
	O 140	14		6	9	108	603	1818	3483	4328
	O 141	14		6	9	108	603	1818	3483	4328
186	P 142	14		2	5	124	615	1790	3475	4360
	P 143	14		2	5	124	615	1790	3475	4360
	P 144	14		2	5	124	615	1790	3475	4360
	P 145	14		2	5	124	615	1790	3475	4360
	P 146	14		2	5	124	615	1790	3475	4360
	Q 147	14		4	5	112	615	1820	3475	4320
	Q 148	14		4	5	112	615	1820	3475	4320
	Q 149	14		4	5	112	615	1820	3475	4320
	Q 150	14		4	5	112	615	1820	3475	4320
	Q 151	14		4	5	112	615	1820	3475	4320
218	R 152	14		6	7	104	609	1834	3479	4304
	R 153	14		6	7	104	609	1834	3479	4304
	S 154	14			7	140	609	1744	3479	4424
	S 155	14			7	140	609	1744	3479	4424
	S 156	14			7	140	609	1744	3479	4424
	R 157	14		6	7	104	609	1834	3479	4304
219	T 158	14		2	11	136	597	1742	3487	4432
	T 159	14		2	11	136	597	1742	3487	4432
	T 160	14		2	11	136	597	1742	3487	4432
220	U 161	14		2	4	122	618	1798	3473	4348
	U 162	14		2	4	122	618	1798	3473	4348
	U 163	14		2	4	122	618	1798	3473	4348
	U 164	14		2	4	122	618	1798	3473	4348
222	V 165	14		4	15	132	585	1740	3495	4440

Table 9. Continued.

tgr	no.	dim	4 32	6 30	8 28	10 26	12 24	14 22	16 20	18	tgr	no.	dim	4 32	6 30	8 28	10 26	12 24	14 22	16 20	18
34	w 77	14	4		61	96	423	1920	3607	4160		W 167	14			15	156	585	1680	3495	4520
	w 78	14	4		61	96	423	1920	3607	4160		W 168	14			15	156	585	1680	3495	4520
35	r 79	14	9		68	96	372	1920	3646	4160		V 169	14		4	15	132	585	1740	3495	4440
36	x 80	14	5		56	96	432	1920	3602	4160		W 170	14			15	156	585	1680	3495	4520
	x 81	14	5		56	96	432	1920	3602	4160	223	X 171	14		2	19	152	573	1678	3503	4528
37	82	14	3		66	96	414	1920	3612	4160		X 172	14		2	19	152	573	1678	3503	4528
38	y 83	14	2	8	55	80	453	1912	3585	4192		X 173	14		2	19	152	573	1678	3503	4528
	y 84	14	2	8	55	80	453	1912	3585	4192		X 174	14		2	19	152	573	1678	3503	4528
	y 85	14	2	8	55	80	453	1912	3585	4192	224	Y 175	14		6	27	144	549	1674	3519	4544
39	86	12			21	24	87	480	915	1040		Y 176	14		6	27	144	549	1674	3519	4544
40	r 87	14	9		68	96	372	1920	3646	4160	225	177	14		10	15	96	585	1830	3495	4320
41	s 88	14	4	12	61	88	423	1844	3607	4304		178	14			15	156	585	1680	3495	4520
	s 89	14	4	12	61	88	423	1844	3607	4304	227	Z 179	14		6		90	630	1890	3465	4220
42	z 90	14	3		44	92	480	1936	3568	4136		Z 180	14		6		90	630	1890	3465	4220

Table 10. Weight enumerators of the codes $\mathcal{C}_{\overline{A}}$ of the twelve $LS(6)$'s.

name	dim	0	4	8	12	16	20	24	28	32	36
ls1	14	1	1	20	1204	6966	6966	1204	20	1	1
ls2	14	1		23	1201	6967	6967	1201	23		1
ls3	14	1		37	1159	6995	6995	1159	37		1
ls4	14	1		29	1183	6979	6979	1183	29		1
ls5	14	1	1	48	1120	7022	7022	1120	48	1	1
ls6	14	1		45	1135	7011	7011	1135	45		1
ls7	14	1	3	18	1198	6972	6972	1198	18	3	1
ls8	12	1		9	360	1539	1944	243			
ls9	12	1		9	360	1539	1944	243			
ls10	12	1			396	1485	1980	234			
ls11	14	1		45	1135	7011	7011	1135	45		1
ls12	12	1		27	288	1647	1872	261			

PROPOSTION 6.1 *If for an* $SRG(35, 16, 6, 8)$ *the code* $\mathcal{C}_A$ *has dimension* 14*, the corresponding regular two-graph does not contain an* $SRG(36, 14, 4, 6)$.

There are 180 $SRG(36, 14, 4, 6)$'s known ([19]). For all corresponding codes, the weight enumerator is displayed in Table 9 (maintaining the ordering of Spence). Theorem 5.1 gives that $\mathbf{1} \in \mathcal{C}_A$, so C_A is self-complementary. The letters indicate which codes are isomorphic. The first one in the table is the smaller subconstituent of the Hall-Janko graph (HJsub). Note that all known regular two-graphs on 36 points occur in the list except those which are excluded by Proposition 6.1, because the code of a descendant has dimension 14.

There are 32548 known $SRG(36, 15, 6, 6)$ and among these are twelve Latin square graphs of order 6 ($LS(6)$). It can be found in [17] that for these twelve graphs $\dim \mathcal{C}_{A+I} = 13$ if the Latin square contains a quotient Latin square of order two and $\dim \mathcal{C}_{A+I} = 14$ otherwise. In the first case, $\dim \mathcal{C}_{\overline{A}} = 12$, so $\mathbf{1} \notin \mathcal{C}_{\overline{A}}$ and hence the two-graph code $\mathcal{C}$ has dimension 13 and is equal to $\mathcal{C}_{\overline{A}} + \langle \mathbf{1} \rangle$. In the other case, $\dim \mathcal{C}_{\overline{A}} = 14$, so $\mathbf{1} \in \mathcal{C}_{\overline{A}}$. By Theorem 5.1 we have that in this case the two-graph code also has dimension 13 and is a subcode of $\mathcal{C}_{\overline{A}}$. For the twelve $LS(6)$'s we generated the codes $\mathcal{C}_{\overline{A}}$. The weight enumerators are shown in Table 10. (We follow the numbering from [8].) The two cases distinguished above can clearly be recognised. In case of equal weight enumerator we give the invariant multisets of the codewords of weight 8. It follows that no two of these codes are isomorphic.

$$\text{ls6: }\{6^6, 10^{18}, 12^{12}\};\ \text{ls11: }\{0^6, 12^{30}\};\ \text{ls8: }\{0^6, 2^{18}, 3^{12}\};\ \text{ls9: }\{2^{36}\}$$

Since $\mathcal{C}_{A+I} = \mathcal{C}_{\overline{A}} + \langle \mathbf{1} \rangle$ it follows that also the twelve codes of the form $\mathcal{C}_{A+I}$ are non-isomorphic. The code $\mathcal{C}_{A+I}$ of ls10, which corresponds to two-graph code 227 of Table 6, has dimension 13 and minimum weight 12. This means that the code is optimal. It is known that the two Latin square graphs corresponding with ls7 and ls9 are switching equivalent. So these graphs define the same two-graph and hence the same two-graph code. As a consequence $\mathcal{C}_{\overline{A}}$ of ls9 is a subcode of $\mathcal{C}_{\overline{A}}$ of ls7.

Let A be the matrix of an $SRG(36, 20, 10, 12)$ (the complement of an $SRG(36, 15, 6, 6)$), then it follows from Proposition 3.2 that the possible values for $\dim \mathcal{C}_A$ are 6, 8, 10, 12, 14 and 16. If we have an $SRG(36, 20, 10, 12)$ with $\dim \mathcal{C}_A = 16$, then we must be in the second case of Theorem 5.1 (since $\dim \mathcal{C}_B \leq 14$) and hence $\mathbf{1} \in \mathcal{C}_A$.

6.4. Case No. 22

Spence [20] proved by complete computer search that there are exactly 28 non-isomorphic $SRG(40, 12, 2, 4)$'s. We use the ordering given in Spence [18] (for the first 27 to be precise; the 28th was found later). Number 3 and 23 are the two point graphs of a generalized quadrangle of order 3 ($GQ(3, 3)$). We see from Table 11 that all codes are non-isomorphic, except possibly for the codes of no. 16 and 17. These cases give the same invariant multiset for every weight that occurred in the weight enumerator. The isomorphism question for these two codes was settled by use of a computer program of Leon [13], which gives the order of the automorphism group of the code. It turns out that the automorphism group of no. 16 has order 9 and the group of no. 17 has order 27. Hence all 28 codes are non-isomorphic. Note that all codes contain the all-one vector. If a $SRG(40, 12, 2, 4)$ contains a 4-clique, the four rows of the adjacency matrix A corresponding with these four vertices add up to the all-one vector. Except for no. 28 every $SRG(40, 12, 2, 4)$ contains a 4-clique. For no. 23 the 40 rows of the adjacency matrix are precisely all codewords of weight 12.

6.5. Case No. 23

There are 120 $SRG(41, 20, 9, 10)$'s known, coming from 18 regular two-graphs, see [9]. By Proposition 3.1 and Theorem 5.1 the two-graph codes all have dimension 21. The weight enumerators of the two-graph codes are in Table 12. We see that no two codes are isomorphic. No. 3 corresponds to the Paley graph, that is, the code is the extended quadratic residue code. We also checked the codes of the 120 strongly regular graphs, which are just the shortened two-graph codes. Only two of them have the same weight enumerator, but no two are isomorphic.

Table 11. Weight enumerators and the relevant invariant multisets of the codes of the 28 $SRG(40, 12, 2, 4)$'s.

no.	dim	0	4	8	12	16	20	24	28	32	36	40
1	16	1		27	1228	15300	32424	15300	1228	27		1
2	16	1		27	1228	15300	32424	15300	1228	27		1
3	16	1		45	1120	15570	32064	15579	1120	45		1
4	16	1		18	1282	15165	32604	15165	1282	18		1
5	16	1		17	1288	15150	32624	15150	1288	17		1
6	16	1		12	1318	15075	32724	15075	1318	12		1
7	16	1		17	1288	15150	32624	15150	1288	17		1
8	16	1		15	1300	15120	32664	15120	1300	15		1
9	16	1		17	1288	15150	32624	15150	1288	17		1
10	16	1		18	1282	15165	32604	15165	1282	18		1
11	16	1		21	1264	15210	32544	15210	1264	21		1
12	16	1		12	1318	15075	32724	15075	1318	12		1
13	16	1		12	1318	15075	32724	15075	1318	12		1
14	16	1		12	1318	15075	32724	15075	1318	12		1
15	16	1		12	1318	15075	32724	15075	1318	12		1
16	16	1		9	1336	15030	32784	15030	1336	9		1
17	16	1		9	1336	15030	32784	15030	1336	9		1
18	14	1		21	304	3690	8352	3690	304	21		1
19	14	1	2	15	312	3698	8332	3698	312	15	2	1
20	14	1	3	9	316	3702	8322	3702	316	9	3	1
21	14	1	2	9	304	3766	8220	3766	304	9	2	1
22	12	1	1	3	96	828	2238	828	96	3	1	1
23	10	1			40	135	672	135	40			1
24	14	1	1	9	336	3638	8412	3638	336	9	2	1
25	14	1		15	308	3728	8280	3728	308	15		1
26	14	1	1	9	324	3702	8310	3702	324	9	1	1
27	12	1		3	100	828	2232	828	100	3		1
28	14	1	1	9	324	3702	8310	3702	324	9	1	1

1: $\{3^6, 5^{18}, 6^{12}, 9^4\}$

2: $\{5^{36}, 9^4\}$

4: $\{3^{36}, 9^4\}$
10: $\{2^9, 3^{15}, 4^9, 6^6, 9^1\}$

5: $\{1^3, 2^9, 3^9, 4^{13}, 5^3, 6^2, 9^1\}$
7: $\{1^2, 2^{10}, 3^{10}, 4^{12}, 5^3, 6^2, 9^1\}$
9: $\{1^2, 2^8, 3^{14}, 4^{10}, 5^3, 6^2, 9^1\}$

6: $\{1^{12}, 2^{12}, 3^{12}, 5^3, 9^1\}$
12: $\{1^9, 2^{18}, 3^{10}, 6^2, 9^1\}$
13: $\{1^{13}, 2^{10}, 3^{10}, 4^6, 9^1\}$
14: $\{1^9, 2^{18}, 3^6, 4^6, 9^1\}$
15: $\{1^{18}, 3^{18}, 5^3, 9^1\}$

26: $\{85^4, 87^2, 91^8, 93^1, 95^9, 103^6, 107^8, 111^2\}$
28: $\{91^{24}, 105^4, 107^{12}\}$

Table 12. Weight enumerators of the codes of the known regular two-graphs on 42 points.

		0	4	6	8	10	12	14	16	18	20
no.	dim	42	38	36	34	32	30	28	26	24	22
1	21	1		14	259	1925	10843	50123	154336	338247	492828
2	21	1	1	62	315	1815	9906	49380	157586	341419	488091
3	21	1				1722	10619	49815	157563	341530	487326
4	21	1		84	168	1722	9359	48387	161259	345730	481866
5	21	1		49	84	1764	9912	48786	159789	344330	483861
6	21	1	7		140	1932	11396	50382	153125	337330	494263
7	21	1		21	112	1512	10164	50438	158137	340830	487361
8	21	1	7	21	91	1309	10367	51439	157192	338863	489286
9	21	1	7	14	231	1743	11060	51026	153426	336455	494613
10	21	1	5	70	279	1755	10046	49576	157278	341055	488511
11	21	1	1	46	219	1431	9826	50612	158418	340555	487467
12	21	1	5	70	199	1419	9758	50312	158894	341391	486527
13	21	1			56	1554	10507	50599	157283	340130	488446
14	21	1		28	105	1575	10101	50025	158550	341705	486486
15	21	1		27	126	1506	10070	50364	158391	341080	487011
16	21	1		15	126	1434	10202	50904	157743	339880	488271
17	21	1		44	120	1626	9895	49515	159339	342930	485106
18	21	1		8	100	1470	10331	50855	157495	339830	488486

7. Soft and Hardware

All of the presented results were obtained on a Intel 486DX 100MHz CPU with 16MB memory, running MS-DOS and Linux. The main program uses the adjacency matrix of the graph as input and produces the following output:

- The dimension of the code.
- The number of codewords.
- The weight enumerator of the code (optional).
- The code itself (optional).
- A basis for the code consisting of rows of the adjacency matrix (optional).
- The codewords of minimum weight not equal to 0 (optional).
- The codewords of any specified weight (optional).
- The binary vectors, indicating which combination of basis vectors gives the codewords of a specific weight (optional)

To speed up the process of enumerating all codewords, we used the Gray Code. The Gray Code enumerates all possible binary vectors of a specific length, with the restriction that two consecutive words only differ in one coordinate. In this way we can compute the next codeword of the code of the graph by adding just one basis vector to the previous codeword instead of constructing each codeword from scratch.

The main program and all other programs (constructing graphs, comparing weight enumerators, computing invariant multisets and comparing them) were written in TURBO PASCAL and run under MS-DOS.

The program for the computation of the order of the automorphism group of a code is written by Leon [13] and can be obtained by anonymous ftp. This program is also implemented in GAP in the package Guava.

Added in Proof: In the mean time Brendan Mckay and Edward Spence have shown by computer search that for case No. 16, 18 and 20 from Table 2, there are no other SRG's (and two graphs) with the given parameters.

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