A joint Initiative of Ludwig-Maximilians-Universität and Ifo Institute for Economic Research

# CESifo Working Papers

# THE OPTIMAL PATH OF ENERGY AND CO2 TAXES FOR INTERTEMPORAL RESOURCE ALLOCATION

Klaus Conrad

CESifo Working Paper No. 552

August 2001

Presented at the CESifo Workshop on Environmental Economics and the Economics of Congestion, Venice, June 2001

CESifo
Center for Economic Studies & Ifo Institute for Economic Research
Poschingerstr. 5, 81679 Munich, Germany
Phone: +49 (89) 9224-1410 - Fax: +49 (89) 9224-1409
e-mail: office@CESifo.de
ISSN 1617-9595

CESifo *Working Paper No. 552*
*August 2001*

# THE OPTIMAL PATH OF ENERGY AND CO2 TAXES FOR INTERTEMPORAL RESOURCE ALLOCATION


## Abstract

The purpose of this paper is to extend the dynamic resource allocation problem by including stock externalities like accumulated CO2 and SO2 emissions as well as flow externalities like waste of energy or pollutants which can be abated (SO2). The objective is to examine how the evolution of energy-, CO2- or SO2-tax rates can address these problems in an optimal way. The concern about the time profile of an energy tax arises from the fact that fossil fuels are an exhaustible resource and that global warming, being a consequence of carbon accumulation in the atmosphere, is a stock externality problem. We use a micro model of a firm, which maximizes profits, uses energy as one of its inputs and is confronted with a varying energy tax. It reacts by substitution, by changing its output level, by investing in energy efficient technology or by purchasing abatement equipment. The government is well aware about firms reaction on price signals. It maximizes a stream of social welfare by choosing an optimal path of its instrument – an energy tax. Our analyses supports the idea of a first rising and later falling tax over time.


JEL Classification: Q32, L72, D99.

*Klaus Conrad*
*Mannheim University*
*Department of Economics*
*Seminargebäude A5*
*68131 Mannheim*
*kconrad@rumms.uni-mannheim.de*

# The Optimal Path of Energy and $CO_2$ Taxes for Intertemporal Resource Allocation

*Klaus Conrad*

*Mannheim University*

## 1. Introduction

When proposing to tax energy, environmental economists have several objectives in mind: (i) the control of $CO_2$ emissions, one of the principal greenhouse gases to global warming; (ii) to raise the scarcity price of an exhaustible resource (fossil fuel), the burning of it is a major source of $CO_2$; (iii) to improve energy efficiency by enhancing R&D activities for a more efficient use of energy; (iv) to reduce other air pollutants like $SO_2$, $NO_X$ or particulates by using less energy or by investing in abatement measures if $SO_2$ is the tax base. Much of the literature on an energy or carbon tax has been concerned with the question of what level of a tax is required to achieve a given goal (e.g. a reduction of $CO_2$ emissions by 20% in 2005). However, in view of the exhaustion of fossil fuel and the tendency towards a catastrophic $CO_2$ accumulation in the atmosphere (whatever comes first), the time path of the tax is of more interest than just its level. The $CO_2$ problem presents a classical problem in intertemporal choice.[1] Should we start with a high $CO_2$ tax to delay depletion and $CO_2$ emissions and then reduce it over time, or should we start with a lower tax level which then should rise over time,[2] or maybe the tax should rise first and then fall.[3]

There is a substantial literature on resource use and economic activity with accumulative pollution.[4] In the prototype of pollution control models economic benefits depend positively on emissions (via production and consumption) and negatively on the pollution stock level. Of interest are time paths that maximize the present value of welfare taking into account the effects of accumulative pollution on production possibilities and utility from consumption.[5] Some papers look at pollution and optimal capital accumulation and others at pollution and optimal investment in cleaning-up activities. Since pollution is essentially a problem of missing markets, the emission fee corresponds to the value of the co-

---

1 See Nordhaus (1982).
2 As is the case in Plourde (1972), Dasgupta (1982), van der Ploeg and Withagen (1991), and Hoel (1993).
3 As is the case in Ulph and Ulph (1994).
4 The seminal article on the dynamics of pollution control is by Keeler et al. (1972).
5 See Plourde (1972), Forster (1977) or Conrad and Olson (1992).

state variable, i.e. to the social price of an additional unit of pollution. Most studies address only the stock externality ($CO_2$) and not the exhaustibility of a resource (fossil fuel). In those studies economic activity and pollution converge to a steady state in which the (constant) flow of additions to the pollution stock is offset by the environment's assimilative capacity.[6] In all studies the shadow price of the resource is interpreted as a pollution tax. The government controls quantities but not the path of a tax to achieve optimal allocation. The price mechanism is not at work to influence producer and consumer behavior.

Our approach differs from the standard approach which can be found in the literature. First, it is not the kind of macro-approach, where the utility of energy enters into a government's objective function and where the country incurs cost of exploration. We consider a profit maximizing firm which uses energy as one of its inputs to produce sectoral output. The firm is therefore not a mining company nor does the government own the mine. The firm has to buy energy from a resource country and has to pay taxes on it to its domestic government. The government maximizes a welfare function where consumer and producer surplus, environmental damage and tax revenues are part of the welfare function. Second, its control variable is not a quantity like consumption or energy consumption but an energy tax. The components of the energy tax and its path depend on the constraints the economy faces. One constraint is that energy use contributes to exhausting the $CO_2$-fill capacity of the atmosphere. Although this is a global problem, the government does not take a free-rider position but favors unilateral actions. If absorption of $CO_2$ is of a significant order, the problem can be addressed in the context of a renewable resource. Then it is possible to study the optimal path of the tax towards a steady state, to explore the property of this steady state and to do comparative statics.

Third, we add the constraint to our model that fossil energy is also a non-renewable resource. The linkage of energy use to two stock variables is not new, of course (e.g. Ulph and Ulph (1994)), but our approach is it because we assume that although the government is neither the owner of the stock of fossil fuel nor is it able to solve the $CO_2$ problem as well as the energy exhaustion problem by a unilateral action, it wishes to contribute to

[6] There are many extensions of the prototype model. In Tahvonen (1995), damage depends also on the rate of the stock accumulation. Tahvonen (1997) extends the stock pollution problem by including a non-polluting backstop technology. The topic in Toman and Palmer (1997) is how an accumulative pollutant should be phased out over time (e.g. toxic waste). Tarcin and Tahvonen (1996) replace the assumption of a constant rate of decay of atmospheric carbon by two different models of decay and show that this extension changes the basic dynamic properties of the optimal carbon tax. In Toman and Withagen (2000) the exhaustion of the assimilation capacity of the environment by pollution accumulation results in a non-convexity of the problem that gives rise to multiple potential optima.

the solution of these problems by introducing an energy tax. It is benevolent in the sense that it is, in addition to the unilateral action with respect to global warming, concerned about the limited availability of fossil fuel resources in the near future. It is an interesting result of our approach that a government of a country without resources of coal or crude oil, knowing only about the profit-maximizing behavior of their firms, will choose the same tax structure as in Ulph and Ulph (1994) where a resource owning government chooses energy consumption in a macro-economic framework.

Instead of carrying out a detailed technical analysis of the properties of an optimal path of the tax, we fourthly introduce in our base model the aspect that part of energy is wasted due to inefficient use of it. More effort in order to reduce the waste contributes to the conservation of the stock of fossil energy and prolongs the time span the $CO_2$-fill capacity is exhausted. We show how the tax has to be modified to become a first-best instrument which is then in addition also an incentive to raise efforts (e.g. R&D expenditure on energy saving equipment). And fifthly, we take into account that fossil fuel is also the source of another air pollutant, $SO_2$. The impact of $SO_2$ on the environment (e.g. acid rain) can be mitigated by investing in abatement measures. We therefore introduce in addition to the energy tax a $SO_2$ tax to cope with the damage from the accumulation of two stock pollutants. Although one could expect a complicated mix of tax interactions, the tax rules turn out to be very simple.

## 2. $CO_2$ Emissions as an Intertemporal Resource Allocation Problem

### 2.1 The capacity of the atmosphere for $CO_2$ as a resource

We treat $CO_2$ emissions as waste disposal which decumulates a resource for $CO_2$ deposits in the atmosphere. With $\overline{DE}$ we denote the critical deposit capacity of the atmosphere for $CO_2$. If $\overline{DE}$ is reached, this is equivalent to waste disposal when the landfill is full. $CO_2$ emissions which are right now in the atmosphere are $P(t)$, and the resource $R(t)$ we want to preserve is

$$R(t) = \overline{DE} - P(t) \tag{1}$$

that is, $R(t)$ is the capacity left for future $CO_2$ emissions. The change in pollution over time is

(2) $$\dot{P} = e \cdot E - G(P) = e \cdot E - G(\overline{DE} - R(t))$$

where $E$ is energy (fossil fuel) and $e$ is a $CO_2$ emission coefficient. $G(P)$ is the natural growth of the $CO_2$ fill capacity due to the absorption from the atmosphere to other carbon sinks (mainly the ocean). It is $G_P(P) > 0$, i.e. the more $CO_2$ emissions are in the atmosphere, the more can be aborted. In terms of $G(\overline{DE} - R(t))$, it is $G_R < 0$ which means that the higher the $CO_2$ storage capacity, the less $CO_2$ exists for pollution decay. Therefore, the pollution variable $P(t)$ causes a permanent reduction of $CO_2$ fill capacity in the atmosphere:

(3) $$P(t) = \int_{-120+t}^{t} (e \cdot E(s) - G(\overline{DE} - R(s)))\, ds$$

where $t_0 = -120$ is the approximate duration time of a $CO_2$-unit in the atmosphere. Because of (1) and (2), $\dot{R}(t)$ is

(4) $$\dot{R} = -\dot{P} = G(\overline{DE} - R) - e \cdot E\,.$$

Since we can not control what has happened in the past, $CO_2$ emissions at $t = 0$ are given,

(5) $$\overline{P}(0) = \int_{-120}^{0} -\dot{R}(t)\, dt$$

and so is the resource stock $R(0) = \overline{DE} - \overline{P}(0)$.[7] The objective of environmental policy is to extend the time span of fossil fuel usage until a final time $T$ where a backstop technology is available. At time $T$ the inequality

(6) $$R(T) = R(0) - P(T) \geq 0$$

[7] We do not intend to model a policy that increase the capacity $R(0)$, that is $E < G(\overline{DE} - R)$.

should hold. When the capacity for $CO_2$ fill is under control, either the capacity is exhausted $(R(T)=0)$ or some capacity is left $(R(T)>0)$, but not used anymore for $CO_2$ emissions. For the years after $T$ either a steady state situation must hold in which $CO_2$ emissions are limited by natural absorption, that is

$$\text{(7)} \qquad \dot{R}(t)=0 \qquad \text{or} \qquad G(\overline{DE}-R(t))=e\cdot E(t) \qquad \text{for} \qquad t>T,$$

or a backstop technology is available. In principle we could think of two state variables, the capacity still available in the atmosphere, $R(t)$, and the pollution stock $P(t)$ that has accumulated over the centuries. However, assuming that extracting capacity equals the net accumulation of $CO_2$ emissions, that is (4), there is thus only one stock variable and $E(t)$ is our control variable to preserve the $CO_2$ fill capacity $R(t)$. Since all countries contribute to the $CO_2$ problem, $E$ should be the aggregated energy consumption. Since we wish to analyze the $CO_2$ tax policy of a single country, we assume that energy consumption of the rest of the world is constant and included in $\overline{DE}$.

### 2.2 Maximizing the performance of the economy by choosing the path of a $CO_2$ tax

In order to find the time profile of the resource price, the government maximizes the present value of the sum of consumers' and producers' surplus and of the "ecological surplus" which consists of $CO_2$ tax revenues minus damage from global warming:

$$\text{(8)} \quad \max_{\{\tau_E(t)\}} \int_0^T e^{-rt}\left\{\int_0^{x(\tau_E)} p(\varsigma)d\varsigma - C\left[x(\tau_E),\overline{DE}-R,PE(\tau_E)\right]+\tau_E\cdot E - D(\overline{DE}-R)\right\}dt.$$

It is $\tau_E$ the energy tax rate, $x(\tau_E)$ is output of the energy intensive industry, $p(x)$ is the market demand function for energy intensive goods, $D(\overline{DE}-R)$ is the damage in money terms caused from global warming $(D_R<0)$,[8] $r$ is the discount factor, and $C(\cdot)$ is a cost

---

[8] The case of a catastrophe if $R\to 0$ could be considered by a damage function $D\left(\frac{\overline{DE}-R}{R}\right)$ where damage and marginal damage become extremely high when $R\to 0$.

function with $PE$ as the price for energy which is the only variable input price. The resource stock variable enters the cost function and represents the aspect that more pollution, that is a reduction in the $CO_2$ fill capacity $R$, will raise the cost of production; hence $C_R < 0$. Possible negative productivity effects follow from the state of health, flooding or hurricanes. If we extend the model by including more industries, than $R$ would also affect the productivity in agriculture.

The government is a Stackelberg leader and has perfect information on producer's behavior, that is, supply $x$ is derived from profit maximizing behavior under price taking behavior:

$$(9) \qquad \max_{x} \quad \pi = p \cdot x - C\left[x, \overline{DE} - R, PE(\tau_E)\right]$$

and cost minimizing energy demand $E$ is derived from $C_{PE} = E\left[x, \overline{DE} - R, PE(\tau_E)\right]$ (Shephard's lemma). From the FOC of (9), i.e. $p = C_x$, follows that supply $x$ depends also on $R$ and $PE(\tau_E)$, hence $x = x(p, \overline{DE} - R, PE(\tau_E))$. The government wishes to control the energy demand by choosing a time path of the control variable $\tau_E(t)$. This tax affects not only $E$, but also substitution of other inputs for $E$, marginal costs, and supply and demand. The current value of the Hamiltonian for the problem (8) subject to (4) is:

$$(10) \qquad H = \int_0^{x(\tau_E)} p(\varsigma)\, d\varsigma - C\left[x(\tau_E), \overline{DE} - R, PE(\tau_E)\right] + \tau_E \cdot E - D(\overline{DE} - R) + q_R \cdot \dot{R}$$

where $q_R$ is the current value of the resource price or user cost.[9] After substitution of the dynamic constraint (4) into (10), the FOC are,[10] assuming an interior solution,

$$(11) \qquad H_{\tau_E} = 0 \qquad \text{and} \qquad \dot{q}_R = r\, q_R - H_R$$

---

[9] Since energy $E$ is derived from Shephard's lemma, it therefore depends on the same variables as the cost function.

[10] We suppress the time subscript $t$ in $H_t, q_R(t)$ and $\tau_E(t)$ although all variables depend on $t$.

and the dynamic constraint, recovered from the Hamiltonian,[11]

(12) $$\dot{R} = H_{q_R} = G(\overline{DE} - R) - e \cdot E .$$

The maximum principle yields

$$H_{\tau_E} = (p - C_x)\,\frac{\partial\, x}{\partial\, \tau_E} - E\,\frac{d\;PE}{d\;\tau_E} + E + \tau_E\,\frac{d\;E}{d\;\tau_E} - q_R \cdot e\;\;\frac{d\;E}{d\;\tau_E} = 0$$

where $\frac{d\;E}{d\;\tau_E} = \frac{\partial\, E}{\partial\, \tau_E} + \frac{\partial\, E}{\partial\, x}\,\frac{\partial\, x}{\partial\, \tau_E} < 0$. Since $p = C_x$ and $PE = PE_0 + \tau_E$, we obtain $\tau_E = e \cdot q_R$; the energy tax is a $CO_2$ tax and proportional to the resource price $q_R$ of the remaining capacity for $CO_2$ storage.[12] Thus $q_R$ represents the shadow cost of $CO_2$ or, equivalently, the benefit from an incremental amount of avoided $CO_2$ (a small increase of the $CO_2$ storage capacity).[13] The portfolio balance condition in (11) yields:[14]

(13) $$\dot{q}_R = (r - G_R)\; q_R + C_R + D_R .$$

There is a tendency of the tax rate to grow at the rate $r + |G_R|$, but there is also a tendency for it to decline over time because an extra amount of $CO_2$ later on would inflict marginal damages over a shorter time horizon.

---

[11] We could also postulate by an additional restriction that the critical concentration $\overline{DE}$ must not be exceeded by time *T* and maintained after that point in time by adding the inequality restriction $R(t) \geq 0 \quad \forall \quad t \geq T$. Then for $t \geq T$ we must form the Lagrangian $L = H + \eta \cdot R$. The term $\eta$ complicates the matter and we neglect this aspect (see Goulder and Mathai (2000)).

[12] It is $x(\tau_E)$ in (8) a short-cut of $x\left[\,p(x), \overline{DE} - R, PE(\tau_E)\right]$ i.e. the government knows via the demand function the effect of controlling the quantity on the price. The first expression in $H_{\tau_E}$ should be $H_{\tau_E} = \left(p - C_x\right)\frac{d\;x}{d\;\tau_E}$ where $\frac{d\;x}{d\;\tau_E} = \frac{\partial\, x}{\partial\, p}\,p' + \frac{\partial\, x}{\partial\, PE}\,\frac{d\;PE}{d\;\tau_E}$.

[13] In most of the literature the shadow value of an unit of $CO_2$ is negative since damage is a function of the stock of accumulated $CO_2$; i.e. *D(P)*, so that $CO_2$ is a "bad" from the social planer's perspective. In this paper, the $CO_2$ capacity of the atmosphere is a "good" and the shadow value $q_R$ is therefore positive.

[14] We omit the term $-(\tau_E - q_R \cdot e)E_R$ which is zero because of $\tau_E = q_R \cdot e$ and the term $(p - C_x)x_R$ because of $p = C_x$.

### 2.3 Basic Characteristics of the $CO_2$ tax

For the interpretation of the slope of the optimal time path of $q_R$ we consider four case:

(i) $r = 0$, $G_R = 0$: The $CO_2$ tax is falling since a unit of $CO_2$ emitted today reduces the $CO_2$ fill capacity at all future dates,and so the increase in the cost of production $(C_R < 0)$ as well as the increase in damage from global warming $(D_R < 0)$ is greater than a unit of $CO_2$ emitted at any later date. Therefore a unit today should be taxed more heavily than a unit tomorrow. The required intertemporal adjustment is brought about if the tax falls at exactly the rate of marginal damage from global warming and from its negative impact on production.

(ii) $r = 0$, $G_R < 0$: If a fraction of $CO_2$ emitted today is absorbed, this reduces future marginal damage. Therefore, the aspect described above is somewhat mitigated, and the tax could fall at a lower rate.

(iii) $r > 0$, $G_R < 0$, $C_R$ and $D_R$ small: This is the situation at the beginning of the planning period. The tax is low and rising. The $CO_2$ emissions contribute to social welfare due to the use of energy in production. If the value $q_R$ of the $CO_2$ fill capacity increases less than postulated in (13), then more $CO_2$ should be emitted to raise today's social welfare from material wellbeing.

(iv) $r > 0$, $G_R < 0$, $C_R$ and $D_R$ high: If a steady state is out of reach then the damage and cost aspects from a low $CO_2$ fill capacity might finally dominate and it is thus possible for the optimal carbon tax to decline over time. As shown by Goulder and Mathai (2000), under standard assumptions on $G(\cdot)$ and $D(\cdot)$ the optimal carbon tax will rise forever, if the optimized path of $CO_2$ slopes upward. The standard assumptions are $G(\overline{DE} - R) = \delta \cdot (\overline{DE} - R)$, $\delta > 0$, $D_R < 0$, $D_{RR} > 0$, $C_R < 0$ and $C_{RR} > 0$. Following Goulder and Mathai, the integration of (13) yields

$$(14) \qquad q_R(t) = \int_t^{T=\infty} -\left[C_R\left(\overline{DE} - R(s)\right) + D_R\left(\overline{DE} - R(s)\right)\right] \cdot e^{-(r+\delta)(s-t)} ds$$

where we have to assume additive separability in $C$ between $R$ and $\tau_E$ (i.e. $C_{R,q_R} = 0$) to make $C_R$ independent of $q_R$. This equation states that the shadow cost of a $CO_2$-unit equals the discounted sum of the two marginal damages to production and the environment that this unit would inflict over all future time. Substituting $q_R(t)$ into (13) yields

(15) $$\dot{q}_R = (r+\delta)e^{(r+\delta)t}\int_t^{\infty} -\left[C_R\left(R(s)\right)+D_R\left(R(s)\right)\right]\cdot e^{-(r+\delta)s}ds + C_R\left(R(t)\right)+D_R\left(R(t)\right).$$

If $C_R$ and $D_R$ were constant, then the first term on the right hand side would reduce to $-\left[C_R + D_R\right]$, and the tax would also be constant since $\dot{q}_R = 0$. If $R(s)$ decreases with $s > t$, i.e. $CO_2$ emissions will increase even under an optimal policy, as environmental economists expect, then the convex damage and cost functions ensure that the first term in the $\dot{q}_R$ equation would be larger than the second term and the slope of the tax path would be positive.

If we substitute $e \cdot q_R$ for $\tau_E$ in the two equations for $\dot{R}$ ((12)) and for $\dot{q}_R$ ((13)), we obtain a system of two differential equations in the two unknowns $R$ and $q_R$. They are necessary conditions that govern the motion of an optimal program. The system must know, in addition, where it is going or where it should start. $R(0)$ is given by the state of the $CO_2$ fill capacity at the beginning. Yet the determination of $q_R(0)$ is obviously crucial. If the terminal values of $R(T)$ and $q_R(T)$ are known, then it would be possible to use the equations of motion to trace back through time to find the appropriate $\tau_E(0)\,(= e \cdot q_R(0))$ to get the $CO_2$ reduction program off to the "right start". For this purpose transversality conditions are used.

We first assume that absorption can be neglected $(G \approx 0)$. Then we can distinguish two cases at the final time *T*:

a) $R(T) = 0$ , $q_R(T) > 0$.

The capacity of the resource is exhausted at *T* and the resource stock must have no value according to the transversality condition $q_R(T) \cdot R(T) = 0$. From the second transversality condition, $H(T) = 0$, we obtain

$$H(T) = \int_0^x p(\zeta)\, d\,\zeta - C\left[\cdot, q_R(T)\right] - D(\overline{DE}) + q_R(T) \cdot G(\overline{DE}) = 0$$

Consumer and producer surplus plus the value of the (minor) absorption capacity is equal to damage. From this condition $q_R(T)$ can be determined. Since no $CO_2$ emissions are permitted after period $T(\dot{R}=0)$, fossil fuel is not an essential input anymore because a carbon-free technology is available at *T*, or energy can be produced from renewable resources (solar-, wind- or hydropower-energy). If we introduce the possibility of some absorption after *T*, then $q_R(T+1)$ follows from

(16) $$e \cdot E\left[x(p,\tau_E),\overline{DE},PE_0 + e \cdot q_R\right] = G(\overline{DE}) .$$

Since supply is a function of $q_R$ as $\tau_E = e \cdot q_R$, $x$ is also determined from (16), given *p*. The tax restricts energy demand such that $\dot{R}=0$.

b) $R(T)>0$ , $q_R(T)=0$.

Now we exclude the case a) (a corner solution) that pollution will ever reach its critical limit, and assume $R(T)>0$. This situation will occur if marginal cost and marginal damage become extremely high if $R \to 0$. When temperature increases up to a threshold value, derived from $\overline{DE}$, damage increases sharply. Then a catastrophe occurs if an (uncertain) level of the stock variable (pollution or aggregated extraction of the $CO_2$-fill capacity) is exceeded. The specification of a damage function $D\left(\frac{\overline{DE}-R}{R}\right)$ would reflect such a situation where $R \to 0$. If we consider the $CO_2$-fill capacity *R* as a non-renewable resource ($G(\cdot)$ is small or zero), then the transversality condition $q_R(T)\cdot R(T)=0$ must be satisfied at a final time *T* if such a time exists. Since $R(T)>0$, the resource price of the remaining stock has fallen to zero, i.e. $q_R(T)=0$. The absorption capacity can not be used anymore because a catastrophe might occur. The second transversality condition postulates that the current value of the performance indicator (*H*) must be zero at *T*; i.e. $H(T)=0$. This implies according to (10) that consumer and producer surplus is equal to damage from global warming (it is $\tau_E(T)=e\cdot q_R(T)=0$ at *T*) and the government terminates the program (no energy tax anymore). $H(T)=0$ can be solved for $R(T)$ and after *T* a backstop technology must be available. Permitting some absorption, a resource price $q_R$ after *T* should control (16) (with $\overline{DE}-R(T)$ instead of $\overline{DE}$).

c) A steady state with $R^*$, $q_R^*$ if $G(\cdot) > 0$.

If we assume that absorption is not irrelevant then we can analyze a steady state for $t > T$, the time where $CO_2$ emissions are limited by natural absorption. The steady state tax $\tau_E^*$ and the resource stock $R^*$ follow from (13) with $\dot{q}_R = 0$ and from (7) with $\dot{R} = 0$, i.e. (16) with $\overline{DE} - R^*$ instead of $\overline{DE}$. The tax is

(17) $$\tau_E^* = e\, q_R^* = -\frac{C_R + D_R}{r - G_R}$$

where $C_R$, $D_R$ and $G_R$ are functions of $R^*$. Since $G_R$ is very small (about –1 percent), the tax rate is the present value of all the increased production and damage costs arising from burning a bit more fossil energy today. The tax captures the present value of the future added costs from energy consumption not incident on a private energy intensive firm, but on the other energy consuming firms and the society.[15]

In models where the social planner maximizes the stream of utility from energy use by choosing an optimal path of fossil fuel extraction, he implicitly also determines the optimal path of a $CO_2$ tax because the co-state variable is equivalent to a $CO_2$ tax if markets are perfectly competitive. In our model the social planer maximizes the performance of the economy by choosing only the optimal path of a $CO_2$ tax. Energy consumption is determined indirectly by the agents' cost-minimizing behavior.

## 2.4 A Phase Diagram for the Steady State

If we believe that in the near future a high percentage of energy is produced from carbon-free resources, then a standard phase diagram could be used to show the dynamics of the stock $R$ and the energy tax around the steady state. Figure 1 presents the result of the analysis.

[15] If the two transversality conditions cannot be satisfied, then the program will have to run forever. This is typically the case for renewable resources that can be characterized in final steady state with $(q_R, R)$ positive. For these, transversality is satisfied in the limit as $T \to \infty$: $e^{-rT}\ q_R(T) \cdot R(T) = 0,\quad e^{-rT}\ H(T) = 0$.

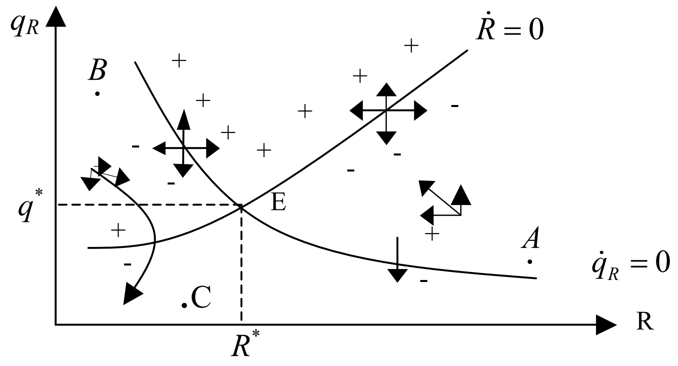


***Fig. 1 The dynamics of the $CO_2$ fill capacity R and of the resource price or $CO_2$ tax $q_R$***

To show the motions of $R$ and $q_R$ off the steady state $(R^*, q_R^*)$, we first determine for the steady state $(\dot{q}_R = d\,\dot{q}_R = \dot{R} = d\,\dot{R} = 0)$ the slope of the $\dot{q}_R = 0$ equation and of the $\dot{R} = 0$ equation:[16]

(18) $\dot{q}_R = 0: \quad \dfrac{d\,q_R}{d\,R} < 0,$

(19) $\dot{R} = 0: \quad \dfrac{d\,q_R}{d\,R} > 0.$

For the motions off the steady state we obtain for the $\dot{q}_R = 0$ equation

(20) $\dfrac{\partial\,\dot{q}_R}{\partial\,q_R} > 0 \quad , \qquad \dfrac{\partial\,\dot{q}_R}{\partial\,R} > 0$

and for the $\dot{R} = 0$ equation

(21) $\dfrac{\partial\,\dot{R}}{\partial\,q_R} > 0 \quad , \qquad \dfrac{\partial\,\dot{R}}{\partial\,R} < 0.$

---

[16] The signs are proven in the Appendix, assuming standard specifications of $C(\cdot), G(\cdot)$ and $D(\cdot)$.

As shown in Figure 1, the control variable $\tau_E$, which corresponds to $e \cdot q_R$, is used to guide *R* from $R(0)$ (given exogenously) to $R^*$ (solved for) in an optimal way. Let us assume that today there is enough capacity for $CO_2$ deposit, i.e. $R(0)$ is to the right of $R^*$. If society is in *A* (i.e. $q_R(0), R(0)$), this corresponds to case (iii) in section 2.3; *R* is high and $D_R$ is negligible. Then a raising tax leads to the steady state. If society is in *B*, this corresponds to case (iv); *R* is low and $D_R$ is high. A falling tax and an increasing *R* (i.e. less $CO_2$ emissions) are in line with Goulder and Mathei's (2000) analysis of a declining tax under less $CO_2$ emissions, and a steady state will be reached. If society is in *C*, the tax is too low and immediate command and control regulations are required to stabilize the situation.

## 2.5 Comparative Statics in the neighborhood of the steady state

We next want to analyze the system, consisting of the maximum principle, $\tau_E = e \cdot q_R$, the dynamic constraint (12) and the portfolio balance condition (13). For that purpose we have to assume functional forms for $G(\overline{DE} - R), \quad C(x, \overline{DE} - R, \tau_E)$ and $D(\overline{DE} - R)$:

(22) $$G(\overline{DE} - R) = \delta(\overline{DE} - R), \qquad \delta > 0$$

(23) $$C(\cdot) = x^{1/h} \cdot c(PE_0 + \tau_E),$$

i.e. we omit the productivity externality from $\overline{DE} - R$. All other input prices are held constant and are sub pressed in $c(\cdot)$.

(24) $$D(\overline{DE} - R) = \frac{1}{2}\gamma(\overline{DE} - R)^2.$$

$\delta$ in (22) is a parameter for natural mitigation, $h < 1$ in (23) is the degree of homogeneity of the production function and expresses decreasing returns to scale. The damage function in (24) is quadratic with $\gamma > 0$ as a damage parameter. From $p = MC(\cdot)$, we obtain

(25) $$x = \left[\frac{h \cdot p}{c}\right]^{\frac{h}{1-h}} .$$

Energy demand $E$ can be derived from Shephard's lemma, using the specification of the cost function in (23) and then replacing $x$ by its expression, derived in (25). This yields:

(26) $$E = \left[\frac{h \cdot p}{c(q_R)}\right]^{1/1-h} c_{PE}(q_R) .$$

Since $C_R = 0$, (13) reduces to

(27) $$\dot{q}_R = (r + \delta)\, q_R - \gamma \cdot (\overline{DE} - R) = q(q_R, R) .$$

The second equation of motion ((27) is the first one) is

(28) $$\dot{R} = \delta(\overline{DE} - R) - e \cdot E(\cdot) = R(q_R, R)$$

with $E(\cdot)$ as given in (26). The steady state solution $(R^*, q_R^*)$ can be obtained by setting $\dot{q}_R = \dot{R} = 0$ in (27) and (28).

For comparative statics with respect to the exogenous variables $r, \delta, \gamma$ and $p$ we consider the two-equation system for the steady state, i.e. (27) and (28):

(29) $$(r + \delta)\, q_R^* - \gamma \cdot (\overline{DE} - R^*) = 0$$

(30) $$-e \cdot E(\cdot, q_R^*) + \delta \cdot (\overline{DE} - R^*) = 0$$

To find $\frac{d\,q^*}{d\,r}$ or $\frac{d\,R^*}{d\,r}$ we have to solve these two equations simultaneously after having injected some $d\,r$. The differentiation of these equations with respect to the two endogenous variables $(q^*, R^*)$ and the exogenous variable in question ($r$) yields:

(31) $$(r+\delta)\, d\, q_R^* + \gamma \cdot d\, R^* = -q_R^*\, d\, r, \qquad -e \cdot E_{q_R}\, d\, q_R^* - \delta \cdot d\, R^* = 0\,.$$

The solution is:[17]

(32) $$\frac{d\, q_R^*}{d\, r} < 0 \quad , \qquad \frac{d\, R^*}{d\, r} < 0\,.$$

A higher interest rate depresses the steady state capital value $q_R^*$ of the $CO_2$-fill capacity and lowers its level. The emphasis is on production and profit now and not later, and therefore on a lower $CO_2$ tax and less $CO_2$-fill capacity.

In a similar way, we can find out how $q_R^*$ and $R^*$ respond to changes in $\delta$ :

(33) $$\frac{d\, q_R^*}{d\, \delta} < 0 \quad , \qquad \frac{d\, R^*}{d\, \delta} > 0\,.$$

More absorption of $CO_2$ by the biosphere will increase the $CO_2$-fill capacity and permits to lower the tax rate.

If damage parameter $\gamma$ changes, we obtain:

(34) $$\frac{d\, q_R^*}{d\, \gamma} > 0 \quad , \qquad \frac{d\, R^*}{d\, \gamma} > 0\,.$$

This again confirms intuition. If damage from $CO_2$ emissions is higher, then $R^*$ and the $CO_2$ tax should be higher too.

## 3. Intertemporal Allocation of Fossil Fuel as a Non-Renewable Resource and as a Source of $CO_2$ Emissions

We next link the problem of controlling $CO_2$ emissions to the problem of controlling the extraction of fossil fuel, the exhaustible source of $CO_2$ emissions. Although the domestic

[17] For technical details see the Appendix.

planer has no real control over resource exhaustion and he can not affect the price path for the exhaustible resource, he can raise the domestic energy price by a fax in order to remind the domestic economy that fossil fuel is an exhaustible resource. We state a dynamic optimization problem with two state variables: $R(t)$, the $CO_2$ fill capacity; $RF(t)$, the level of the stock of fossil fuel at time $t$ (crude oil, coal, natural gas). The government selects values for the control variable $\tau_E(t)$ for $t = 0, 1, ...T$ so as to maximize

$$\max_{\{\tau_E(t)\}} W = \int_0^T e^{-rt} \left\{ \int_0^{x(\tau_E)} p(\zeta) d\,\zeta - C(x, \overline{DE} - R, PE_0 + \tau_E) + \tau_E \cdot E - D(\overline{DE} - R) \right\} d\,t$$

subject to the constraints

$$(36) \qquad \dot{R} = G(\overline{DE} - R) - e \cdot E(\cdot)$$

$$(37) \qquad \dot{RF} = -E(\cdot) \quad , \qquad RF(0) \geq \int_0^T E\,dt\,.$$

Ulph and Ulph (1994) construct such a model with a $CO_2$ stock externality which is related to the use of a non-renewable resource. But in their model an agency maximizes the present value of a stream of utility where utility depends on consumption of energy. It also exploits its own resource by choosing an optimal time path of extraction. In our model the government is concerned about the negative externalities from the derived demand for energy. Although it does not own the stock of the non-renewable resource, it knows that production of goods in its country contributes via energy use to the exhaustion of world wide fossil fuel reserves. The energy tax therefore addresses not only the problem of global warming but also the problem that fossil fuels will be exhausted in the near future.[18]

The current-value Hamiltonian is

$$H = \int_0^{x(\tau_E)} p(\zeta)\, d\,\zeta - C(\cdot) + \tau_E \cdot E - D(\overline{DE} - R) + q_R\, \dot{R} + q_{RF} \cdot \dot{RF}$$

---

[18] As the rest of the world also consumes energy, we had to subtract also this energy use in (37). Since we are not interested in doing numerical simulation of a world model we assume this consumption to be constant and for simplicity, zero.

and the maximum principle yields

$$H_{\tau_E} = \frac{d\,E}{d\,\tau_E}(\tau_E - q_R \cdot e - q_{RF}) = 0\,.$$

The energy tax consists of the shadow price of $CO_2$ pollution, $q_R \cdot e$, and of the shadow price of fossil fuel, $q_{RF}$, which in our model is received by the resource-concerned government but not by the owner of the resource:

(38) $$\tau_E = q_R \cdot e + q_{RF}\,.$$

From the portfolio balance condition $\dot{q}_R = r\,q_R - H_R$ we obtain

(39) $$\dot{q}_R = r\,q_R + C_R + D_R - q_R\,G_R + (-\tau_E + q_R \cdot e + q_{RF})E_R$$

which reduces to the same condition as in (13). From the second portfolio condition, $\dot{q}_{RF} = r\,q_{RF} - H_{RF}$, we obtain

(40) $$\dot{q}_{RF} = r \cdot q_{RF}\,.$$

The optimal tax $\tau_E$ in (38) derived from our approach, is in principle the same as that obtained by Ulph and Ulph (1994). It is known that perfect competition on the resource market with price taking resource suppliers will result in the same price and quantity paths like those chosen by a planer. In our model, however, we consider firms under perfect competition which produce anything except of being a mining firm. They use energy at a price $PE_0$ which could be supplied under any kind of market structure.

The change of the tax over time is $\dot{\tau}_E = \dot{q}_R \cdot e + \dot{q}_{RF}$ or, if we use (39), (40), and (38):

(41) $$\dot{\tau}_E = e\big[(r - G_R)\,q_R + C_R + D_R\big] + r\,q_{RF} = r \cdot \tau_E + e\big[-G_R\,q_R + C_R + D_R\big].$$

As for the slope of the energy tax we could repeat the argument by Goulder and Mathai (2000), presented in (14) and (15), by saying that the tax path will be upward sloping because

$q_{RF}$ rises according to Hotelling's rule and $q_R$ rises because we expect that accumulated $CO_2$ emissions will steadily increase. This sufficient condition for an upward sloping path of $\tau_E$ might not hold if a model takes into account that the source of $CO_2$ emissions is exhaustible fossil fuel.[19] Since it can be shown that $\dot{x} < 0$ and $\dot{E} < 0$, it is possible that finally $R(s) > R(t) \quad \forall \quad s > t$ and the tax path then becomes downward sloping, a possibility also described by Ulph and Ulph (1994). Since it is difficult to derive general conclusions for this kind of model, we therefore discuss only some cases that might occur.

a) $R(T) = 0 \quad , \qquad q_R(T) > 0 \quad , \qquad G(\cdot) = 0.$

As the $CO_2$-fill capacity is exhausted at *T*, no fossil fuel can be used after *T*. Therefore, an optimal extraction path would require that $RF(T) = 0$. However, as the country can not sell fossil fuel because it does not own the stock, it is not in its interest that the stock of fossil fuel is exhausted. From (3), (6) and (37) follows that obviously $e \cdot RF(0) = P(T) = R(0)$ holds. The tax $\tau_E(T)$ solves the second transversality condition:

(42) $$H(T) = \int_0^{x(\tau_E)} p(\zeta)\, d\, \zeta - C(\cdot, \tau_E) - D(\overline{DE}) = 0\,;$$

the sum of producer and consumer surplus at *T* is equal to damage. After *T*, energy is produced by a backstop technology.

b) $R(T) > 0 \quad , \qquad q_R(T) = 0 \quad , \qquad G(\cdot) = 0$.

In this case, damage becomes catastrophic and emissions are not permitted anymore. Again, fossil fuel should be exhausted at *T*, i.e. $RF(T) = 0$. This time, from (3), (7) and (37) follows

19 In a dynamic game of $CO_2$ emissions Hoel (1993) studies the intertemporal properties of an international carbon tax. He shows that an optimal tax rate should be increasing over time, but at a rate lower than the interest rate, like our result in (41). But he did not add the aspect that the source of $CO_2$ emissions is a non-renewable resource.

(43) $$\int_0^T E(\cdot)dt = RF(0) \quad , \qquad \int_0^T e \cdot E(\cdot)dt = \overline{DE} - R(T)$$

and therefore $R(T) = \overline{DE} - e \cdot RF(0)$. Then the tax $\tau_E(T)$ solves $H(T) = 0$ with $D(\overline{DE} - R(T))$ in (42).

c) $RF(T) = 0 \quad , \qquad q_{RF}(T) > 0 \quad , \qquad R(T) > 0 \quad , \qquad G(\cdot) = 0$.

This case represents the aspect that the stock of fossil fuel is exhausted before global warming becomes a serious problem. $R(T)$ can be determined similarly as in case b), and so can $\tau_E(T)$.

d) A steady state $R^*, q_R^*$ with $G(\cdot) > 0$ and $RF > 0$.

With absorption, the $CO_2$ problem has reached a steady state before the resource *RF* is exhausted. Then $q_R^* = \dfrac{-D_R(\overline{DE} - R^*)}{r - G_R(\overline{DE} - R^*)}$ from (39) $(\dot{q}_R = 0)$ together with

(44) $$G(\overline{DE} - R^*) = e \cdot E\left[ x(q_R^* \cdot e + q_{RF}), q_R^* \cdot e + q_{RF} \right]$$

have to be solved for $R^*$ and $q_R^*$.[20] It is $R^* = R(q_{RF})$ and $q_{RF}$ must keep $R^*$ constant. The tax component $q_{RF}$ is therefore no longer on its optimal path.

We could also think of introducing two control variables, an energy tax $\tau_{EN}$ and a $CO_2$ tax $\tau_{CO}$. We have to modify the welfare function by defining $PE = PE_0 + \tau_{EN} + e \cdot \tau_{CO}$ and by introducing two types of tax revenues. From $H_{\tau_{EN}} = 0$ we obtain $\tau_{EN} + \tau_{CO} \cdot e = q_R \cdot e + q_{RF}$, and from $H_{\tau_{CO}} = 0$ we obtain this condition again. Since $q_R$ and $q_{RF}$ follow the same path in each FOC, there is no need to have an energy tax as well as a $CO_2$-tax. Welfare can not be improved by choosing two instruments.

---

[20] We assume here that $C(\cdot)$ does not depend on $\overline{DE} - R$ anymore.

It is important to note that our restriction to one industry is not a restriction of generality of the results. If we affix an index *i* for industry *i* to *p*, *C*, *x* and *E*, and introduce a sum over *i* in the welfare function at the beginning of this section and in (36) and (37), the tax formula (38) is the same, i.e. independent on the number and size of industries. If $C_{i,R} = 0$, then also the equation of motion of $q_R$ is the same and so is (40). If $C_{i,R} < 0$, then

$$\dot{q}_R = q_R(r - G_R) + \sum_i C_{i,R} + D_R \,. \tag{45}$$

The stock externalities on the costs of production lower the slope of $\frac{\dot{q}_R}{q_R}$, that is, the initial level of the $CO_2$ tax should be higher.[21]

To our knowledge, all authors look at a mining firm or at a government which owns the resource. If all countries are interested in reducing $CO_2$ and in conserving fossil fuel, however, then it is irrelevant whether the use of fossil fuel is taxed or the extraction of it. The path of the tax is the same. Not the same is the distributional effect between the suppliers and users of fossil fuel.[22] If producers charge the $CO_2$ tax and the royalty, they obtain the revenues. If the user country charges a $CO_2$ tax, then the consumers of the user countries benefit from the recycling of the tax revenue. The resource supplier receives only the price he can receive on the resource market ($PE_0$ in our model). We have not modeled the first phase of a resource model, i.e. the policy of the country that owns the resource. If extraction costs do not depend on the stock *RF*, then the price path $\dot{PE}_0 = r \cdot PE_0$ follows the Hotelling rule. With mitigation policies in the user countries, the price $PE_0$ does not depend on $\tau_E$. This can lead to the curious result that the governments of the user countries have higher revenues from taxing oil and gas than the producer countries earn from selling their resources. In our model the government of a resource importing country raises the tax on the stock externality, the revenue of it is recycled to the consumers. There is therefore a rent-transfer of $RF(0) \cdot \tau_E$ in favor of the importers. If the oil producer would internalise at the well the stock externality in their price and extraction policy of fossil fuel, this would change the distributional outcome under a $CO_2$ mitigation policy.

---

[21] See also the higher value of $q_R(t)$ in (14) if marginal damage to production increases.

[22] See Blanck and Ströbele (1995) on this point.

## 4. On the Improvement of Energy Efficiency in an Intertemporal Setting

The analysis of this section describes the effect of a carbon tax on induced technical change. We first consider an industry which uses "gross" energy $E$ as an input for producing output $x$. By gross energy we mean energy input with a byproduct "waste of energy" which reduces the efficiency of the production process. We distinguish between gross energy $E$ and net energy input $NE$ where

(46) $$NE = (1-\alpha(ef)) \cdot E \quad , \qquad 0 \le \alpha(ef) \le 1 \quad , \quad \text{and} \qquad WE = \alpha(ef) \cdot E$$

with $WE$ as waste of energy, $ef$ as effort and $\alpha(ef)$ as the waste coefficient or coefficient of energy inefficiency. To improve energy efficiency, i.e. to reduce the waste coefficient, is costly because it requires time and effort, and hence causes costs of labor and material. We denote with $ef$ the effort to reduce the waste of energy and assume $\alpha(ef)$ to be a function of this effort $(\alpha'(ef) < 0,\ \alpha''(ef) > 0)$. Effort $ef$ could be the intensity of R&D activities to improve energy efficiency. The cost of energy, including the energy tax, is $(PE_0 + \tau_E) \cdot E$ which is equivalent to $\dfrac{PE_0 + \tau_E}{1-\alpha(ef)} \cdot NE$, using (46). Energy $NE$ is the input quantity which enters a production function for $x$ as a frontier for efficient production, and the tax $\tau_E$ can be used as a policy instrument to raise energy efficiency.

Next we define the price of energy to be

(47) $$PE(ef, \tau_E) = \frac{PE_0 + \tau_E}{1-\alpha(ef)}$$

with

(48) $$PE_{ef} = \frac{\alpha'}{(1-\alpha)^2}(PE_0 + \tau_E) < 0$$

i.e. effort $ef$ reduces the price for the efficient energy input $NE$. If we choose a cost function with $PE$ as one of the input prices (the other prices are constant), then the problem of the firm or industry under perfect competition is

(49) $$\max_{x,ef} \quad \pi = p \cdot x - C(x, PE(ef, \tau_E), ef)$$

with $PE$ as defined in (47). Lower output on the production side from the *WE*-reducing effect of effort *ef* is expressed in terms of $C_{ef} > 0$ and $C_{ef,ef} > 0$ on the cost side. An environmentally friendly production process with emphasis on energy conservation increases the cost for producing *x*. The benefit is a lower price *PE* due to energy efficiency of the input energy.

The FOC with respect to *x* is:

(50) $$p - C_x(x, PE, ef) = 0$$

and the FOC with respect to *e* is:

(51) $$-NE \cdot PE_{ef} - C_{ef} = 0$$

because of Shephard's Lemma $(NE = C_{PE})$. According to (51), the level of *ef* is optimal if marginal savings in the cost of energy justifies exactly the increase in the cost of producing output *x* with a more energy efficient technology. $PE_{ef}$ is negative because an increase in effort reduces waste of energy and this energy augmenting effect raises net energy *NE*. In order to determine the effect of a change in the energy tax $\tau_E$ on production, effort and energy wasted, the equations (50) and (51) have to be differentiated totally. [23] The effect we are interested in is:

(52) $$\frac{d\ ef}{d\ \tau_E} > 0 \ .$$

We assume that the government knows this positive effect of an energy tax and write $ef = ef(\tau_E)$ as the solution of (51).[24]

We next are interested in the structure and the optimal path of the energy tax $\tau_E$

[23] See Conrad (2000).
[24] We assume a homothetic production process so that *x* in (51) cancels out.

which should take into account the accumulation of $CO_2$, the scarcity of the non-renewable resource energy, and in addition the potential of being an incentive for improving energy efficiency. The objective function of the government is

$$
\begin{aligned}
(53)\qquad & \max_{\{\tau_E(t)\}} \int_0^T e^{-rt} \left[ \int_0^{x(\tau_E)} p(\zeta)d\,\zeta - C(x, PE(\tau_E, ef(\tau_E)), ef) \right. \\
& \left. + \tau_E \cdot E - PE_0 \cdot WE - D(\overline{DE} - R) \right] dt
\end{aligned}
$$

subject to (36) and (37).[25] Since the use of "gross" energy $E$ produces $CO_2$ emissions and contributes to the exhaustion of fossil fuels, the dynamic constraints (36) and (37) do not reflect the benefit of a better fuel efficiency. This aspect enters the objective function in (53) by subtracting the cost of wasted energy as a "bad" for the economy. The task of the energy tax is to reduce also waste because that would contribute to mitigate the $CO_2$ emission problem and the energy scarcity. The Hamiltonian is

$$
\begin{aligned}
(54)\qquad H = & \int_0^{x(\tau_E)} p(\zeta)d\,\zeta - C(x, PE(\tau_E, ef(\tau_E)); ef) + \tau_E \cdot E \\
& - PE_0 \cdot WE - D(\overline{DE} - R) + q_R \cdot \dot{R} + q_{RF} \cdot \dot{RF}
\end{aligned}
$$

and the maximum principle requires[26]

$$
(55)\qquad \frac{d\,H}{d\,\tau_E} = \tau_E \frac{d\,E}{d\,\tau_E} - PE_0 \frac{d\,WE}{d\,\tau_E} - e \cdot q_R \frac{d\,E}{d\,\tau_E} - q_{RF} \frac{d\,E}{d\,\tau_E} = 0 .
$$

This yields the components of an optimal path of an energy tax $\tau_E$:

[25] We assume that $R$ does not enter the cost function.

[26] We made use of (50) and (51) in order to end up with this shorter expression; for further details see the Appendix.

$$(56) \qquad \tau_E = e \cdot q_R + q_{RF} + PE_0 \frac{\dfrac{d\,(\alpha(ef) \cdot E)}{d\,\tau_E}}{\dfrac{d\,E}{d\,\tau_E}}$$

or, written in terms of the elasticity $\varepsilon_{\alpha,ef} < 0$ of energy inefficiency with respect to effort, the elasticity $\eta_{ef,\tau_E} > 0$ of effort with respect to the tax and of the elasticity $\varepsilon_{E,\tau_E} < 0$ of gross energy demand with respect to the tax:[27]

$$(57) \qquad \tau_E = e \cdot q_R + q_{RF} + \alpha \cdot PE_0 \left[ 1 + \frac{\varepsilon_{\alpha,ef} \cdot \eta_{ef,\tau_E}}{\varepsilon_{E,\tau_E}} \right].$$

If we neglect the aspect of waste $(\alpha = 0)$, then the tax is the same as in (38). If energy is wasted then the tax rate should address in addition to the pollution and resource problem also the inefficiency problem. The cost of wasted energy, $\alpha \cdot PE_0$, is multiplied by a factor greater than one. It would be one only, if the tax has no effect on effort $(\eta_{ef,\tau_E} = 0)$ or if effort can not reduce the waste coefficient $(\varepsilon_{\alpha,ef} = 0)$. It is intuitively obvious, that the tax should be higher the more it can influence effort, which in turn is successful in reducing inefficiency in energy use. If the tax elasticity for energy demand is low, then the factor behind $\alpha \cdot PE_0$ must be higher in order to address the waste problem.[28] The portfolio balance conditions are the same as in (13) and (40) except that $C_R = 0$ because we have dropped here the aspect that *R* has an impact on costs. To obtain the time path of the tax rate $\tau_E$ we treat the elasticity term in (57) as a constant and denote it by *A*. Then

$$\dot{\tau}_E = e \cdot \dot{q}_R + \dot{q}_{RF} + \alpha' \frac{d\,ef}{d\,\tau_E} \dot{\tau}_E \cdot PE_0 \left[ 1 + A \right]$$

---

[27] See the Appendix.

[28] Using the definition $E = \dfrac{NE}{1 - \alpha(ef(\tau_E))}$, the elasticity $\varepsilon_{E,\tau_E}$ can be decomposed as $\varepsilon_{E,\tau_E} = \varepsilon_{NE,\tau_E} + \dfrac{\alpha}{1-\alpha} \varepsilon_{\alpha,ef} \cdot \eta_{ef,\tau_E}$. The demand for efficient energy use (*NE*) will be reduced by the tax and the tax affects effort positively, which lowers the waste of energy.

or

(58) $$\dot{\tau}_E = \frac{e \cdot \dot{q}_R + \dot{q}_{RF}}{1 - \alpha' \dfrac{d\, ef}{d\, \tau_E} PE_0(1+A)}.$$

Since the numerator is greater than one, the growth rate of the tax is smaller than in the case $\alpha = 0$. The energy tax starts at a higher level for the benefit of a higher energy efficiency. Since the equations of the evolution (i.e. for $\dot{q}_R$ and $\dot{q}_{RF}$) are the same as derived in section 3, we can repeat the arguments to make the tax rise over time. This implies steadily increasing efforts to improve energy efficiency by reducing waste of energy because of $\dfrac{d\, ef}{d\, \tau_E} > 0$ and $\alpha'(ef) < 0$. Since $\alpha'' > 0$, the numerator in (58) becomes smaller and an inverted U-shape of the tax is a likely outcome.

## 5. On the Improvement of Abatement Measures in an Intertemporal Setting

In this section we include also pollutants of energy use which can be abated like $SO_2$ and $NO_X$. The firm can react to standards and/or emission taxes by factor substitution or by abatement activities or by both. It has an abatement function and determines the level of the abatement activity by equating marginal abatement costs to the uniform tax rate on emissions. The firm produces output *x* at cost $C\left[x, PE(a, \tau_E, \tau_{SO})\right]$ where $PE(\cdot)$ consists of the basic price $PE_0$, the cost of abatement *ca*, the cost from taxing non-abated emissions (e.g. $SO_2$) and the energy tax $\tau_E$

(59) $$PE(a, \tau_E, \tau_{SO}) = PE_0 + \tau_E + ca \cdot a \cdot e_{SO} + \tau_{SO}(1-a)e_{SO}$$

$ca = ca(a)$ is the unit cost of abatement which depends on the degree of abatement activity, $a$, $(0 \le a \le 1)$, $e_{SO}$ is the emission coefficient of the input energy (e.g. tons of $SO_2$ per ton of input) and $\tau_{SO}$ is an $SO_2$ emission tax rate. We assume $ca_a > 0$ and $ca_{aa} > 0$. The firm maximizes profit choosing output *x* and the degree of abatement:

$$\max_{x,a} \quad \pi = p \cdot x - C\big(x, PE(a, \tau_E, \tau_{SO})\big). \tag{60}$$

From the FOC $p = C_x$; output is a function of the tax rates $x = x(\tau_E, \tau_{SO})$. Maximizing (60) with respect to the degree *a* is equivalent to $\min\limits_{a} \quad PE(a, \tau_E, \tau_{SO})$. The FOC is

$$\frac{d\ PE}{d\ a} = \big(ca_a \cdot a + ca - \tau_{SO}\big) \cdot e_{SO} = 0; \tag{61}$$

that is, marginal cost of abatement is equal to the tax rate. Comparative statics for (61) yields $\dfrac{d\ a}{d\ \tau_{SO}} > 0$; the government can stimulate abatement by raising the tax.

Similarly to section 3 we are interested in the structure and the optimal path of the energy tax $\tau_E$. It affects substitution and lowers output (*p* is fixed) in order to cope with the $CO_2$ pollution problem and with the resource scarcity. In order to stimulate abatement, the government introduces another tax $\tau_{SO}$. Its path has an impact on substitution, on output (marginal cost $C_x$ will change) and on abatement. The objective function of the government is

$$\begin{aligned} \max_{\{\tau_E(t), \tau_{SO}(t)\}} \quad W = \int_0^T e^{-rt} \Bigg[ \int_0^x p(\zeta) d\,\zeta - C\big(x, PE(a, \tau_E, \tau_{SO})\big) + \tau_E \cdot E \\ + \tau_{SO} \cdot (1-a) \cdot e_{SO} \cdot E - D_1(\overline{DE} - R) - D_2(\overline{DE}_{SO} - R_{SO}) \Bigg] dt \end{aligned} \tag{62}$$

where analogously to the $CO_2$ accumulation problem $\overline{DE}_{SO}$ denotes the critical deposit capacity of the environment for $SO_2$. If the $SO_2$-fill capacity $R_{SO}$ is exhausted, the acidification of land, forest and lakes will become a very serious problem with huge damages to the economy. The dynamic constraints are (36), (37) and

$$\dot{R}_{SO} = G_2(DE_{SO} - R_{SO}) - (1-a) \cdot e_{SO} \cdot E. \tag{63}$$

We note that $E = C_{PE}$ and that the government knows that *a*, *x* and *E* are functions of the tax rates with the corresponding reaction of the firm on their levels $\left(a(\tau_{SO}), x(\tau_E, \tau_{SO}) \right.$ and $\left. E(\tau_E, \tau_{SO})\right)$. The Hamiltonian is

$$H = \int_0^x p(\zeta) d\,\zeta - C(x, PE(\cdot)) + \tau_E \cdot E + \tau_{SO} \cdot (1-a) \cdot e_{SO} \cdot E - D_1(\overline{DE} - R)$$

$$- D_2(\overline{DE}_{SO} - R_{SO}) + q_R \cdot \dot{R} + q_{R_{SO}} \cdot \dot{R}_{SO} + q_{RF} \cdot \dot{RF}$$

Using $p = C_x$ and (61), the FOC $H_{\tau_{SO}} = 0$ yields

(64) $$\tau_E - q_R \cdot e_{CO} - (q_{R_{SO}} - \tau_{SO}) \cdot (1-a) e_{SO} - q_{RF} + \frac{\frac{d\,a}{d\,\tau_{SO}}}{\frac{d\,E}{d\,\tau_{SO}}} \left[ q_{R_{SO}} - \tau_{SO} \right] e_{SO} \cdot E = 0 .$$

The second FOC $H_{\tau_E} = 0$ yields

(65) $$\tau_E + (1-a) \cdot e_{SO} \cdot \left( \tau_{SO} - q_{R_{SO}} \right) - q_R \cdot e_{CO} - q_{RF} = 0 .$$

A solution of this system (64) and (65) in the two unknown variables $\tau_E$ and $\tau_{SO}$ is:

(66) $$\tau_{\mathrm{SO}} = q_{R_{SO}} \quad , \qquad \tau_E = q_R \cdot e_{CO} + q_{RF} .$$

As before, the energy tax takes care of the $CO_2$ problem and of the resource scarcity, whereas the $SO_2$ tax $\tau_{SO}$ addresses only the problem of acidification. As we have introduced this time two instruments instead of only one, we have assigned two problems of market failure to one tax and the other one to the other tax. In section 4 we had introduced only one tax (see (57)) which had to address three problems all together. Finally, there are three equations of motion for the resource prices. Since that for $q_{R_{SO}}$ is similar to the one for $q_R$, all three shadow prices rise over time if both stocks of pollutant increase. Since $\frac{d\,a}{d\,\tau_{SO}} > 0$, the degree of abatement will then rise over time. If energy consumption has finally dropped over time to a low level, and fossil fuel is close to exhaustion, the tax components $q_R$ and $q_{R_{SO}}$ might become downward sloping as the stocks of pollutants do not increase anymore. This then mitigates the effort to abate.

## 6. Concluding remarks

Our main intention was to use a basic micro model of an industry which maximizes profits, uses energy as one of its inputs and is confronted with a varying energy tax. The firm reacts by substitution, changing its output level, by investing in energy efficient technology or by purchasing abatement equipment. The government is well aware about firms reaction on price signals. It maximizes a stream of social welfare by choosing an optimal path of its instrument – an energy tax. We find that our tax rate is the same as the one derived from the model of an energy extracting benevolent government.

In this paper we extend the Hotelling model by including stock externalities ($CO_2$, $SO_2$) and flow externalities like waste of energy and pollutants which can be abated ($SO_2$) and examine how an energy ($CO_2/SO_2$) tax can be used to address these problems in an optimal way. The concern about the time profile of an energy tax arises from the facts that fossil fuels are an exhaustible resource and that global warming, being a consequence of carbon accumulation in the atmosphere, is a stock externality problem. It has been in response to these concerns that several authors have addressed the question of the optimal time path of a tax to control the two stock problems. The objective in these papers has been to design a tax, both in level and in time profile, as to bring about socially desirable paths of fossil fuel consumption and carbon accumulation. Unfortunately – and economists are used to that – these studies differ in their conclusions about the shape of the time path of a carbon tax. The optimal time path is either increasing over time (Hoel (1993)) (but at a rate lower than the interest rate), or inverted U-shaped (e.g. Ulph and Ulph (1994)), or monotonically decreasing (e.g. Sinclair (1994)). In addition, by specifying a more complex decay of carbon, the tax may as well be constant through time, increase monotonically, or have a U-shape (Farzin and Tahvonen (1996)). The objective of this paper has been to emphasize not just the level of an energy tax but to think about the time path of such a tax. The evolution of the tax should reflect the problem that $CO_2$ emissions accumulate in the atmosphere over time. It should also incorporate the aspect that the source of $CO_2$ emissions, fossil fuel, is an exhaustible resource. In addition, we have shown that the slope of the tax rate on its optimal path should be higher if fossil fuels are not used in an efficient manner. We finally addressed the problem of $CO_2$ accumulation, $SO_2$ accumulation, the exhaustibility of fossil fuels and the need of abatement measures by introducing two taxes. The message turned out to be rather simple; do tax energy because of its exhaustibility and source of $CO_2$ emissions and use a separate tax on $SO_2$ emissions to enhance its abatement. We also explored the idea of a first rising and later falling tax over time, first favored by Ulph and Ulph (1994). With positive discounting producers need to face a higher specific tax in the future than today to bring about the same reduction in energy use. If the $CO_2$-fill capacity is nearly exhausted and $CO_2$ emissions are not permitted in the years to come, a unit of $CO_2$ emitted at a later date contributes less to marginal damage and should be taxed less, especially as a backstop technology is close anyway.

**References**

Blank, J. and W. Ströbele, 1995, Das $CO_2$ Problem aus ressourcenökonomischer Sicht, *WiSt*, Heft 6, 283-289.

Conrad, K., 1993, Taxes and Subsidies for Pollution-Intensive Industries as Trade Policy, *Journal of Environmental Economics and Management*, 25, 121-135.

Conrad, K., 2000, An Econometric Model of Production with Endogenous Improvement in Energy Efficiency, 1970-1995, *Applied Economics*, 32, 1153-1160.

Conrad, J.M., Olson, L.J., 1992, The economics of stock pollutant: Aldicarb on Long Island, *Environmental and Resource Economics*, 3, 245-258.

Dasgupta, P., 1982, The Control of Resources, *Basil Blackwell*, Oxford.

Farzin, Y., and O. Tahvonen, 1996, The global carbon cycle and the optimal time path of a carbon tax, *Oxford Economic Papers*, 48, 515-536.

Forster, B., 1977, On a One State Variable Control Problem – Consumption Pollution Trade-offs, in J.D. Pitchford and S. Turnovsky (eds.), *Applications of Control Theory to Economic Analysis*, North-Holland, Amsterdam.

Goulder, L.H. and K. Mathai, 2000, Optimal $CO_2$ Abatement in the Presence of Induced Technological Change, *Journal of Environmental Economics and Management*, 39 (1), 1-38.

Hoel, M., 1993, Intertemporal Properties of an International Carbon Tax, *Resource and Energy Economics*, 15, 51-70.

Keeler, E., Spence, M., Zeckhauser, R., 1972, The Optimal Control of Pollution, *Journal of Economic Theory*, 4, 19-34.

van der Ploeg, F., Withagen, C., 1991, Pollution Control and the Ramsey Problem, *Environmental and Resource Economics*, 1, 215-236.

Plourde, C.G., 1972, A Model of Waste Accumulation and Disposal, *Canadian Journal of Economics*, 5, 199-225.

Sinclair, P., 1994, On the Trend of Fossil Fuel Taxation, *Oxford Economic Papers*, 46, 869-877.

Tahvonen, O., 1995, Dynamics of Pollution Control when Damage is Sensitive to the Rate of Pollution Accumulation, *Environmental and Resource Economics*, 5, 9-27.

Tahvonen, O., 1997, Fossil Fuels, Stock Externalities, and Backstop Technology, *Canadian Journal of Economics*, 30, 855-874.

Toman, M., Palmer, K., 1997, How should an Accumulative Toxic Substance be Banned? *Environmental and Resource Economics*, 9 (1), 83-102.

Toman, M.A., and C. Withagen, 2000, Accumulative Pollution, “clean technology”, and Policy Design, *Resource and Energy Economics*, 22, 367-384.

Ulph, A., and D. Ulph, 1994, The Optimal Path of a Carbon Stock, *Oxford Economic Papers*, 46, 857-868.

**Appendix**

**Proof of (18)-(21):**

In order to evaluate the slopes of the two equations of motion (13) and (14) near the steady state we choose the specification (22)-(24) given in section 2. We then differentiate each of the equations of motion (26) and (28) with respect to each of the included variables. For example, for the $q_R(q_R, R)$ equation,

$$d\,\dot{q}_R = (r+\delta)\,d\,q_R + \gamma\,d\,R$$

and for the $R(q_R, R)$ equation

$$d\,\dot{R} = -\delta \cdot d\,R - e \cdot E_{q_R} \cdot d\,q_R$$

where $E_{q_R} < 0$, using (27).

Next we verify (18) as the slope of the $\dot{q}_R = 0$ equation at the steady state:

(18') $$\frac{d\,q_R}{d\,R} = -\frac{\gamma}{r+\delta} < 0$$

and of the $\dot{R} = 0$ equation

(19') $$\frac{d\,q_R}{d\,R} = -\frac{\delta}{e \cdot E_{q_R}} > 0\,.$$

In order to verify (20), i.e. the motions off the steady state, we obtain for the $\dot{q}_R = 0$ equation

(20') $$\frac{\partial\,\dot{q}_R}{\partial\,q_R} = r+\delta > 0, \qquad \frac{\partial\,\dot{q}_R}{\partial\,R} = \gamma > 0$$

and for the $\dot{R} = 0$ equation,

(21') $$\frac{\partial \dot{R}}{\partial q_R} = -e \cdot E_{q_R} > 0, \qquad \frac{\partial \dot{R}}{\partial R} = -\delta < 0 .$$

**Proof of (32) to (35):**

The system (31), but with total differentiation with respect to all three exogenous variables, is:

(A1) $$\begin{aligned} (r+\delta)\, d\, q_R^* + \gamma\, d\, R^* &= -q_R^*\, d\, r - q_R^*\, d\, \delta + (\overline{DE} - R) d\, \gamma \\ -E_{q_R}\, d\, q_R^* - \delta\, d\, R^* &= -\overline{DE}\, d\, \delta \qquad (e = 1). \end{aligned}$$

The determinant of the characteristic matrix of this system is

$$D = -(r+\delta) \cdot \delta + E_{q_R} \cdot \gamma < 0 .$$

By Cramer's rule $(d\, \delta = d\, \gamma = 0)$:

$$\frac{d\, q_R^*}{d\, r} = \frac{1}{D} \begin{vmatrix} -q_R^* & \gamma \\ 0 & -\delta \end{vmatrix} = \frac{1}{D}\left(q_R^*\, \delta\right) < 0$$

and

$$\frac{d\, R^*}{d\, r} = \frac{1}{D} \begin{vmatrix} r+\delta & -q_R^* \\ -E_{q_R} & 0 \end{vmatrix} = \frac{1}{D}\left(-E_{q_R} \cdot q_R^*\right) < 0$$

which proves (32). Similarly $(d\, r = d\, \gamma = 0)$:

$$\frac{d\, q_R^*}{d\, \delta} = \frac{1}{D} \begin{vmatrix} -q_R^* & \gamma \\ -\overline{DE} & -\delta \end{vmatrix} = \frac{1}{D}\left(q_R^* \cdot \delta + \overline{DE} \cdot \gamma\right) < 0$$

because the positive term $\gamma \cdot \overline{DE}$ dominates the small negative term $q_R^*\, \delta$. Furthermore,

$$\frac{d\,R^*}{d\,\delta} = \frac{1}{D} \begin{vmatrix} r+\delta & -q_R^* \\ -E_{q_R} & -\overline{DE} \end{vmatrix} = \frac{-1}{D}\Big((r+\delta)\,\overline{DE} + E_{q_R}\cdot q_R\Big) > 0$$

because the positive term $(r+\delta)\,\overline{DE}$ dominates the small negative term $E_{q_R}\cdot q_R$. This proves (33). Next $(d\,r = d\,\delta = 0)$:

$$\frac{d\,q_R^*}{d\,\gamma} = \frac{1}{D} \begin{vmatrix} \overline{DE}-R & \gamma \\ 0 & -\delta \end{vmatrix} = \frac{1}{D}\Big(-\delta(\overline{DE}-R)\Big) > 0$$

and

$$\frac{d\,R^*}{d\,\gamma} = \frac{1}{D} \begin{vmatrix} r+\delta & \overline{DE}-R \\ -E_{q_R} & 0 \end{vmatrix} = \frac{1}{D}\Big(E_{q_R}(\overline{DE}-R)\Big) > 0$$

which proves (34).

**Proof of (56)**

$$H_{\tau_E} = (p(x)-C_x)\,\frac{d\,x}{d\,\tau_E} - NE\cdot\left(\frac{\partial\,PE}{\partial\,\tau_E} + \frac{\partial\,PE}{\partial\,ef}\,\frac{d\,ef}{d\,\tau_E}\right) - C_{ef}\,\frac{d\,ef}{d\,\tau_E} + E + \tau_E\,\frac{d\,E}{d\,\tau}$$

$$-PE_0\left(\frac{d\,\alpha}{d\,ef}\,\frac{d\,ef}{d\,\tau_E}\,E + \alpha\,\frac{d\,E}{d\,\tau_E}\right) - q_R\cdot e\,\frac{d\,E}{d\,\tau_E} - q_{RF}\,\frac{d\,E}{d\,\tau_E} = 0$$

Because of $p = C_x$, (51), $\dfrac{\partial\,PE}{\partial\partial\,\tau_E} = \dfrac{1}{1-\alpha}$ and $E = \dfrac{NE}{1-\alpha}$, we obtain

$$\tau_E = e\cdot q_R + q_{RF} + PE_0\cdot\frac{\left(\alpha'\,\dfrac{d\,ef}{d\,\tau_E}\,E + \alpha\,\dfrac{d\,E}{d\,\tau_E}\right)}{\dfrac{d\,E}{d\,\tau_E}}$$

which can be written as in (57) if we use the definition of the three elasticities given in (57).