Hierarchies of beliefs for compact possibility models

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Abstract

In this paper, we construct a universal type space for a class of possibility models by imposing topological restrictions on the players' beliefs. Along the lines of Mertens and Zamir (1985) or Brandenburger and Dekel (1993), we show that the space of all hierarchies of compact beliefs that satisfy common knowledge of coherency (types) is canonically homeomorphic to the space of compact beliefs over the state of nature and the types of the other players. The resulting type space is universal, in the sense that any compact and continuous possibility structure can be uniquely represented within it. We show how to extend our construction to conditional systems of compact beliefs.

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1. Introduction

In models of interactive beliefs, a state of the world should contain a specification of the state of nature, and a specification of the players' epistemic state, that is, their beliefs about the state of nature and the other players' beliefs. The specification of the players' beliefs can lead to an infinite regress or circularity problem: the beliefs of a player, say player a, are in part defined over the beliefs of the other players, which are in turn defined over the beliefs of player a, and so on. A key question raised by the infinite regress problem is whether it is possible to construct a complete model of a situation of incomplete information, in which each state is an exhaustive description of the state of nature and of the other players' epistemic states.

As argued by Aumann (1976), a model that is not common knowledge among the players is necessarily incomplete, as an exhaustive description of a state should include the informational partitions and beliefs of all players in that state. Starting from the information partitions or, alternatively, Kripke structures (Fagin, Halpern, Moses and Vardi (1995, Chapter 2)), to define hierarchies of beliefs presumes that the information partitions are common knowledge in an "informal" sense. Of course, if the players' types fully specify the players' knowledge, one should be able to incorporate common knowledge of information partitions into the players' types. One is thus led to ask how, starting from a given incomplete information situation, a commonly known model can be specified (see Dekel and Gul (1997) for an overview of this issue).

In the context of Harsanyi's (1967-1968) model of games with incomplete information, Mertens and Zamir (1985) and Brandenburger and Dekel (1993) have solved this problem for the case of probabilistic beliefs by showing the existence of a universal type space consisting of all hierarchies of beliefs (types) for which coherency is common knowledge. In particular, a type of a player determines unambiguously a joint belief over the state of nature and the other players' types, and any such joint belief is represented by a type. The hierarchical construction of this canonical homeomorphism accomplishes the desired task: common knowledge of coherency is the formal equivalent of the "informal" hypothesis that information partitions are common knowledge. Battigalli and Siniscalchi (1999) have extended this result to the case of conditional beliefs, and Epstein and Wang (1996) to an important class of non Bayesian beliefs.

The infinite regress problem persists in knowledge and possibility models where the beliefs of a player are represented by a *possibility set* consisting of all states regarded possible. Instead of assuming the existence of a commonly known model, Fagin, Halpern and Vardi (1991) attempt to construct such a model from primitive terms and syntactic operators. However, it turns out that, in general, hierarchies of beliefs and their representation by (commonly known) knowledge structures do not necessarily characterize fully the players' knowledge. Indeed, even if an epistemic situation can be represented by a "local" (Kripke) structure, it does not follows that such an epistemic situation is a complete representation of the knowledge of the players. As pointed out by Dekel and Gul (1997), the problem is that the same hierarchy of beliefs can in principle be associated with several beliefs over the beliefs of the other players. An "informal" common knowledge assumption may then help to uniquely determine the players' beliefs. Specifically, Fagin, Halpern and Vardi (1991) have shown that countable hierarchies of knowledge are not sufficient to describe adequately the interactive knowledge of the players, and Fagin (1994) and Heifetz and Samet (1998) that indeed no ordinal level in the hierarchy of knowledge suffices. Furthermore, Brandenburger (1998) and Brandenburger and Keisler (1999) have shown that, except in degenerate cases, any purely set-theoretic type model of beliefs is necessarily incomplete in that it is always possible to construct a belief that is not held by any type. A positive result is shown in Fagin, Geanakoplos, Halpern and Vardi (1999): if a countable description of the players' interactive beliefs satisfies a "continuity" condition, all higher levels of knowledge are determined unambiguously.

A related concern with possibility models is the nonexistence of a universal space for the agents' beliefs. Heifetz and Samet (1998) have shown that, given at least two agents and two states of nature, there is no universal knowledge space to which any knowledge space can be mapped in a knowledge-preserving way. This result has been extended recently to every class of Kripke structures that contains all knowledge spaces (Meier (2003)). Although not every knowledge space is a possibility structure, and hence the results of Meier (2003) do not apply directly, these results provide strong evidence that there is no universal possibility structure for the class of all possibility structures.

These negative results suggest that a natural strategy to obtain a complete and universal representation of players' interactive knowledge in possibility models is to impose further restrictions on beliefs. The objective of the present paper is to provide such a construction for a class of possibility models by assuming that players' beliefs satisfy some topological requirements. Specifically, we assume that the basic uncertainty space is a compact Hausdorff space and we construct hierarchies of compact possibility sets. Endowing spaces of compact subsets with the standard Hausdorff topology ensures that this procedure can be repeated from any stage of the hierarchy to the next. We show that the space of all hierarchies of compact beliefs satisfying common knowledge of coherency (types) is canonically homeomorphic to the space of compact beliefs over the state of nature and the types of the other players, and is universal for the class of compact and continuous possibility structures. A natural concern raised by any construction that imposes restrictions on player's beliefs is how limitative these restrictions are. In that respect, the topological assumptions made in the present paper are not overly restrictive. Indeed, finite possibility structures are commonly used in applications, and every such structure, appropriately endowed with the discrete topology, yields a compact and continuous possibility structure.

Our construction enables to establish which hierarchies of beliefs are complete descriptions of the knowledge of the players. The existence of a universal space delivers completeness in two senses. First, common knowledge of coherency ensures that a hierarchy of beliefs of a player is just a belief about the state of nature and the hierarchies of beliefs of the other players. In particular, every type of a player exactly pins down a partitional model, as in the probabilistic setups of Mertens and Zamir (1985) or Brandenburger and Deckel (1993). Second, common knowledge of coherency also ensures that the possibility correspondence is onto: any belief about the state of nature and the hierarchies of the other players corresponds to a hierarchy of beliefs, so that there is no loss of generality in representing player's types by hierarchies. Of course, other representations are possible. But the universality of the resulting type space implies that any compact and continuous possibility structure can be uniquely represented within it, and thus can essentially be viewed as a subspace of it. Thus the universal type space is (internally) complete but also (externally) without loss of generality.

An important motivation for introducing hierarchical models of beliefs is to provide epistemic conditions for solution concepts in games (Tan and Werlang (1988)). For normal-form games, our construction can be used to provide an axiomatization of Bernheim's (1984) concept of point-rationalizability (Mariotti (2003)). The recent debate on backward induction has also prompted the development of epistemic models for extensive-form games. Battigalli and Siniscalchi (1999) provide a complete model of conditional probability systems for the epistemic analysis of games. As pointed out by Brandenburger and Keisler (1999), however, epistemic conditions for solution concepts in games of perfect information should carry over to non probabilistic frameworks. In the last part of the paper, we extend our analysis to conditional systems of compact beliefs by adjusting the construction of Battigalli and Siniscalchi (1999) to our framework.

The paper is organized as follows. The model is introduced in Section 2, and the universal type space is constructed in Section 3. The main assumptions and results are discussed in Section 4. Conditional compact beliefs are studied in Section 5. All proofs are in the Appendices.

2. The model

2.1. Possibility structures

Definition. Consider two individuals[1], a and b, facing uncertainty over a space S. The concept of a *possibility structure* delivers the simplest model of interactive beliefs over S. Specifically, for any set X, let $\mathcal{P}(X)$ denote the set of non-empty subsets of X. The following definition is borrowed from Brandenburger and Keisler (1999):

Definition 1 *An S–based possibility structure is a pair (T, R) consisting of:*

(i) A non-empty set T;

(ii) A mapping $R : T \to \mathcal{P}(S \times T)$.

Elements of T are called *types*. R is a *possibility mapping* that assigns to each type $t \in T$ of a player a set-theoretic belief $R(t) \subset S \times T$ about the basic uncertainty parameter and the type of the other player. For any $t \in T$, $R(t)$ represents the pairs in $S \times T$ considered possible by type t, and is called the *possibility set* of type t.

An incompleteness result. Possibility models highlight a conceptual difficulty for the notion of interactive beliefs. Suppose, for example, that T represents all possible beliefs of the players. Then, any subset of $S \times T$ should be, in principle, a possible belief. The following result shows that this is impossible.

[1]This is without loss of generality. Our results easily extend to an arbitrary number of agents.

Lemma 1 *Let (T, R) be an S–based possibility structure. If R is onto, then S and T are singletons.*

The intuition is simple. If S contains at least two elements, then so must T. But then, a simple adaptation of Cantor's theorem implies that $\mathcal{P}(T)$, and thus $\mathcal{P}(S \times T)$, must have a strictly larger cardinality than T. This result shows that, in contrast with probabilistic models, there does not exist a complete possibility structure: one can always find a belief over S and the beliefs of the opponent that is not represented within the model. (See Brandenburger (1998) for a similar result.)

Remark 1 *Lemma 1 relies on a cardinality argument. Brandenburger and Keisler (1999) prove a stronger result: if S is not a singleton then, for any S–based possibility structure (T, R), it is possible to define a subset of $S \times T$ that is not a possibility set using only the resources of the first-order language induced by (T, R).*

To derive a universal type space, it is necessary to impose structural conditions on S and T, and impose restrictions on the possibility mapping R. This is the objective of the next two subsections.

2.2. Technical preliminaries

Given a topological space X, let $\mathcal{K}(X)$ denote the set of non-empty compact subsets of X. We endow $\mathcal{K}(X)$ with the Hausdorff topology. This is the topology generated by all subsets of the form $\{K \in \mathcal{K}(X) \mid K \subset U\}$ and $\{K \in \mathcal{K}(X) \mid K \cap U \neq \emptyset\}$ for U open in X. For the remainder of this paper, $\mathcal{K}(X)$ will always be endowed with the Hausdorff topology. The following lemma is standard; a proof is available in Appendix A.

Lemma 2 *Let X, Y be topological spaces, and $f : X \to Y$ a continuous map. Then,*

(i) If X is Hausdorff, $\mathcal{K}(X)$ is Hausdorff;

(ii) If X is compact, $\mathcal{K}(X)$ is compact;

(iii) The mapping $f^{\mathcal{K}} : \mathcal{K}(X) \to \mathcal{K}(Y) : K \mapsto f(K)$ is continuous;

(iv) If Z is a topological space and $g : Y \to Z$ a continuous map, then $(g \circ f)^{\mathcal{K}} = g^{\mathcal{K}} \circ f^{\mathcal{K}}$.

For future reference, define the space $\mathcal{O}(X)$ of open subsets of X distinct from X itself. Its Hausdorff topology is generated by all subsets of the form $\{O \in \mathcal{O}(X) \mid F \subset O\}$ and $\{O \in \mathcal{O}(X) \mid O \cup F \neq X\}$ for F closed in X. By taking complement sets, it is immediate that if X is compact and Hausdorff, $\mathcal{O}(X)$ is homeomorphic to $\mathcal{K}(X)$.

2.3. A topological model of beliefs

Compact possibility structures. The restrictions that we shall impose on S, T and R are topological. The following assumption will be maintained throughout the paper.

Assumption 1 *S is a compact Hausdorff space.*

The basic drawback of the pure set-theoretic model of Subsection 2.1 is that players can distinguish too finely between different subsets of $S \times T$. An intuitive restriction is that players cannot distinguish very "close" subsets. Specifically, given a topology on $S \times T$, we shall assume that players cannot tell apart two subsets of $S \times T$ with the same closure. This leads to the following definition.

Definition 2 *An S–based compact possibility structure is a pair (T, R) consisting of:*

(i) A non-empty Hausdorff space T;

(ii) A mapping $R : T \to \mathcal{K}(S \times T)$.

As in Definition 1, a player's type specifies a set-theoretic belief about the basic uncertainty parameter and the type of the other player. The difference is that we restrict the possibility sets of each player to be compact subsets of $S \times T$. If T is a compact Hausdorff space, this is equivalent to considering the quotient space $\mathcal{P}(S \times T)/\sim$ where any two sets $X, Y \in \mathcal{P}(S \times T)$ are $\sim$–equivalent if and only if they have the same closure. This follows from the fact that compact and closed subsets of a compact Hausdorff space coincide.

An S–based compact possibility structure (T, R) is *complete* if the possibility mapping R is onto, and is *continuous* if R is continuous. The set of S–based compact continuous possibility structures is denoted by $\mathcal{C}$.

Define Id_S to be the identity map on S and consider a family $\mathcal{F}$ of S–based compact possibility structures. An S–based compact possibility structure $(T^{\mathcal{F}}, R^{\mathcal{F}})$ is *universal* for

$\mathcal{F}$, if for any (T, R) in $\mathcal{F}$, there exists a unique mapping φ such that the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & T^{\mathcal{F}} \\ \Big\downarrow R & & \Big\downarrow R^{\mathcal{F}} \\ \mathcal{K}(S \times T) & \xrightarrow{(\mathrm{Id}_S;\varphi)^{\mathcal{K}}} & \mathcal{K}(S \times T^{\mathcal{F}}) \end{array}.$$

In the terminology of Mertens and Zamir (1985) or Battigalli and Siniscalchi (1999), $(T^{\mathcal{F}}, R^{\mathcal{F}})$ is universal if there is a unique *belief morphism* φ from (T, R) to $(T^{\mathcal{F}}, R^{\mathcal{F}})$.

3. The universal possibility structure

In this section, we show the existence of a universal possibility structure for $\mathcal{C}$. We first construct a canonical homeomorphism between hierarchies of beliefs that satisfy common knowledge of coherency and beliefs over S and such hierarchies. The line of argument is similar to that in Brandenburger and Dekel (1993). We then show that this space of hierarchies of beliefs together with the canonical homeomorphism form a universal compact continuous possibility structure.

3.1. The canonical homeomorphism

We restrict players' beliefs at any stage of the hierarchy to be compact. A player first-order belief over S will be represented by an element of $\mathcal{K}(S)$. A player's second-order belief will be represented by an element of $\mathcal{K}(S \times \mathcal{K}(S))$, a joint belief over S and his opponent's first-order belief. In general, define inductively the sets

$$X_0 = S, \qquad X_{n+1} = X_n \times \mathcal{K}(X_n); \qquad n \geq 0.$$

A player's *type* is a hierarchy of beliefs $\{\kappa_n\}_{n\geq 1} \in \prod_{n\geq 0} \mathcal{K}(X_n)$. We denote by T_0 the set of all possible types.

As in Bayesian models, a natural coherency condition on a player's type is that different levels of beliefs do not contradict each other. For any sets X and Y, let us denote by Proj_X be the projection of elements in $X \times Y$ on X. The following definition provides the coherency condition.

Definition 3 *A type* $\{\kappa_n\}_{n\geq 1} \in T_0$ *is coherent if and only if*

$$\kappa_n = \mathrm{Proj}_{X_{n-1}}(\kappa_{n+1}); \qquad n \geq 1.$$

The next lemma, a topological version of Kolmogorov's extension theorem, states that a compact subset of a countable product of compact Hausdorff spaces is determined by its projections on the cylinder sets.

Lemma 3 *Let* $\{Z_n\}_{n\geq 0}$ *be a collection of compact Hausdorff spaces. For each* $n \geq 0$*, let* $Z^n = \prod_{0\leq\nu\leq n} Z_\nu$ *and* $Z^\infty = \prod_{\nu\geq 0} Z_\nu$*. Define*

$$D = \left\{ \{\kappa_n\}_{n\geq 1} \mid \kappa_n \in \mathcal{K}(Z^{n-1}) \text{ and } \mathrm{Proj}_{Z^{n-1}}(\kappa_{n+1}) = \kappa_n \right\}.$$

There exists a homeomorphism $f : D \to \mathcal{K}(Z^\infty)$ *such that, for any* $\{\kappa_n\}_{n\geq 1} \in D$*,*

$$\mathrm{Proj}_{Z^{m-1}}\left(f(\{\kappa_n\}_{n\geq 1})\right) = \kappa_m; \qquad m \geq 1.$$

Let T_1 be the set of coherent types. The following proposition is an immediate consequence of Lemma 3. It states that a coherent type for a player is equivalent to a belief over S and the type of the other player.

Proposition 1 *There exists a homeomorphism* $f : T_1 \to \mathcal{K}(S \times T_0)$ *such that, for any* $\{\kappa_n\}_{n\geq 1} \in T_1$*,*

$$\mathrm{Proj}_{X_{m-1}}\left(f(\{\kappa_n\}_{n\geq 1})\right) = \kappa_m; \qquad m \geq 1.$$

It should be noted that the homeomorphism f is *canonical* in that it preserves the marginal beliefs associated to any level of the hierarchy by any coherent type. As is customary in this literature, we close the model by imposing common knowledge of coherency. Define inductively the sets

$$T_{k+1} = \{t \in T_1 \mid f(t) \subset S \times T_k\}; \qquad k \geq 1,$$

and let $T_\infty = \bigcap_{k\geq 1} T_k$. $T_\infty \times T_\infty$ is interpreted as the set of players' types such that each player believes that the other player's type is coherent, believes that the other player believes that his type is coherent, and so on. Hereafter, we shall refer to T_∞ as the *universal type space.* We can now state our main result.

Proposition 2 T_∞ *is non-empty. Moreover, the restriction of* f *to* T_∞ *induces a homeomorphism* $g : T_\infty \to \mathcal{K}(S \times T_\infty)$*.*

It should be noted that the non-emptiness of T_∞ depends crucially on the compactness of the sets $\{T_k\}_{k\geq 1}$. As the homeomorphism f, g preserves the beliefs associated by a player's type to any level of the hierarchy. The reader familiar with the results in Fagin, Geanakoplos, Halpern and Vardi (1999) mentioned in the Introduction will surely notice that restricting the beliefs to belong to $\mathcal{K}(S \times T_\infty)$ ensures that appropriate "continuity" conditions are satisfied.

3.2. Universality

In possibility structures, continuity is not only a desirable property but also a crucial one, as we shall argue, for the representation of hierarchical knowledge.

Preliminaries. To investigate the relationship between hierarchical beliefs and compact possibility models, it is convenient to construct a second hierarchy of beliefs, hereafter called $*$–beliefs. The starting point of this hierarchy is the same as in Subsection 3.1, i.e., a belief over S represented by an element of $\mathcal{K}(S)$. However, for any $n \geq 2$, a player's nth order $*$–belief consists of a joint belief over S and his opponent's $(n-1)$th order $*$–belief. Define inductively the sets

$$X_0^* = S, \qquad X_{n+1}^* = S \times \mathcal{K}(X_n^*); \qquad n \geq 0.$$

A $*$–hierarchy is a sequence $\{\kappa_n^*\}_{n\geq 1} \in \prod_{n\geq 0} \mathcal{K}(X_n^*)$. We denote by T_0^* the set of all possible $*$–hierarchies. As for hierarchies of beliefs, a coherency condition is imposed in order to ensure that the different levels of $*$–beliefs do not contradict each other. Define the family $\{P_n\}_{n\geq 1}$ of operators $P_n : \mathcal{K}(X_n^*) \to \mathcal{K}(X_{n-1}^*)$ inductively as

$$P_1 = \mathrm{Proj}_S^{\mathcal{K}}, \qquad P_{n+1} = (\mathrm{Id}_S; P_n)^{\mathcal{K}}; \qquad n \geq 1.$$

Since $\mathrm{Proj_S}$ is continuous, Lemma 2(iii) guarantees that the operator P_n is well defined and continuous for any $n \geq 1$. The following definition states a coherency condition for $*$–hierarchies analogous to that given for hierarchies in Definition 3.

Definition 4 *A $*$–hierarchy $\{\kappa_n^*\}_{n\geq 1} \in T_0^*$ is coherent if and only if*

$$\kappa_n^* = P_{n+1}(\kappa_{n+1}^*); \qquad n \geq 1.$$

A player's coherent $*$–hierarchy generates the same beliefs over S at all levels, and coherent hierarchical $*$–beliefs about the other player's lower order $*$–beliefs. Let T_1^* be the set of coherent $*$–hierarchies.

We shall later show formally that T_1^* is homeomorphic to T_∞. To see this intuitively, consider, for instance, the space:

$$X_3 = S \times \mathcal{K}(S) \times \mathcal{K}(S \times \mathcal{K}(S)) \times \mathcal{K}(S \times \mathcal{K}(S) \times \mathcal{K}(S \times \mathcal{K}(S)))$$

used to model fourth-order beliefs in the construction of the original hierarchies. If a player, say a, believes that the beliefs of the opponent b are coherent, the second and third elements, $\mathcal{K}(S)$ and $\mathcal{K}(S \times \mathcal{K}(S))$, of the Cartesian product that defines X_3 are redundant since they can be derived by projection of $\mathcal{K}(S \times \mathcal{K}(S) \times \mathcal{K}(S \times \mathcal{K}(S)))$. By the same token, if a believes that b believes that a's beliefs are coherent, $\mathcal{K}(S)$ inside $\mathcal{K}(S \times \mathcal{K}(S) \times \mathcal{K}(S \times \mathcal{K}(S)))$ is also redundant. Deleting these redundant spaces, one obtains X_3^*.

A representation result. We now return to the problem of representation of hierarchies by types. Consider a compact possibility structure (T, R) and define inductively the sets

$$\hat{X}_0 = S \times T, \qquad \hat{X}_{n+1} = S \times \mathcal{K}(\hat{X}_n); \qquad n \geq 0.$$

We then have the following lemma.

Lemma 4 *Suppose that (T, R) is in $\mathcal{C}$. Then, the inductive family of functions*

$$R_0 = R, \qquad R_{n+1} = (\mathrm{Id}_S; R_n)^{\mathcal{K}}; \qquad n \geq 0$$

is well defined and, for each $n \geq 0$, the mapping $R_n : \mathcal{K}(\hat{X}_{n-1}) \to \mathcal{K}(\hat{X}_n)$ is continuous.

The continuity of R guarantees that compactness is preserved in the iteration. The mappings $\{R_n\}_{n\geq 0}$ operate on their respective domains by "expanding" a type into the set of hierarchies consistent with it. Specifically, if R is continuous, one can associate to any type $t \in T$ a sequence $\{\hat{\kappa}_n\}_{n\geq 1} \in \prod_{n\geq 0} \mathcal{K}(\hat{X}_n)$ by setting $\hat{\kappa}_n = R_{n-1} \circ \ldots \circ R_0(t)$ for any $n \geq 1$.

Any sequence $\{\hat{\kappa}_n\}_{n\geq 1} \in \prod_{n\geq 0} \mathcal{K}(\hat{X}_n)$ can be transformed into a $*$–hierarchy. First, define the family $\{Q_n\}_{n\geq 0}$ of operators $Q_n : \mathcal{K}(\hat{X}_n) \to \mathcal{K}(X_n^*)$ inductively as

$$Q_0 = \mathrm{Proj}_S^{\mathcal{K}}, \qquad Q_{n+1} = (\mathrm{Id}_S; Q_n)^{\mathcal{K}}; \qquad n \geq 0.$$

Again, since Proj_S is continuous, Lemma 2(iii) guarantees that Q_n is well defined and continuous for any $n \geq 0$. The following lemma clarifies the relationships between the families of operators $\{P_n\}_{n\geq 1}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$.

Lemma 5 *Suppose that (T, R) is in $\mathcal{C}$. Then, the following diagram commutes:*

$$
\begin{array}{ccccccccc}
T & \xrightarrow{R_0} & \mathcal{K}(\hat{X}_0) & \xrightarrow{R_1} & \mathcal{K}(\hat{X}_1) & \xrightarrow{R_2} & \mathcal{K}(\hat{X}_2) & \xrightarrow{R_3} & \cdots \\
 & & \downarrow Q_0 & & \downarrow Q_1 & & \downarrow Q_2 & & \\
 & & \mathcal{K}(X_0^*) & \xleftarrow{P_1} & \mathcal{K}(X_1^*) & \xleftarrow{P_2} & \mathcal{K}(X_2^*) & \xleftarrow{P_3} & \cdots
\end{array}
$$

The proof is by induction, and to be found in the Appendix. Composing the operators Q_n and R_n for any $n \geq 0$, one can map a type to a $*$–hierarchy.

Definition 5 *Consider (T, R) in $\mathcal{C}$. A type $t \in T$ generates a $*$–hierarchy $\{\kappa_n^*\}_{n\geq 1} \in T_0^*$ if*

$$\kappa_n^* = Q_{n-1} \circ R_{n-1} \circ \ldots \circ R_0(t); \qquad n \geq 1.$$

Now denote by $\varphi_{T,R}$ the mapping $T \to T_0^* : t \mapsto \{Q_{n-1} \circ R_{n-1} \circ \ldots \circ R_0(t)\}_{n\geq 1}$.

Proposition 3 *Suppose that (T, R) is in $\mathcal{C}$. Then,*

(i) $\varphi_{T,R}$ is continuous and maps T into T_1^*;

(ii) $\varphi_{T_\infty,g} : T_\infty \to T_1^*$ is a homeomorphism.

Proposition 3 shows that continuity of the possibility mapping in a compact possibility structure (T, R) allows to map any type in T to a coherent $*$–hierarchy, or, equivalently by (ii), to an infinite hierarchy in T_∞. This transformation is possible in general only if R is continuous (See Mertens and Zamir (1985) or Battigalli and Siniscalchi (1999) for similar results in the Bayesian setting).

The homeomorphisms $g : T_\infty \to \mathcal{K}(S \times T_\infty)$ and $\varphi_{T_\infty,g} : T_\infty \to T_1^*$ induce a homeomorphism $g^* = (\mathrm{Id}_S; \varphi_{T_\infty,g})^{\mathcal{K}} \circ g \circ \varphi_{T_\infty,g}^{-1}$ between T_1^* and $\mathcal{K}(S \times T_1^*)$. Using the canonicity of g, it follows that g^* is also canonical in the sense that, for any coherent $*$–hierarchy $\{\kappa_n^*\}_{n\geq 1} \in T_1^*$,

$$\mathrm{Proj}_S^{\mathcal{K}} \circ g^*(\{\kappa_n^*\}_{n\geq 1}) = \kappa_1^*,$$

$$(\mathrm{Id}_S; \mathrm{Proj}_{\mathcal{K}(X_m^*)})^{\mathcal{K}} \circ g^*(\{\kappa_n^*\}_{n\geq 1}) = \kappa_{m+1}^*; \qquad m \geq 1.$$

That is, g^* preserves the marginal beliefs associated to any level of the hierarchy by any coherent $*$–hierarchy. Naturally, $\varphi_{T_1^*,g^*} = \mathrm{Id}_{T_1^*}$, since any coherent $*$–hierarchy coincides with its own representation. It should be noted that $\varphi_{T_\infty,g}^{-1} : T_1^* \to T_\infty$ is the mapping that associates to any coherent $*$–hierarchy the universal type that generates it.

Take $(T, R) \in \mathcal{C}$. It is easy to verify that the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\varphi_{T,R}} & T_1^* \\ \Big\downarrow{\scriptstyle R} & & \Big\downarrow{\scriptstyle g^*} \\ \mathcal{K}(S \times T) & \xrightarrow{(\mathrm{Id}_S;\varphi_{T,R})^{\mathcal{K}}} & \mathcal{K}(S \times T_1^*) \end{array} .$$

Hence $\varphi_{T,R}$ is a belief morphism from (T, R) to (T_1^*, g^*). To prove that (T_1^*, g^*), and therefore (T_∞, g) by homeomorphism, is universal for $\mathcal{C}$, we need only to check that $\varphi_{T,R}$ is the unique belief morphism from (T, R) to (T_1^*, g^*). We only sketch the argument. Given a compact, continuous possibility structure (T, R), construct the mappings $\{R_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ as above. The same procedure applied to (T_1^*, g^*) instead of (T, R) yields the analogous mappings $\{R_n^*\}_{n\geq 0}$, and $\{Q_n^*\}_{n\geq 0}$. If $\phi : T \to T_1^*$ is a belief morphism from (T, R) to (T_1^*, g^*), then it is immediate by construction that the following diagram commutes:

$$\begin{array}{ccccc} T & & \xrightarrow{\phi} & & T_1^* \\ \Big\downarrow{\scriptstyle R_0} & & & & \Big\downarrow{\scriptstyle R_0^*} \\ \mathcal{K}(S \times T) & & \xrightarrow{(\mathrm{Id}_S;\phi)^{\mathcal{K}}} & & \mathcal{K}(S \times T_1^*) \\ & \overset{Q_0}{\searrow} & & \overset{Q_0^*}{\swarrow} & \\ \Big\downarrow{\scriptstyle R_1} & & \mathcal{K}(X_0^*) & & \Big\downarrow{\scriptstyle R_1^*} \\ \mathcal{K}(S \times \mathcal{K}(S \times T)) & & \xrightarrow{(\mathrm{Id}_S;(\mathrm{Id}_S;\phi)^{\mathcal{K}})^{\mathcal{K}}} & & \mathcal{K}(S \times \mathcal{K}(S \times T_1^*)) \\ & \overset{Q_1}{\searrow} & & \overset{Q_1^*}{\swarrow} & \\ \Big\downarrow{\scriptstyle R_2} & & \mathcal{K}(X_1^*) & & \Big\downarrow{\scriptstyle R_2^*} \\ \vdots & & \cdots & & \vdots \end{array} .$$

We leave it to the reader to verify that the commutativity of this diagram and the fact that $\varphi_{T_1^*,g^*} = \mathrm{Id}_{T_1^*}$ imply that $\phi = \varphi_{T,R}$.

4. Discussion of the results

We shall now discuss the results proved in the previous section.

4.1. *On the compactness assumption*

It is not *a priori* clear why one should restrict the possibility sets to be compact. Indeed, since $\mathcal{K}(X)$ is homeomorphic to $\mathcal{O}(X)$ for any compact Hausdorff space X, our construction implies the existence of a type space $\tilde{T}_\infty$ and of a homeomorphism $\tilde{g} : \tilde{T}_\infty \to \mathcal{O}(S \times \tilde{T}_\infty)$. This seems to indicate that one could choose the open sets as well as the closed sets as a basis for the construction of a complete interactive model. Intuitively, this would correspond to a situation in which players cannot distinguish different subsets of the type space with the same interior.[2] The problem is that, contrary to the space of compact subsets, removing the empty set from the space of open sets undermines the compactness of the resulting space.[3] A type space based on the open sets would then have to include the empty set as a possible belief. Intuitively, $\mathcal{O}(S \times \tilde{T}_\infty)$ should instead be interpreted as the set of subsets regarded as *impossible* by a player, and the resulting *impossibility structure* $(\tilde{T}_\infty, \tilde{g})$ as the mirror image of (T_∞, g). Of course, one could eliminate the empty set from $\mathcal{O}(S \times \tilde{T}_\infty)$ *after* constructing the type space $\tilde{T}_\infty$. This procedure, however, would not prevent the empty set from being "possible" for lower order beliefs. Moreover, the resulting model would not exhibit common knowledge of "impossibility" of "possible" empty sets. The arguments used above to obtain common knowledge of coherency are of little use since, as we have shown, removing the empty set from $\mathcal{O}(S \times \tilde{T}_\infty)$ does not preserve compactness.

4.2. *Continuity and the Bayesian model*

Complete possibility structures can also be derived from the universal Bayesian type space constructed by Mertens and Zamir (1985) or Brandenburger and Dekel (1993). Specifically, let S be a compact metric space. Brandenburger and Dekel show that there exists a non-empty compact metric space Θ_∞ (the "universal Bayesian type space") and a homeomorphism γ from Θ_∞ to $\Delta(S \times \Theta_\infty)$, the set of Borel probability measures on $S \times \Theta_\infty$ endowed with the weak* topology. Denote by $\text{Supp} : \Delta(S \times \Theta_\infty) \to \mathcal{K}(S \times \Theta_\infty)$ the support mapping. It is easy to check that it is onto. Since γ is a homeomorphism, it follows that the mapping $\text{Supp} \circ \gamma : \Theta_\infty \to \mathcal{K}(S \times \Theta_\infty)$ is onto, and therefore that $(\Theta_\infty, \text{Supp} \circ \gamma)$ is an

[2] The fact that the trivial belief $S \times \tilde{T}_\infty$ does not belong to $\mathcal{O}(S \times \tilde{T}_\infty)$ might seem disturbing at first. However, for any compact Hausdorff space X, $\mathcal{K}(X) \cup \{\emptyset\}$ is compact and homeomorphic to $\mathcal{O}(X) \cup \{X\}$. Hence, adding $\emptyset$ to $\mathcal{K}(X)$ allows to extend the homeomorphism $\tilde{g}$ to include the trivial belief.

[3] Formally, when X is an infinite compact Hausdorff space, $\mathcal{O}(X) \setminus \{\emptyset\}$ is not compact even if $\mathcal{O}(X)$ is. To see why, note that the collection $\{O \in \mathcal{O}(X) \mid \{x\} \subset O\}_{x \in X}$ is an open cover for $\mathcal{O}(X) \setminus \{\emptyset\}$ which, since X is Hausdorff, does not have a finite subcover.

S–based complete compact possibility structure.

However, the support mapping is not continuous and hence, the possibility mapping $\mathrm{Supp} \circ \gamma$ is not continuous either. Indeed, two probabilistic types θ_1, $\theta_2 \in \Theta_\infty$ can be arbitrarily close to each other and thus induce arbitrarily close beliefs $\gamma(\theta_1)$ and $\gamma(\theta_2)$ (in the sense of the weak* topology on $\Delta(S \times \Theta_\infty)$), whereas the probability measures $\gamma(\theta_1)$ and $\gamma(\theta_2)$ have very different supports (in the sense of the Hausdorff topology on $\mathcal{K}(S \times \Theta_\infty)$). More importantly, the construction of the previous section does not apply: one cannot in general translate a Bayesian type $\theta \in \Theta_\infty$ into an explicit hierarchy of compact beliefs. The reason is that even if the possibility mapping $\mathrm{Supp} \circ \gamma$ maps each type $\theta \in \Theta_\infty$ to a compact subset of $S \times \Theta_\infty$, the map $(\mathrm{Id}_S, \mathrm{Supp} \circ \gamma)$ does not necessarily map a compact subset of $S \times \Theta_\infty$ to a compact subset of $S \times \mathcal{K}(S \times \Theta_\infty)$.

Example 1 *Suppose $S = [0, 1]$, $\theta \in \Theta_\infty$, and $\{s_n\}_{n \geq 1}$ a sequence in $(0, 1]$ converging to 0. For any $n \geq 1$, define $\theta_n = \gamma^{-1}(1/n\ \delta_{(s_n,\theta)} + (1 - 1/n)\, \delta_{(1,\theta)})$ and $\theta_\infty = \gamma^{-1}(\delta_{(1,\theta)})$, where for any $x \in S \times \Theta_\infty$, δ_x is the point mass at x. Clearly, $\{\theta_n\}_{n \geq 1}$ converges weakly to θ_∞, so $\{\theta_n \mid 1 \leq n \leq \infty\} \in \mathcal{K}(\Theta_\infty)$. Hence, since $\mathrm{Supp} \circ \gamma$ is onto, there exists $\vartheta \in \Theta_\infty$ such that $R_0(\vartheta) \equiv \mathrm{Supp} \circ \gamma(\tilde{\theta}) = \{(0, \theta_n) \mid 1 \leq n \leq \infty\}$. But*

$$(\mathrm{Id}_S, R_0) \circ R_0(\vartheta) = \{(0, \{(s_n, \theta) \mid n \geq 1\})\} \cup \{(0, \{(1, \theta)\})\}$$

is not a compact subset of $S \times \mathcal{K}(S \times \Theta_\infty)$ as $\{s_n\}_{n \geq 1}$ converges to 0, not 1. In particular, its projection on $S \times \mathcal{K}(S)$, $\{(0, \{s_n \mid n \geq 1\})\} \cup \{(0, \{1\})\}$, does not belong to $\mathcal{K}(X_1^*)$.

Example 1 can be easily generalized to any compact Hausdorff space S with at least one accumulation point, and to any discontinuous mapping $S \to \mathcal{K}(S \times \Theta_\infty)$, but not to a finite set S, for instance. However, it should be noted that, even if S is finite, in which case any $\theta \in \Theta_\infty$ induces a hierarchy in T_1^*, closeness of two types in the summary structure Θ_∞ does not imply closeness of the induced hierarchies in T_1^*.

One may also argue that this lack of continuity is an artifact of using the support mapping in the above construction, and that a different mapping could associate compact subsets of $S \times \Theta_\infty$ to elements of Θ_∞ continuously. However, the following example shows that, in general, such mapping does not exist.

Example 2 *Suppose that S is not connected, and let (T, R) be an S–based complete compact possibility structure such that R is continuous. Since S is not connected, it can be* *partitioned into two disjoint open sets S_1 and S_2. This yields a partition of $\mathcal{K}(S \times T)$ into two disjoint open sets $\{K \in \mathcal{K}(S \times T) \mid K \subset S_1 \times T\}$ and $\{K \in \mathcal{K}(S \times T) \mid K \cap S_2 \times T \neq \emptyset\}$. Therefore, $\mathcal{K}(S \times T)$ is not connected. Since R is onto and continuous, it follows that T is not connected. On the other hand, $\Delta(S \times \Theta_\infty)$ and hence Θ_∞ by homeomorphism are arcwise connected, hence connected. Thus if (Θ_∞, R) is an S–based complete compact possibility structure, R cannot be continuous.*

It follows from our previous observations that if S is disconnected and has an accumulation point, and if (Θ_∞, R) is an S–based complete compact possibility structure, there always exists a type $\theta \in \Theta_\infty$ that has no explicit representation in T_1^*. Hence, the probabilistic (support) model cannot be embedded in a model of compact set-theoretic beliefs.

4.3. Comparison with Epstein and Wang (1996)

The existence of a space T homeomorphic to $\mathcal{K}(S \times T)$ can be directly derived from the embedding result in Theorem 6.1 of Epstein and Wang (1996). Our objective, however, is not restricted to showing the existence of such a homeomorphism. Rather, our main focus is to show that any compact and continuous possibility structure can be uniquely and canonically represented in T_∞. We also wish to remark that, in general, a type space T in a compact continuous possibility structure (T, R) where R is a homeomorphism between T and $\mathcal{K}(S \times T)$ is not necessarily homeomorphic to the universal type space T_∞. To see this, define inductively the sets

$$Y_0 = [0, 1], \qquad Y_{n+1} = \mathcal{K}(Y_n); \qquad n \geq 0.$$

Consider the mapping $p_0 : Y_1 \to Y_0 : y_1 \mapsto \inf\{y_0 \mid y_0 \in y_1\}$. Using the usual metric for the Hausdorff topology on $\mathcal{K}([0, 1])$, it is easy to verify that p_0 is onto and continuous. For any $n \geq 1$, define the mapping $p_n : Y_{n+1} \to Y_n : y_{n+1} \mapsto \{p_{n-1}(y_n) \mid y_n \in y_{n+1}\}$. Together with Lemma 2(iii), the fact that p_0 is onto and continuous implies by induction that all the mappings $\{p_n\}_{n \geq 0}$ are also onto and continuous and that $p_{n+1} = p_n^{\mathcal{K}}$ for any $n \geq 0$. Define

$$Y_\infty = \{\{y_n\}_{n \geq 0} \mid y_n \in Y_n \text{ and } y_n = p_n(y_{n+1});\ n \geq 0\}.$$

Following the same lines as in the proof of Lemma 3, it is easy to check that Y_∞ is a non-empty compact subset of $\prod_{n \geq 0} Y_n$ endowed with the product topology. Moreover:

Lemma 6 *There exists a homeomorphism $p : Y_\infty \to \mathcal{K}(Y_\infty)$.*

We are now ready to complete our example. Suppose that S is a singleton. Then p induces a homeomorphism from Y_∞ to $\mathcal{K}(S \times Y_\infty)$. However, T_∞ is then a singleton, whereas Y_∞ is not. Hence Y_∞ and T_∞ cannot be homeomorphic.[4]

5. Infinite hierarchies of conditional compact beliefs

In this section, we show how the infinite hierarchies of Section 3 can be generalized to compact conditional belief systems by adapting the construction provided by Battigalli and Siniscalchi (1999) for probabilistic beliefs.

5.1. Conditional belief systems

Consider an agent facing uncertainty over the space S. As in Section 2, a belief of the agent is a possibility set, that is, an element of $\mathcal{P}(S)$. In addition, let $\mathcal{B} \subset \mathcal{P}(S)$ be a non-empty collection representing the events that are observable by the agent. A conditional belief system assigns to any observable event $B \in \mathcal{B}$ a belief about the basic uncertainty parameter, representing the states in S considered possible by the agent conditional on B. The following definition is adapted from Brandenburger (1997):

Definition 6 *A conditional belief system on $(S, \mathcal{B})$ is a mapping $\xi : \mathcal{B} \to \mathcal{P}(S)$ such that:*

(i) For all $B \in \mathcal{B}$, $\xi(B) \subset B$;

(ii) For all $A, B \in \mathcal{B}$, if $A \subset B$ and $\xi(B) \cap A \neq \emptyset$, then $\xi(A) = \xi(B) \cap A$.

Condition (i) means that an observable event is self-evident whenever it occurs. Condition (ii) can be interpreted as a set-theoretic version of Bayes' rule. It captures the idea that the agent maintains his beliefs as long as they are not contradicted by further evidence. It implies in particular that if A and B are two observable events such that A refines B and the agent believes that A obtains conditional on B, then he must have the same beliefs at A and B. It should be noted that a conditional belief system need not be monotonic. For instance, if $A \subset B$ and $\xi(B) \cap A = \emptyset$, that is, the agent thinks that A is impossible conditional on B, then necessarily $\xi(A) \cap \xi(B) = \emptyset$.

[4]See Mertens, Sorin and Zamir (1994, Chapter III, Theorem 1.2, Remark 2) for a related point in the Bayesian setup.

5.2. Topological restrictions

To construct a universal space of conditional belief systems, we need to impose conditions on S and $\mathcal{B}$, as well as on conditional belief systems. We maintain that S is a compact Hausdorff space. In addition, we shall impose the following condition on $\mathcal{B}$. (Recall that a subset of a topological space is clopen if it is both closed and open.)

Assumption 2 *$\mathcal{B}$ is a collection of non-empty clopen subsets of S such that $S \in \mathcal{B}$.*

As in Battigalli and Siniscalchi (1999), Assumption 2 is crucial for our results. Together with Assumption 1, it implies that every event $B \in \mathcal{B}$ is compact as well as its complementary. Although this might seem a very strong assumption, it holds in a variety of situations. The less interesting case is when $\mathcal{B} = \{S\}$, that is, the only observable event is the state space itself. More interestingly, Assumption 2 is also satisfied when S is a product space $S' \times T$, where S' is a finite set, and $\mathcal{B}$ is composed of events of the form $B' \times T$, where B' is any non-empty subset of S'. It should be noted that, since separability plays no role in our analysis, there is no need to restrict the family $\mathcal{B}$ to be at most countable, as Battigalli and Sinischalchi (1999) do. For instance, one can take S to be the product of an uncountable number of copies of a finite set endowed with the discrete topology, and let $\mathcal{B}$ be the family of non-empty finite cylinders, which is uncountable.

Definition 7 *A conditional belief system ξ on $(S, \mathcal{B})$ is compact if for any $B \in \mathcal{B}$, $\xi(B) \in \mathcal{K}(S)$.*

We shall denote by $\mathcal{K}(S, \mathcal{B})$ the set of compact conditional belief systems on $(S, \mathcal{B})$. Note that $\mathcal{K}(S, \mathcal{B})$ can be seen as a subset of the product space $\mathcal{K}(S)^{\mathcal{B}}$, which is a compact Hausdorff space when endowed with the product topology. We then have the following lemma.

Lemma 7 $\mathcal{K}(S, \mathcal{B}) \in \mathcal{K}(\mathcal{K}(S)^{\mathcal{B}})$.

This result provides us with the recursivity necessary to any hierarchical construction. Indeed, since $\mathcal{K}(S, \mathcal{B})$ is itself compact and Hausdorff, so is $S \times \mathcal{K}(S, \mathcal{B})$. We can then endow this latter set with the family of compact events inherited from $\mathcal{B}$, that is, the family $\mathcal{C}(\mathcal{B})$ of cylinders $\mathcal{C}(B) = B \times \mathcal{K}(S, \mathcal{B})$, $B \in \mathcal{B}$. These cylinders form a family of clopen events in $S \times \mathcal{K}(S, \mathcal{B})$. Hence the space of conditional systems of compact beliefs over $S \times \mathcal{K}(S, \mathcal{B})$, $\mathcal{K}(S \times \mathcal{K}(S, \mathcal{B}), \mathcal{C}(\mathcal{B}))$ is a compact subset of $\mathcal{K}(S \times \mathcal{K}(S, \mathcal{B}))^{\mathcal{C}(\mathcal{B})}$, and the construction can be iterated again.

Definition 8 *An* $(S, \mathcal{B})$*–based compact conditional possibility structure is a pair* (T, R) *consisting of:*

(i) A non-empty Hausdorff space T*;*

(ii) A mapping $R : T \to \mathcal{K}(S \times T, \mathcal{B}^c)$*, where* $\mathcal{B}^c = \{B \times T \mid B \in \mathcal{B}\}$*.*

As before, the elements of T are called *types*. R is a *conditional possibility mapping* that assigns to each type $t \in T$ of a player a conditional belief system $R(t)$ about the basic uncertainty parameter and the type of the other player. An $(S, \mathcal{B})$–based compact conditional possibility structure (T, R) is *complete* if the possibility mapping R is onto, and is *continuous* if R is continuous. Consider a family $\mathcal{F}$ of $(S, \mathcal{B})$–based compact conditional possibility structures. An $(S, \mathcal{B})$–based compact conditional possibility structure $(T^{\mathcal{F}}, R^{\mathcal{F}})$ is *universal* for $\mathcal{F}$ if for any (T, R) in $\mathcal{F}$, there exists a unique mapping φ such that, for any $B \in \mathcal{B}$, the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & T^{\mathcal{F}} \\ \Big\downarrow {\scriptstyle R_B} & & \Big\downarrow {\scriptstyle R_B^{\mathcal{F}}} \\ \mathcal{K}(S \times T) & \xrightarrow{(\mathrm{Id}_S;\varphi)^{\mathcal{K}}} & \mathcal{K}(S \times T^{\mathcal{F}}) \end{array} ,$$

where R_B and $R_B^{\mathcal{F}}$ denote respectively the B-components of R and $R^{\mathcal{F}}$.

5.3. The infinite hierarchy

We are now ready to construct the universal space of compact conditional belief systems over $(S, \mathcal{B})$. For notational simplicity, we assume that there are only two agents, a and b, facing the same basic uncertainty space S, and sharing the same collection $\mathcal{B}$ of observable events. These agents have beliefs about S and about each other's beliefs conditional on each observable event $B \in \mathcal{B}$. Let $X_0 = S$, $\mathcal{B}_0 = \mathcal{B}$, and construct inductively the sets

$$\begin{aligned} X_{n+1} &= X_n \times \mathcal{K}(X_n, \mathcal{B}_n), \\ \mathcal{B}_{n+1} &= \{C \subset X_{n+1} \mid \exists B \in \mathcal{B}_n : C = B \times \mathcal{K}(X_n, \mathcal{B}_n)\}\,; \qquad n \geq 0. \end{aligned}$$

A player's $(n+1)$th-order conditional system of compact belief is an element ξ_{n+1} of $\mathcal{K}(X_n, \mathcal{B}_n)$. A nth-order conditioning event is an element of $\mathcal{B}_n$, that is, given some $B \in \mathcal{B}$, a cylinder set of the form

$$\mathcal{C}_n(B) = B \times \prod_{0\leq m\leq n-1} \mathcal{K}(X_m, \mathcal{B}_m).$$

For each $B \in \mathcal{B}$, any $\xi_{n+1} \in \mathcal{K}(X_n, \mathcal{B}_n)$ determines a compact belief $\xi_{n+1}(\mathcal{C}_n(B))$ on X_n conditional on B. Since $\mathcal{B}_n$ is essentially a copy of $\mathcal{B}$ in X_n, we can without risk of confusion write $\mathcal{K}(X_n, \mathcal{B})$ instead of $\mathcal{K}(X_n, \mathcal{B}_n)$. A player's *type* is a hierarchy of beliefs $\{\xi_n\}_{n\geq 1} \in \prod_{n\geq 0} \mathcal{K}(X_n, \mathcal{B})$. We denote by $T_0(\mathcal{B})$ the set of all possible types. By Lemma 7, it is clear that $T_0(\mathcal{B})$ is compact and Hausdorff in the product topology. It follows also that $\mathcal{K}(S\times T_0, \mathcal{B})$ is compact Hausdorff as well. For any $B \in \mathcal{B}$, we set $\mathcal{C}_\infty(B) = B \times T_0(\mathcal{B})$.

5.4. The universal possibility structure

The coherency condition in Definition 3 easily extends to compact conditional beliefs systems.

Definition 9 *A type $\{\xi_n\}_{n\geq 1} \in T_0(\mathcal{B})$ is coherent if and only if, for any $B \in \mathcal{B}$,*

$$\xi_n(\mathcal{C}_{n-1}(B)) = \mathrm{Proj}_{X_{n-1}}(\xi_{n+1}(\mathcal{C}_n(B))); \qquad n \geq 1.$$

Let $T_1(\mathcal{B})$ be the set of coherent types. The following proposition states that a coherent type for a player is equivalent to a system of conditional compact beliefs over S and the type of the other player.

Proposition 4 *There exists a homeomorphism $f(\mathcal{B}) : T_1(\mathcal{B}) \to \mathcal{K}(S \times T_0(\mathcal{B}), \mathcal{B})$ such that, for any $\{\xi_n\}_{n\geq 1} \in T_1(\mathcal{B})$ and $B \in \mathcal{B}$,*

$$\mathrm{Proj}_{X_{m-1}}(f(\mathcal{B})(\{\xi_n\}_{n\geq 1})(\mathcal{C}_\infty(B))) = \xi_m(\mathcal{C}_{m-1}(B)); \qquad m \geq 1.$$

Finally, we close the model by imposing common knowledge of coherency. This is done in a manner similar to the unconditional model. Define inductively the sets:

$$T_{k+1}(\mathcal{B}) = \{t \in T_1(\mathcal{B}) \mid f(\mathcal{B})(t)(\mathcal{C}_\infty(B)) \subset B \times T_k(\mathcal{B});\ B \in \mathcal{B}\}\,; \qquad k \geq 1,$$

and let $T_\infty(\mathcal{B}) = \bigcap_{k\geq 1} T_k(\mathcal{B})$.[5] The following result parallels Proposition 2.

Proposition 5 *$T_\infty(\mathcal{B})$ is non-empty. Moreover, the restriction of $f(\mathcal{B})$ to $T_\infty(\mathcal{B})$ induces a homeomorphism $g(\mathcal{B}) : T_\infty(\mathcal{B}) \to \mathcal{K}(S \times T_\infty(\mathcal{B}), \mathcal{B})$.*

[5] As noted by Battigalli and Siniscalchi (1999) in their construction of hierarchies of conditional probabilistic beliefs, there cannot be any inconsistency in assuming that there is common knowledge of coherency conditional on any event $B \in \mathcal{B}$, since these "external" events are defined on the basic uncertainty space S, and therefore do not convey any restriction about the epistemic types of the players.

Finally, $(T_\infty(\mathcal{B}), g(\mathcal{B}))$ can be shown to be an universal compact continuous conditional possibility structure by arguments analogous to those in Section 3.

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Appendix A

Proof of Lemma 1. It is immediate that if S and T are singletons, the mapping R is onto. Conversely, suppose there exists an onto mapping $R : T \to \mathcal{P}(S \times T)$. Then, for any $s \in S$, there exists an onto mapping $R_s : T \to \{s\} \times \mathcal{P}(T)$. Then the mapping $f : T \to \mathcal{P}(T) : t \mapsto \text{Proj}_{\mathcal{P}(T)}(t)$ is onto. Let $A = \{t \in T \mid t \notin f(t)\}$. If $A \neq \emptyset$, the usual diagonalization argument applies. Hence $A = \emptyset$, and $f^{-1}(\{\{t\}\}) = \{t\}$ for any $t \in T$. If T is not a singleton, there does not exists a $t \in T$ such that $f(t) = T$, a contradiction. (This argument follows Brandenburger (1998), Lemma 4.4.) Thus T is a singleton, hence R is one-to-one, and therefore a bijection. It follows that S is a singleton as well. □

Proof of Lemma 2. (i) Let $K_1, K_2 \in \mathcal{K}(X)$, $K_1 \neq K_2$, and, without loss of generality, take $x_1 \in K_1 \backslash K_2$. Since X is Hausdorff, there exist two disjoint open subsets of X, U and V, such that $x_1 \in U$ and $K_2 \subset V$. Let $\mathcal{U} = \{K \in \mathcal{K}(X) \mid K \cap U \neq \emptyset\}$ and $\mathcal{V} = \{K \in \mathcal{K}(X) \mid K \subset V\}$. Then $\mathcal{U}$ and $\mathcal{V}$ are open subsets of $\mathcal{K}(X)$ such that $K_1 \in \mathcal{U}$, $K_2 \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. Hence $\mathcal{K}(X)$ is Hausdorff.

(ii) $\mathcal{S}^{\mathcal{K}}(X) = \{\{K \in \mathcal{K}(X) \mid K \subset U\}, \{K \in \mathcal{K}(X) \mid K \cap U \neq \emptyset\}, U \text{ open in } X\}$ forms a subbase of the Hausdorff topology on $\mathcal{K}(X)$. By Alexander's subbase theorem, $\mathcal{K}(X)$ is compact if and only if every $\mathcal{S}^{\mathcal{K}}(X)$–cover of $\mathcal{K}(X)$ has a finite subcover. Let $\mathcal{C} = \{\{K \in \mathcal{K}(X) \mid K \subset U_i\}, \{K \in \mathcal{K}(X) \mid K \cap V_j \neq \emptyset\}\}_{i \in I, j \in J}$ be an $\mathcal{S}^{\mathcal{K}}(X)$–cover of $\mathcal{K}(X)$ associated to collections $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ of open subsets of X. Suppose first that $X = \bigcup_{j \in J} V_j$. Then since X is compact, one can extract a finite subcover $\{V_j\}_{j \in J'}$ of X from $\{V_j\}_{j \in J}$, and $\mathcal{C}' = \{\{K \in \mathcal{K}(X) \mid K \cap V_j \neq \emptyset\}\}_{j \in J'}$ is a finite subcover of $\mathcal{K}(X)$. Suppose next that $X \neq \bigcup_{j \in J} V_j$. Then $\bigcap_{j \in J} V_j^c$ is closed, hence compact in X. Since $\mathcal{C}$ is an $\mathcal{S}^{\mathcal{K}}(X)$–cover of $\mathcal{K}(X)$, there must exists $i_0 \in I$ such that $\bigcap_{j \in J} V_j^c \subset U_{i_0}$. Note that $U_{i_0}^c$ is closed, hence compact in X, and that $U_{i_0}^c \subset \bigcup_{j \in J} V_j$. Hence there exists a finite subcover $\{V_j\}_{j \in J'}$ of $U_{i_0}^c$. It follows that if $K \in \mathcal{K}(X)$ is such that $K \not\subset U_{i_0}$, then there exists $j \in J'$ such that $K \cap V_j \neq \emptyset$. Hence $\mathcal{C}' = \{\{K \in \mathcal{K}(X) \mid K \subset U_{i_0}\}, \{K \in \mathcal{K}(X) \mid K \cap V_j \neq \emptyset\}\}_{j \in J'}$ is a finite subcover of $\mathcal{K}(X)$.

(iii) Let $\mathcal{S}^{\mathcal{K}}(Y) = \{\{L \in \mathcal{K}(Y) \mid L \subset V\}, \{L \in \mathcal{K}(Y) \mid L \cap V \neq \emptyset\}, V \text{ open in } Y\}$ and $\mathcal{S}^{\mathcal{K}}(X)$ as in (ii). It is immediate to check that $(f^{\mathcal{K}})^{-1}(\mathcal{S}^{\mathcal{K}}(Y)) \subset \mathcal{S}^{\mathcal{K}}(X)$ provided f is continuous. Since $\mathcal{S}^{\mathcal{K}}(X)$ and $\mathcal{S}^{\mathcal{K}}(Y)$ are respectively subbases of the Hausdorff topology on $\mathcal{K}(X)$ and $\mathcal{K}(Y)$, the result follows.

(iv) Simply remark that for any $A \in \mathcal{K}(X)$, $g^{\mathcal{K}}(f^{\mathcal{K}}(A)) = g^{\mathcal{K}}(\{f(x) \mid x \in A\}) = \{g(f(x)) \mid x \in A\}$. The result follows immediately. □

Appendix B

Proof of Lemma 3. Let $\{\kappa_n\}_{n\geq 1} \in D$, and let $K_n = \kappa_n \times \prod_{\nu \geq n} Z_\nu \subset Z^\infty$. It is obvious from the definition of D that $\{K_n\}_{n\geq 1}$ is a nested sequence of sets. Let $K = \bigcap_{n\geq 1} K_n$. For any $n \geq 0$ and $k_n \in \kappa_n$, there exists $k_{n+1} \in \kappa_{n+1}$ such that $\text{Proj}_{Z^{n-1}}(k_{n+1}) = k_n$. In particular $(\text{Proj}_{Z_0}(k_1), \ldots, \text{Proj}_{Z_n}(k_{n+1}), \ldots) \in K \neq \emptyset$. Tychonoff theorem implies that the sets Z^∞ and $\{K_n\}_{n\geq 1}$ are compact in the product topology, hence K is compact as well. By construction, $\text{Proj}_{Z^{n-1}}(K) = \kappa_n$ for all $n \geq 1$. It follows that $f : D \to \mathcal{K}(Z^\infty) : \{\kappa_n\}_{n\geq 1} \mapsto K$ is one-to-one and onto. By Lemma 2(i), $\mathcal{K}(Z^{n-1})$ is Hausdorff for each $n \geq 1$; hence D is Hausdorff in the relative product topology. By Lemma 2(ii), $\mathcal{K}(Z^\infty)$ is compact. Since f^{-1} is a bijection from a compact space into a Hausdorff space, it is a homeomorphism if and only if it is continuous. It is thus sufficient to prove that for each $n \geq 1$, the mapping $\text{Proj}^{\mathcal{K}}_{Z^n} : \mathcal{K}(Z^\infty) \to \mathcal{K}(Z^{n-1}) : K \mapsto \text{Proj}_{Z^n}(K)$, is continuous. This follows from Lemma 2(iii) and the definition of the product topology on Z^∞. □

Proof of Proposition 1. Set $Z_0 = X_0$, and $Z_n = \mathcal{K}(X_{n-1})$ for each $n \geq 1$. Thus $Z^n = X_n$ and $Z^\infty = S \times T_0$. By Lemma 2(i)-(ii), the sets $\{Z_n\}_{n\geq 1}$ are compact Hausdorff provided S is. The set of coherent types is exactly D. The result follows then from Lemma 3. □

Proof of Proposition 2. Using the fact that f is a homeomorphism, it is immediate to check by induction that the sets $\{T_k\}_{k\geq 2}$ are non-empty and nested. We now prove that for each $k \geq 1$, $T_k \in \mathcal{K}(T_1)$. For $k = 1$, this follows from Proposition 1. Suppose next that $T_{k-1} \in \mathcal{K}(T_1)$, for some $k \geq 2$. By compactness of S and T_{k-1}, $\mathcal{K}(S \times T_{k-1})$ is a compact topological subspace of $\mathcal{K}(S \times T_1)$. Since $t \in T_k$ if and only if $f(t)$ is a compact subset of $S \times T_{k-1}$ in the relative topology induced by $S \times T_1$, it follows that $T_k = f^{-1}(\mathcal{K}(S \times T_{k-1}))$, and hence that T_k is compact since f is a homeomorphism. Since the sets $\{T_k\}_{k\geq 1}$ are nested, $T_\infty \neq \emptyset$. One has $T_\infty = \{t \in T_1 \mid f(t) \subset S \times T_\infty\}$, so $f(T_\infty) = \{K \in \mathcal{K}(S \times T_0) \mid K \subset S \times T_\infty\}$ since f is onto. But $f(T_\infty)$ is homeomorphic to T_∞ and $\{K \in \mathcal{K}(S \times T_0) \mid K \subset S \times T_\infty\}$ is homeomorphic to $\mathcal{K}(S \times T_\infty)$, hence the result. □

Proof of Lemma 4. If the mapping $R_0 = R$ is continuous, then so is the mapping $(\text{Id}_S; R_0) : \hat{X}_0 \to S \times \mathcal{K}(\hat{X}_0)$. Therefore $R_1 = (\text{Id}_S; R_0)^{\mathcal{K}}$ is continuous by Lemma 2(iii). The claim follows by iterated applications of this argument. □

Proof of Lemma 5. We have to prove that, for any $n \geq 0$, $P_{n+1} \circ Q_{n+1} \circ R_{n+1} = Q_n$. We proceed by induction. First, note that by construction, $Q_1 \circ R_1 = (\text{Id}_S; \text{Proj}_\text{S})^{\mathcal{K}} \circ (\text{Id}_S; R_0)^{\mathcal{K}}$, which in turn is equal to $(\text{Id}_S; \text{Proj}_\text{S} \circ R_0)^{\mathcal{K}}$ by Lemma 2(iv). Since $P_1 = \text{Proj}_S^{\mathcal{K}}$ and the first component of $Q_1 \circ R_1$ is Id_S, $P_1 \circ Q_1 \circ R_1 = \text{Proj}_S^{\mathcal{K}} = Q_0$ by construction. This proves the claim for $n = 0$. Suppose now that the claim holds for some $n \geq 0$. By construction, $P_{n+2} \circ Q_{n+2} \circ R_{n+2} = (\text{Id}_S; P_{n+1})^{\mathcal{K}} \circ (\text{Id}_S; Q_{n+1})^{\mathcal{K}} \circ (\text{Id}_S; R_{n+1})^{\mathcal{K}}$, which is equal to $(\text{Id}_S; P_{n+1} \circ Q_{n+1} \circ R_{n+1})^{\mathcal{K}}$ by Lemma 2(iv), and therefore to $(\text{Id}_S; P_n)^{\mathcal{K}} = P_{n+1}$ by the induction step and the definition of P_{n+1}. This completes the proof. □

Proof of Proposition 3. (i) Let $\hat{\kappa} : T \to \prod_{n \geq 0} \mathcal{K}(\hat{X}_n) : t \mapsto \{R_{n-1} \circ \ldots \circ R_0(t)\}_{n \geq 1}$ and $\kappa^* : \prod_{n \geq 0} \mathcal{K}(\hat{X}_n) \to T_0^* : \{\hat{\kappa}_n\}_{n \geq 1} \mapsto \{Q_{n-1}(\hat{\kappa}_n)\}_{n \geq 0}$. By Lemma 4, $R_{n-1} \circ \ldots \circ R_0$ is continuous for any $n \geq 1$. By definition of the product topology on $\prod_{n \geq 1} \mathcal{K}(\hat{X}_n)$, this implies that $\hat{\kappa}$ is continuous. Next, Q_n is continuous for each $n \geq 0$. Since κ^* maps $\prod_{n \geq 0} \mathcal{K}(\hat{X}_n)$ component by component into T_0^*, κ^* is continuous, as well as $\varphi_{T,R} = \kappa^* \circ \hat{\kappa}$. Finally, it is immediate from Lemma 5 that $\varphi_{T,R}(t)$ is a coherent $*$–hierarchy for any $t \in T$, and therefore that φ maps T into T_1^*.

(ii) Note first that if T is a compact space and (T, R) is a complete possibility structure, then the mappings $\{R_n\}_{n \geq 0}$ are onto. Indeed, if (T, R) is complete, $R_0 = R$ is onto by definition. If R_0 is continuous, the inverse image by R_0 of any compact $\hat{\kappa}_1 \in \mathcal{K}(\hat{X}_0)$ is closed in T, hence compact if T is compact. (Note that since $\hat{X}_0$ is Hausdorff, any compact subset of $\hat{X}_0$ is closed.) The claim follows by iterated applications of this argument. Next, we show that if T is compact and (T, R) is complete, then $\varphi_{T,R} : T \to T_1^*$ is onto. Indeed, fix $\{\kappa_n^*\}_{n \geq 1} \in T_1^*$. Note that since Q_{n-1} is continuous and $\mathcal{K}(\hat{X}_{n-1})$ is compact, $Q_{n-1}^{-1}(\kappa_n^*)$ is a non-empty compact subset of $\mathcal{K}(\hat{X}_{n-1})$ for any $n \geq 1$. Consider $\hat{\kappa}_n \in Q_{n-1}^{-1}(\kappa_n^*)$ and $\hat{\kappa}_{n-1} \in R_{n-1}^{-1}(\hat{\kappa}_n)$. If $n = 1$, $\hat{\kappa}_{n-1} \in T$. If $n \geq 2$, it follows from the previous claim and the coherency of $\{\kappa_n^*\}_{n \geq 1}$ that $Q_{n-2}(\hat{\kappa}_{n-1}) = P_{n-1}(\kappa_n^*) = \kappa_{n-1}^*$. This implies in particular that $R_{n-1}^{-1}(Q_{n-1}^{-1}(\kappa_n^*)) \subset Q_{n-2}^{-1}(\kappa_{n-1}^*)$. Repeated applications of this argument using the mappings $R_{n-1}, ..., R_0$ allow us no construct a sequence $(t, \hat{k}_1, \ldots k_n) \in T \times \prod_{0 \leq m \leq n-1} \mathcal{K}(\hat{X}_m)$ such that $\hat{k}_1 = R_0(t)$, $R_m(\hat{\kappa}_m) = \hat{\kappa}_{m+1}$ and $Q_{m-1}(\hat{\kappa}_m) = \kappa_m^*$ for each $m \in \{1, \ldots, n-1\}$. For any $n \geq 1$, let $T_n = \{t \in T \mid \kappa_n^* = Q_{n-1} \circ R_{n-1} \circ ... \circ R_0(t)\}$. By the previous claim, T_n is a non-empty and compact subset of T for any $n \geq 1$. Moreover, using the fact that $R_{n-1}^{-1}(Q_{n-1}^{-1}(\kappa_n^*)) \subset Q_{n-2}^{-1}(\kappa_{n-1}^*)$ for each $n \geq 2$, it is easy to check that $T_{n+1} \subset T_n$ for any $n \geq 1$. Therefore $\bigcap_{n\geq 1} T_n \neq \emptyset$. To complete the proof, remark that by construction $\varphi_{T,R}(t) = \{\kappa_n^*\}_{n\geq 1}$ for any $t \in \bigcap_{n\geq 1} T_n$. Hence $\varphi_{T,R}$ is onto, as claimed. Now, suppose that $(T, R) = (T_\infty, g)$. From (i) and the above argument, the mapping $\varphi_{T_\infty,g}$ is continuous and onto. Since T_∞ is compact and T_1^* is Hausdorff, it is sufficient to prove that $\varphi_{T_\infty,g}$ is one-to-one. For any $n \geq 1$, let $\mathcal{K}_n = \{\kappa_n \in \mathcal{K}(X_{n-1}) \mid \exists t \in T_\infty \text{ s.t. } \kappa_n = \text{Proj}_{\mathcal{K}(X_{n-1})}(t)\}$ and $\mathcal{K}^n = \{\{\kappa_m\}_{1\leq m\leq n} \in \prod_{1\leq m\leq n} \mathcal{K}_m \mid \exists t \in T_\infty \text{ s.t. } \kappa_m = \text{Proj}_{\mathcal{K}(X_{m-1})}(t);\ m \in \{1, \ldots, n\}\}$. We shall now show that, for any $n \geq 1$, there exists a one-to-one and continuous mapping $f_n : \mathcal{K}_n \to \mathcal{K}(X_{n-1}^*)$ such that, for any $t \in T_\infty$, $Q_{n-1} \circ R_{n-1} \circ \ldots \circ R_0(t) = f_n(\text{Proj}_{\mathcal{K}(X_{n-1})}(t))$. Note first that, for any $t \in T_\infty$, $Q_0 \circ R_0(t) = \text{Proj}_{X_0}^{\mathcal{K}}(g(t)) = \text{Proj}_{\mathcal{K}(X_0)}(t)$ by canonicity of g. Hence $f_1 = \text{Id}_{\mathcal{K}(S)} = \text{Id}_{\mathcal{K}_1}$, which proves the claim for $n = 1$. Next, suppose that the claim holds for $n \geq 1$. Consider the mapping $\tau_n : \mathcal{K}^n \to \mathcal{K}_n : (\kappa_1, \ldots, \kappa_n) \mapsto \kappa_n$, i.e., the restriction to $\mathcal{K}^n$ of $\text{Proj}_{\mathcal{K}(X_{n-1})} : \prod_{1\leq m\leq n} \mathcal{K}(X_{m-1}) \to \mathcal{K}(X_{n-1})$. Since any type in T_∞ is coherent, τ_n is a continuous bijection, hence a homeomorphism from $\mathcal{K}^n$ to $\mathcal{K}_n$. Then, for each $t \in T_\infty$,

$$
\begin{aligned}
Q_n \circ R_n \circ \ldots \circ R_0(t) &= (\text{Id}_S; Q_{n-1})^{\mathcal{K}} \circ (\text{Id}_S; R_{n-1})^{\mathcal{K}} \circ \ldots \circ (\text{Id}_S; g)^{\mathcal{K}} \circ g(t) \\
&= (\text{Id}_S; Q_{n-1} \circ R_{n-1} \circ \ldots \circ g)^{\mathcal{K}} \circ g(t) \\
&= \big\{(\tilde{s}, f_n(\text{Proj}_{\mathcal{K}(X_{n-1})}(\tilde{t})) \mid (\tilde{s}, \tilde{t}) \in g(t)\big\} \\
&= \big\{(\tilde{s}, f_n \circ \tau_n(\text{Proj}_{\mathcal{K}(X_0)}(\tilde{t}), \ldots, \text{Proj}_{\mathcal{K}(X_{n-1})}(\tilde{t})) \mid (\tilde{s}, \tilde{t}) \in g(t)\big\} \\
&= (\text{Id}_S; f_n \circ \tau_n)^{\mathcal{K}}\big(\big\{(\tilde{s}, \text{Proj}_{\mathcal{K}(X_0)}(\tilde{t}), \ldots, \text{Proj}_{\mathcal{K}(X_{n-1})}(\tilde{t})) \mid (\tilde{s}, \tilde{t}) \in g(t)\big\}\big) \\
&= (\text{Id}_S; f_n \circ \tau_n)^{\mathcal{K}}\big(\text{Proj}_{X_n}(g(t))\big) \\
&= (\text{Id}_S; f_n \circ \tau_n)^{\mathcal{K}}\left(\text{Proj}_{\mathcal{K}(X_n)}(t)\right),
\end{aligned}
$$

where the second equality follows from Lemma 2(iv), the third from the induction hypothesis, the fourth from the definition of τ_n together with the coherency of $\tilde{t}$, and the seventh from the canonicity of g. Since f_n is continuous and one-to-one, and τ_n is a homeomorphism, it follows that $f_{n+1} \equiv (\text{Id}_S; f_n \circ \tau_n)^{\mathcal{K}} : \mathcal{K}_{n+1} \to \mathcal{K}(X_n^*)$ is continuous and one-to-one. Consider two types $t \neq \tilde{t} \in T_\infty$, and let $n(t, \tilde{t}) = \inf\{n \geq 1 \mid \text{Proj}_{\mathcal{K}(X_{n-1})}(t) \neq \text{Proj}_{\mathcal{K}(X_{n-1})}(\tilde{t})\} < \infty$. From the above argument, it follows that for any $n \geq 1$, t and $\tilde{t}$ have different images under

$Q_{n-1} \circ R_{n-1} \circ \ldots \circ R_0$ if and only if $n \geq n(t, \tilde{t})$. Thus $\varphi_{T_\infty,g}$ is one-to-one, as claimed. $\square$

Proof of Lemma 6. For any $\{y_n\}_{n\geq 0} \in Y_\infty$, define inductively $H_1 = \prod_{n\geq 1} y_n$ and, for any $n \geq 1$, $H_{n+1} = \{\upsilon \in H_n \mid \upsilon_{n-1} = p_{n-1}(\upsilon_n)\}$. Along the same lines as in the proof of Lemma 3, it is easy to check that $H = \bigcap_{n\geq 1} H_n$ is a non-empty compact subset of Y_∞ and that the mapping $p : Y_\infty \to \mathcal{K}(Y_\infty) : \{y_n\}_{n\geq 0} \mapsto H$ is one-to-one and continuous. Now, let $\kappa \in \mathcal{K}(Y_\infty)$. For each $n \geq 0$, $\mathrm{Proj}_{Y_n}(\kappa)$ is a compact subset of Y_n, hence an element of Y_{n+1}. Define a sequence $\{y_n(\kappa)\}_{n\geq 0} \in \prod_{n\geq 0} Y_n$ by $y_0(\kappa) = \inf\{y_0 \mid y_0 \in \mathrm{Proj}_{Y_0}(\kappa)\}$, and $y_n(\kappa) = \mathrm{Proj}_{Y_{n-1}}(\kappa)$ for each $n \geq 1$. Note first that $p_0(y_1(\kappa)) = y_0(\kappa)$ by construction. Next, for any $n \geq 1$, $p_n(y_{n+1}(\kappa)) = \{p_{n-1}(y_n) \mid y_n \in \mathrm{Proj}_{Y_n}(\kappa)\} = \mathrm{Proj}_{Y_{n-1}}(\kappa) = y_n(\kappa)$ since any sequence in κ belongs to Y_∞. Hence $\{y_n(\kappa)\}_{n\geq 0} \in Y_\infty$. Since $p(\{y_n(\kappa)\}_{n\geq 0}) = \kappa$ by construction, this implies that p is onto, which completes the proof. $\square$

Proof of Lemma 7. Note first that $\xi = \mathrm{Id}_{\mathcal{B}}$ trivially satisfies Definition 6, so $\mathcal{K}(S, \mathcal{B})$ is non-empty. The set of $\xi \in \mathcal{K}(S)^{\mathcal{B}}$ that satisfy Definition 6(i) is homeomorphic to $\prod_{B\in\mathcal{B}} \mathcal{K}(B)$, hence compact. Now fix $A, B \in \mathcal{B}$, $A \subset B$, and let $\Xi_{A,B}$ be the set of $\xi \in \mathcal{K}(S)^{\mathcal{B}}$ that satisfy Definition 6(ii) for the pair A, B, that is, $\Xi_{A,B} = \{\xi \in \mathcal{K}(S)^{\mathcal{B}} \mid \xi(B) \cap A = \emptyset\} \cup \{\xi \in \mathcal{K}(S)^{\mathcal{B}} \mid \xi(B) \cap A \neq \emptyset \text{ and } \xi(B) \cap A = \xi(A)\}$. The first set in this union is homeomorphic to $\mathcal{K}(S)^{\mathcal{B}\setminus\{B\}} \times \mathcal{K}(A^c)$ and is therefore compact as A is open. The second set is homeomorphic to $\mathcal{K}(S)^{\mathcal{B}\setminus\{A,B\}} \times \{(K \cap A, K) \mid K \in \mathcal{K}(S),\, K \cap A \neq \emptyset\}$. Note that, as A is closed, $\{K \in \mathcal{K}(S) \mid K \cap A \neq \emptyset\}$ is compact. Moreover, $\{(K \cap A, K) \mid K \in \mathcal{K}(S),\, K \cap A \neq \emptyset\}$ is homeomorphic to $\mathrm{Graph}(\phi)$, where $\phi : \{K \in \mathcal{K}(S) \mid K \cap A \neq \emptyset\} \to \mathcal{K}(S) : K \mapsto K \cap A$. Using the fact that A is clopen, it is easy to check that ϕ is continuous. Hence, since $\mathcal{K}(S)$ is Hausdorff, $\mathrm{Graph}(\phi)$ is closed, and therefore compact as a subset of the compact space $\{K \in \mathcal{K}(S) \mid K \cap A \neq \emptyset\} \times \mathcal{K}(S)$. Thus $\Xi_{A,B}$ is compact, as the union of two compact sets. Finally, $\mathcal{K}(S, \mathcal{B})$ can be written as $\prod_{B\in\mathcal{B}} \mathcal{K}(B) \cap \bigcap_{A,B\in\mathcal{B},A\subset B} \Xi_{A,B}$ and hence is compact. $\square$

Proof of Proposition 4. For any $B \in \mathcal{B}$, $\{\xi_n\}_{n\geq 1} \mapsto \{\xi_n(\mathcal{C}_{n-1}(B))\}_{n\geq 1}$ is a continuous map from $T_1(\mathcal{B})$ into $D \equiv \{\{\kappa_n\}_{n\geq 1} \mid \kappa_n \in \mathcal{K}(X_{n-1}) \text{ and } \mathrm{Proj}_{X_{n-1}}(\kappa_{n+1}) = \kappa_n;\ n \geq 1\}$. Define $Z_0 = S$, $Z_n = \mathcal{K}(X_{n-1}, \mathcal{B})$ and note that $X_{n-1} = S \times \prod_{0\leq m\leq n-2} \mathcal{K}(X_m, \mathcal{B}) = \prod_{0\leq m\leq n-1} Z_m$ for any $n \geq 1$. Since the spaces $\{Z_n\}_{n\geq 0}$ are compact Hausdorff, it follows from Lemma 3 that there exists a homeomorphism $f : D \to \mathcal{K}(S \times T_0(\mathcal{B}))$ such that, for all $\{\kappa_n\}_{n\geq 1} \in D$ and $m \geq 1$, $\mathrm{Proj}_{X_{m-1}}(f(\{\kappa_n\}_{n\geq 1})) = \kappa_m$. Thus for any $B \in \mathcal{B}$, the mapping $f_B : T_1(\mathcal{B}) \to \mathcal{K}(S \times T_0(\mathcal{B})) : \{\xi_n\}_{n\geq 1} \mapsto f(\{\xi_n(\mathcal{C}_{n-1}(B))\}_{n\geq 1})$ is well defined and continuous, and $\mathrm{Proj}_{X_{m-1}}(f_B(\{\xi_n\}_{n\geq 1})) = \xi_m(\mathcal{C}_{m-1}(B))$ for each $\{\xi_n\}_{n\geq 1} \in T_1(\mathcal{B})$. This implies that the mapping $f(\mathcal{B}) = (f_B)_{B\in\mathcal{B}} : T_1(\mathcal{B}) \to \mathcal{K}(S \times T_0(\mathcal{B}))^{\mathcal{B}}$ is continuous and one-to-one, and satisfies the condition required by the result. Hence, since $T_1(\mathcal{B})$ is compact and $\mathcal{K}(S \times T_0(\mathcal{B}))^{\mathcal{B}}$ is a Hausdorff space, we only have to prove that $f(\mathcal{B})(T_1(\mathcal{B})) = \mathcal{K}(S \times T_0(\mathcal{B}), \mathcal{B})$. To prove that $f(T_1(\mathcal{B})) \subset \mathcal{K}(S \times T_0(\mathcal{B}), \mathcal{B})$, note first that for any $B \in \mathcal{B}$ and $\{\xi_n\}_{n\geq 1} \in T_1(\mathcal{B})$, $\mathrm{Proj}_{X_0}(f_B(\{\xi_n\}_{n\geq 1})) = \xi_1(B) \subset B$ since ξ_1 is a compact conditional belief system over $(S, \mathcal{B})$. It follows that $f(\mathcal{B})(\{\xi_n\}_{n\geq 1})(\mathcal{C}_\infty(B)) \subset \mathcal{C}_\infty(B) = B \times T_0(\mathcal{B})$ and thus Definition 6(i) is satisfied. Next, let $A, B \in \mathcal{B}$ such that $A \subset B$ and $f(\mathcal{B})(\{\xi_n\}_{n\geq 1})(\mathcal{C}_\infty(B)) \cap \mathcal{C}_\infty(A) \neq \emptyset$. Then, for each $m \geq 0$, $\mathrm{Proj}_{X_m}(f(\mathcal{B})(\{\xi_n\}_{n\geq 1})(\mathcal{C}_\infty(B))) \cap \mathrm{Proj}_{X_m}(\mathcal{C}_\infty(A)) \neq \emptyset$. By projection, we get that $\xi_{m+1}(\mathcal{C}_m(B)) \cap \mathcal{C}_m(A) \neq \emptyset$, and since Definition 6(ii) applies to ξ_{m+1}, $\xi_{m+1}(\mathcal{C}_m(A)) = \xi_{m+1}(\mathcal{C}_m(B)) \cap \mathcal{C}_m(A)$. Since this is true for each m, it follows that $f(\mathcal{B})(\{\xi_n\}_{n\geq 1})(\mathcal{C}_\infty(A)) = f(\mathcal{B})(\{\xi_n\}_{n\geq 1})(\mathcal{C}_\infty(B)) \cap \mathcal{C}_\infty(A)$ and thus Definition 6(ii) is satisfied. It remains to prove that $\mathcal{K}(S \times T_0(\mathcal{B}), \mathcal{B}) \subset f(\mathcal{B})(T_1(\mathcal{B}))$. Let $\xi \in \mathcal{K}(S \times T_0(\mathcal{B}), \mathcal{B})$, and for any $B \in \mathcal{B}$ and $n \geq 1$, define $\xi_n(\mathcal{C}_{n-1}(B)) = \mathrm{Proj}_{X_{n-1}}(\xi(\mathcal{C}_\infty(B))) \in \mathcal{K}(X_{n-1})$. It is sufficient to prove that $\{\xi_n\}_{n\geq 1} \in T_1(\mathcal{B})$. First, it is clear that, by construction, $\xi_n(\mathcal{C}_{n-1}(B)) = \mathrm{Proj}_{X_{n-1}}(\xi_{n+1}(\mathcal{C}_n(B)))$ for any $n \geq 1$. Hence, we need only to prove that for each $n \geq 1$, $\xi_n \in \mathcal{K}(X_{n-1}, \mathcal{B})$. First, since $\xi \in \mathcal{K}(S \times T_0(\mathcal{B}), \mathcal{B})$, one must have $\xi(\mathcal{C}_\infty(B)) \subset \mathcal{C}_\infty(B)$ for any $B \in \mathcal{B}$, hence $\xi_n(\mathcal{C}_{n-1}(B)) \subset \mathcal{C}_{n-1}(B)$ by projection on X_{n-1}. Thus Definition 6(i) holds. Now suppose that for some $A, B \in \mathcal{B}$, $A \subset B$, we have $\xi_n(\mathcal{C}_{n-1}(B)) \cap \mathcal{C}_{n-1}(A) \neq \emptyset$. Then by projection on $X_0 = S$, we have that $\xi_1(\mathcal{C}_0(B)) \cap A \neq \emptyset$ which implies that $\xi(\mathcal{C}_\infty(B)) \cap \mathcal{C}_\infty(A) \neq \emptyset$. Hence, by Definition 6(ii), $\xi(\mathcal{C}_\infty(A)) = \xi(\mathcal{C}_\infty(B)) \cap \mathcal{C}_\infty(A)$. It follows then by projection on X_{n-1} that $\xi_n(\mathcal{C}_{n-1}(A)) = \xi_n(\mathcal{C}_{n-1}(B)) \cap \mathcal{C}_{n-1}(A)$, and thus Definition 6(ii) is satisfied. □

Proof of Proposition 5. First, if $t \in T_1(\mathcal{B})$ and for each $B \in \mathcal{B}$, $f(\mathcal{B})(t)(B) \subset B \times T_\infty(\mathcal{B})$, then $f(\mathcal{B})(t)(B) \subset B \times T_k(\mathcal{B})$ for each $k \geq 1$ and therefore $t \in \bigcap_{k\geq 1} T_k(\mathcal{B}) = T_\infty(\mathcal{B})$. Reciprocally, if $t \in T_\infty(\mathcal{B})$, then for each $B \in \mathcal{B}$ and $k \geq 1$, $f(\mathcal{B})(t)(B) \subset B \times T_k(\mathcal{B})$ and therefore $f(\mathcal{B})(t)(B) \subset B \times \bigcap_{k\geq 1} T_k(\mathcal{B}) = B \times T_\infty(\mathcal{B})$. Thus $T_\infty(\mathcal{B}) = \{t \in T_1(\mathcal{B}) \mid f(\mathcal{B})(t)(B) \subset B \times T_\infty(\mathcal{B});\ B \in \mathcal{B}\}$. It follows therefore that $f(\mathcal{B})(T_\infty(\mathcal{B})) = \{\xi \in \mathcal{K}(S \times T_0(\mathcal{B}), \mathcal{B}) \mid \xi(B \times T_0(\mathcal{B})) \subset B \times T_\infty(\mathcal{B});\ B \in \mathcal{B}\} = \mathcal{K}(S \times T_\infty(\mathcal{B}), \mathcal{B})$, which implies the result. □

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