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Abstract

We decompose the generalized Lorenz order into a size and a distribution component. The former is represented by stochastic dominance, the latter by the standard Lorenz order. We show that it is always possible, given generalized Lorenz dominance between two distributions F and G, to find distributions H_1 and H_2 such that F stochastically dominates H_1 and H_1 Lorenz-dominates G, and such that F Lorenz-dominates H_2 and H_2 stochastically dominates G. We also show that generalized Lorenz dominance is characterized by this property and discuss the implications of these results for choice under risk.

Keywords: Income distribution, welfare dominance, Lorenz order, stochastic dominance, decisions under risk

JEL Classification: D31, D63, D81

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1 Introduction

It is well-known from Atkinson (1970), Shorrocks (1983) and Kakwani (1984) that the standard and generalized Lorenz orderings, respectively, allow important judgements concerning economic welfare. If F and G have equal means and the Lorenz curve of distribution F is nowhere below the Lorenz curve of distribution G, then F is preferred to G by all utilitarian social welfare functionals with increasing and concave utility. If the generalized Lorenz curve of F is nowhere below the generalized Lorenz curve of G, then F is preferred to G even if the means μ_F and μ_G are different.

Shorrocks (1983) shows that the conditions (i) $\mu_F \geq \mu_G$ and (ii) F Lorenz-dominates G ensure that F dominates G in the generalized Lorenz sense. Another pair of sufficient conditions is provided by Ramos et al. (2000), who show that (i) $\mu_F \geq \mu_G$ and (iii) unimodality of the ratio of the density functions likewise imply generalized Lorenz dominance. While condition (i) is also necessary for generalized Lorenz dominance, conditions (ii) and (iii) are not. Both sets of sufficient conditions however suggest that the welfare increase from G to F can be factored into two components, one related to an increase in the mean (the efficiency component, represented by first-order stochastic dominance below), and one due to an increase in equality (the equity component, represented by the standard Lorenz order). So far, this has only been solved for the very restrictive case of empirical distributions based on an equal number of income recipients (Saposnik, 1993); it is extended here to arbitrary income distributions with finite expectations. We also show that the ability to be factorized like this is unique to the generalized Lorenz order, so the generalized Lorenz order is in fact characterized by such a factorization, which does not seem to have been noted before. Moreover, we discuss the implications of this factorization for decisions under risk.

2 Lorenz dominance and generalized Lorenz dominance

The Lorenz curve L_F of an income distribution F is defined as $L_F(p) = \frac{1}{\mu_F}\int_0^p F^{-1}(u)\,du$, for $p \in [0,1]$, where $F^{-1}(u) = \sup\{x|F(x) \leq u\}$, $u \in [0,1]$, is the quantile function of F and μ_F is its mean. We assume throughout that incomes are non-negative and that $\mu_F, \mu_G < \infty$. Also, for random variables X and Y with distribution functions F and G, we use $F \succeq G$ and $X \succeq Y$ interchangeably, where '$\succeq$' denotes some partial order.

An income distribution F is preferred to an income distribution G in the sense of Lorenz, denoted $F \geq_L G$, if its Lorenz curve is nowhere below the Lorenz curve of G. The Lorenz criterion is scale-free; apart from a scale factor a distribution F is uniquely determined by its Lorenz curve (Iritani and Kuga, 1983).

Empirical Lorenz curves sometimes intersect. The question arises how F and G can be ranked in such a situation. Shorrocks (1983) and Kakwani (1984) introduce generalized Lorenz curves, defined as $GL_F(p) = \mu_F \cdot L_F(p)$, $p \in [0,1]$, and suggest to prefer F to G if its generalized Lorenz curve is nowhere below the generalized Lorenz curve of G, denoted as $F \geq_{GL} G$. Generalized Lorenz curves are non-decreasing, continuous and convex, with $GL_F(0) = 0$ and $GL_F(1) = \mu_F < \infty$. Thistle (1989a) shows that a distribution is uniquely determined by its generalized Lorenz curve. Also, from Thistle (1989b), generalized Lorenz dominance is equivalent to second-order stochastic dominance (SSD), where $F \geq_{SSD} G$ if and only if $\int_0^x F(t)\, dt \leq \int_0^x G(t)\, dt$ for all $x \in \mathbb{R}_+$. This, in turn, is equivalent to preference of F to G by all additively separable individualistic social welfare functionals with increasing and concave utility. In particular, generalized Lorenz dominance implies $\mu_F \geq \mu_G$. The welfare increase from G to F engendered by generalized Lorenz dominance may therefore be thought of as having an equity component (captured by the Lorenz criterion) and an efficiency component (due to the increase in the mean).

This factoring into an equity component and an efficiency component is done via first-order stochastic dominance (FSD), also known as 'rank dominance' (Saposnik, 1981) in the inequality literature. $F \geq_{FSD} G$, defined as $F(x) \leq G(x)$ for all $x \in \mathbb{R}_+$, is an efficiency criterion and is equivalent to preference of F over G for all additively separable individualistic social welfare functionals with increasing utility. First-order stochastic dominance implies second-order stochastic dominance, but the converse is not true. The following lemma gives a condition under which both criteria coincide (see also Thistle, 1989b).

Lemma 1 *$F \geq_{FSD} G$ if and only if $GL_F(p) - GL_G(p)$ is increasing.*

Proof: Suppose $GL_F(p) - GL_G(p)$ is increasing. This means that the integrand in

$$GL_F(p) - GL_G(p) = \int_0^p F^{-1}(u)du - \int_0^p G^{-1}(u)du = \int_0^p \left\{F^{-1}(u) - G^{-1}(u)\right\} du$$

is nonnegative. But this is just $F^{-1} \geq G^{-1}$ or, equivalently, $F \leq G$. Hence we have $F \geq_{FSD} G$. The other implication is obvious. ∎

A widening gap between non-intersecting generalized Lorenz curves therefore implies that the distributions are ranked even according to the stronger FSD criterion.

3 Decomposing generalized Lorenz dominance

Drawing on majorization theory, Saposnik (1993) factors the generalized Lorenz criterion for discrete income distributions with bounded support and an equal number of income recipients. Using a different approach, the following theorem generalizes this result to arbitrary income distributions with finite expectations. It also provides a converse, thereby giving a characterization of generalized Lorenz dominance.

Theorem 2 *Suppose F, G are income distributions supported on the positive halfline with finite expectations. Then the following are equivalent:*

(a) $F \geq_{GL} G$.

(b) There is an income distribution H_1, with $\mu_{H_1} = \mu_G$, such that $F \geq_{FSD} H_1 \geq_L G$.

(c) There is an income distribution H_2, with $\mu_{H_2} = \mu_F$, such that $F \geq_L H_2 \geq_{FSD} G$.

Proof: To prove (b) $\Longrightarrow$ (a), observe that $\mu_{H_1} = \mu_G$ is equivalent to $GL_{H_1}(1) = GL_G(1)$, so that standard and generalized Lorenz dominance coincide. Since $F \geq_{FSD} H_1$ implies $F \geq_{GL} H_1$, we have $F \geq_{GL} H_1 \geq_{GL} G$, i.e. (a). The proof of (c) $\Longrightarrow$ (a) is similar.

For proof of (a) $\implies$ (b) and (a) $\implies$ (c) we assume without loss of generality that $\mu_F > \mu_G$, i.e. $GL_F(1) > GL_G(1)$.

Consider first (a) $\implies$ (b). Define GL_{H_1} in terms of GL_F and a line segment connecting GL_F and GL_G as follows:

$$GL_{H_1}(p) = \begin{cases} GL_F(p), & p \leq p_0 \\ \beta \cdot p + \mu_G - \beta, & p > p_0. \end{cases}$$

Here, $\beta \in [D^- GL_F(p_0), D^+ GL_F(p_0)]$, where D^- and D^+ denote left and right derivatives, respectively, which exist by convexity of GL_F. The existence of p_0 and β with the required properties follows from monotonicity and continuity of generalized Lorenz curves. In particular, if F is supported on an interval the linear segment is defined via the tangent T_{p_0} to GL_F at p_0 and $\beta = F^{-1}(p_0)$.

By construction, GL_{H_1} is increasing, continuous and convex, with $GL_{H_1}(0) = 0$ and $GL_{H_1}(1) = \mu_G$, hence GL_{H_1} is a proper generalized Lorenz curve. As $GL_F - GL_{H_1}$ is also nondecreasing, Lemma 1 implies $F \geq_{FSD} H_1$. On the other hand, $GL_{H_1} \geq GL_G$. But $\mu_{H_1} = \mu_G$, so that generalized Lorenz dominance and ordinary Lorenz dominance coincide. This gives the second inequality.

(a) $\implies$ (c). We construct GL_{H_2} by adding a suitable function Δ to GL_G. Define δ_n, for $n = 1, 2, 3, \ldots$, by

$$\delta_n := \max \left\{ \delta \;\middle|\; GL_F(\delta) - GL_G(\delta) \leq \mu_F - \mu_G - \frac{\mu_F - \mu_G}{2^n} \right\}.$$

Such δ's exist by continuity of GL_F and GL_G. Set

$$\Delta(p) := \begin{cases} 0, & p < \delta_1, \\ \Delta_1(p), & \delta_1 \leq p < \delta_2, \\ \vdots & \vdots \\ \Delta_n(p), & \delta_n \leq p < \delta_{n+1}, \\ \vdots & \vdots \end{cases}$$

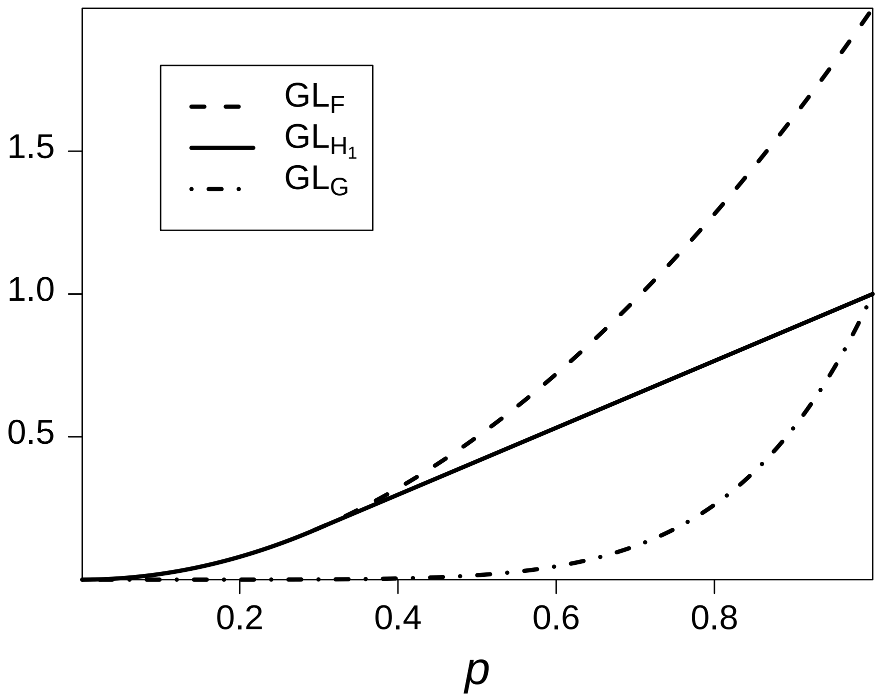
where $\Delta_n(p) := \Delta_{n-1}(p) + (p - \delta_n) \cdot \frac{\mu_F - \mu_G}{2^n\,(1-\delta_n)}$, for $n \geq 2$, and $\Delta_1(p) := (p - \delta_1) \cdot \frac{\mu_F - \mu_G}{2\,(1-\delta_1)}$. That is, $\Delta(p) := \sum_{n=1}^{\infty} (p - \delta_n) \cdot \frac{\mu_F - \mu_G}{2^n\,(1-\delta_n)} \cdot \mathbb{1}_{[\delta_n,1]}(p)$, where $\mathbb{1}_A$ is the indicator function of the set A.

Now $\Delta(0) = 0$, $\Delta(1) = \mu_F - \mu_G$, and Δ is increasing and convex, so that $GL_G + \Delta$ is a proper generalized Lorenz curve, GL_{H_2}. As Δ equals $GL_{H_2} - GL_G$ by construction, Lemma 1 gives $H_2 \geq_{FSD} G$. On the other hand, $GL_F \geq GL_{H_2}$, with $GL_{H_2}(1) = GL_F(1) = \mu_F$, which yields $F \geq_L H_2$. ∎

Theorem 2 covers finite populations with an unequal number of income recipients, but also continuous approximations to empirical income distributions. The proof of (a) $\Longrightarrow$ (b) parallels Müller (1996), who derives a similar result for stop-loss ordering, a dominance concept from actuarial science which is related to generalized Lorenz dominance.

The construction of the generalized Lorenz curves GL_{H_1} and GL_{H_2} is illustrated in Figures 1 and 2.

Figure 1: Construction of GL_{H_1}



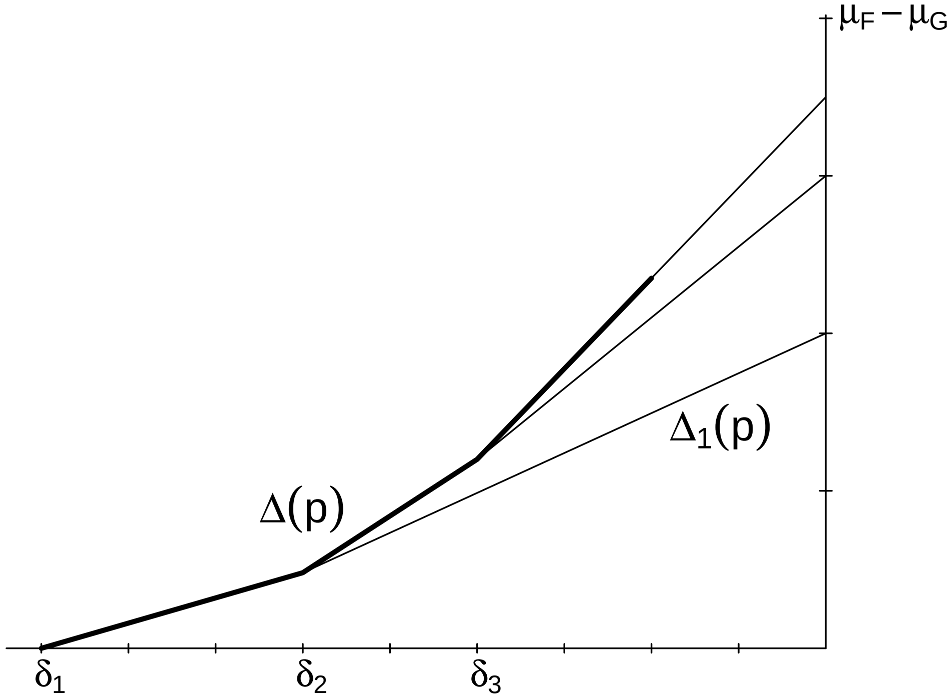
The 'intermediate' distributions H_1 and H_2 are not unique:

Example: Salem and Mount (1974) suggest the gamma distribution, with density

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\lambda x}\,,$$

where $x \geq 0, \lambda > 0, \alpha > 0$, as a descriptive model for the size distribution of income. Let F, G follow gamma distributions, denoted as $\text{Ga}(\alpha, \lambda)$ and $\text{Ga}(\beta, \nu)$, respectively. From Taillie (1981) we know that $F \geq_L G$ if and only if $\alpha \geq \beta$. From Stoyan (1983, p. 202; see also Ramos et al., 2000, pp. 290-291) we moreover have that $\lambda > \nu$ and $\alpha/\lambda \geq \beta/\nu$ imply $F \geq_{SSD} G$ (or, equivalently, $F \geq_{GL} G$), whereas $\lambda \leq \nu$ and $\alpha \geq \beta$ imply $F \geq_{FSD} G$. (Stoyan calls FSD and SSD stochastic ordering and concave ordering, respectively.)

Figure 2: Construction of Δ



Now suppose $F \sim$ Ga(20,5) and $G \sim$ Ga(10,4), hence $F \geq_{GL} G$. Then H_1 may be chosen as Ga(15,6) or Ga(12.5,5), H_2 could be Ga(12,3) or Ga(16,4), for example. Also, both choices of H_1 are different from the distribution H_1 constructed in the proof above: as $D^{-}GL_{H_1}(1) = H_1^{-1}(1) < \infty$ there, the latter distribution has bounded support.

4 Implications for decisions under risk

Welfare comparisons of income distributions have well-known parallels to decisions under risk (Atkinson, 1970, Rothschild and Stiglitz, 1970). In an expected utility framework, the preferences of risk-averse individuals can be conveniently expressed in terms of second-order stochastic dominance (Hadar and Russell, 1969, Hanoch and Levy, 1969, and Rothschild and Stiglitz, 1970). More specifically, any risk-averse expected utility maximizer facing two (random) investment alternatives X and Y with distribution functions F and G will prefer X to Y if and only if $F \geq_{SSD} G$. Here risk aversion corresponds to an increasing and concave utility function, hence the expected return of the preferred investment X is at least as large as the expected return on investment Y. If X and Y have the same mean and $E\,u(X) \geq E\,u(Y)$ for all concave (not necessarily increasing) u, then X is preferred to Y in the Rothschild-Stiglitz sense, denoted as $X \geq_{RS} Y$; meaning that X is less variable than Y. It is well known that, for distributions with equal means, the Rothschild-Stiglitz criterion is equivalent to Lorenz dominance. Finally, preference of X over Y by all expected utility maximizers who value more over less, i.e. who exhibit increasing utility functions, is equivalent to $X \geq_{FSD} Y$.

As $F \geq_{SSD} G$ is equivalent to $F \geq_{GL} G$ we immediately have the following implication of our Theorem 2:

Theorem 3 *Suppose X, Y are investment alternatives with $E(X), E(Y) < \infty$. Then the following are equivalent:*

(a) $X \geq_{SSD} Y$.

(b) There is an investment Z_1, with $E(Z_1) = E(Y)$, such that $X \geq_{FSD} Z_1 \geq_{RS} Y$.

(c) There is an investment Z_2, with $E(Z_2) = E(X)$, such that $X \geq_{RS} Z_2 \geq_{FSD} Y$.

Theorem 3 may be thought of as separating the 'return aspects' and the 'risk aspects' of the preferences of risk-averse individuals in an expected utility framework: If $X \geq_{SSD} Y$, there is an investment alternative Z_1 which will be considered inferior to X by all expected utility maximizers with increasing utility functions and is at the same time less risky than Y in the Rothschild-Stiglitz sense. Also, there is an investment alternative Z_2 which will be preferred to Y by all expected utility maximizers with increasing utility functions and is at the same time more risky than X in the Rothschild-Stiglitz sense.

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