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CESifo Working Paper No. 3394
Category 11: Industrial Organisation
March 2011

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Abstract

We propose a smooth multibidding mechanism for environments where a group of agents have to choose one out of several projects. Our proposal is related to the multibidding mechanism (Pérez-Castrillo and Wettstein, 2002) but it is "smoother" in the sense that small variations in an agent's bids do not lead to dramatic changes in the probability of selecting a project. This mechanism is shown to possess several interesting properties. First, the equilibrium outcome is unique. Second, it ensures an equal sharing of the surplus that it induces. Finally, it enables reaching an outcome as close to efficiency as is desired.

JEL-Code: D780, D720.

Keywords: mechanism design, NIMBY.

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March 16, 2011
We thank Inés Macho-Stadler, Pedro Rey-Biel, Robert Veszteg and David Wettstein for helpful remarks. Part of this research was conducted while the second author visited the Institute of Economic Analysis (IAE) and the INRA-LAMETA. The hospitality of these institutions is greatly appreciated. The first author acknowledges the financial support from ECO2009-7616, Consolider-Ingenio CSD2006-16, 2009SGR-169, Barcelona Economics-Xarxa CREA and ICREA Academia. David Pérez-Castrillo is MOVE fellow.

1 Introduction

1.1 Contribution

A mechanism designed to help agents reach (efficient) decisions on contentious issues typically requires information about agents' preferences for each possible decision. The *multibidding mechanism*, proposed by Pérez-Castrillo and Wettstein (2002) allows the agents to express their relative preference for projects. It proceeds as follows. Each agent submits a vector of bids, one for each project, with the sole restriction that the sum of each agent's bids is zero. Therefore, bids measure relative rather than absolute valuation. Each agent also nominates one of the projects specifically. The project with the highest aggregate bid (sum of bids made for this project) is chosen. In case there is more than one such project, there is a rule that gives priority to projects that have been nominated by some agent. The winning project is carried out, agents pay the promised bid corresponding to this project, and any surplus is shared among the agents, so that the mechanism is budget-balanced.

The main property of the multibidding mechanism is that all its Nash (and strong Nash) equilibrium outcomes are efficient. However, in general environments, the mechanism has one major weak aspect that we address in the current paper. Specifically, the set of equilibrium outcomes is quite large, as it consists of all the outcomes where each agent's payoff is at least the expected payoff he would obtain in a situation where all the projects have the same probability of being developed. Therefore, almost any ("reasonable") sharing of the surplus is an equilibrium outcome.

In the present paper, we tackle the issue highlighted above by proposing a *smooth multibidding mechanism*. It is close to the original proposal but ours is "smoother" in the sense that small variations of an agent's bids do not lead to dramatic changes in the probability of selecting a project. In the smooth mechanism, each agent only submits a vector of bids, without nominating any project. All projects can be selected, with each project's probability being a function of its aggregate bid as well as the aggregate bids of the rest of the projects. Projects with a negative aggregate bid have a very low, but positive, fixed probability of being selected (a function of some parameter ε). Each project with a positive aggregate bid is selected with a probability that is a function of the level of its (and others') positive aggregate bid. We highlight that the present mechanism does not require the use of a tiebreaking rule. Such a rule plays a crucial role in the initial mechanism. As such, it is immune to the criticism raised by Ehlers (2009).[1]

We first show that, for a given value of ε, *the equilibrium outcome is unique*. Therefore, there is no coordination issue with respect to agents' expectation about the final outcome. We then characterize the equilibrium outcome. Although there may be several equilibrium strategies, the differences among them only concern bids for those projects that, at equilibrium, end up with negative aggregate bids. We identify the set of projects with positive equilibrium bids as well as each agent's bids to any project in this set. Only projects that are efficient, or whose total valuation is very close to the efficient one, ultimately receive a positive aggregate bid. In case some non-efficient project receives a positive aggregate bid, its level reflects the degree of inefficiency.

Second, the smooth multibidding mechanism ensures an *equal sharing of the surplus* that it induces. Indeed, an agent's equilibrium payoff in the mechanism is the sum of the value of the average project plus his fair share of the remaining surplus. That is, agents obtain the same level of utility as in the original multibidding mechanism, and the surplus is divided in equal parts among the agents. Since efficiency and equity often have the same importance in collective decision-making, this fairness property is a sensible advantage of the present mechanism.

We also show that each agent's expected payoff increases as the value of the parameter ε decreases; therefore, the distance to efficient outcomes decreases as well. Moreover, the probability of choosing an inefficient project converges to zero as the value of the parameter ε becomes small. We can bound the level of expected inefficiency as a function of the parameter ε: the maximum level of inefficiency of a project that receives a positive aggregate bid is a linear function of the square root of ε. Therefore, the smooth multibidding mechanism *gets as close to efficiency as one wishes*

To summarize, the present mechanism exhibits the interesting properties of uniqueness and fairness of its equilibrium outcome. Moreover, it gets as close to an efficient outcome

[1] In the mechanism developed by Perez-Castrillo and Wettstein (2002) tiebreaking rules play a crucial role. Indeed, at equilibrium, such a rule is always used because all projects' equilibrium aggregate bids are zero. Ehlers (2009) points out that without tiebreaking rules equilibria may fail to exist.

as wished. Therefore, this mechanism constitutes an interesting alternative to the original multibidding mechanism in situations where efficiency and equity are policy objectives.

1.2 Applicability of the mechanism and related literature

There are many economic situations where the smooth multibidding mechanism can be successfully used. A first case concerns the complex problem of the location of noxious facilities, such as prisons, dump sites, nuclear waste repositories, or airports. Many authors address this type of problem; we can refer among other papers to Kunreuther and Kleindorfer, 1986; Rob, 1989; O'Sullivan, 1993; Ingberman, 1995; Pérez-Castrillo and Wettstein, 2002; Minehart and Neeman, 2002; and Laurent-Lucchetti and Leroux, forthcoming.[2] Whereas the construction of such facilities may provide large global benefits, their cost is usually borne by the hosting agent. The sitting problems are so severe and so common that an acronym is used to refer to them: NIMBY (Not In My Back Yard).

Another sensitive decision problem concerns the location of large international research infrastructures. The decision about the city that should host such a facility is always the subject of hot debate among the candidates and other interested countries and institutions. In 2002, the European Commission started the European Strategy Forum on Research Infrastructures (ESFRI) to support and facilitate multilateral initiatives leading to a better use and development of research infrastructures, including biological archives, communication networks, research vessels, satellite and aircraft observation facilities, tele-

[2]Kunreuther and Kleindorfer (1986) showed that sealed-bid mechanisms lead to efficient outcomes in incomplete information environments where each agent is indifferent as to all the outcomes, as long as he is not the host, when agents use max-min strategies. O'Sullivan (1993) proved that efficiency is also reached in Bayes-Nash equilibria when there are two agents whose cost parameters are independently drawn. Ingberman (1995) highlighted the impossibility of reaching efficient majority decisions through an auction when cost to the agents of using a common facility is related to their distance from it. Rob (1989) studied mechanisms where a randomized decision rule and an expected compensation for each location are associated to each cost vector reported by the locations. He showed that the resulting mechanism could lead to inefficient outcomes. In a complete information scenario, Laurent-Lucchetti and Leroux (2009) proposed a two-stage mechanism that selects the efficient site and any individually rational division of the hosting provided the profile of benefits is known to the planner (otherwise, the mechanism should be extended to a more complex action space)..

scopes, synchrotrons, and particle accelerators. Although its 2006 Report presented a first roadmap identifying 35 projects with the scientific needs for the next 10-20 years, ESFRI is silent about how the interested countries should determine the location of the facility. However, this is a very difficult decision that involves many scientific, economic, and social issues. For each project, supporting countries should work out a procedure to choose the host of the facility. Therefore, they must first decide on a mechanism and then use the procedure to elect the hosting city.

The previous examples belong to a general class of problems in which a group of agents has to choose one out of several projects. In some situations, the set of projects coincides with the set of agents, as is the case if a group of municipalities meet to choose one of them to host a dump site or a hospital. In another context, the set of agents is larger than the set of projects, as is typically the case when countries or institutions build a large international research infrastructure: in such a situation, all countries may not have an own proposal regarding the specifics of the project to be carried out. The main objective of a mechanism in such situations would be to maximize the aggregate welfare of all the agents (efficiency). Moreover, such decisions typically require to compensate (some) agents with monetary transfers. The protocol defined in the present contribution can be considered a valuable option to be considered.

Our proposal is also related to papers that look for mechanisms that agents can use to choose whether to develop a project and which one to develop (see, for instance, Moulin, 1984, and Jackson and Moulin, 1992); to reach good allocations in economic environments with public goods and externalities (Varian, 1994a and 1994b); to dissolve a partnership (McAfee, 1992); to sell (or not) a project to one agent when it affects many (Jehiel *et al.*, 1996); or to award an indivisible good to one agent (in the spirit of King Solomon's dilemma; see, for instance, Glazer and Ma, 1989, and Perry and Reny, 1999).

Our contribution can also expand the set of applications of the multibidding mechanism as part of more complex mechanisms implementing solution concepts. Indeed, variants of the multibidding mechanism have been used in several environments; see Pérez-Castrillo and Wettstein (2001), Bergantiños and Vidal-Puga (2003, 2010), Macho-Stadler *et al.* (2006), Porteiro (2007), Slikker (2007), Ju *et al.* (2007), Kamijo (2008), Ehlers (2009), Ju and Wettstein, (2009), and Veszteg (2010).[3] In the mechanisms proposed in these papers, the multibidding procedure is generally followed by stages where some agents make proposals to other agents, who can either accept or reject them. The proposers always have incentives to put forward efficient allocations, since they are the residual claimants of the surplus. Given that efficiency is ensured independently of the identity of the proposer, there is no need to resort to the tiebreaking rule in these applications.

Finally, our paper can also be related to the literature on virtual (or $\varepsilon-$) implementation (Matsushima, 1988, and Abreu and Sen, 1991) in the sense that our objective is not to achieve an exact implementation of an efficient and fair outcome but to get as "close" as wished to that allocation.

The paper is organized as follows. In Section 2, we present the environment and the smooth multibidding mechanism. The equilibrium strategies and outcome are stated in Section 3. Section 4 studies the main properties of the equilibrium outcome, including the convergence properties when the parameter ε goes to zero. We provide a simple example in Section 5. Finally, Section 6 concludes the paper. All proofs are included in the Appendix.

2 The environment and the mechanism

We consider a set of agents $N = \{1, \ldots, n\}$ which have to choose which project will be carried out of a set of possible projects $K = \{1, \ldots, k\}$. The utility (payoff) of agent i if project q is selected is given by v_q^i.

We denote by $V_q \equiv \sum_{i \in N} v_q^i$ the sum of agents' utilities if project q is implemented. Project q is *efficient* if $V_q \geq V_p$ for all $p \in K$. We denote by E the set of efficient projects, that is,

$$E = \{q \in K / V_q \geq V_p \text{ for all } p \in K\}.$$

Information about all the values v_q^i is complete among the agents; that is, each agent knows not only the value he assigns to the projects but also the values assigned by

[3] For further discussions and applications, see Pérez-Castrillo and Veszteg (2007) and Veszteg (2010).

the other agents. However, the planner does not have information about these values. Alternatively, even if she did have some information, she would not want to use it. The planner is interested in designing an impartial mechanism that will treat all the agents in a symmetric manner.

We propose a *smooth multibidding mechanism* through which agents influence the probability that projects are selected. We now describe the mechanism, which has a unique stage.

Each agent $i \in N$ makes a vector of bids $b^i \equiv \left(b^i_q\right)_{q \in K}$ in $\mathbb{R}^k$, one bid for each q in K with $\sum_{q \in K} b^i_q = 0$. All agents make their decision simultaneously. Once the agents have chosen their bids, the outcome of the smooth multibidding mechanism is the following.

For each $q \in K$, $B_q \equiv \sum_{i \in N} b^i_q$ denotes the aggregate bid for project q and $B \equiv (B_q)_{q \in K}$ denotes the vector of aggregate bids. The probability that project q be carried out if the vector of aggregate bids is B is

$$f_q(B) = \frac{g(B_q)}{\sum_{p \in K} g(B_p)},$$

where we consider the following function $g(.)$:

$$g(B_p) = \begin{array}{cc} \varepsilon & \text{for all } B_p < 0 \\ \varepsilon + B_p & \text{for all } B_p \geq 0 \end{array},$$

with $\varepsilon > 0$. That is, the "weight" of each project is a fixed, positive amount ε plus its aggregate bid, in case it is positive. Finally, if project q is chosen, each agent $i \in N$ pays his bid for that project, b^i_q, and he receives a fair share of the aggregate bid, B_q. Therefore, agent i's utility if project q is implemented is

$$v^i_q - b^i_q + \frac{1}{n} B_q.$$

If we go back to the example of a set of countries that must select the location of an international research infrastructure identified by ESFRI, it is usually the case that the set of possible locations K is a subset of the set of interested countries N. The value v^i_q represents the utility that country i obtains if the infrastructure is located in country q. Therefore, we expect v^i_q to be quite high when $q = i$ and much lower if the location q is far from country i. The bid b^i_q can be interpreted as the extra contribution (in addition to its "proportional" share) that country i is ready to provide if the facility is located in country q. Therefore, b_i^i may reflect country i's willingness to pay to host the infrastructure and $-b_q^i$ is the compensation demanded if location q is inconvenient for this country.

The smooth multibidding mechanism borrows from the multibbiding mechanism of Pérez-Castrillo and Wettstein (2002) the idea of allowing the agents to express their relative preference for projects through a vector of bids. However, under the original mechanism, the probability of selecting any project jumps from 0 to 1 as the aggregate bid for this project just passes the maximum aggregate bid for the other projects. Under our proposal, a higher (positive) aggregate bid for a project increases the probability that it is selected, but the increase is "smooth". This feature allows us to offer a mechanism that does not require ad hoc tiebreaking rules.

3 The equilibria of the smooth multibidding mechanism

In this section, we first derive several properties that are necessarily satisfied by the Nash equilibria (NE) of the smooth multibidding mechanism. Second, we use these properties to provide a characterization of the set of NE.

Consider a vector of agents' bids $(b^i)_{i\in N}$ and let A denote the set of projects for which the aggregate bid is positive under this vector of strategies, that is, $A \equiv \{q \in K / B_q > 0\}$. Similarly, denote by $D \equiv \{q \in K / B_q < 0\}$ and $O \equiv \{q \in K / B_q = 0\}$ so we have $D \cup O = K \backslash A$. Additionally, we denote a the number of projects in A.[4] The probability that project $q \in K$ is chosen is given by

$$f_q(B) = \begin{array}{ll} \frac{\varepsilon + B_q}{k\varepsilon + \sum_{d\in A} B_d} & \text{for all } q \in A \\ \frac{\varepsilon}{k\varepsilon + \sum_{d\in A} B_d} & \text{for all } q \in K\backslash A. \end{array}$$

Agent i chooses his vector of bids b^i to maximize his expected profits given the bids

[4] Although the sets A, D, and O depend on the the vector of aggregate bids B, we avoid using the notations $A(B)$, $D(B)$, $O(B)$, and $a(B)$ for simplicity.

chosen by the rest of the agents. Agent i's profits are

$$\Pi^i(b^i, b^{-i}) = \sum_{p \in K} f_p(B) \left[v_p^i - b_p^i + \frac{1}{n} B_p \right].$$

Therefore, agent i chooses b^i to solve the following program, which we denote by $[P^i]$:

$$Max_{b^i} \sum_{p \in K} f_p(B) \left[v_p^i - b_p^i + \frac{1}{n} B_p \right]$$

$$\text{s.t.} \sum_{p \in K} b_p^i = 0.$$

We note first, that agent i's program $[P^i]$ is well behaved except that the derivative on the right of function $f_p(B)$ with respect to B_q (hence, with respect to b_q^i as well) is different from its derivative on the left, at the point $B_q = 0$.

Denoting by λ the Lagrange multiplier of the constraint, the First-Order Conditions (FOC) of $[P^i]$ for any $q \in D$ are:

$$\frac{\partial L}{\partial b_q^i} = \frac{\partial \Pi^i}{\partial b_q^i}(b^i, b^{-i}) + \lambda = -\frac{(n-1)}{n} f_q(B) + \lambda = 0, \tag{1}$$

where we have taken into account that $\frac{\partial f_p(B)}{\partial B_q} = 0$ for all $p \in K$ and $q \in D$. It is worthwhile to notice that $f_q(B)$ is the same for all $q \in D$, which supports the following property: increasing agent i's bid to a project in D and decreasing another of this agent's bids to a different project in D does not matter, as long as both projects still receive a negative aggregate bid after the changes.

The FOC for any $q \in A$ is

$$\frac{\partial L}{\partial b_q^i} = \frac{\partial f_q(B)}{\partial B_q} \left[v_q^i - b_q^i + \frac{1}{n} B_q \right] - \frac{(n-1)}{n} f_q(B) + \sum_{p \in A \setminus \{q\}} \frac{\partial f_p(B)}{\partial B_q} \left[v_p^i - b_p^i + \frac{1}{n} B_p \right] + \sum_{p \in K \setminus A} \frac{\partial f_p(B)}{\partial B_q} \left[v_p^i - b_p^i + \frac{1}{n} B_p \right] + \lambda = 0 \tag{2}$$

where

$$\frac{\partial f_q(B)}{\partial B_q} = \frac{(k-1)\varepsilon + \sum_{d \in A \setminus \{q\}} B_d}{\left(k\varepsilon + \sum_{d \in A} B_d\right)^2} \tag{3}$$

$$\frac{\partial f_p(B)}{\partial B_q} = -\frac{\varepsilon + B_p}{\left(k\varepsilon + \sum_{d \in A} B_d\right)^2} \text{ for all } p \in A \setminus \{q\} \tag{4}$$

$$\frac{\partial f_p(B)}{\partial B_q} = -\frac{\varepsilon}{\left(k\varepsilon + \sum_{d\in A} B_d\right)^2} \text{ for all } p \in K\backslash A. \tag{5}$$

Finally, for any $q \in O$, it needs to be the case that $\frac{\partial L}{\partial b_q^i} \geq 0$ on the left and $\frac{\partial L}{\partial b_q^i} \leq 0$ on the right. In fact, the derivative on the left is the same as the left-hand side of equation (1), which is independent of q. Therefore, the derivative $\frac{\partial L}{\partial b_q^i} \geq 0$ on the left always holds (it holds with equality). Therefore, we only have to add the following condition:

$$\frac{\partial L}{\partial b_q^i} = \frac{\partial f_q(B)}{\partial B_q}\left[v_q^i - b_q^i + \frac{1}{n}B_q\right] - \frac{(n-1)}{n} f_q(B) + \\ \sum_{p\in A} \frac{\partial f_p(B)}{\partial B_q}\left[v_p^i - b_p^i + \frac{1}{n}B_p\right] + \sum_{p\in K\backslash(A\cup\{q\})} \frac{\partial f_p(B)}{\partial B_q}\left[v_p^i - b_p^i + \frac{1}{n}B_p\right] + \lambda \leq 0, \tag{6}$$

for any $q \in O$, where

$$\frac{\partial f_q(B)}{\partial B_q} = \frac{(k-1)\varepsilon + \sum_{d\in A} B_d}{\left(k\varepsilon + \sum_{d\in A} B_d\right)^2}, \tag{7}$$

and $\frac{\partial f_p(B)}{\partial B_q}$ is given by (4) for any $p \in A$, and it is given by (5) for any $p \in K\backslash(A \cup q)$.

The previous FOCs are necessary (although not sufficient) to characterize the NE of the proposed mechanism given that any equilibrium must be interior.

Next, we use the FOCs of each agent's program to characterize the set A and the NE aggregate and individual bids to these projects. Lemma 1 conveys useful information about the equilibrium aggregate bids for the projects in A.

Lemma 1 *In any NE of the smooth multibidding mechanism, if $q, q' \in A$ then*

$$B_q = B_{q'} + \frac{1}{(n-1)}\left(V_q - V_{q'}\right). \tag{8}$$

Lemma 1 implies that differences in bids among those projects that receive positive aggregate bids directly reflect the differences in total values of the projects.

We now use the previous result and the FOCs to characterize the aggregate bid received by any project in A.

Proposition 1 *In any NE of the smooth multibidding mechanism, if $q \in A$ then B_q satisfies $h(B_q) = 0$, where*

$$h(B_q) \equiv -(n-1)\, aB_q^2 - \left[\varepsilon k\,(n-1) + 2\sum_{d\in A} V_d - 2aV_q\right] B_q + \\ \varepsilon k V_q - \varepsilon \sum_{p\in K} V_p - \frac{1}{(n-1)} \sum_{d\in A} \left(V_q - V_d\right)^2. \tag{9}$$

The above characterization of the aggregate bids follows from a straightforward rewriting of the optimality conditions satisfied necessarily by the agents' individual bids. We notice that the concave function $h(.)$ starts (at $B_q = 0$) at a positive value only if V_q is close enough to the valuation of the projects in A, that is, if V_q is high enough. The derivative of $h(.)$ also depends on V_q: it is larger when V_q is larger. This derivative $h'(.)$ can be positive or negative at $B_q = 0$, depending on V_q and it converges to $-\infty$ as B_q becomes very large. Therefore, $h(.)$ always crosses (once) the horizontal axis if it starts with a positive value. We also prove in the Appendix that it never crosses the horizontal axis if $h(.)$ starts at a negative value.

Before we continue the characterization of the equilibrium agents' bids, we turn to the analysis of the set A, that is, the set of projects that receive positive aggregate bid. Lemma 1 showed that, for projects in A, aggregate bids increase with total valuation. The same logic suggests that any project in A should have higher total valuation than any project outside A (as they receive non-positive aggregate bids). Lemma 2 shows that this intuition is indeed correct.

Lemma 2 *In any NE of the smooth multibidding mechanism, if project q satisfies $V_q \geq V_t$ for some $t \in A$, then $q \in A$.*

We can now provide a simple condition to check whether a project receives, at equilibrium, a positive aggregate bid. That is, Proposition 2 characterizes the set of projects A.

Proposition 2 *In any NE of the smooth multibidding mechanism, $q \in A$ if and only if the following condition holds:*

$$\varepsilon k V_q - \varepsilon \sum_{p \in K} V_p - \frac{1}{(n-1)} \sum_{d \in S_q} (V_d - V_q)^2 > 0, \tag{10}$$

where $S_q \equiv \{p \in K / V_p \geq V_q\}$.

The above result enables to conclude that the valuation of any particular project in A cannot be too far from that of projects whose total valuation is higher. Only projects whose total valuation is higher than the average valuation can be in A (otherwise, the left-hand side of equation (10) is negative). Moreover, for a "better-than-average" project to be in A, it is also necessary that the sum of the differences (to the power 2) between the value of this project and the value of the projects that are more efficient should be small enough. In particular, only efficient projects will receive positive (aggregate) bids in situations where the difference between the total values of any efficient project and the least inefficient project is sufficiently large. Finally, the set A only contains efficient projects if the parameter ε is small enough.

Proposition 2 characterizes the set of projects A, whose equilibrium aggregate bids satisfy $h(B_q) = 0$. Additionally, equation $h(B_q) = 0$ characterizes a unique value for the aggregate bid of any project in A. On the other hand, one implication of the FOCs (specifically, condition (1)) is that any switch in agents' strategy concerning bids to projects outside A is irrelevant, as long as the set A is not changed. Therefore, we already have the main information concerning the characteristics of the NE strategies of the smooth multibbiding mechanism. Theorem 1 provides the full description of the equilibria through a complete characterization of the equilibrium bids.

Theorem 1 *Denote* $A = \left\{ q \in K / \varepsilon k V_q - \varepsilon \sum_{p \in K} V_p - \frac{1}{(n-1)} \sum_{d \in S_q} (V_d - V_q)^2 > 0 \right\}$. *The set of bids* $(b^i)_{i \in N}$, *with* $\sum_{q \in K} b^i_q = 0$ *for all* $i \in N$, *constitutes a NE of the smooth multibidding mechanism if and only if it satisfies the following properties:*
(a) $h(B_q) = 0$ *whenever* $q \in A$,
(b) $b^i_q = v^i_q + \frac{1}{n} B_q - \frac{1}{k} \sum_{p \in K} v^i_p - \frac{1}{n} \left(V_q - \frac{1}{k} \sum_{p \in K} V_p \right)$ *for all* $i \in N$, *whenever* $q \in A$, *and*
(c) $b^i_q \geq B_q + v^i_q + \frac{n-1}{n} B_t - \frac{1}{k} \sum_{p \in K} v^i_p - \frac{1}{n} (V_t - \frac{1}{k} \sum_{p \in K} V_p)$ *for all* $i \in N$ *and any given project* $t \in A$, *whenever* $q \notin A$.

There is an intuitive progression from the first to the last property in the above theorem. The set A identifies the set of projects that shall receive positive bids. Property (a) then characterizes the aggregate bids of such projects, which will indeed be positive. Property (b) uses (a) to characterize the corresponding individual bids. Finally, property (c) follows from (b) and provides a lower bound on the individual bids for projects outside A, whose aggregate bid shall be non positive.

There are three important remarks regarding the last property. First, property (c) prevents any agent from having incentives to increase his bid for a project whose aggregate bid is non positive. Second, the statement of this property is actually independent from the specific choice of project $t \in A$ since the difference $\frac{n-1}{n}B_t - \frac{1}{n}V_t$ is independent from this choice. Third, property (c) is stronger that the FOC (6) as it concerns not only projects in O but also in D. Therefore, not all strategies that satisfy the FOCs constitute a NE.

Theorem 1 enables us to construct a particular vector of bids that satisfy conditions (a) to (c), which also shows that the set of NE of the smooth multibidding mechanism is always non empty:

Proposition 3 *The set of NE of the smooth multibidding mechanism is non empty. In particular, the following set of bids $(b^i)_{i\in N}$ is a NE:*
(I) b^i_q is constructed using (b) and (a), for all $q \in A$, where A is the non-empty set identified in Theorem 1;
(II) $b^i_q = v^i_q + \frac{n-1}{n}B_t - \frac{1}{k}\sum_{p\in K} v^i_p - \frac{1}{n}(V_t - \frac{1}{k}\sum_{p\in K} V_p)$ for all $q \in K\backslash (A \cup \{d\})$, where d is a particular element of $K\backslash A$;
(III) $b^i_d = -\sum_{q\in K\backslash d} b^i_q$.

Therefore, we have an easy way to construct a NE for any environment. The proof of Proposition 3 only requires that condition (c) is also satisfied for the particular project $d \in K\backslash A$.

The characterization of equilibrium bids is an important point of the analysis, but we have little information on the practical properties satisfied by the mechanism (in terms of equilibrium payoffs, degree of efficiency). These properties will be provided in the next section.

4 Properties of the smooth multibidding mechanism

In the present section we provide several properties satisfied by the equilibrium outcome of the proposed mechanism.

4.1 Fairness of equilibrium payoffs

We start with the characterization of the agents' equilibrium payoffs.

Proposition 4 *In any NE of the smooth multibidding mechanism, agent i's profits are*

$$\Pi^i = \frac{1}{k}\sum_{p\in K} v_p^i + \frac{1}{n}\left[\sum_{q\in K} f_q(B)V_q - \frac{1}{k}\sum_{p\in K} V_p\right].$$

The above equality highlights that an agent's equilibrium payoff is made of two parts. The first part amounts to the value of the average project, $\frac{1}{k}\sum_{p\in K} v_p^i$. This would correspond to the payoff associated with the random assignment mechanism, that is, a mechanism that would choose any project with the same probability. We notice that this mechanism is a benchmark that is used often in practical situations. It basically corresponds to a process where agents would "throw a die" to determinate which specific project would be implemented in case they would be indifferent between all the potential projects. It is also the utility level that each agent can secure himself when he plays the multibidding mechanism (Pérez-Castrillo and Wettstein, 2002). The second part of the equilibrium payoff is the fair share of the surplus. Therefore, not only is the equilibrium payoff of the smooth multibidding mechanism unique (in contrast with the multibidding mechanism, whose outcome set can be large) but it also implies a fair share of the surplus (i.e., the additional payoff obtained with respect of the average value of the projects).

One implication of Proposition 4 is that agents' interests are aligned: when one agent's payoff increases, the payoffs of the other agents also increase. Therefore, there is no conflict between total profits and individual profits.

4.2 Monotonicity of equilibrium payoffs

The characterization of the agents' expected payoffs obtained in the previous sub-section is also useful because it enables us to check whether the parameter ε has a monotone effect on the optimal payoffs. Intuitively, one would think that payoffs should increase with a decrease in the value of the parameter. This is confirmed by the next result.

Proposition 5 *Any agent's optimal expected profits increase with a decrease in the value of the parameter* ε*.*

A smaller value of ε leads to a higher efficiency level attained by the mechanism and each agent having higher expected payoffs. This provides a clear implication for a practical implementation of the mechanism: the value of the parameter ε should be positive and chosen as small as possible, as this would ensure that the agents' expected payoffs will come close to their highest possible values.

4.3 Convergence of the mechanism to full efficiency

Before analyzing the convergence of the outcome of the mechanism to the Pareto efficient outcome, it is interesting to characterize the degree of inefficiency (that is, the distance to the efficient outcome) that can be sustained in projects with positive equilibrium aggregate bids. The next result provides an upper bound on this degree.

Proposition 6 *In any NE of the smooth multibidding mechanism, for any $q \in A$, the following inequality holds:*

$$\frac{V^* - V_q}{\sqrt{V^*}} \leq \sqrt{(n-1)(k-1)}\sqrt{\varepsilon},$$

where V^* denotes the value of an efficient project.

Proposition 6 implies that the potential degree of inefficiency of any project with positive aggregate bid at equilibrium is a linear function of the square root of ε. Therefore, the mechanism will only select efficient projects for situations where the degree of heterogeneity in the value of the projects is sufficiently large. As soon as the difference between the value of an efficient project V_e and that of a second-best project V_s is large enough, then the mechanism will select efficient projects only. Moreover, as the value of the parameter ε becomes arbitrarily small the degree of heterogeneity required converges to zero. Therefore, small values of ε will ensure that the outcome implemented by the mechanism approximates a Pareto efficient outcome. This is consistent with the conclusion resulting from the monotonicity of the agents' expected payoffs as described in Proposition 5.

Next, we provide additional information on the agents' optimal bids when all selected projects are efficient. In such a situation, the optimal aggregate bids can be easily characterized, as highlighted by the following result.

Proposition 7 *In any NE of the smooth multibidding mechanism, if $A = E$ then*

$$B_q = \frac{-\varepsilon k + \sqrt{\varepsilon^2 k^2 + 4\varepsilon \frac{a}{(n-1)} \left(k V_q - \sum_{p \in K} V_p \right)}}{2a}$$

for any $q \in A$. Moreover, B_q converges to 0 and $f_q(B)$ converges to $1/a$ as ε tends towards 0.

When only efficient projects are selected by the mechanism, the form of the aggregate bids is simple. According to this expression, the aggregate bid will be higher as the difference between the value of an efficient project and those of the other projects increases. Moreover, all efficient projects will be selected with equal probability approximately equal to $\frac{1}{a}$ as the parameter ε becomes arbitrarily small. In particular, the probability that an efficient project is selected converges to 1 as ε tends toward 0.

Propositions 6 and 7 enable us to provide a final result on the relative efficiency of the mechanism as the value of the parameter ε becomes small. Specifically, we show that, for any equilibrium, the probability of implementing an inefficient project converges to zero. Therefore, the outcome of the mechanism gets as close to efficiency as one wishes as the parameter ε tends towards zero.

Proposition 8 *The outcome of the smooth multibidding mechanism converges to the efficient outcome as the parameter ε converges to zero. In other words, if project $q \in K$ denotes an inefficient project, its probability to be implemented at the equilibrium converges to zero as ε becomes small.*

The above result confirms that the effect of a variation of the parameter ε is intuitive. Regarding the actual implementation of the mechanism, small values of this parameter will ensure that the chance of choosing an inefficient project comes close to zero.

5 Example

Before concluding the paper, it might be useful to highlight the main properties of the mechanism with a simple example. Let us consider the following situation.

Two agents (1 and 2) have to make a collective decision on the implementation of a project. There are four potential choices corresponding to the set $K = \{1, 2, 3, 4\}$ where the agents' benefits are: $v_1^1 = 6$, $v_1^2 = 3$; $v_2^1 = 4$, $v_2^2 = 6$; $v_3^1 = 2$, $v_3^2 = 1$; and $v_4^1 = 8$, $v_4^2 = 2$, respectively. The weighting parameter ε is positive; we will highlight how its value influences the outcome of the mechanism.

At equilibrium of the smooth multibidding mechanism, Project 3 will receive a negative aggregate bid for any possible ε (this follows from Theorem 1 (a)). Project 1 will also receive negative aggregate bid as soon as $\varepsilon < 1/2$. In this case, Proposition 7 provides the expression for the aggregate bids of projects 2 and 4 (the efficient projects):

$$B_2 = B_4 = -\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon} > 0,$$

and Theorem 1 (c) enables one to find the equilibrium individual bids for projects 1 and 2. For example, the bids that agents submit for project 2 are

$$b_2^1 = -2 + \frac{1}{2}\left[-\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon}\right]$$
$$b_2^2 = 2 + \frac{1}{2}\left[-\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon}\right].$$

Notice that, since agents have different individual valuations for this project, individual bids are different too. Now, the probability that projects 2 and 4 are selected at equilibrium is

$$f_2(B) = f_4(B) = \frac{\sqrt{4+\varepsilon}}{2\left[\sqrt{\varepsilon} + \sqrt{4+\varepsilon}\right]},$$

therefore, each $f_2(B)$ and $f_4(B)$ converges to $1/2$ as ε converges to zero.

Finally, regarding the agents' equilibrium payoffs, we know from Proposition 4 that, for instance, agent 1's payoff is given by the following expressions:

$$\Pi^1 = 5 + \frac{1}{2}\left[\frac{1}{\left[\sqrt{\varepsilon} + \sqrt{4+\varepsilon}\right]}\left(10\sqrt{4+\varepsilon} + 6\sqrt{\varepsilon}\right) - 8\right]$$

which corresponds to this agent's value of the average project (5) plus his fair share of the collective benefits. The collective benefits converge towards the total value 10 of an efficient project minus the total value of the average project 8. Therefore, Π^1 converges to 6 as ε converges to 0.

6 Conclusion

Relying on the main characteristics of the multibidding mechanism (Pérez-Castrillo and Wettstein, 2002), we developed a new procedure for choosing efficient projects in situations where the social planner does not have information on the agents' preferences.

The present protocol implements a *unique* equilibrium outcome, satisfies a *fairness property* and is immune to the problems highlighted by Ehlers (2009) as *the use of tiebreaking rules is avoided* (by making the probability to select a given project continuous). Moreover, it may come *as close to the Pareto efficient outcome as the social planner wishes.*

The resulting outcome of the present mechanism is unique, and satisfies an interesting trade off between efficiency and equity considerations. As such, this mechanism seems to be appropriate in situations of collective decision-making where economic efficiency is not the only relevant property, that is, situations where equity is an equally important criterion.

7 Appendix

Proof of Lemma 1. Assume A contains at least two projects, otherwise the lemma holds trivially. The derivative of the payoff to any agent i, when adding an infinitesimal δ to b_q^i and substracting δ from $b_{q'}^i$ is

$$\frac{1}{\left(k\varepsilon+\sum_{d\in A}B_d\right)}\left[v_q^i-b_q^i+\frac{1}{n}B_q\right]-\frac{(n-1)}{n}\frac{(\varepsilon+B_q)}{\left(k\varepsilon+\sum_{d\in A}B_d\right)}-$$
$$\frac{1}{\left(k\varepsilon+\sum_{d\in A}B_d\right)}\left[v_{q'}^i-b_{q'}^i+\frac{1}{n}B_{q'}\right]+\frac{(n-1)}{n}\frac{(\varepsilon+B_{q'})}{\left(k\varepsilon+\sum_{d\in A}B_d\right)}.$$

The previous derivative must be zero at the optimum, that is,

$$v_q^i-b_q^i-\frac{(n-2)}{n}B_q=v_{q'}^i-b_{q'}^i-\frac{(n-2)}{n}B_{q'}. \tag{11}$$

Summing over N we get $V_q-(n-1)\,B_q=V_{q'}-(n-1)\,B_{q'}$, which is equivalent to (8). ■

Proof of Proposition 1. The FOC for any $q\in D$ implies $\lambda=\frac{(n-1)}{n}\frac{\varepsilon}{\left(k\varepsilon+\sum_{d\in A}B_d\right)}$. Therefore, we write the FOC with respect to $q\in A$ (equation (2)) as (after easy simplifications)

$$\frac{1}{\left(k\varepsilon+\sum_{d\in A}B_d\right)^2}\left[\left(k\varepsilon+\sum_{d\in A}B_d\right)\left[v_q^i-b_q^i+\frac{1}{n}B_q\right]-\sum_{p\in A}B_p\left[v_p^i-b_p^i+\frac{1}{n}B_p\right]\right]-$$
$$\frac{\varepsilon}{\left(k\varepsilon+\sum_{d\in A}B_d\right)^2}\sum_{p\in K}v_p^i-\frac{(n-1)}{n}\frac{B_q}{\left(k\varepsilon+\sum_{d\in A}B_d\right)}=0, \tag{12}$$

or

$$\left(k\varepsilon+\sum_{d\in A}B_d\right)\left[v_q^i-b_q^i-\frac{(n-2)}{n}B_q\right]-\sum_{p\in A}B_p\left[v_p^i-b_p^i+\frac{1}{n}B_p\right]-\varepsilon\sum_{p\in K}v_p^i=0. \quad (13)$$

Summing (13) over $i\in N$ we obtain

$$\left(k\varepsilon+\sum_{d\in A}B_d\right)\left[V_q-(n-1)\,B_q\right]-\sum_{p\in A}B_pV_p-\varepsilon\sum_{p\in K}V_p=0,$$

i.e.,

$$\varepsilon kV_q-\varepsilon\sum_{p\in K}V_p-\varepsilon k\,(n-1)\,B_q+\sum_{d\in A}B_d\,(V_q-V_d)-(n-1)\,B_q\sum_{d\in A}B_d=0. \quad (14)$$

Note that we can write the last two terms in (14) as

$$\sum_{d\in A}B_d\left[(V_q-V_d)-(n-1)\,B_q\right]=-\frac{1}{n-1}\sum_{d\in A}\left[(V_q-V_d)-(n-1)\,B_q\right]^2=$$
$$-\frac{1}{n-1}\sum_{d\in A}(V_q-V_d)^2-(n-1)\,aB_q^2+2aV_qB_q-2B_q\sum_{d\in A}V_d,$$

where we have used equation (8). Therefore, (14) can be written as $h(B_q)=0$. ■

Proof of Lemma 2. First, suppose $K\backslash A$ contains at least two projects. Take projects $q\in K\backslash A$ and $t\in A$ satisfying $V_q\geq V_t$. We know that for any agent $i\in N$, changes in $\left(b_p^i\right)_{p\in K\backslash A}$ do not influence his profits as long as $B_p\leq 0$ for all $p\in K\backslash A$ is maintained. Therefore, if $B_q<0$, then agent i can increase b_q^i to $\underline{b}_q^i=b_q^i-B_q$ and decrease b_p^i to $\underline{b}_p^i=b_p^i+B_q$ for some other $p\in K\backslash A$. The derivative of the payoff to any agent i, when adding a positive infinitesimal δ to $\underline{b}_q^i$ and substracting δ from b_t^i is

$$\frac{1}{\left(k\varepsilon+\sum_{d\in A}B_d\right)}\left[v_q^i-\left(b_q^i-B_q\right)\right]-\frac{(n-1)}{n}\frac{\varepsilon}{\left(k\varepsilon+\sum_{d\in A}B_d\right)}-$$
$$\frac{1}{\left(k\varepsilon+\sum_{d\in A}B_d\right)}\left[v_t^i-b_t^i+\frac{1}{n}B_t\right]+\frac{(n-1)}{n}\frac{(\varepsilon+B_t)}{\left(k\varepsilon+\sum_{d\in A}B_d\right)}.$$

The previous derivative can not be positive at the optimum, that is,

$$\left[v_q^i-b_q^i+B_q\right]-\left[v_t^i-b_t^i-\frac{(n-2)}{n}B_t\right]\leq 0\text{ for all }i\in N.$$

Summing the previous equation over N, we get

$$V_q-V_t+(n-1)B_t\leq 0. \quad (15)$$

However, the last inequality cannot hold if $V_q \geq V_t$ and $B_t > 0$.

Second, suppose $K \backslash A = \{q\}$ and pick t such that V_t is the lowest among the elements in A. Taking into account that $V_t \leq V_q$, then $V_t \leq V_p$ for all $p \in K$. For this project t,

$$h'(B_t)\mid_{B_t=0} = -\left[\varepsilon k(n-1) + 2\sum_{d \in A} V_d - 2aV_t\right] \leq -\varepsilon k(n-1) < 0$$

and

$$h(B_t)\mid_{B_t=0} = \varepsilon k V_t - \varepsilon \sum_{p \in K} V_p - \frac{1}{(n-1)} \sum_{d \in A} (V_t - V_d)^2 \leq 0.$$

Therefore, $B_t > 0$ is not possible. ■

Proof of Proposition 2. We first prove by contradiction that project q does not belong to A if (10) does not hold. We know that, according to Lemma 2, $S_q \subset A$ if $q \in A$. Denote by t the project in A with the lowest total valuation: $V_t \leq V_p$ for all $p \in A$. Then

$$h(B_t)\mid_{B_t=0} = \varepsilon k V_t - \varepsilon \sum_{p \in K} V_p - \frac{1}{(n-1)} \sum_{d \in A} (V_d - V_t)^2 \leq$$

$$\varepsilon k V_q - \varepsilon \sum_{p \in K} V_p - \frac{1}{(n-1)} \sum_{d \in S_q}^{2} (V_d - V_q) \leq 0.$$

Also, $h'(B_t)\mid_{B_t=0} = -\left[\varepsilon k(n-1) + 2\sum_{d \in A} V_d - 2aV_t\right] < 0$ which, together with $h''(B_t) < 0$, implies that $h(B_t) < 0$ for all positive B_t. However, this is not possible at equilibrium.

Second, we prove that (10) does not hold if $q \in K \backslash A$. Note that (10) cannot happen for q if $\{q\} = K \backslash A$. Therefore, we take $q \in K \backslash A$ and and suppose that there are at least two projects outside A. Consider some $t \in A$. By the same calculations as in the proof of Lemma 2, we obtain (see (15)) $V_q - V_t + (n-1)B_t \leq 0$, that is, $B_t \leq \frac{1}{(n-1)}(V_t - V_q)$, or, $h(B_t)\mid_{B_t = \frac{1}{(n-1)}(V_t - V_q)} \leq 0$. This is equivalent to

$$-a\frac{1}{(n-1)}(V_t - V_q)^2 - \frac{1}{(n-1)}\left[\varepsilon k(n-1) + 2\sum_{d \in A} V_d - 2aV_t\right](V_t - V_q) +$$

$$\varepsilon k V_t - \varepsilon \sum_{p \in K} V_p - \frac{1}{(n-1)} \sum_{d \in A} (V_d - V_t)^2 \leq 0, \quad (16)$$

i.e.,

$$\varepsilon k V_q - \varepsilon \sum_{p\in K} V_p - \frac{1}{(n-1)} \sum_{d\in A} (V_t - V_d)^2 - \frac{2}{(n-1)} \sum_{d\in A} V_d (V_t - V_q) + \frac{2}{(n-1)} a V_t (V_t - V_q) - \frac{1}{(n-1)} a (V_q - V_t)^2 \leq 0.$$

Using that $-(V_t - V_d)^2 - 2V_d (V_t - V_q) = -V_t^2 - V_d^2 + 2V_d V_q$ and $2V_t (V_t - V_q) - (V_t - V_q)^2 = V_t^2 - V_q^2$, the previous inequality is equivalent to

$$\varepsilon k V_q - \varepsilon \sum_{p\in K} V_p - \frac{1}{(n-1)} a V_t^2 - \frac{1}{(n-1)} \sum_{d\in A} V_d^2 + \frac{2}{(n-1)} V_q \sum_{d\in A} V_d + \frac{1}{(n-1)} a V_t^2 - \frac{1}{(n-1)} a V_q^2 \leq 0,$$

i.e.,

$$\varepsilon k V_q - \varepsilon \sum_{p\in K} V_p - \frac{1}{(n-1)} \sum_{d\in A} (V_d - V_q)^2 \leq 0. \tag{17}$$

Given that $S_q \supset A$ for any $q \notin A$, it is necessarily the case that equation (10) cannot hold, as we wanted to prove. ■

Proof of Theorem 1. The necessity of part (a) comes from propositions 1 and 2. For part (b), note that from (11), we know that

$$v_p^i - b_p^i + \frac{1}{n} B_p = v_q^i - b_q^i + \frac{1}{n} B_q + \frac{(n-1)}{n} (B_p - B_q)$$

for any $q, p \in A$. Therefore, we can write (12) as

$$\left(k\varepsilon + \sum_{d\in A} B_d\right) \left[v_q^i - b_q^i + \frac{1}{n} B_q\right] - \sum_{p\in A} B_p \left[v_q^i - b_q^i + \frac{1}{n} B_q + \frac{(n-1)}{n} (B_p - B_q)\right] - \varepsilon \sum_{p\in K} v_p^i - \frac{n-1}{n} B_q \left(k\varepsilon + \sum_{d\in A} B_d\right) = 0,$$

i.e.,

$$k\varepsilon \left[v_q^i - b_q^i + \frac{1}{n} B_q\right] - \varepsilon \sum_{p\in K} v_p^i - \frac{1}{n} \left[(n-1) \sum_{d\in A} B_d^2 + k\varepsilon (n-1) B_q\right] = 0. \tag{18}$$

We use (8) to show that

$$\sum_{d\in A} B_d^2 = \sum_{d\in A} \left[B_q + \frac{1}{(n-1)} (V_d - V_q)\right]^2 = a B_q^2 + \frac{2}{(n-1)} B_q \sum_{d\in A} V_d - \frac{2}{(n-1)} a B_q V_q + \frac{1}{(n-1)^2} \sum_{d\in A} (V_d - V_q)^2.$$

Therefore, (18) is equivalent to

$$k\varepsilon\left[v_q^i - b_q^i + \frac{1}{n}B_q\right] - \varepsilon\sum_{p\in K} v_p^i -$$

$$\frac{1}{n}\left[(n-1)\,aB_q^2 + 2B_q\sum_{d\in A} V_d - 2aB_qV_q + \frac{1}{(n-1)}\sum_{d\in A}(V_d - V_q)^2 + k\varepsilon\,(n-1)\,B_q\right] = 0$$

and, using that $h(B_q) = 0$, we obtain

$$k\varepsilon\left[v_q^i - b_q^i + \frac{1}{n}B_q\right] - \varepsilon\sum_{p\in K} v_p^i - \frac{1}{n}\left[k\varepsilon V_q - \varepsilon\sum_{p\in K} V_p\right] = 0,$$

and part (b) follows. For part (c), from the same calculations as in the proof of Lemma 3, it follows that, for any agent i, any project $q \in O$ and any $t \in A$ we have

$$\frac{\partial L}{\partial b_q^i} = \frac{1}{\left(k\varepsilon + \sum_{d\in A} B_d\right)}\left[v_q^i - b_q^i\right] - \frac{1}{\left(k\varepsilon + \sum_{d\in A} B_d\right)}\left[v_t^i - b_t^i - \frac{(n-2)}{n}B_t\right] \leq 0,$$

as agents do not have incentives to deviate. This implies the following inequality:

$$v_q^i - b_q^i - \left[v_t^i - b_t^i - \frac{(n-2)}{n}B_t\right] \leq 0. \tag{19}$$

Using (b) and rewriting, we check that part (c) follows for any $q \in O$. Part (c) is also implied by (c) for any $q \in D$ when $K\backslash A$ is a singleton, $K\backslash A = \{q\}$, using $b_q^i = -\sum_{d\in A} b_d^i$. Finally, when $K\backslash A$ contains at least two projects, any agent can unilaterally amend his bids regarding the projects in K/A to make any project with an initially negative aggregate bid get one equal to zero. Moreover, the resulting situation is payoff equivalent to the initial one. This implies that condition (19) must hold for all projects $q \in D$ once we increase b_q^i to $\underline{b}_q^i$ so that the $\underline{B}_q = 0$, that is, $\underline{b}_q^i = b_q^i - B_q$. Therefore, condition (c) must hold.

We now prove that any vector of bids $(b^i)_{i\in N}$ satisfying (a) to (c) is indeed a NE by showing that b^i is a best response to b^{-i}.

Any best response $\underline{b}^i$ to b^{-i} must satisfy the FOCs. We denote by $\underline{B}_q = \underline{b}_q^i + \sum_{j\in N\backslash i} b_q^j$ for any $q \in K$, and by $\underline{A}$, $\underline{a}$ the set and number corresponding to the vector of bids $(\underline{b}^i, b^{-i})$. Following the same calculations as in the proof of Lemma 1, FOCs imply

$$v_q^i - \underline{b}_q^i - \frac{(n-2)}{n}\underline{B}_q = v_{q'}^i - \underline{b}_{q'}^i - \frac{(n-2)}{n}\underline{B}_{q'} \text{ for any } q, q' \in \underline{A}. \tag{20}$$

Also, when $K\backslash\underline{A}$ has at least two elements, calculations similar to those in Lemma 2 imply

$$v_q^i - \underline{b}_q^i + \underline{B}_q \leq v_{q'}^i - \underline{b}_{q'}^i - \frac{(n-2)}{n}\underline{B}_{q'} \text{ for any } q \in K\backslash\underline{A}, q' \in \underline{A}. \tag{21}$$

When $K\backslash\underline{A}$ only contains one element, that is, $K\backslash\underline{A} = \{d\}$ for some $d \in K$, then (21) also holds as it is implied by (20). Indeed, summing (20) over $q \in K\backslash\underline{A}$ and taking into account that $\underline{b}_d^i = -\sum_{q\in\underline{A}} \underline{b}_q^i$ and $\underline{B}_d = -\sum_{q\in\underline{A}} \underline{B}_q$, we obtain

$$\sum_{q\in\underline{A}} v_q^i + \underline{b}_d^i + \frac{(n-2)}{n}\underline{B}_d = (k-1)v_{q'}^i - (k-1)\underline{b}_{q'}^i - (k-1)\frac{(n-2)}{n}\underline{B}_{q'} \text{ for any } q' \in \underline{A},$$

that is,

$$\begin{aligned} v_d^i - \underline{b}_d^i + \underline{B}_d = v_{q'}^i - \underline{b}_{q'}^i - \frac{(n-2)}{n}\underline{B}_{q'} + & \\ \sum_{q\in K} v_q^i + 2\frac{(n-1)}{n}\underline{B}_d - kv_{q'}^i + k\underline{b}_{q'}^i + k\frac{(n-2)}{n}\underline{B}_{q'} & \text{ for any } q' \in \underline{A}. \end{aligned}$$

Therefore, (21) holds if

$$v_{q'}^i - \underline{b}_{q'}^i + \frac{1}{n}\underline{B}_{q'} \geq \frac{1}{k}\sum_{q\in K} v_q^i + \frac{(n-1)}{n}\left[\frac{2}{k}\underline{B}_d + \underline{B}_{q'}\right] \text{ for some } q' \in \underline{A}.$$

Since $\underline{B}_d = -\sum_{q\in K\backslash\underline{A}} \underline{B}_q$, it is necessarily the case that $\frac{2}{k}\underline{B}_d + \underline{B}_{q'} \leq 0$ for some $q' \in K\backslash\underline{A}$. Moreover, $v_{q'}^i - \underline{b}_{q'}^i + \frac{1}{n}\underline{B}_{q'} \geq \frac{1}{k}\sum_{q\in K} v_q^i$ for any $q' \in K\backslash\underline{A}$.[5] Therefore, (21) also holds for d when $K\backslash\underline{A} = \{d\}$.

Now, we take any $q' \in \underline{A}$ and rewrite (20) and (21) as

$$v_q^i + \sum_{j\in N\backslash i} b_q^j - \frac{(2n-2)}{n}\underline{B}_q = v_{q'}^i + \sum_{j\in N\backslash i} b_{q'}^j - \frac{(2n-2)}{n}\underline{B}_{q'} \text{ for any } q, q' \in \underline{A}, \tag{22}$$

$$v_q^i + \sum_{j\in N\backslash i} b_q^j \leq v_{q'}^i + \sum_{j\in N\backslash i} b_q^j - \frac{(2n-2)}{n}\underline{B}_{q'} \text{ for any } q \in K\backslash\underline{A}, q' \in \underline{A}. \tag{23}$$

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[5] Any best response $\underline{b}^i$ must ensure expected profits higher or equal than $\frac{1}{k}\sum_{q\in K} v_q^i$, which is the level that agent i can secure with the "safe" strategy $\underline{\underline{b}}_q^i = -\sum_{j\in N\backslash i} b_q^j$: profits under $\underline{\underline{b}}^i$ are $\frac{1}{k}\sum_{q\in K}\left[v_q^i - b_q^i\right] = \frac{1}{k}\sum_{q\in K} v_q^i$ because $\underline{\underline{B}}_q = 0$ for all $q \in K$. Therefore, all the projects $q \in \underline{A}$ must provide this level of profits in case they are chosen; otherwise, agent i would decrease all the bids on those projects which provide less profits (he would also increase $\underline{b}_d^i$, still ensuring that $\underline{B}_d$ is negative); this would increase the probability of success of those projects whose profits in case there are chosen is higher or equal than $\frac{1}{k}\sum_{q\in K} v_q^i$.

Equation (23) is a necessary condition for q to be in $K\backslash\underline{A}$. Similarly, because $\underline{B}_q$ is positive if $q \in \underline{A}$, a necessary condition for q to be in $\underline{A}$ is (following (22))

$$v_q^i + \sum_{j\in N\backslash i} b_q^j > v_{q'}^i + \sum_{j\in N\backslash i} b_{q'}^j - \frac{(2n-2)}{n}\underline{B}_{q'}. \tag{24}$$

Therefore, if $q' \in \underline{A}$, then $q \in \underline{A}$ if and only if (24) holds. Equation (24) implies that if $q' \in \underline{A}$, then $q \in \underline{A}$ if $v_q^i + \sum_{j\in N\backslash i} b_q^j$ is larger than $v_{q'}^i + \sum_{j\in N\backslash i} b_{q'}^j$. An implication is that $q \in \underline{A}$ if and only if $v_q^i + \sum_{j\in N\backslash i} b_q^j$ is larger than some threshold. Also notice that this is also necessarily true for the set A (possibly with a different threshold). Therefore, either $A \subset \underline{A}$ or $\underline{A} \subset A$.

We go back to (20), which we rewrite as (25)

$$\frac{(2n-2)}{n}\left(\underline{b}_{q'}^i - \underline{b}_q^i\right) = v_{q'}^i - v_q^i - \frac{(n-2)}{n}\sum_{j\in N\backslash i}\left(\underline{b}_{q'}^j - \underline{b}_q^j\right) \text{ for any } q, q' \in \underline{A}. \tag{25}$$

Taking into account that (25) also holds for b^i (instead of $\underline{b}^i$) if $q, q' \in A$, then

$$\underline{b}_{q'}^i - \underline{b}_q^i = b_{q'}^i - b_q^i \text{ for any } q, q' \in \underline{A} \cap A, \tag{26}$$

that is, $\underline{b}_d^i = b_d^i + \delta$ (and also $\underline{B}_d = B_d + \delta$), for some $\delta \in \mathbb{R}$, for all $d \in \underline{A} \cap A$.

Take some $q \in \underline{A}$. The FOC with respect to $\underline{b}_q^i$ is (see (13))

$$\left(k\varepsilon + \sum_{d\in\underline{A}} \underline{B}_d\right)\left[v_q^i - \underline{b}_q^i - \frac{(n-2)}{n}\underline{B}_q\right] - \sum_{d\in\underline{A}} \underline{B}_d\left[v_d^i - \underline{b}_d^i + \frac{1}{n}\underline{B}_d\right] - \varepsilon\sum_{p\in K} v_p^i = 0. \tag{27}$$

First, suppose that $\delta \leq 0$. Then, (24) is more limiting for $\underline{b}^i$ than for b^i; therefore, $\underline{A} \subset A$. Equation (27) becomes

$$\left(k\varepsilon + \sum_{d\in\underline{A}} B_d + \underline{a}\delta\right)\left[v_q^i - b_q^i - \frac{(n-2)}{n}B_q - 2\frac{(n-1)}{n}\delta\right] -$$
$$\sum_{d\in\underline{A}}(B_d + \delta)\left[v_d^i - b_d^i + \frac{1}{n}B_d - \frac{(n-1)}{n}\delta\right] - \varepsilon\sum_{p\in K} v_p^i = 0,$$

which we write as

$$\left(k\varepsilon+\sum_{d\in A}B_d\right)\left[v_q^i-b_q^i-\frac{(n-2)}{n}B_q\right]-\sum_{d\in A}B_d\left[v_d^i-b_d^i+\frac{1}{n}B_d\right]-\varepsilon\sum_{p\in K}v_p^i-$$
$$\left(\sum_{d\in A\backslash\underline{A}}B_d\right)\left[v_q^i-b_q^i-\frac{(n-2)}{n}B_q\right]+\sum_{d\in A\backslash\underline{A}}B_d\left[v_d^i-b_d^i+\frac{1}{n}B_d\right]+$$
$$\delta\left(\underline{a}\left[v_q^i-b_q^i-\frac{(n-2)}{n}B_q\right]-2\frac{(n-1)}{n}\left(k\varepsilon+\sum_{d\in\underline{A}}B_d\right)-\sum_{d\in\underline{A}}\left[v_d^i-b_d^i+\frac{1}{n}B_d\right]+\frac{(n-1)}{n}\sum_{d\in\underline{A}}B_d\right)+$$
$$\delta^2\left(-2\underline{a}\frac{(n-1)}{n}+\underline{a}\frac{(n-1)}{n}\right)=0. \quad (28)$$

The sum of the first three terms in (28) is equal to zero, as it corresponds to the FOC of b^i. Then after some calculations, (28) becomes

$$-\left(\frac{(n-1)}{n}[\underline{a}\,(B_q+\delta)+2k\varepsilon+\sum_{d\in\underline{A}}B_d]+\sum_{d\in\underline{A}}\left(\left[v_d^i-b_d^i+\frac{1}{n}B_d\right]-\left[v_q^i-b_q^i+\frac{1}{n}B_q\right]\right)\right)\delta+$$
$$\sum_{d\in A\backslash\underline{A}}B_d\left(\left[v_d^i-b_d^i+\frac{1}{n}B_d\right]-\left[v_q^i-b_q^i+\frac{1}{n}B_q\right]\right)+\left(\sum_{d\in A\backslash\underline{A}}B_d\right)\frac{(n-1)}{n}B_q=0. \quad (29)$$

We notice that because of condition (c), $\left[v_d^i-b_d^i+\frac{1}{n}B_d\right]-\left[v_q^i-b_q+\frac{1}{n}B_q\right]=\frac{1}{n}\left[V_d-V_q\right]$ for any $q,d\in A$ (in particular, this is also true if $q,d\in\underline{A}$). Moreover, Lemma 1 implies that $V_d-V_q+(n-1)B_q=(n-1)B_d$ or any $q,d\in A$. Therefore, (29) can be written as

$$\frac{(n-1)}{n}\sum_{d\in A\backslash\underline{A}}B_d^2-\frac{(n-1)}{n}\left[\underline{a}\delta+2k\varepsilon+2\sum_{d\in\underline{A}}B_d\right]\delta=0. \quad (30)$$

The first term in (30) is non-negative; in fact, it is zero if and only if $A\backslash\underline{A}$ is empty. Moreover, $\underline{a}\delta+2k\varepsilon+2\sum_{d\in\underline{A}}B_d$ is positive. Taking into account that $\delta\leq0$, (29) only holds if $A\backslash\underline{A}$ is empty, that is, $A=\underline{A}$ and $\delta=0$.

Second, suppose that $\delta\geq0$, which implies (following (24)) that $\underline{A}\supset A$. We take $q\in A$ and we rewrite (27):

$$\left(k\varepsilon+\sum_{d\in A}B_d+a\delta+\sum_{d\in\underline{A}\backslash A}\underline{B}_d\right)\left[v_q^i-b_q^i-\frac{(n-2)}{n}B_q-2\frac{(n-1)}{n}\delta\right]-$$
$$\sum_{d\in A}(B_d+\delta)\left[v_d^i-b_d^i+\frac{1}{n}B_d-\frac{(n-1)}{n}\delta\right]-\sum_{d\in\underline{A}\backslash A}\underline{B}_d\left[v_d^i-\underline{b}_d^i+\frac{1}{n}\underline{B}_d\right]-\varepsilon\sum_{p\in K}v_p^i=0,$$

i.e.,

$$\left(k\varepsilon+\sum_{d\in A}B_d\right)\left[v_q^i-b_q^i-\frac{(n-2)}{n}B_q\right]-\sum_{d\in A}B_d\left[v_d^i-b_d^i+\frac{1}{n}B_d\right]-\varepsilon\sum_{p\in K}v_p^i-$$
$$\left(\sum_{d\in\underline{A}\setminus A}\underline{B}_d\right)\left[v_q^i-\underline{b}_q^i-\frac{(n-2)}{n}\underline{B}_q\right]-\sum_{d\in\underline{A}\setminus A}\underline{B}_d\left[v_d^i-\underline{b}_d^i+\frac{1}{n}\underline{B}_d\right]+$$
$$a\left[v_q^i-b_q^i-\frac{(n-2)}{n}B_q-2\frac{(n-1)}{n}\delta\right]\delta-2\frac{(n-1)}{n}\left(k\varepsilon+\sum_{d\in A}B_d\right)\delta-$$
$$\sum_{d\in A}\left[v_d^i-b_d^i+\frac{1}{n}B_d-\frac{(n-1)}{n}\delta\right]\delta=0, \quad (31)$$

which, because $q\in A$, and after following steps similar to those in the first case, gives

$$\sum_{d\in\underline{A}\setminus A}\underline{B}_d\left(\left[v_q^i-\underline{b}_q^i-\frac{(n-2)}{n}\underline{B}_q\right]-\left[v_d^i-\underline{b}_d^i-\frac{(n-2)}{n}\underline{B}_q+\frac{(n-1)}{n}\underline{B}_q\right]\right)-$$
$$\frac{(n-1)}{n}\left[a\delta+2k\varepsilon+\sum_{d\in A}B_d\right]\delta=0. \quad (32)$$

Using (20), we deduce that the first term is equal to $-\frac{(n-1)}{n}\sum_{d\in\underline{A}\setminus A}\underline{B}_d\underline{B}_q$. Therefore, taking into account that $\delta\geq 0$, (32) only holds if $\underline{A}=A$ and $\delta=0$.

This concludes the proof. ■

Proof of Proposition 3. We show that b_d^i satisfies condition (c).

$$b_d^i=-\sum_{q\in K\setminus d}b_q^i=-\sum_{q\in A}b_q^i-\sum_{q\in K\setminus(A\cup\{d\})}b_q^i=$$
$$-\sum_{q\in A}v_q^i-\frac{1}{n}\sum_{q\in A}B_q+\frac{a}{k}\sum_{p\in K}v_p^i+\frac{1}{n}\sum_{q\in A}V_q-\frac{1}{n}\frac{a}{k}\sum_{p\in K}V_p-$$
$$\sum_{q\in K\setminus(A\cup\{d\})}B_q-\sum_{q\in K\setminus(A\cup\{d\})}v_q^i-(k-a-1)\left[\frac{(n-1)}{n}B_t-\frac{1}{k}\sum_{p\in K}v_p^i-\frac{1}{n}V_t+\frac{1}{nk}\sum_{p\in K}V_p\right].$$

Therefore, $b_d^i\geq B_d+v_d^i+\frac{n-1}{n}B_t-\frac{1}{k}\sum_{p\in K}v_p^i-\frac{1}{n}(V_t-\frac{1}{k}\sum_{p\in K}V_p)$ if and only if

$$-\frac{1}{n}\sum_{q\in A}B_q-\sum_{q\in K\setminus A}B_q-(k-a)\frac{(n-1)}{n}B_t-\frac{1}{n}\sum_{p\in K}V_p+\frac{1}{n}\sum_{q\in A}V_q+(k-a)\frac{1}{n}V_t\geq 0,$$

which, after easy calculations, gives

$$\frac{1}{n}\sum_{q\in K\setminus A}\left[V_t-V_q-(n-1)\,B_t-(n-1)\,B_q\right]\geq 0. \quad (33)$$

Notice that the previous inequality is independent of i and holds if $V_t - V_q - (n-1)\,B_t \geq 0$ for all $q \in K\backslash A$ (as $B_q \leq 0$ for $q \in K\backslash A$), that is, if $h\left(\frac{1}{(n-1)}\left(V_t - V_q\right)\right) \leq 0$ (where we take the function $h(.)$ corresponding to project t). Therefore, (33) holds if equation (16) is satisfied. In the proof of Proposition 2 we have shown that (16) is equivalent to (17):

$$\varepsilon k V_q - \varepsilon \sum_{p \in K} V_p - \frac{1}{(n-1)} \sum_{d \in A} (V_d - V_q)^2 \leq 0.$$

Equation (17) holds for the project with the highest V_q among the projects in $K\backslash A$, as, for this project, $\sum_{d \in S_q} (V_d - V_q)^2 = \sum_{d \in A} (V_d - V_q)^2$ and (17) is equivalent to the condition that q does not belong to A. Additionally, $V_t - V_q - (n-1)\,B_t \geq 0$ as well for the other projects whose V_q is smaller. This concludes the proof. ■

Proof of Proposition 4. Denote $f^{\varepsilon} = \frac{1}{\left(k\varepsilon + \sum_{d \in A} B_d\right)}$ the probability that any of the projects outside A is selected. Using Theorem 1, we derive:

$$\Pi^i = \sum_{q \in K} f_q(B) \left[v_q^i - b_q^i + \frac{1}{n} B_q\right] =$$

$$\sum_{q \in A} f_q(B) \left[\frac{1}{k} \sum_{p \in K} v_p^i + \frac{1}{n}\left(V_q - \frac{1}{k} \sum_{p \in K} V_p\right)\right] + f^{\varepsilon} \sum_{q \in K\backslash A} \left[v_q^i - b_q^i + \frac{1}{n} B_q\right]. \quad (34)$$

We elaborate on the second term of (34), also using Theorem 1:

$$\sum_{q \in K\backslash A} \left[v_q^i - b_q^i + \frac{1}{n} B_q\right] = \sum_{q \in K\backslash A} v_q^i + \sum_{q \in A} b_q^i - \frac{1}{n} \sum_{q \in A} B_q =$$

$$\sum_{q \in K\backslash A} v_q^i + \sum_{q \in A} v_q^i + \frac{1}{n} \sum_{q \in A} B_q - \frac{1}{k} a \sum_{p \in K} v_p^i - \frac{1}{n} \sum_{q \in A} V_q + \frac{1}{n}\frac{1}{k} a \sum_{p \in K} V_p - \frac{1}{n} \sum_{q \in A} B_q =$$

$$\frac{(k-a)}{k} \sum_{q \in K} v_q^i - \frac{1}{n} \sum_{q \in A} V_q + \frac{1}{n}\frac{1}{k} a \sum_{p \in K} V_p.$$

Therefore,

$$\Pi^i = \sum_{q \in A} f_q(B) \frac{1}{k} \sum_{p \in K} v_p^i + \frac{1}{n} \sum_{q \in A} f_q(B) V_q - \frac{1}{k} \sum_{q \in A} f_q(B) \sum_{p \in K} V_p +$$

$$f^{\varepsilon} \frac{(k-a)}{k} \sum_{q \in K} v_q^i - f^{\varepsilon} \frac{1}{n} \sum_{q \in A} V_q + f^{\varepsilon} \frac{1}{n}\frac{1}{k} a \sum_{p \in K} V_p.$$

Using $(k-a) f^{\varepsilon} + \sum_{d \in A} f_q(B) = 1$, we obtain

$$\Pi^i = \frac{1}{k} \sum_{p \in K} v_p^i - \frac{1}{n}\frac{1}{k} \sum_{p \in K} V_p + \frac{1}{n} \sum_{q \in A} f_q(B) V_q - f^{\varepsilon} \frac{1}{n} \sum_{q \in A} V_q + f^{\varepsilon} \frac{1}{n}\frac{1}{k} k \sum_{p \in K} V_p,$$

which, after simplification, is the expression in the Proposition. ■

Proof of Proposition 5. Given Proposition 4, Proposition 5 is equivalent to the property that the function $P(\varepsilon) \equiv \sum_{q\in K} f_q(B(\varepsilon);\varepsilon)V_q$ is decreasing with ε. We rewrite the continuously differentiable function $P(\varepsilon)$ as

$$P(\varepsilon) = \sum_{q\in A} \frac{[\varepsilon + B_q(\varepsilon)]}{[k\varepsilon + \sum_{d\in A} B_d(\varepsilon)]} V_q + \sum_{q\in K\backslash A} \frac{\varepsilon}{[k\varepsilon + \sum_{d\in A} B_d(\varepsilon)]} V_q.$$

We deduce that $B'_q(\varepsilon)$ is the same for any $q \in A$ from Lemma 1, and we denote such a derivative by $B'(\varepsilon)$. Then the sign of $P'(\varepsilon)$ is the same as that of the following expression:

$$\sum_{d\in A} B_d(\varepsilon) \sum_{q\in A} V_q + k\varepsilon B'(\varepsilon) \sum_{q\in A} V_q + B'(\varepsilon) \sum_{d\in A} B_d(\varepsilon) \sum_{q\in A} V_q - k \sum_{q\in A} B_q(\varepsilon) V_q - \varepsilon a B'(\varepsilon) \sum_{q\in A} V_q -$$
$$aB'(\varepsilon) \sum_{q\in A} B_q(\varepsilon) V_q + \left[\sum_{d\in A} B_d(\varepsilon) - \varepsilon a B'(\varepsilon)\right] \sum_{q\in K\backslash A} V_q,$$

which, after some easy calculations, can be written as

$$D(\varepsilon) \equiv \left[\sum_{d\in A} [B_d(\varepsilon) - \varepsilon B'(\varepsilon)]\right] \left[\sum_{q\in K\backslash A} V_q - \frac{(k-a)}{a} \sum_{q\in A} V_q\right] +$$
$$\left[B'(\varepsilon) + \frac{k}{a}\right] \left[\sum_{d\in A} B_d(\varepsilon) \sum_{q\in A} V_q - a \sum_{q\in A} B_q(\varepsilon) V_q\right].$$

We now analyze the sign of the four elements of $D(\varepsilon)$.

First, $\sum_{q\in K\backslash A} V_q - \frac{(k-a)}{a} \sum_{q\in A} V_q < 0$ given that $V_q > V_d$ for every $q \in A$ and $d \in K\backslash A$.

Second, using Lemma 1 and denoting by $B(\varepsilon)$ and V^* the aggregate bid and the value of any project in E, we find

$$\sum_{d\in A} B_d(\varepsilon) \sum_{q\in A} V_q - a \sum_{q\in A} B_q(\varepsilon) V_q =$$
$$\sum_{d\in A} \left[B(\varepsilon) - \frac{(V^* - V_d)}{(n-1)}\right] \sum_{q\in A} V_q - a \sum_{q\in A} \left[B(\varepsilon) - \frac{(V^* - V_q)}{(n-1)}\right] V_q = \sum_{d\in A} V_d \sum_{q\in A} V_q - a \sum_{q\in A} V_q^2 \leq 0,$$

where the inequality is strict whenever A is larger than E and the proof of the inequality can be easily done by induction.

Third, we prove that $B'(\varepsilon) + \frac{k}{a} > 0$. Using Proposition 1 we obtain

$$B_q(\varepsilon) = \frac{-\varepsilon k(n-1) - [2\sum_{d\in A} V_d - 2aV_q] + \sqrt{\Delta_q(\varepsilon)}}{2(n-1)a}$$

where

$$\Delta_q(\varepsilon) = \left[\varepsilon k(n-1) + 2\sum_{d\in A} V_d - 2aV_q\right]^2 + 4(n-1)a\left[\varepsilon k V_q - \varepsilon\sum_{p\in K} V_p - \frac{1}{(n-1)}\sum_{d\in A}(V_q - V_d)^2\right].$$

Therefore,

$$B'(\varepsilon) + \frac{k}{a} = \frac{1}{2(n-1)a}\left[-k(n-1) + \frac{1}{2}\Delta_q'(\varepsilon)\frac{1}{\sqrt{\Delta_q(\varepsilon)}}\right] + \frac{k}{a} = \frac{k}{2a} + \frac{1}{4(n-1)a}\Delta_q'(\varepsilon)\frac{1}{\sqrt{\Delta_q(\varepsilon)}} > 0$$

since

$$\Delta_q'(\varepsilon) = 2k(n-1)\left[\varepsilon k(n-1) + 2\sum_{d\in A} V_d - 2aV_q\right] + 4(n-1)a\left[kV_q - \sum_{p\in K} V_p\right] =$$
$$2\varepsilon k^2(n-1)^2 + 4(n-1)\left[k\sum_{d\in A} V_d - a\sum_{p\in K} V_p\right] > 0,$$

because the average value of V_d in A is higher than (or equal to) that of V_p in K.

Finally, we check whether $\sum_{q\in A}[B_q(\varepsilon) - \varepsilon B'(\varepsilon)] > 0$. The inequality is equivalent to

$$\sum_{q\in A}\left(-\varepsilon k(n-1) - [2\sum_{d\in A} V_d - 2aV_q] + \sqrt{\Delta_q(\varepsilon)} - \varepsilon\left[-k(n-1) + \frac{1}{2}\Delta_q'(\varepsilon)\frac{1}{\sqrt{\Delta_q(\varepsilon)}}\right]\right) > 0$$

i.e.,

$$\sum_{q\in A}\left[-2[\sum_{d\in A} V_d - aV_q] + \sqrt{\Delta_q(\varepsilon)} - \frac{\varepsilon\Delta_q'(\varepsilon)}{2\sqrt{\Delta_q(\varepsilon)}}\right] = \sum_{q\in A}\frac{1}{\sqrt{\Delta_q(\varepsilon)}}\left[\Delta_q(\varepsilon) - \varepsilon\frac{1}{2}\Delta_q'(\varepsilon)\right] > 0.$$

which, given that $\sqrt{\Delta_q(\varepsilon)} > 0$ for all $q \in A$, holds if and only if $\Delta_q(\varepsilon) - \varepsilon\frac{1}{2}\Delta_q'(\varepsilon) > 0$, i.e.,

$$\left[\varepsilon k(n-1) + 2\sum_{d\in A} V_d - 2aV_q\right]^2 + 4(n-1)a\left[\varepsilon kV_q - \varepsilon\sum_{p\in K} V_p - \frac{1}{(n-1)}\sum_{d\in A}(V_q - V_d)^2\right] -$$
$$\varepsilon k(n-1)\left[\varepsilon k(n-1) + 2\sum_{d\in A} V_d - 2aV_q\right] - 2\varepsilon(n-1)a\left[kV_q - \sum_{p\in K} V_p\right] =$$
$$\left[2\sum_{d\in A} V_d - 2aV_q\right]\left[\varepsilon k(n-1) + 2\sum_{d\in A} V_d - 2aV_q\right] - 4a\sum_{d\in A}(V_q - V_d)^2 + 2\varepsilon(n-1)a\left[kV_q - \sum_{p\in K} V_p\right] =$$
$$2\varepsilon(n-1)\left[k\sum_{d\in A} V_d - a\sum_{p\in K} V_p\right] + 4\left[\left(\sum_{d\in A} V_d\right)^2 - a\sum_{d\in A}(V_d)^2\right] > 0, \quad (35)$$

which is independent of q.

If (35) is satisfied then we conclude that $B - \varepsilon B' > 0$, which implies that $P'(\varepsilon) < 0$, and Proposition 5 is proven. $B - \varepsilon B' > 0$ holds, for example, when $A = E$ (in which case, the second term is zero), but it does not necessarily hold otherwise. Now, suppose that $B - \varepsilon B' \leq 0$, that is,

$$\varepsilon \left[k \sum_{d\in A} V_d - a \sum_{p\in K} V_p \right] \leq \frac{2}{n-1} \left[a \sum_{d\in A} (V_d)^2 - \left(\sum_{d\in A} V_d \right)^2 \right]. \tag{36}$$

We then rewrite $D(\varepsilon)$ as:

$$D(\varepsilon) = \sum_{d\in A} B_d(\varepsilon) \left[\sum_{q\in K\setminus A} V_q - \frac{(k-a)}{a} \sum_{q\in A} V_q \right] +$$
$$\varepsilon a B'(\varepsilon) \left[- \sum_{q\in K\setminus A} V_q + \frac{(k-a)}{a} \sum_{q\in A} V_q \right] + \left[B'(\varepsilon) + \frac{k}{a} \right] \left[\left(\sum_{d\in A} V_d \right)^2 - a \sum_{q\in A} (V_q)^2 \right].$$

It is easily checked that the following equality holds:

$$a \left[- \sum_{q\in K\setminus A} V_q + \frac{(k-a)}{a} \sum_{q\in A} V_q \right] = (k-a) \sum_{q\in A} V_q - a \sum_{q\in K\setminus A} V_q = k \sum_{q\in A} V_q - a \sum_{p\in K} V_p.$$

Plugging this last equality into the expression of $D(\varepsilon)$ and then using (36), we conclude that the following inequality is satisfied:

$$D(\varepsilon) \leq \sum_{d\in A} B_d(\varepsilon) \left[\sum_{q\in K\setminus A} V_q - \frac{(k-a)}{a} \sum_{q\in A} V_q \right] +$$
$$B'(\varepsilon) \frac{2}{(n-1)} \left[a \sum_{q\in A} (V_q)^2 - \left(\sum_{d\in A} V_d \right) \right] + \left[B'(\varepsilon) + \frac{k}{a} \right] \left[\left(\sum_{d\in A} V_d \right)^2 - a \sum_{q\in A} (V_q)^2 \right] =$$
$$\sum_{d\in A} B_d(\varepsilon) \left[\sum_{q\in K\setminus A} V_q - \frac{(k-a)}{a} \sum_{q\in A} V_q \right] + \frac{k}{a} \left[\left(\sum_{d\in A} V_d \right)^2 - a \sum_{q\in A} (V_q)^2 \right] +$$
$$B'(\varepsilon) \left[a \sum_{q\in A} (V_q)^2 - (\sum_{d\in A} V_d)^2 \right] \left[\frac{2}{n-1} - 1 \right].$$

We already checked that the first two terms of the last expression are negative. We thus can conclude that the sign of $D(\varepsilon)$ (and that of $P'(\varepsilon)$) is negative if the last term of the expression is non positive. Provided that $n \geq 3$ this is always satisfied. Finally, notice that for $n = 2$ and $A \neq E$ (if $A = E$, we already know that $P'(\varepsilon) < 0$), $\sum_{q \in K \setminus A} V_q - \frac{(k-a)}{a} \sum_{q \in A} V_q = 0$; therefore $P'(\varepsilon) < 0$ if and only if

$$\left[B'(\varepsilon) + \frac{k}{a}\right] \left[\sum_{d \in A} B_d(\varepsilon) \sum_{q \in A} V_q - a \sum_{q \in A} B_q(\varepsilon) V_q\right] < 0,$$

which always holds. We can thus conclude the proof. ■

Proof of Proposition 6. Following Proposition 2, for any $q \in A$ the value V_q satisfies:

$$(n-1)\varepsilon \left[kV_s - \sum_{d \in K} V_d\right] - \sum_{d \in S_q} (V_q - V_d)^2 > 0,$$

which, denoting as V^* the value of an efficient project, implies that

$$(n-1)\varepsilon \left[kV_q - \sum_{d \in K} V_d\right] - (V^* - V_q)^2 =$$

$$-{V_q}^2 + [2V^* + \varepsilon(k-1)(n-1)] V_q - \left[\varepsilon(n-1) \sum_{d \in K/\{q\}} V_d + V^{*2}\right] > 0. \quad (37)$$

The left-hand side of (37) is a polynomial expression of degree two. Let us denote

$$\Delta \equiv [2V^* + \varepsilon(k-1)(n-1)]^2 - 4\left[\varepsilon(n-1) \sum_{d \in K/\{q\}} V_d + V^{*2}\right] =$$

$$\varepsilon(k-1)(n-1)\left(4\left[(k-1)V^* - \sum_{d \in K/\{q\}} V_d\right] + 1\right).$$

One can notice that Δ is positive, as $V^* \geq V_d$ for any project d. This implies that condition (37) (which holds for any $q \in A$) is equivalent to the property that V_q lies in the interval I:

$$I \equiv \left]V^* + \frac{\varepsilon(k-1)(n-1)}{2} - \frac{\sqrt{\Delta}}{2}, V^* + \frac{\varepsilon(k-1)(n-1)}{2} + \frac{\sqrt{\Delta}}{2}\right[.$$

We write the condition $V_q \geq V^* + \frac{\varepsilon(k-1)(n-1)}{2} - \frac{\sqrt{\Delta}}{2}$ as

$$V^* - V_q \leq U \equiv \frac{\sqrt{\Delta}}{2} - \frac{\varepsilon(k-1)(n-1)}{2}.$$

Bound U can be rewritten as follows:

$$U = \frac{\varepsilon(k-1)(n-1)}{2}\left[\sqrt{\frac{4}{\varepsilon(k-1)^2(n-1)}\left[(k-1)V^* - \sum_{d\in K/\{q\}} V_d\right] + 1} - 1\right] \leq$$

$$\frac{\varepsilon(k-1)(n-1)}{2}\sqrt{\frac{4}{\varepsilon(k-1)^2(n-1)}\left[(k-1)V^* - \sum_{d\in K/\{q\}} V_d\right]} \leq$$

$$\overline{U} \equiv \frac{\varepsilon(k-1)(n-1)}{2}\sqrt{\frac{4}{\varepsilon(k-1)^2(n-1)}(k-1)V^*}.$$

Therefore, $V^* - V_q \leq \overline{U}$, which gives the expression stated in the Proposition. ■

Proof of Proposition 7. The expression for B_q follows immediately from $h(B_q) = 0$ once we take into account that $V_q = V_d$ for any $q, d \in A$ when $A = E$. It is also immediate that B_q converges to 0 as ε tends towards 0. Finally,

$$f_q(B) = \frac{\varepsilon + Bq}{k\varepsilon + \sum_{d\in A} B_d} = \frac{1}{a}\frac{(2a-k)\varepsilon + \sqrt{\varepsilon^2 k^2 + 4\varepsilon\frac{a}{(n-1)}\left(kV_q - \sum_{p\in K} V_p\right)}}{k\varepsilon + \sqrt{\varepsilon^2 k^2 + 4\varepsilon\frac{a}{(n-1)}\left(kV_q - \sum_{p\in K} V_p\right)}},$$

which converges to $1/a$ as ε tends towards 0. ■

Proof of Proposition 8. Let $s \in K$ denote a second-best project and $\gamma \equiv V^* - V_s > 0$ denote the difference between the value of an efficient project and that of s. We have $\frac{V^*-V_s}{V^*} = \frac{\gamma}{V^*} > 0$. Let us consider that the parameter ε takes values such that

$$\varepsilon < \left(\frac{\gamma}{V^*}\right)^2 \frac{1}{(n-1)(k-1)}.$$

Then, by Proposition 6 we deduce that project s does not belong to A for the above values of the parameter ε, which implies that any inefficient project is in $K\backslash A$ as well. Therefore, for small enough values of ε, $A = E$ and, according to Proposition 7, the probability of selecting an efficient project converges to 1 as the parameter ε tends to zero, which ensures convergence to an efficient outcome as ε tends to zero. ■

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