Credit derivatives pricing with default density term structure modelled by Lévy random fields

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December 13, 2011

Abstract

We model the term structure of the forward default intensity and the default density by using Lévy random fields, which allow us to consider the credit derivatives with an after-default recovery payment. As applications, we study the pricing of a defaultable bond and represent the pricing kernel as the unique solution of a parabolic integro-differential equation. Finally, we illustrate by numerical examples the impact of the contagious jump risks on the defaultable bond price in our model.

Key words: default density, Lévy random field, credit derivatives pricing, parabolic integro-differential equation

1 Introduction

The term structure modelling in the interest rate and in the credit risk modelling has been widely adopted and extended since the original paper of Heath-Jarrow-Morton [16]. Notably, there have appeared many important papers (e.g. [1, 4, 8, 9, 11, 12]) incorporating jump diffusions to describe the family of bond prices or the forward curves as a generalization of the classic HJM model.

In the credit risk modelling, the conditional survival probability associated to the default time is an important quantity for measuring default risk and studying valuation of credit derivatives. Let τ be a nonnegative random variable defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ satisfying the usual conditions. The conditional survival probability (CSP) is defined as $S_t(\theta) = \mathbb{P}(\tau > \theta|\mathcal{F}_t)$, $t, \theta \geqslant 0$. To describe the term structure of the CSP, we can use both the density and the intensity point of view. On the one hand, as in El Karoui et al [10], we assume that there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions $(\omega, \theta) \to \alpha_t(\omega, \theta)$ such that the CSP has the following additive representation:

$$S_t(\theta) = \int_\theta^\infty \alpha_t(v)\mathrm{d}v. \tag{1.1}$$

The family of random variables $\alpha_t(\cdot)$ is called the conditional *density* of the default time τ given $\mathcal{F}_t$. On the other hand, similarly to the definition of forward rate, we can use the “intensity” point of view and the

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following multiplicative representation:

$$S_t(\theta) = \exp\left(-\int_0^\theta \lambda_t(v)\mathrm{d}v\right) \tag{1.2}$$

where the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $(\omega, \theta) \to \lambda_t(\omega, \theta)$ is called the *forward intensity*. It is equivalent to assume the existence of the density or the intensity for all positive t and θ. We have the relationship:

$$\alpha_t(\theta) = S_t(\theta)\lambda_t(\theta). \tag{1.3}$$

In the interest rate models, the time θ is always larger than t and the forward rate has no economic interpretation for $\theta < t$. However, it is noted in [10] that to study what happens after a default event, we need the whole term structure of the conditional survival probability, that is, for all positive t and θ. One typical example is a defaultable bond where the recovery payment is effectuated at a given maturity later than the economic default date.

In this paper, we consider the whole term structure modelling of CSP and the applications to the credit derivative pricing. In the credit risk models, the default contagion phenomenon is often modelled by positive jumps in the intensity process. We take this point into modelling consideration and propose a forward intensity driven by Lévy random fields. In the existing Lévy term structure models, in Filipović et al. [11, 12], the authors consider forward curve evolutions as solutions of the infinite dimensional Musiela parametrization first-order hyperbolic stochastic differential equations driven by n-independent Lévy processes or driven by a Wiener process together with an independent Poisson measure. In [9], Eberlein and Raible present a class of bond price models that can be driven by a wide range of Lévy processes with finite exponential moments. This model was further applied to describe the defaultable Lévy term structure and explore ratings and restructuring of the defaultable market. The driving process of the Lévy term structure model in [9] was further extended to non-homogeneous Lévy processes in [8].

Motivated by those existing Lévy term structure models and the random field models which are widely used to model various stochastic dynamics (e.g. [1, 5, 6, 7, 15, 17, 18]), we suppose that the Lévy random field in our model is a combination of a kernel-correlated Gaussian field and an independent (central) Poisson random measure. The jump component described as Poisson measure is similar to that used in [12], but it is not necessary to assume the exponential integrability condition for the characteristic measure under our framework (see Section 2). The kernel-correlated Gaussian field is more flexible compared to the Gaussian components without kernel-correlation considered in [15, 17, 18]. In fact, we can choose appropriate correlated-kernels of the Gaussian field so that the models considered in [9, 11, 12] can be covered (see Remark 2.3). Note that it is not genetically tractable for pricing of defaultable bonds under infinite dimensional framework as in [11, 12]. Although we do not intend to consider the forward intensity under infinite dimensional framework as in [11, 12], it has a close relationship between the (infinite dimensional) Wiener process and the kernel-correlated Gaussian field. Indeed, the kernel-correlated Gaussian field can product a cylindrical Wiener process by establishing appropriate Hilbert spaces (see Proposition 2.5 in [7]). We deduce the dynamics of the CSP and the associated density in this setting. In particular, we emphasize on a martingale condition, which can be viewed as an analogue of the non-arbitrage condition in the classical HJM model.

For the pricing of credit derivatives, we follow the standard general framework in Bielecki and Rutkowski [3]. The global market information contains both the default information and the "default-free" market information represented by the filtration $\mathbb{F}$, which is obtained by an enlargement of filtration. We are particularly interested in an economic default case, that is, the default does not lead to the total bankruptcy of the underlying firm and a partial recovery value is repaid at the maturity date of the bond in case of default prior to the maturity. To evaluate this "after-default" payment, we use the density approach in [10] and obtain that the key quantities for the pricing of a defaultable bond are two pricing kernels, one depending on the interest rate and the default density, and the other depending on additionally the recovery rate. We assume that both the short interest rate and the default density are modelled by the Lévy random field model and are correlated between them. For the recovery rate, we analyze firstly the simple case where the recovery rate is deterministic and then the random recovery case. We show that the pricing kernel is related to the solution of a second-order parabolic integro-differential equation and we prove, based on a result of Garroni and Menaldi [14], the existence and the uniqueness of the solution to the equation.

The rest of the paper is organized as follows. We present our model setting in Section 2 and give the martingale condition. We then analyze the dynamics of the CSP and the conditional density in Section 3. In Section 4, we discuss the pricing of credit derivatives and in particular the defaultable zero-coupon bond. The two sections 5 and 6 focus on the pricing kernels. Finally, we present some numerical illustrations in the last section 7.

2 Forward intensity driven by Lévy random field

In this paper, we adopt a random field point of view to model the forward intensity $\lambda_t(\theta)$ where both t and θ are positive. We consider a Lévy random field on $\mathbb{R}_+ \times \mathbb{R}^d$ which is a combination of a Gaussian random field Y^G and a compensated Poisson random measure Y^P independent to Y^G. Here $\mathbb{R}_+$ denotes the time space and $\mathbb{R}^d$ is considered as a parameter space.

We assume that the covariance of the Gaussian random field Y^G is given by a kernel measure c on $\mathbb{R}^d$ which has a continuous and symmetric density on $\mathbb{R}^d \setminus \{0\}$ with respect to the Lebesgue measure and such that $c(\{0\}) > 0$. Namely for $(\phi_1, \phi_2) \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)^2$,

$$\mathbb{E}[Y^G(\phi_1)Y^G(\phi_2)] = \int_0^\infty \int_{\mathbb{R}^{2d}} \phi_1(t,\xi_1)\phi_2(t,\xi_2)c(\xi_1-\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}t,$$

where by abuse of language $c(\xi_1 - \xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2$ denotes the measure on $\mathbb{R}^d \times \mathbb{R}^d$ the inverse image of the measure c by the mapping from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}^d$ which sends (ξ_1, ξ_2) to $\xi_1 - \xi_2$. The Gaussian random field Y^G defines a worthy martingale measure (see [20, p.289] and [6, p.190]). Let $\mathbb{F}^G = (\mathcal{F}_t^G)_{t\geqslant 0}$ be the filtration satisfying the usual conditions which is generated by

$$\sigma(Y^G([0,u]\times A),\ u \leqslant t,\ A \in \mathcal{B}_b(\mathbb{R}^d)), \quad t \geqslant 0,$$

where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the set of all bounded Borel subsets of $\mathbb{R}^d$. Let $\mathcal{P}^G$ be the predictable σ-algebra on $\Omega \times \mathbb{R}_+$ associated to $\mathbb{F}^G$ and Φ_c be the linear space of all $\mathcal{P}^G \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions h such that

$$\|h\|_{c,T} := \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^{2d}} |h(t,\xi_1)h(t,\xi_2)|c(\xi_1-\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}t\Big]^{1/2} < +\infty$$

for any $T > 0$. The stochastic integral $h \cdot Y^G$ is well defined for any $h \in \Phi_c$. When c is the Dirac distribution concentrated on the origin, the stochastic integral:

$$B(t_0,\ldots,t_d) = Y^G([0,t_0]\times\cdots\times[0,t_d]), \quad (t_0,\ldots,t_d) \in \mathbb{R}_+^{d+1} \tag{2.1}$$

defines a $(d+1)$-parameter Brownian sheet. If in particular $d = 0$, it becomes a standard Brownian motion.

Denote the intensity measure of the compensated Poisson field Y^P by $\nu(\mathrm{d}\xi)\mathrm{d}t$, $(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^d$, where ν is a σ-finite measure on $\mathbb{R}^d$. Let $\mathbb{F}^P = (\mathcal{F}_t^P)_{t\geqslant 0}$ be the filtration satisfying the usual conditions generated by

$$\sigma(Y^P([0,u]\times A),\ u \leqslant t,\ A \in \mathcal{B}_b(\mathbb{R}^d)), \quad t \geqslant 0$$

and $\mathcal{P}^P$ be the predictable σ-algebra on $\Omega \times \mathbb{R}_+$ associated to $\mathbb{F}^P$. Denote by Ψ_ν the linear space of all $\mathcal{P}^P \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions such that

$$\|g\|_{\nu,T} := \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^d} |g(t,\xi)|^2 \nu(\mathrm{d}\xi)\mathrm{d}t\Big] < +\infty$$

for any $T > 0$. The stochastic integral $g \cdot Y^P$ is well defined for any $g \in \Psi_\nu$.

Let $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ be the natural filtration generated by the Lévy random field, namely $\mathbb{F} := \mathbb{F}^G \vee \mathbb{F}^P$. We describe the forward intensity by using the Lévy random filed as the following additive HJM type model:

$$\mathrm{d}\lambda_t(\theta) = \mu_t(\theta)\mathrm{d}t + \int_{\mathbb{R}^d} \sigma_t(\theta,\xi) Y^G(\mathrm{d}t, \mathrm{d}\xi) + \int_{\mathbb{R}^d} \gamma_{t-}(\theta,\xi) Y^P(\mathrm{d}t,\mathrm{d}\xi), \tag{2.2}$$

where

(1) $\boldsymbol{\mu} = (\mu_t(\theta);\ (t,\theta) \in \mathbb{R}_+^2)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable and $\int_0^T \mathbb{E}\left[|\mu_t(\theta)|\right] \mathrm{d}t < \infty$, where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times \mathbb{R}_+$ associated to the filtration $\mathbb{F}$,

(2) $\boldsymbol{\sigma} = (\sigma_t(\theta,\xi);\ (t,\theta,\xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d)$ is $\mathcal{P}^G \otimes \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$-measurable and for any $\theta \in \mathbb{R}_+$, $\sigma_\cdot(\theta,\cdot) \in \Phi_c$,

(3) $\boldsymbol{\gamma} = (\gamma_t(\theta,\zeta) \geqslant 0;\ (t,\theta,\zeta) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d)$ is $\mathcal{P}^P \otimes \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$-measurable and for any $\theta \in \mathbb{R}_+$, $\gamma_\cdot(\theta,\cdot) \in \Psi_\nu$.

The model (2.2) can also be written in the integral form as

$$\lambda_t(\theta) = \lambda_0(\theta) + \int_0^t \mu_s(\theta)\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \sigma_s(\theta,\xi) Y^G(\mathrm{d}s,\mathrm{d}\xi) + \int_0^t \int_{\mathbb{R}^d} \gamma_{s-}(\theta,\xi) Y^P(\mathrm{d}s,\mathrm{d}\xi) \tag{2.3}$$

where both stochastic integrals with respect to Y^G and Y^P are $\mathbb{F}$-martingales with mean zero, $\lambda_0(\cdot)$ is a deterministic Borel function on $\mathbb{R}_+$.

Similarly to the classical HJM model, in the above Lévy field model (2.2), there exists a relationship between the drift coefficient $\boldsymbol{\mu}$ and the diffusion coefficients $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ due to the fact that, for any $\theta \geqslant 0$, the conditional survival probability process $\big(S_t(\theta) = \exp\big(-\int_0^\theta \lambda_t(v)\mathrm{d}v\big),\ t \geqslant 0\big)$ should be an $\mathbb{F}$-martingale. We call this relationship the martingale condition (**MC**). Let us introduce the following notation:

$$I_\mu(t,\theta) := \int_0^\theta \mu_t(v)\mathrm{d}v, \quad I_\sigma(t,\theta,\xi) := \int_0^\theta \sigma_t(v,\xi)\mathrm{d}v, \quad \text{and} \quad I_\gamma(t,\theta,\xi) := \int_0^\theta \gamma_t(v,\xi)\mathrm{d}v,$$

where $(t,\theta,\xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$.

Theorem 2.1 *For all $\theta \geqslant 0$ and $T > 0$, one has*

$$\int_0^T \mathbb{E}\left[|I_\mu(t,\theta)|\right] \mathrm{d}t < \infty, \quad I_\sigma(\cdot,\theta,\cdot) \in \Phi_c, \quad I_\gamma(\cdot,\theta,\cdot) \in \Psi_\nu, \quad \text{and} \quad e^{-I_\gamma(\cdot,\theta,\cdot)} - 1 \in \Psi_\nu, \tag{2.4}$$

Moreover, the process family $\big(S_t(\theta) = \exp\big(-\int_0^\theta \lambda_t(v)\mathrm{d}v\big),\ t \geqslant 0\big)$ is a family of $\mathbb{F}$-martingales if and only if the following condition is satisfied:

$$\begin{aligned}(\mathbf{MC}) \qquad \forall\, \theta \geqslant 0,\ \ \mu_t(\theta) = &\int_{\mathbb{R}^{2d}} \sigma_t(\theta,\xi_1) I_\sigma(t,\theta,\xi_2) c(\xi_1 - \xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ &+ \int_{\mathbb{R}^d} \gamma_t(\theta,\xi)(1 - e^{-I_\gamma(t,\theta,\xi)})\nu(\mathrm{d}\xi).\end{aligned} \tag{2.5}$$

Proof. The proofs for the first three statements in (2.4) are similar. We only provide the details for the third one. For any $T > 0$, we have

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \mathbb{E}|I_\gamma(t,\theta,\xi)|^2 \nu(\mathrm{d}\xi)\mathrm{d}t &= \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\left|\int_0^\theta \gamma_t(v,\xi)\mathrm{d}v\right|^2 \nu(\mathrm{d}\xi)\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\left[\int_0^\theta \gamma_t(v_1,\xi)\mathrm{d}v_1 \int_0^\theta \gamma_t(v_2,\xi)\mathrm{d}v_2\right] \nu(\mathrm{d}\xi)\mathrm{d}t \\ &\leq \frac{1}{2}\int_0^T \int_0^\theta \int_0^\theta \int_{\mathbb{R}^d} \mathbb{E}\left[|\gamma_t(v_1,\xi)|^2 + |\gamma_t(v_2,\xi)|^2\right] \nu(\mathrm{d}\xi)\mathrm{d}v_1\mathrm{d}v_2\mathrm{d}t \\ &= \theta \int_0^\theta \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\left[|\gamma_t(v,\xi)|^2\right] \nu(\mathrm{d}\xi)\mathrm{d}t\mathrm{d}v,\end{aligned}$$

which is finite since $\gamma_\cdot(v,\cdot) \in \Psi_\nu$ for any $v \geqslant 0$. For the last assertion in (2.4), note that $\gamma_t(\theta,\xi) \geqslant 0$ and thus

$$\left|e^{-I_\gamma(t,\theta,\xi)} - 1\right| \leq |I_\gamma(t,\theta,\xi)|,$$

for all $(t,\theta,\xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$.

We now prove that the condition $(\mathbf{MC})$ is equivalent to the martingale condition for $(S_t(\theta),\ t \geqslant 0)$. In fact

$$\begin{aligned}\frac{\mathrm{d}S_t(\theta)}{S_{t-}(\theta)} = &-I_\mu(t,\theta)\mathrm{d}t - \int_{\mathbb{R}^d} I_\sigma(t,\theta,\xi)Y^G(\mathrm{d}t,\mathrm{d}\xi) \\ &+ \frac{1}{2}\int_{\mathbb{R}^{2d}} I_\sigma(t,\theta,\xi_1)I_\sigma(t,\theta,\xi_2)c(\xi_1-\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}t \\ &+ \int_{\mathbb{R}^d} (e^{-I_\gamma(t^-,\theta,\xi)} - 1)Y^P(\mathrm{d}t,\mathrm{d}\xi) \\ &+ \int_{\mathbb{R}^d} (e^{-I_\gamma(t,\theta,\xi)} - 1 + I_\gamma(t,\theta,\xi))\nu(\mathrm{d}\xi)\mathrm{d}t,\end{aligned} \tag{2.6}$$

so the martingale condition of $(S_t(\theta),\ t \geqslant 0)$ is thus equivalent to the following equality

$$\begin{aligned}I_\mu(t,\theta) = &\frac{1}{2}\int_{\mathbb{R}^{2d}} I_\sigma(t,\theta,\xi_1)I_\sigma(t,\theta,\xi_2)c(\xi_1-\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ &+ \int_{\mathbb{R}^d} (e^{-I_\gamma(t,\theta,\xi)} - 1 + I_\gamma(t,\theta,\xi))\nu(\mathrm{d}\xi),\end{aligned}$$

which is equivalent to $(\mathbf{MC})$. □

Remark 2.2 Consider the particular case where $d = 0$, c is the Dirac measure, and $\nu = 0$. The condition (**MC**) becomes

$$\forall\, \theta \geqslant 0, \quad \mu_t(\theta) = \sigma_t(\theta)\int_0^\theta \sigma_t(v)\mathrm{d}v.$$

This corresponds to the non-arbitrage condition in the classical HJM model where the forward intensity is driven by a standard Brownian motion.

Remark 2.3 There exist random field models in the literature. We make below some comparisons. The forward intensity model (2.2) can be extended to the following form:

$$\begin{aligned} \mathrm{d}\lambda_t(\theta) = \mu_t(\theta)\mathrm{d}t + \int_{\mathbb{R}^d} \sigma_t(\theta,\zeta)Y^G(\mathrm{d}t,\mathrm{d}\zeta) + \int_{0<|\xi|\leqslant 1} \gamma_{t-}(\theta,\xi)Y^P(\mathrm{d}t,\mathrm{d}\xi) \\ + \int_{|\xi|>1} \hat{\gamma}_{t-}(\theta,\xi)(Y^P(\mathrm{d}t,\mathrm{d}\xi) + \nu(\mathrm{d}\xi)\mathrm{d}t), \end{aligned} \tag{2.7}$$

where $\sigma_\cdot(\theta,\cdot) \in \Phi_c$, $\gamma_\cdot(\theta,\cdot)\mathbb{1}_{\{|\cdot|\leqslant 1\}}$ and $\hat{\gamma}_\cdot(\theta,\cdot)\mathbb{1}_{\{|\cdot|>1\}} \in \Psi_\nu$, for each fixed $\theta \geqslant 0$. Under the model (2.7), the corresponding martingale condition ($\mathbf{MC}$) will be changed accordingly. We next consider a special form of the predictable random filed with separable variables:

$$\sigma_t(\theta,\zeta) = \widetilde{\sigma}_t(\theta)\widetilde{\phi}(\zeta), \;\; \gamma_t(\theta,\xi) = \hat{\gamma}_t(\theta,\xi) = \langle \widetilde{\gamma}_t(\theta), \xi\rangle, \;\; \zeta \in \mathbb{R}^d, \; \xi \in \mathbb{R}^d_+$$

where $(\widetilde{\sigma}_t(\theta);\ (t,\theta) \in \mathbb{R}^2_+)$ is a real-valued predictable random field, $(\widetilde{\gamma}_t(\theta) = (\widetilde{\gamma}^1_t(\theta),\ldots,\widetilde{\gamma}^d_t(\theta));\ (t,\theta) \in \mathbb{R}^2_+)$ is a $\mathbb{R}^d_+$-valued predictable field and $\widetilde{\phi}(\zeta)$ is a deterministic measurable function on $\mathbb{R}^d$. In this case, the extended model (2.7) can be rewritten as

$$\mathrm{d}\lambda_t(\theta) = (\mu_t(\theta) - a)\mathrm{d}t + \sigma_t(\theta)Y^G(\mathrm{d}t, \widetilde{\phi}(\star)) + \langle \widetilde{\gamma}_t(\theta), \mathrm{d}L_t\rangle, \tag{2.8}$$

where $a \in \mathbb{R}$, $\langle \cdot,\cdot\rangle$ denotes the inner-product on $\mathbb{R}^d$ and

$$\mathrm{d}L_t = a\mathrm{d}t + \int_{0<|\xi|\leqslant 1} \xi Y^P(\mathrm{d}t,\mathrm{d}\xi) + \int_{|\xi|>1} \xi(Y^P(\mathrm{d}t,\mathrm{d}\xi) + \nu(\mathrm{d}\xi)\mathrm{d}t)$$

is a non-Gaussian Lévy process if the characteristic measure ν is a Lévy measure. If $\widetilde{\phi} \equiv 1$, then $Y^G(\mathbb{1}_{[0,t]} \times \widetilde{\phi}(\star))$ becomes a Brownian motion when the correlated-kernel is Dirac. Choose appropriate smooth function $\widetilde{\phi}$ as in the proof of Proposition 2.5 in [7], then $Y^G(\mathbb{1}_{[0,t]} \times \widetilde{\phi}(\star))$ becomes a cylindrical Wiener process. Thus we recover the Lévy interest rate term structure models considered in [9, 11, 12], if the Lévy measure ν satisfies the exponential integrability condition. We next give a comparison of our Lévy random field $Y^G + Y^P$ introduced previously in this section with existing Lévy fields in literature.

1. As in (2.1), the field $Y^G + Y^P$ can be reduced to a Brownian sheet in Walsh [20], when the kernel c is Dirac and the characteristic measure $\nu = 0$ (hence $Y^P = 0$);

2. the field $Y^G + Y^P$ becomes a so-called “colored” space-time white noise model established by [5], when the kernel $c(\xi) = |\xi|^{-\alpha}$ with $0 < \alpha < d$ and $\nu = 0$;

3. the fractional space-time white noise (fractional in space and time in white) used in [19] corresponds to the field $Y^G + Y^P$ with the kernel $c(\xi) = h(2h-1)|\xi|^{2h-2}$ with $\frac{1}{2} < h < 1$, $d = 1$ and $\nu = 0$;

4. the Poisson sheet in [1] corresponds to the field $Y^G + Y^P + \nu(\mathrm{d}\xi)\mathrm{d}t$ with $c = 0$ and $\nu(\xi) = z\delta_1(\mathrm{d}\xi)$ where $z > 0$ is single point and δ_1 is the Dirac measure concentrated at 1. The Gamma sheet in [1] is the field $Y^G + Y^P + \nu(\mathrm{d}\xi)\mathrm{d}t$ with $c = 0$ and $\nu(\mathrm{d}\xi) = \frac{e^{-\xi}}{\xi}\mathbb{1}_{\{\xi>0\}}z\mathrm{d}\xi$ where $d = 1$ and $z > 0$ is a single point.

3 Conditional survival probability and density

In this section, we concentrate on the conditional survival probability $(S_t(\theta),\ t \geqslant 0)$ and the conditional density $(\alpha_t(\theta),\ t \geqslant 0)$. Here we specify a càdlàg version of the martingale $(S_t(\theta),\ t \geqslant 0)$ for any $\theta \geqslant 0$.

In fact, to show that the integral $\int_0^\theta \lambda_t(v)\mathrm{d}v$ defines a càdlàg process, we need a stronger assumption on the process $\lambda(\theta)$ in order to apply Lebesgue's theorem. The càdlàg version of $S(\theta)$, if well defined, should have a universal version of its predictable projection as follows:

$$S_{t-}(\theta) = S_t^{(p)}(\theta) = \exp\left(-\int_0^\theta \lambda_{t-}(v)\mathrm{d}v\right).$$

Thus

$$S_t(\theta) := \lim_{\substack{q\in\mathbb{Q}^+\\ q\downarrow t}} \exp\left(-\int_0^\theta \lambda_{q-}(v)\mathrm{d}v\right) \tag{3.1}$$

defines a universal càdlàg version of the martingale $S(\theta)$.

We observe from the equality (2.6) that, under the condition $(\mathbf{MC})$, the conditional survival probability admits the following dynamics:

$$\frac{\mathrm{d}S_t(\theta)}{S_{t-}(\theta)} = -\int_{\mathbb{R}^d} I_\sigma(t,\theta,\xi)Y^G(\mathrm{d}t,\mathrm{d}\xi) + \int_{\mathbb{R}^d}(e^{-I_\gamma(t^-,\theta,\xi)}-1)Y^P(\mathrm{d}t,\mathrm{d}\xi), \tag{3.2}$$

where $S_0(\theta) = \exp(-\int_0^\theta \lambda_0(v)\mathrm{d}v)$.

For $\theta \geqslant 0$, we denote by $M(\theta)$ the martingale defined as

$$\mathrm{d}M_t(\theta) = -\int_{\mathbb{R}^d} I_\sigma(t,\theta,\xi)Y^G(\mathrm{d}t,\mathrm{d}\xi) + \int_{\mathbb{R}^d}(e^{-I_\gamma(t-,\theta,\xi)}-1)Y^P(\mathrm{d}t,\mathrm{d}\xi), \quad M_0(\theta)=0. \tag{3.3}$$

With this notation, $S(\theta)/S_0(\theta)$ is the Doléans-Dade exponential of the martingale $M(\theta)$. Moreover, denote by $m(\theta)$ the martingale defined by the dynamics:

$$\mathrm{d}m_t(\theta) = -\int_{\mathbb{R}^d}\sigma_t(\theta,\xi)Y^G(\mathrm{d}t,\mathrm{d}\xi) - \int_{\mathbb{R}^d}\gamma_{t-}(\theta,\xi)e^{-I_\gamma(t-,\theta,\xi)}Y^P(\mathrm{d}t,\mathrm{d}\xi), \quad m_0(\theta)=0. \tag{3.4}$$

Observe that the following relation holds

$$M_t(\theta) = \int_0^\theta m_t(u)\mathrm{d}u.$$

We then consider the dynamics of the conditional density of default given in (1.1).

Proposition 3.1 *The conditional density process $\alpha(\theta)$ admits the dynamics:*

$$\mathrm{d}\alpha_t(\theta) \quad = \quad \alpha_{t-}(\theta)\mathrm{d}M_t(\theta) - S_{t-}(\theta)\mathrm{d}m_t(\theta) \tag{3.5}$$

or equivalently

$$\frac{\mathrm{d}\alpha_t(\theta)}{\alpha_{t-}(\theta)} = \mathrm{d}M_t(\theta) - \frac{1}{\lambda_{t-}(\theta)}\mathrm{d}m_t(\theta).$$

Proof. Keep the martingale condition (**MC**) in mind. The dynamics (3.5) is derived by employing Itô's formula to $\alpha(\theta)=\lambda(\theta)S(\theta)$ for each positive θ fixed. □

An important property in the credit analysis is the immersion property, or the so called (H)-hypothesis, which means that an $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale. The (H)-hypothesis is satisfied if and only if $\alpha_t(\theta)=\alpha_\theta(\theta)$ or equivalently $\lambda_t(\theta)=\lambda_\theta(\theta)$ for any $t\geqslant\theta$. In the random field setting, by (2.3), this is equivalently to

$$\int_\theta^t\int_{\mathbb{R}^d}\sigma_s(\theta,\xi)Y^G(\mathrm{d}s,\mathrm{d}\xi) = \int_\theta^t\int_{\mathbb{R}^d}\gamma_{s-}(\theta,\xi)Y^P(\mathrm{d}s,\mathrm{d}\xi) = 0$$

for $t \geqslant \theta$, or equivalently

$$\sigma_t(\theta,\xi) = 0 \quad \text{and} \quad \gamma_t(\theta,\xi) = 0 \quad \nu(\mathrm{d}\xi)\text{-a.e.}$$

for $t > \theta$. Note that the martingale condition (**MC**) then implies that $\mu_t(\theta) = 0$ for $t > \theta$.

We recall that the $\mathbb{F}$-intensity process λ of the default time τ coincides with the diagonal forward intensity, i.e. $\lambda_t = \lambda_t(t)$. It is closely related to the Azéma supermartingale:

$$S_t = S_t(t) = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t),$$

which is also called the survival process of τ.

Proposition 3.2 *Let M be the $\mathbb{F}$-martingale having the dynamics*

$$\mathrm{d}M_t = -\int_{\mathbb{R}^d} I_\sigma(t,t,\xi) Y^G(\mathrm{d}t,\mathrm{d}\xi) + \int_{\mathbb{R}^d} \Big(e^{-I_\gamma(t-,t,\xi)} - 1\Big) Y^P(\mathrm{d}t,\mathrm{d}\xi).$$

Then

$$S_t = \exp\Big(-\int_0^t \lambda_s \mathrm{d}s\Big)\mathcal{E}(M)_t,$$

where $\mathcal{E}(M)$ is the Doléans-Dade exponential of M.

Proof. The Azéma supermartingale S has a multiplicative decomposition of the form $S_t = L_t \exp(-\int_0^t \lambda_s \mathrm{d}s)$ (see [10, Proposition 4.1]), where L is an $\mathbb{F}$-martingale having the following dynamics

$$\mathrm{d}L_t = \exp\Big(\int_0^t \lambda_s \mathrm{d}s\Big)\,\mathrm{d}\widehat{L}_t,$$

with

$$\widehat{L}_t = -\int_0^t \alpha_t(u) - \alpha_u(u)\mathrm{d}u.$$

By Proposition 3.1, together with (3.3) and (3.4),

$$\mathrm{d}\widehat{L}_t = -\int_{\mathbb{R}^d}\int_0^t A(t,\theta,\xi)\mathrm{d}\theta Y^G(\mathrm{d}t,\mathrm{d}\xi) - \int_{\mathbb{R}^d}\int_0^t B(t,\theta,\xi)\mathrm{d}\theta Y^P(\mathrm{d}t,\mathrm{d}\xi),$$

where

$$A(t,\theta,\xi) = -\alpha_{t-}(\theta) I_\sigma(t,\theta,\xi) + S_{t-}(\theta)\sigma_t(\theta,\xi),$$
$$B(t,\theta,\xi) = \alpha_{t-}(\theta)(e^{-I_\gamma(t-,\theta,\xi)} - 1) + S_{t-}(\theta)\gamma_{t-}(\theta,\xi)e^{-I_\gamma(t-,\theta,\xi)}.$$

By integration by part, we obtain

$$-\int_0^t A(t,\theta,\xi)\mathrm{d}\theta = -S_{t-}(t) I_\sigma(t,t,\xi),$$

$$-\int_0^t B(t,\theta,\xi)\mathrm{d}\theta = S_{t-}(t)(e^{-I_\gamma(t-,t,\xi)} - 1).$$

Moreover, the Doob-Meyer decomposition of S is given by

$$S_t = 1 + \widehat{L}_t - \int_0^t \alpha_u(u)\mathrm{d}u,$$

which implies that

$$\frac{\mathrm{d}L_t}{L_{t-}} = \frac{\mathrm{d}\widehat{L}_t}{S_{t-}} = \frac{\mathrm{d}S_t}{S_{t-}} + \lambda_t \mathrm{d}t.$$

By (3.1), one has $S_{t-} = S_{t-}(t)$. Hence the martingale L is the Doléans-Dade exponential of M and the assertion follows. □

4 The pricing of defaultable bonds

In this section, we focus on the pricing of credit derivatives. In general, a credit sensitive contingent claim can be represented by a triplet (C, G, R) (see Bielecki and Rutkowski [3]) where the $\mathcal{F}_T$-measurable random variable C_T represents the maturity payment if no default occurs before the maturity T, and G is an $\mathbb{F}$-adapted continuous process of finite variation such that $G_0 = 0$ and represents the coupon payment. Differently from the case where the default payment occurs at τ immediately, we assume that in the economic default case, the default (or the recovery) payment takes place, after a period of legal proceedings, at the maturity date T later than the economic default date τ and admits the form $R_T(\tau)$ where $R_T(\cdot)$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

The global market information is described by the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geqslant 0}$, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$, which is made to satisfy the usual conditions. The value at time $t \leqslant T$ of the contingent claim (C, G, R) is given by the following $\mathcal{G}_t$-conditional expectation:

$$V_t = \mathbb{E}_{\mathbb{Q}}\left[\Big(C_T 1\!\!1_{\{\tau>T\}} + \int_t^T 1\!\!1_{\{\tau>u\}} e^{-\int_t^u r_s \mathrm{d}s} \mathrm{d}G_s + 1\!\!1_{\{\tau\leqslant T\}} R_T(\tau)\Big) e^{-\int_t^T r_s \mathrm{d}s} \,\Big|\, \mathcal{G}_t\right], \tag{4.1}$$

where $\mathbb{Q}$ denotes a risk-neutral pricing probability measure and the interest rate $r = (r_t;\ t \geqslant 0)$ is an $\mathbb{F}$-adapted process. The following result computes V_t using $\mathcal{F}_t$-conditional expectations. The first two terms result from [3] and the third one from [10]. With an abuse of notation, we denote in the following the $\mathbb{F}$-conditional density of τ under the risk-neutral probability $\mathbb{Q}$ by $(\alpha_t(\theta), t \geqslant 0)$. The general result on the density under a change of probability measure is given in [10, Theorem 6.1].

Proposition 4.1 *We suppose that the economic default time τ admits a conditional density w.r.t. the filtration $\mathbb{F}$, denoted by $\alpha_t(\cdot)$ under the risk-neutral probability measure $\mathbb{Q}$. Then the value of the credit sensitive contingent claim (C, G, R) is given by*

$$\begin{aligned} V_t = 1\!\!1_{\{\tau>t\}} \frac{B_t}{S_t} \mathbb{E}_{\mathbb{Q}}\left[\Big(C_T S_T + \int_t^T R_T(u)\alpha_T(u)\mathrm{d}u\Big) B_T^{-1} + \int_t^T S_u B_u^{-1} \mathrm{d}G_u \,\Big|\, \mathcal{F}_t\right] \\ + 1\!\!1_{\{\tau\leqslant t\}} B_t \mathbb{E}_{\mathbb{Q}}\left[R_T(\theta) \frac{\alpha_T(\theta)}{\alpha_t(\theta)} B_T^{-1} | \mathcal{F}_t\right]\Big|_{\theta=\tau} \end{aligned} \tag{4.2}$$

where $S_t = \mathbb{Q}(\tau > t|\mathcal{F}_t) = \int_t^\infty \alpha_t(\theta)\mathrm{d}\theta$ and $B_t = \exp(\int_0^t r_s \mathrm{d}s)$.

Proof. The $\mathcal{G}_t$-measurable random variable V_t can be decomposed in two parts $V_t = 1\!\!1_{\{\tau>t\}}\overline{V}_t + 1\!\!1_{\{\tau\leq t\}}\widetilde{V}_t(\tau)$ where $\overline{V}_t$ is $\mathcal{F}_t$-measurable and $\widetilde{V}_t(\cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. On the set $\{\tau > t\}$, we use Jeulin-Yor's lemma (see [3]) and the conditional density to obtain

$$\begin{aligned} \overline{V}_t &= \frac{1}{S_t}\mathbb{E}_{\mathbb{Q}}\left[\Big(C_T 1\!\!1_{\{\tau>T\}} + 1\!\!1_{\{t<\tau\leqslant T\}} R_T(\tau)\Big) e^{-\int_t^T r_s \mathrm{d}s} + \int_t^T 1\!\!1_{\{\tau>u\}} e^{-\int_t^u r_s \mathrm{d}s} \mathrm{d}G_s \,\Big|\, \mathcal{F}_t\right] \\ &= \frac{1}{S_t}\mathbb{E}_{\mathbb{Q}}\left[\Big(C_T S_T + \int_t^T R_T(\theta)\alpha_T(\theta)\mathrm{d}\theta\Big) e^{-\int_t^T r_s \mathrm{d}s} + \int_t^T S_u e^{-\int_t^u r_s \mathrm{d}s} \mathrm{d}G_s \,\Big|\, \mathcal{F}_t\right]. \end{aligned}$$

On the set $\{\tau \leqslant t\}$, by [10, Thm 3.1], we have

$$\widetilde{V}_t(\tau) = \mathbb{E}_{\mathbb{Q}}\left[R_T(\theta)\frac{\alpha_T(\theta)}{\alpha_t(\theta)} \exp\left(-\int_t^T r_s \mathrm{d}s\right) \,\Big|\, \mathcal{F}_t\right]\Big|_{\theta=\tau},$$

which complete the proof. □

Note that for the pricing of the two "before default" payment terms (C,G), the quantity

$$\frac{S_u}{S_t} = \exp\left(-\int_t^u \lambda_s \mathrm{d}s\right)\frac{\mathcal{E}(M)_u}{\mathcal{E}(M)_t}, \quad u > t$$

and hence the intensity λ play an important role. However, for the default recovery payment R (which depends on τ), the "after-default" density $\alpha_t(\theta)$ where $t \geqslant \theta$ is needed. This point has been discussed in [10]. In the following of this paper, we adopt the density approach for both the before-default and after-default pricing.

We consider in particular a defaultable zero-coupon bond of maturity T with $C = 1$ and $G = 0$. Its price at $t \leqslant T$ is given by

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\left(1\!\!1_{\{\tau>T\}} + 1\!\!1_{\{\tau\leqslant T\}}R_T(\tau)\right)\exp\left(-\int_t^T r_s\mathrm{d}s\right)\,\Big|\,\mathcal{G}_t\right]. \tag{4.3}$$

The following result is a direct consequence of the previous proposition. We first introduce the following price kernels:

$$K_1(t,\theta) = \frac{1}{S_t}\mathbb{E}_{\mathbb{Q}}\left[\alpha_T(\theta)\exp\left(-\int_t^T r_s\mathrm{d}s\right)\,\Big|\,\mathcal{F}_t\right] \tag{4.4}$$

$$K_2(t,\theta) = \frac{1}{\alpha_t(\theta)}\mathbb{E}_{\mathbb{Q}}\left[R_T(\theta)\alpha_T(\theta)\exp\left(-\int_t^T r_s\mathrm{d}s\right)\,\Big|\,\mathcal{F}_t\right], \tag{4.5}$$

where $t \leqslant T$, and $\theta \geqslant 0$.

Corollary 4.2 *Using the conditional density of τ under $\mathbb{Q}$, the price* (4.3) *of the defaultable zero-coupon bond at time $t \leqslant T$ has the following representation:*

$$P(t,T) \;=\; 1\!\!1_{\{\tau>t\}}\left[\int_T^\infty K_1(t,\theta)\mathrm{d}\theta + \int_t^T K_2(t,\theta)\frac{\alpha_t(\theta)}{S_t}\mathrm{d}\theta\right] + 1\!\!1_{\{\tau\leqslant t\}}K_2(t,\tau). \tag{4.6}$$

We will identify the above price kernels in the next two sections with different settings.

5 The first pricing kernel

In this section, we study in detail the pricing kernels (4.4) and (4.5) when the random interest rate is described as an extended Vasicek model. We suppose in this section the after-default recovery payment is deterministic. The case where the after-default recovery payment is random will be considered in the next section.

Firstly we recall the forward intensity model (2.2) and assume that the $\mathbb{F}$-predictable random fields $(\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\gamma})$ are deterministic in (2.2). We then express the instantaneous interest rate process $r = (r_t, t \geq 0)$ as the following extended Vasicek model under the risk-neutral pricing measure $\mathbb{Q}$:

$$\mathrm{d}r_t = \kappa(\delta - r_t)\mathrm{d}t + \int_{\mathbb{R}^d}\rho_t(\xi)Y^G(\mathrm{d}t,\mathrm{d}\xi) + \int_{\mathbb{R}^d}\phi_t(\xi)Y^P(\mathrm{d}t,\mathrm{d}\xi), \tag{5.1}$$

where $\kappa > 0$, $\delta > 0$, and $\rho.(\cdot)$ and $\phi.(\cdot)$ are deterministic volatility functions, assumed to belong to Φ_c and Ψ_ν respectively. In the particular case where $d = 0$, $\phi_t(\xi) \equiv 0$ and the volatility function $\rho.(\cdot) \equiv \rho > 0$ is constant, the interest rate r satisfies the classical Brownian-driven Vasicek model:

$$\mathrm{d}r_t = \kappa(\delta - r_t)\mathrm{d}t + \rho\mathrm{d}W_t, \tag{5.2}$$

where W is a standard Brownian motion.

Similarly to the solution form of the Ornstein-Uhlenbeck stochastic differential equation, the extended Vasicek model (5.1) also admits an explicit expression as follows:

$$\begin{aligned} r_t = r_0 e^{-\kappa t} + \delta(1-e^{-\kappa t}) + \int_0^t \int_{\mathbb{R}^d} e^{-\kappa(t-u)} \rho_u(\xi) Y^G(\mathrm{d}u, \mathrm{d}\xi) \\ + \int_0^t \int_{\mathbb{R}^d} e^{-\kappa(t-u)} \phi_u(\xi) Y^P(\mathrm{d}u, \mathrm{d}\xi), \end{aligned} \tag{5.3}$$

where $r_0 > 0$ denotes the deterministic initial interest rate value.

Next we compute the first pricing kernel in (4.4). For $\theta \geqslant 0$, we introduce the following integro-differential operator $\boldsymbol{A}_\theta$ acting on functions with three variables t, x and y which are differential in t and second-order differentiable in (x, y) :

$$\begin{aligned} \boldsymbol{A}_\theta K(t,x,y) &= \kappa(\widehat{\delta}_t(\theta) - x)\frac{\partial K}{\partial x}(t,x,y) + a(t,\theta)\frac{\partial K}{\partial y}(t,x,y) + a_{11}(t)\frac{\partial^2 K}{\partial x^2}(t,x,y) \\ &+ a_{22}(t,\theta)\frac{\partial^2 K}{\partial y^2}(t,x,y) + a_{12}(t,\theta)\frac{\partial^2 K}{\partial x \partial y}(t,x,y) \\ &+ \int_{\mathbb{R}^d} \Big[K(t, x+\phi_t(\xi), y+\gamma_t(\theta,\xi)) - K(t,x,y) \\ &\quad - \phi_t(\xi)\frac{\partial K}{\partial x}(t,x,y) - \gamma_t(\theta,\xi)\frac{\partial K}{\partial y}(t,x,y)\Big]\nu(\mathrm{d}\xi), \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} \widehat{\delta}_t(\theta) &= \delta + \kappa^{-1}\int_{\mathbb{R}^{2d}} \rho_t(\xi) I_\sigma(t,\theta,\xi) c(\zeta-\xi)\mathrm{d}\zeta\mathrm{d}\xi + \kappa^{-1}\int_{\mathbb{R}^d} \phi_t(\xi)(e^{-I_\gamma(t,\theta,\xi)} - 1)\nu(\mathrm{d}\xi), \\ a(t,\theta) &= \mu_t(\theta) - \int_{\mathbb{R}^d} \sigma_t(\theta,\zeta) I_\sigma(t,\theta,\xi) c(\zeta-\xi)\mathrm{d}\zeta\mathrm{d}\xi - \int_{\mathbb{R}^d} \gamma_{t^-}(\theta,\zeta)(1-e^{-I_\gamma(t,\theta,\xi)})\nu(\mathrm{d}\xi), \\ a_{11}(t) &= \frac{1}{2}\int_{\mathbb{R}^{2d}} \rho_t(\xi_1)\rho_t(\xi_2) c(\xi_1-\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2, \\ a_{22}(t,\theta) &= \frac{1}{2}\int_{\mathbb{R}^{2d}} \sigma_t(\theta,\xi_1)\sigma_t(\theta,\xi_2) c(\xi_1-\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2, \\ a_{12}(t,\theta) &= \int_{\mathbb{R}^{2d}} \sigma_t(\theta,\xi_1)\rho_t(\xi_2) c(\xi_1-\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2. \end{aligned}$$

Remark 5.1 Recall the martingale condition (**MC**) given by (2.5) which has been assumed throughout the paper. We have the coefficient $a(t,\theta) = 0$ for the partial derivative $\frac{\partial K}{\partial y}$ under (**MC**).

We introduce the following assumptions where we have fixed $\theta \geqslant 0$.

Assumption 5.2 (1) There exists $q \in (0,1)$ such that

(i) the functions $a_{11}(\cdot)$, $a_{22}(\cdot,\theta)$ and $a_{12}(\cdot,\theta)$ are $\frac{q}{2}$-Lipschitz on $[0,T]$,

(ii) there exists a Borel function J_q on $\mathbb{R}^d$ (which could depend on θ) such that

$$\max\left\{|\phi_t(\xi) - \phi_s(\xi)|, |\gamma_t(\theta,\xi) - \gamma_s(\theta,\xi)|\right\} \leq J_q(\xi)|t-s|^{q/2}$$

and

$$\int_{\mathbb{R}^d} \frac{J_q(\xi)^2}{1+J_q(\xi)}\nu(\mathrm{d}\xi) < \infty.$$

(2) $|\phi_t(\xi)|$ and $|\gamma_t(\theta,\xi)|$ are uniformly bounded from above by a Borel function $J_0(\xi)$ such that

$$\int_{\mathbb{R}^d} \frac{J_0(\xi)^2}{1+J_0(\xi)}\nu(\mathrm{d}\xi) < +\infty.$$

(3) There exists a constant $\beta(\theta) > 0$ such that, for any $(x,y) \in \mathbb{R}^2$ and any $t \in [0,T]$, one has

$$a_{11}(t)x^2 + 2a_{12}(t,\theta)xy + a_{22}(t,\theta)y^2 \geqslant \beta(\theta)(x^2+y^2).$$

Then we have the main result of this section.

Theorem 5.3 *Let $\theta \geqslant 0$ be fixed. Under Assumption* 5.2*, the Cauchy problem*

$$\frac{\partial K}{\partial t}(t,x,y) - xK(t,x,y) + \boldsymbol{A}_\theta K(t,x,y) = 0, \quad K(T,x,y) = y \tag{5.5}$$

has a unique solution $\breve{K}$, where the integro-differential operator $\boldsymbol{A}_\theta$ is defined in (5.4)*. Moreover, the following equality holds*

$$\mathbb{E}_{\mathbb{Q}}\left[\alpha_T(\theta)\exp\left(-\int_t^T r_s \mathrm{d}s\right)\,\middle|\,\mathcal{F}_t\right] = S_t(\theta)\breve{K}(t,r_t,\lambda_t(\theta)), \tag{5.6}$$

where $S_t(\theta) = \mathbb{Q}(\tau > \theta|\mathcal{F}_t)$ is CSP and $\lambda_t(\theta)$ is the corresponding forward intensity under the pricing measure $\mathbb{Q}$.

Proof. The first assertion comes from a general result of Garroni and Menaldi [14, Theorem II.3.1]. Let q be as in Assumption 5.2. We shall actually prove that the Cauchy problem (5.5) with a terminal condition[1] $K(T,\cdot,\cdot) = \psi \in C^q(\mathbb{R}^2)$ has a unique solution in the Hölder space $C^{1+\frac{q}{2},2+q}([0,T]\times\mathbb{R}^2)$ by constructing a contractible operator. The case of (5.5) with unbounded terminal function $\varphi(t,x,y) = y$ will be treated by taking limits. We recall that $C^{1+\frac{q}{2},2+q}([0,T]\times\mathbb{R}^2)$ denotes the vector subspace of[2] $C^{1,2}([0,T]\times\mathbb{R}^2)$ of functions f such that

$$\|f\|_{1+\frac{q}{2},2+q} := \|f\|_{1,2} + \sum_{1\le a+b+2c\le 2} \langle \partial_t^c\partial_x^a\partial y^b\rangle_{t,\frac{1}{2}(q+a+b+2c-1)} + \sum_{a+b+2c=2}\langle \partial_t^c\partial_x^a\partial y^b\rangle_{(x,y),q} < +\infty,$$

where for any function $g : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ and any $\beta \in (0,1)$,

$$\langle g\rangle_{t,\beta} := \sup_{z\in\mathbb{R}^2}\sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{|g(s,z)-g(t,z)|}{|s-t|^\beta},\ \langle g\rangle_{(x,y),\beta} := \sum_{t\in[0,T]}\sup_{\substack{z,w\in\mathbb{R}^2\\ z\neq w}}\frac{|g(t,z)-g(t,w)|}{|z-w|^\beta}.$$

[1]The expression $C^q(\mathbb{R}^2)$ denotes the vector space of all bounded functions f on $\mathbb{R}^2$ which are Hölder continuous of order q (namely, such that $\|f\|_{\sup} + \|f\|_q < +\infty$), where

$$\|f\|_q := \sup_{\substack{z,w\in\mathbb{R}^2\\ z\neq w}}\frac{|h(z)-h(w)|}{|z-w|^q}.$$

[2]The expression $C^{1,2}([0,T]\times\mathbb{R}^2)$ denotes the vector space of all continuous functions f on $[0,T]\times\mathbb{R}^2$ such that

$$\|f\|_{1,2} := \sum_{a+b+2c\le 2}\|\partial_t^c\partial_x^a\partial_y^b f\|_{\sup} < +\infty.$$

The vector space $C^{1,2}([0,T]\times\mathbb{R}^2)$ together with the norm $\|\cdot\|_{1,2}$ form a Banach space.

The vector space $C^{1+\frac{q}{2},2+q}([0,T]\times\mathbb{R}^2)$ together with the norm $\|\cdot\|_{1+\frac{q}{2},2+q}$ form a Banach space.

Let $\boldsymbol{I}_\theta$ be the integro-differential operator defined as

$$(\boldsymbol{I}_\theta K)(t,x,y)=\int_{\mathbb{R}^d}\left[K(t,x+\phi_t(\xi),y+\gamma_t(\theta,\xi))-K(t,x,y)-\phi_t(\xi)\frac{\partial K}{\partial x}-\gamma_t(\theta,\xi)\frac{\partial K}{\partial y}\right]\nu(\mathrm{d}\xi).$$

For $K\in C^{1+\frac{q}{2},2+q}([0,T]\times\mathbb{R}^2)$ and $\psi\in C^q(\mathbb{R}^2)$, let $\Theta_\psi(K)$ be the unique solution of the Cauchy problem

$$\frac{\partial F}{\partial t}-xF+\widehat{\boldsymbol{A}}_\theta(F)=\boldsymbol{I}_\theta(K),\quad F(T,x,y)=\psi(x,y), \tag{5.7}$$

where $\widehat{\boldsymbol{A}}_\theta$ denotes the differential operator

$$a_{11}(t)\frac{\partial^2}{\partial x^2}+a_{12}(t,\theta)\frac{\partial^2}{\partial x\partial y}+a_{22}(t,\theta)\frac{\partial^2}{\partial y^2}+\kappa(\widehat{\delta}_t(\theta)-x)\frac{\partial}{\partial x}+a(t,\theta)\frac{\partial}{\partial y}.$$

Denote by $C^{\frac{q}{2},q}([0,T]\times\mathbb{R}^2)$ the vector space of functions f on $[0,T]\times\mathbb{R}^2$ such that

$$\|f\|_{\frac{q}{2},q}:=\|f\|_{\sup}+\sup_{(x,y)\in\mathbb{R}^2}\|f(\cdot,x,y)\|_{\frac{q}{2}}+\sup_{t\in[0,T]}\|f(t,\cdot,\cdot)\|_q<+\infty$$

which is a Banach space with respect to the norm $\|\cdot\|_{\frac{q}{2},q}$. Since $K\in C^{1+\frac{q}{2},2+q}([0,T]\times\mathbb{R}^2)$, by the Assumption 5.2 (1.ii) and (2), we obtain that $\boldsymbol{I}_\theta(K)\in C^{\frac{q}{2},q}([0,T]\times\mathbb{R}^2)$ (see [14, Lemma II.1.5]). Therefore the existence and uniqueness of the solution $\Theta_\psi(K)\in C^{1+\frac{q}{2},2+q}([0,T]\times\mathbb{R}^2)$ to (5.4) comes from the classical theory of parabolic partial differential equations (e.g. [13]). Moreover, the solution verifies the following Hölder estimate ([14, Theorem I.2.1])

$$\|\Theta_\psi(K_1)-\Theta_\psi(K_2)\|_{1+\frac{q}{2},2+q}\leqslant C_1\|\boldsymbol{I}_\theta(K_1-K_2)\|_{\frac{q}{2},q} \tag{5.8}$$

which holds for all $K_1,K_2\in C^{1+\frac{q}{2},2+q}([0,T]\times\mathbb{R}^2)$ such that $K_1(T,\cdot,\cdot)=K_2(T,\cdot,\cdot)=\psi$, where C_1 is a constant independent of ψ.

For arbitrary $\varepsilon>0$, the following estimate holds for any $K\in C^{1+\frac{q}{2},2+q}([0,T]\times\mathbb{R}^2)$ (see [14, Lemma II.1.5])

$$\|\boldsymbol{I}_\theta(K)\|_{\frac{q}{2},q}\leqslant\varepsilon\|\nabla^2_{(x,y)}K\|_{\frac{q}{2},q}+C(\varepsilon)\Big(\|K\|_{\frac{q}{2},q}+\|\nabla_{(x,y)}(K)\|_{\frac{q}{2},q}\Big), \tag{5.9}$$

where the constant $C(\varepsilon)$ only depends on ε. Denote by C^q_ψ the subset of functions in $C^{1+\frac{q}{2},2+q}([0,T]\times\mathbb{R}^2)$ whose restriction on $\{T\}\times\mathbb{R}^2$ coincides with ψ. By choosing $\varepsilon>0$ small enough, we obtain from (5.8) and (5.9) that Θ_ψ is a contracting operator on the complete metric space C^q_ψ, provided that T is sufficiently small. Hence for sufficiently small T, the operator Θ_ψ has a unique fixed point and therefore the Cauchy problem

$$\frac{\partial K}{\partial t}(t,x,y)-xK(t,x,y)+\boldsymbol{A}_\theta K(t,x,y)=0,\quad K(T,x,y)=\psi(x,y)$$

has a unique solution. For general T, it suffices to divide $[0,T]$ into a finite union of small intervals and resolve the Cauchy problem progressively.

For the terminal function $\varphi(x,y)=y$, we can take, for each integer $n\geq 1$, a function $\psi_n\in C_0^\infty(\mathbb{R}^2)$ which coincides with φ on the ball B_n of radius n centered at $(0,0)$. For any $n\geqslant 1$, let K_n be the unique solution of the equation (5.5) with terminal condition $K_n(T,x,y)=\psi_n(x,y)$. By a maximum principle for the equation (5.5) (see [14, Theorem II.2.15]), for $n\geqslant m$, K_n coincides with K_m on the ball B_m. By taking $\breve{K}=K_n$ on $[0,T]\times B_n$ we obtain a global solution to the Cauchy problem (5.5). The uniqueness of $\breve{K}$ also results from the maximum principle.

We now prove the second assertion that $\breve{K}$ satisfies the equality (5.6). To this end, we compute the denominator of the pricing kernel (4.4) by introducing a change of probability measure:

$$\frac{\mathrm{d}\mathbb{Q}^\theta}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}_t} = \frac{S_t(\theta)}{S_0(\theta)}. \tag{5.10}$$

By Bayes' formula and (1.3), we have

$$\mathbb{E}_{\mathbb{Q}}\left[\alpha_T(\theta)\exp\left(-\int_t^T r_s\mathrm{d}s\right)\Big|\mathcal{F}_t\right] = S_t(\theta)\mathbb{E}_{\mathbb{Q}^\theta}\left[\lambda_T(\theta)\exp\left(-\int_t^T r_s\mathrm{d}s\right)\Big|\mathcal{F}_t\right].$$

Note that, by Girsanov's theorem (see [4, Theorem 3.3]), under the probability measure $\mathbb{Q}^\theta$,

$$\widehat{Y}^G(\mathrm{d}t,\mathrm{d}\xi) := Y^G(\mathrm{d}t,\mathrm{d}\xi) + \left(\int_{\mathbb{R}^d} I_\sigma(t,\theta,\xi)c(\zeta-\xi)\mathrm{d}\zeta\right)\mathrm{d}\xi\mathrm{d}t$$

defines a Gaussian field with correlated kernel c on $\mathbb{R}^d$, and

$$\widehat{Y}^P(\mathrm{d}t,\mathrm{d}\xi) := Y^P(\mathrm{d}t,\mathrm{d}\xi) + (1-e^{-I_\gamma(t,\theta,\xi)})\nu(\mathrm{d}\xi)\mathrm{d}t$$

defines a compensated Poisson random measure with predictable compensator $e^{-I_\gamma(t,\theta,\xi)}\nu(\mathrm{d}\xi)\mathrm{d}t$. Then the dynamics (5.1) of the interest rate r can be rewritten as

$$\mathrm{d}r_t = \kappa(\widehat{\delta}_t(\theta)-r_t)\mathrm{d}t + \int_{\mathbb{R}^d}\rho_t(\xi)\widehat{Y}^G(\mathrm{d}t,\mathrm{d}\xi) + \int_{\mathbb{R}^d}\phi_t(\xi)\widehat{Y}^P(\mathrm{d}t,\mathrm{d}\xi),$$

where

$$\widehat{\delta}_t(\theta) = \delta + \kappa^{-1}\int_{\mathbb{R}^{2d}}\rho_t(\xi)I_\sigma(t,\theta,\xi)c(\zeta-\xi)\mathrm{d}\zeta\mathrm{d}\xi + \kappa^{-1}\int_{\mathbb{R}^d}\phi_t(\xi)(e^{-I_\gamma(t,\theta,\xi)}-1)\nu(\mathrm{d}\xi),$$

and the dynamics (2.2) of the forward intensity rate can be rewritten as

$$\mathrm{d}\lambda_t(\theta) = \widehat{\mu}_t(\theta)\mathrm{d}t + \int_{\mathbb{R}^d}\sigma_t(\theta,\xi)\widehat{Y}^G(\mathrm{d}t,\mathrm{d}\xi) + \int_{\mathbb{R}^d}\gamma_{t-}(\theta,\xi)\widehat{Y}^P(\mathrm{d}t,\mathrm{d}\xi)$$

where

$$\widehat{\mu}_t(\theta) = \mu_t(\theta) - \int_{\mathbb{R}^d}\sigma_t(\theta,\zeta)I_\sigma(t,\theta,\xi)c(\zeta-\xi)\mathrm{d}\zeta\mathrm{d}\xi - \int_{\mathbb{R}^d}\gamma_{t-}(\theta,\xi)(1-e^{-I_\gamma(t,\theta,\xi)})\nu(\mathrm{d}\xi).$$

Note that the forward intensity process $\lambda(\theta)$ is a $(\mathbb{Q}^\theta,\mathbb{F})$-martingale for each θ fixed. Assume that

$$\mathbb{E}_{\mathbb{Q}^\theta}\left[\lambda_T(\theta)\exp\left(-\int_t^T r_s\mathrm{d}s\right)\Big|\mathcal{F}_t\right] = K(t,r_t,\lambda_t(\theta)),$$

where the function $K(t,x,y)$ is sufficiently regular. Then Itô's formula applied to the $(\mathbb{Q}^\theta,\mathbb{F})$-martingale

$$\exp\left(-\int_0^t r_s\mathrm{d}s\right)K(t,r_t,\lambda_t(\theta))$$

yields

$$\begin{aligned}
&-r_tK(t,r_t,\lambda_t(\theta))+\frac{\partial K}{\partial t}(t,r_t,\lambda_t(\theta))+\kappa(\widehat{\delta}_t(\theta)-r_t)\frac{\partial K}{\partial x}(t,r_t,\lambda_t(\theta))+\widehat{\mu}_t(\theta)\frac{\partial K}{\partial y}(t,r_t,\lambda_t(\theta))\\
&+a_{11}(t)\frac{\partial^2 K}{\partial x^2}(t,r_t,\lambda_t(\theta))+a_{22}(t,\theta)\frac{\partial^2 K}{\partial y^2}(t,r_t,\lambda_t(\theta))+a_{12}(t,\theta)\frac{\partial^2 K}{\partial x\partial y}(t,r_t,\lambda_t(\theta))\\
&+\int_{\mathbb{R}^d}\Big[K(t,r_t+\phi_t(\xi),\lambda_t(\theta)+\gamma_t(\theta,\xi))-K(t,r_t,\lambda_t(\theta))\\
&\qquad-\phi_t(\xi)\frac{\partial K}{\partial x}(t,r_t,\lambda_t(\theta))-\gamma_t(\theta,\xi)\frac{\partial K}{\partial y}(t,r_t,\lambda_t(\theta))\Big]\nu(\mathrm{d}\xi)=0.
\end{aligned}$$

Conversely, if $\check{K}$ is the solution to

$$\frac{\partial K}{\partial t}-xK+\boldsymbol{A}_\theta K=0,\quad K(T,x,y)=y,$$

then one has

$$\mathbb{E}_{\mathbb{Q}}\left[\alpha_T(\theta)\exp\left(-\int_t^T r_s\mathrm{d}s\right)\,\bigg|\,\mathcal{F}_t\right]=S_t(\theta)\check{K}(t,r_t,\lambda_t(\theta)).$$

Thus we complete the proof of the theorem. □

Accordingly the pricing kernels (4.4) and (4.5) are given by

$$K_1(t,\theta)\;=\;\frac{S_t(\theta)}{S_t}\check{K}(t,r_t,\lambda_t(\theta)) \tag{5.11}$$

$$K_2(t,\theta)\;=\;\frac{S_t(\theta)}{\alpha_t(\theta)}R_T(\theta)\check{K}(t,r_t,\lambda_t(\theta)) \tag{5.12}$$

where $t\leqslant T$ and $\theta\geqslant 0$. By Corollary 4.2, we obtain immediately the pricing formula for the defaultable zero-coupon bond.

Remark 5.4 Concerning the pricing kernel at the left side of the equality (5.6), one possible alternative way is to solve it directly by using the dynamics of the density $\alpha_t(\theta)$. However, in view of (3.5), the corresponding solution $K(t,r_t,S_t(\theta),\lambda_t(\theta))$ will include three variables apart from time variable. The main advantage of the change of probability method (5.10) is that we obtain the solution function in the form $K(t,r_t,S_t(\theta),\lambda_t(\theta))=S_t(\theta)\check{K}(t,r_t,\lambda_t(\theta))$. This indeed decreases the dimension of variables for our pricing kernel function and is important in the numerical computation.

Remark 5.5 If the interest rate r is independent of the forward intensity, hence independent of the density, then the computation of the pricing kernels is easier. Denote by $B(t,T)$ the price of the standard zero-coupon bond, i.e. $B(t,T)=\mathbb{E}_{\mathbb{Q}}[\exp(-\int_t^T r_s\mathrm{d}s)|\mathcal{F}_t]$. Recall that we have assumed the recovery rate deterministic in this section. Then

$$K_1(t,\theta)=\frac{\alpha_t(\theta)B(t,T)}{S_t},\quad K_2(t,\theta)=R_T(\theta)B(t,T)$$

which implies that the time-t value (4.6) of defaultable zero-coupon bond has the following representation:

$$\frac{P(t,T)}{B(t,T)}=1\!\!1_{\{\tau>t\}}\left(1-\frac{\int_t^T(1-R_T(\theta))\alpha_t(\theta)\mathrm{d}\theta}{S_t}\right)+1\!\!1_{\{\tau\leqslant t\}}R_T(\tau). \tag{5.13}$$

This quantity serves to measure the default risk including both the default probability and the loss given default. We also notice in (5.13) that the recovery corresponds to a “recovery of face value” since it can be written as the quotient between the defaultable bond and an equivalent default-free bond.

Remark 5.6 The zero coupon price $B(t,T) := \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s \mathrm{d}s\Big)\Big|\mathcal{F}_t\Big]$ can be given in the form $K(t,r_t)$ where $K(\cdot,\cdot)$ is the unique solution to the following integro-differential equation:

$$-xK+\frac{\partial K}{\partial t}+\kappa(\delta-x)\frac{\partial K}{\partial x}+a_{11}(t)\frac{\partial^2 K}{\partial x^2}+\int_{\mathbb{R}^d}\Big[K(t,x+\phi_t(\xi))-K(t,x)-\phi_t(\xi)\frac{\partial K}{\partial x}(t,x)\Big]\nu(\mathrm{d}\xi)=0$$

with the terminal condition $K(T,x)=1$. If there is no jumps in r_t, i.e., $\phi_t(\xi)\equiv 0$, then the above equation becomes

$$-xK+\frac{\partial K}{\partial t}+\kappa(\delta-x)\frac{\partial K}{\partial x}+a_{11}(t)\frac{\partial^2 K}{\partial x^2}=0.$$

Its unique solution is

$$\widehat{K}(t,x)=\exp\bigg(\frac{1-e^{-\kappa(T-t)}}{\kappa}(\delta-x)-\delta(T-t)+\int_t^T a_{11}(u)\Big(\frac{1-e^{-\kappa(T-u)}}{\kappa^2}\Big)^2\mathrm{d}u\bigg),$$

where $a_{11}(t)$ is given in (5.4). Thus we obtain the following equality $B(t,T)=\widehat{K}(t,r_t)$, which is similar to the classical case.

6 Random recovery rate and the second pricing kernel

In this section, we consider the general case for the pricing kernel (4.5), where the after-default recovery payment is random as an extension to the previous section.

Bakshi et al. [2] assumed that the recovery rate is related to the underlying intensity as the following form: $R_t = w_0 + w_1 e^{-\lambda_t}$, $w_0, w_1 \geqslant 0$, $w_0+w_1\leqslant 1$ and λ is the intensity process of default. In a similar manner, we assume that $R_T(\theta)$ is of the form

$$R_T(\theta)=w_0+w_1 e^{-f(\lambda_T(\theta))},\quad \theta\geqslant 0 \tag{6.1}$$

where $\lambda_T(\theta)$ is the forward intensity implied by (1.2) under the pricing measure $\mathbb{Q}$, w_0, w_1 satisfy the same condition as above and f is a non-negative function which is locally Hölder continuous of positive order.

Proposition 6.1 *Let $\theta\geqslant 0$ be fixed. Under the Assumption* 5.2, *the pricing kernel* (4.5) *is given by*

$$K_2(t,\theta)=\frac{w_0}{\lambda_t(\theta)}\breve{K}(t,r_t,\lambda_t(\theta))+\frac{w_1}{\lambda_t(\theta)}\widetilde{K}(t,r_t,\lambda_t(\theta)), \tag{6.2}$$

where $\breve{K}$ and $\widetilde{K}$ are respectively solutions to the partial integro-differential equation:

$$\frac{\partial K}{\partial t}(t,x,y)-xK(t,x,y)+\boldsymbol{A}_\theta K(t,x,y)=0 \tag{6.3}$$

under the terminal conditions $\breve{K}(T,x,y)=y$ and $\widetilde{K}(T,x,y)=ye^{-f(y)}$.

Proof. Similarly to Theorem 5.5, the equation (6.3) with the terminal condition $K(T,x,y)=ye^{-f(y)}$ admits a unique solution $\widetilde{K}$. Moreover, by a change of probability measure we obtain

$$\mathbb{E}_{\mathbb{Q}}\bigg[\alpha_T(\theta)e^{-f(\lambda_T(\theta))}\exp\bigg(-\int_t^T r_s\mathrm{d}s\bigg)\bigg|\mathcal{F}_t\bigg]=S_t(\theta)\widetilde{K}(t,r_t,\lambda_t(\theta)).$$

Hence the formula (6.2) follows from the following relation (see (4.4), (4.5) and (6.1)) :

$$K_2(t,\theta)=\frac{w_0 S_t}{\alpha_t(\theta)}K_1(t,\theta)+\frac{w_1}{\alpha_t(\theta)}\mathbb{E}\bigg[\alpha_T(\theta)e^{-f(\lambda_T(\theta))}\exp\bigg(-\int_t^T r_s\mathrm{d}s\bigg)\bigg|\mathcal{F}_t\bigg],$$

where $K_1(t,\theta)$ is the first price kernel (4.4). □

Corollary 6.2 *Under the Assumption* 5.2*, the price of the defaultable zero-coupon bond is given by*

$$P(t,T) = \mathbb{1}_{\{\tau>t\}}\Big[\int_T^\infty \frac{S_t(\theta)}{S_t}\breve{K}(t,r_t,\lambda_t(\theta))\mathrm{d}\theta + \int_t^T \frac{S_t(\theta)}{S_t}\Big(w_0\breve{K}(t,r_t,\lambda_t(\theta)) + w_1\widetilde{K}(t,r_t,\lambda_t(\theta))\Big)\mathrm{d}\theta\Big]$$
$$+ \mathbb{1}_{\{\tau\leqslant t\}}\frac{1}{\lambda_t(\tau)}\Big[w_0\breve{K}(t,r_t,\lambda_t(\tau)) + w_1\widetilde{K}(t,r_t,\lambda_t(\tau))\Big],$$

where $\breve{K}$ *and* $\widetilde{K}$ *are given in Proposition* 6.1 *respectively.*

7 Numerical illustrations

In this section, we illustrate our previous results by numerical examples. We are particularly interested in the contagion phenomenon. More precisely, we shall analyze in detail the the jump part in the default density dynamics and its impact on the defaultable bond pricing.

In the numerical example, we consider the dynamics of the default density described by (3.5) and we let the martingale $m(\theta)$ be given by

$$\mathrm{d}m_t(\theta) = -\sigma_t(\theta)\mathrm{d}W_t + \int_{\mathbb{R}_+}\gamma_{t-}(\theta)\xi e^{-\xi\int_0^\theta \gamma_{t-}(v)\mathrm{d}v}Y^P(\mathrm{d}t,\mathrm{d}\xi),\quad m_0(\theta)=0, \tag{7.1}$$

with $W = (W_t;\ t \geqslant 0)$ being a standard Brownian motion independent of the Poisson measure Y^P. Compared with (3.4), the corrected kernel c of the Gaussian field Y^G is the Dirac measure and $d = 1$, the volatility coefficient $\sigma_t(\theta,\xi) = \sigma_t(\theta)$ does not depend on ξ and the jump amplitude coefficient is given by $\gamma_t(\theta,\xi) = \gamma_t(\theta)\xi\mathbb{1}_{\{\xi>0\}}$ where $\gamma_t(\theta) > 0$. Recall in addition that $M_t(\theta) = \int_0^\theta m_t(u)\mathrm{d}u$ and

$$\mathrm{d}\alpha_t(\theta) = \alpha_{t-}(\theta)\mathrm{d}M_t(\theta) - S_{t-}(\theta)\mathrm{d}m_t(\theta).$$

To illustrate the impact of the jump part on the defaultable bond price $P(t,T)$ given by (4.6), we first consider the case when the martingale $m(\theta)$ has no jumps, i.e., $\gamma = 0$. We then include the jump part in the density dynamics. We use the initial default density given by $\alpha_0(\theta) = \lambda e^{-\lambda\theta}$ with λ being a positive constant.

In the coming tests, we suppose that $\sigma_t(\theta)$ and $\gamma_t(\theta)$ are deterministic and we use the following forms of the coefficients and the characteristic measure in (7.1),

$$\begin{cases}\sigma_t(\theta) = \sigma(\theta-t)^+, & \sigma>0,\\ \gamma_t(\theta) = b(\theta-t)^+, & b>0,\\ \nu(\mathrm{d}\xi) = \frac{\zeta}{\varpi}e^{-\xi/\varpi}\mathbb{1}_{\{\xi>0\}}\mathrm{d}\xi, & \zeta>0,\varpi>0.\end{cases}$$

We assume that both the recovery rate $R \in [0,1]$ and the interest rate r are constants and define $B(t,T) = e^{-r(T-t)}$ for $0 \leqslant t \leqslant T$. By Remark 5.5, the defaultable bond price $P(t,T)$ given by (4.6) admits an explicit form. Since the quotient $P(t,T)/B(t,T)$ equals the constant R on the set $\{\tau \leqslant t\}$ in this case, we only study the pre-default part on $\{\tau > t\}$ in (5.13), which is denoted by $P(t,T)$ henceforth and is given by

$$P(t,T) \;=\; B(t,T)\left(1-(1-R)\frac{\int_t^T\alpha_t(\theta)\mathrm{d}\theta}{\int_t^\infty\alpha_t(\theta)\mathrm{d}\theta}\right). \tag{7.2}$$

The main task is then to approximate the integral $\int_t^\infty \alpha_t(\theta)\mathrm{d}\theta$ by a finite sum $\sum_{i=t/\Delta+1}^{N/\Delta}\Delta * \alpha_t(i*\Delta)$. Here we choose $\Delta = 1/100$ and $N = 10/\lambda$. We perform 10^4 experiments to compute the $\mathcal{F}_t$-measurable random variable $P(t,T)$. In each experiment, we first generate the underlying Brownian motion and the central compound Poisson process. Then for each $\theta \in \{i\Delta; i = 1, 2, \ldots, N/\Delta\}$, we compute $\alpha_{t_i}(\theta)$ on $\{t_i = i\Delta_t; i = 1, 2, \ldots, t/\Delta_t\}$ with $\Delta_t = 1/100$.

The preferred parameter values are as follows:

$$t = 0.5,\ T = 1,\ r = 0.05,\ R = 0.4,\ b = 1,\ \zeta = 10,\ \lambda = 0.1.$$

Figure 1 plots the kernel estimations of the densities of $P(t,T)$ given by

$$f_P(x) := \frac{1}{k}\sum_{i=1}^{k} f_h(x - P_i(t,T)), \tag{7.3}$$

where $P_i(t,T)$ is the price obtained in the i-th experiment, $f_h(x) = \frac{1}{\sqrt{2\pi h}}\exp\left(-\frac{x^2}{2h^2}\right)$, and $h = 1.06 s_k k^{-1/5}$ is the bandwidth, with s_k being the sample standard deviation. From Figure 1, we find that the existence of the jump risk will increases the decentrality of the price. The right tail of the price distribution becomes fatter and fatter as the mean jump size ϖ increasing.

Figure 1: The (normal) kernel estimations of the price densities for $\varpi = 0, 0.0002, 0.0006, 0.001, 0.002$ and $\sigma = 0.001$.

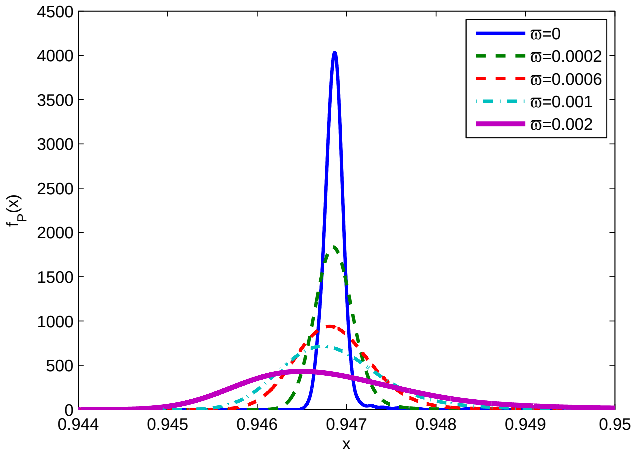
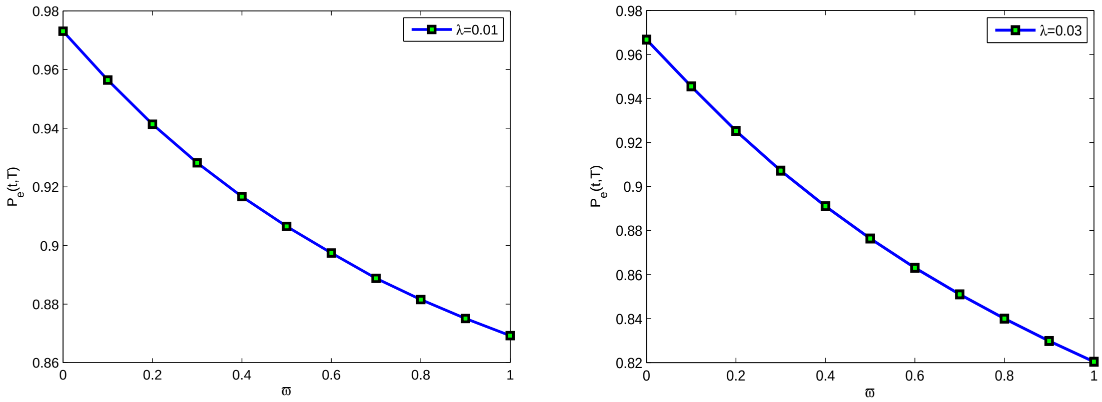


Figure 2: $P_e(0.5,1) := \mathbb{E}[P(0.5,1)]$ as a function of ϖ with $\lambda = 0.01, 0.03, 0.1, 0.3$ and $\sigma = 0.001$.



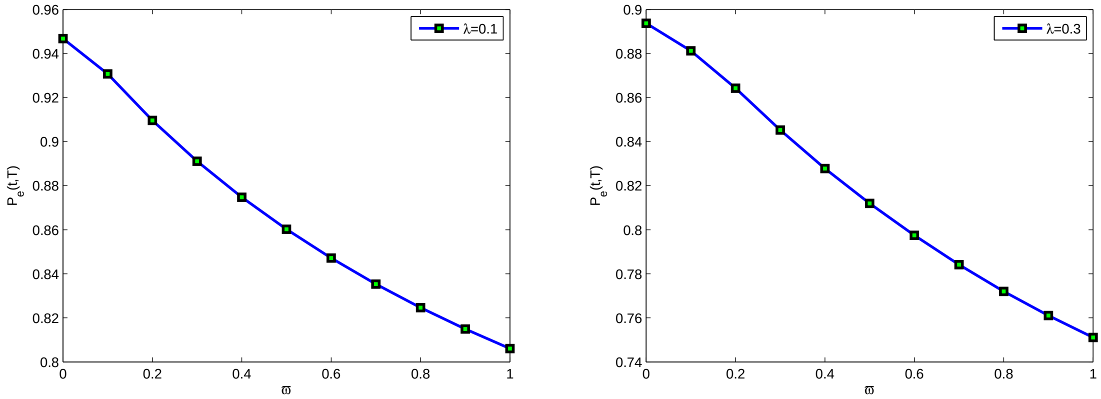
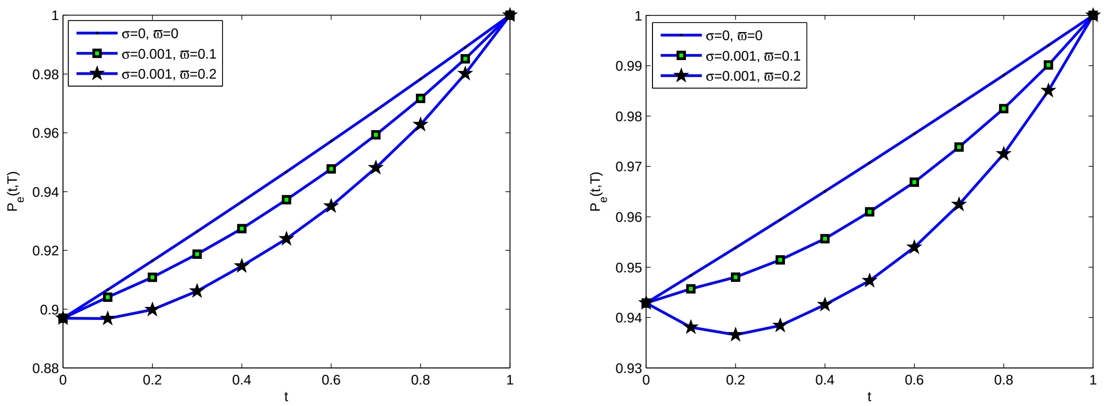


Figure 2 shows the mean of the price $P_e(0.5, 1) := \mathbb{E}[P(0.5, 1)]$ as a function of ϖ for different values of λ. We observe that the defaultable bond price is a decreasing function of the intensity λ, and also of the mean jump size ϖ. Hence, when there is larger default risk of the underlying asset itself (with larger λ), the corresponding bond price is smaller. Furthermore, when there is more significant counterparty risks, that is, when there is a larger contagious jump in the density (larger ϖ), then the bond price will also decrease. Both observations correspond to the reality on the market.

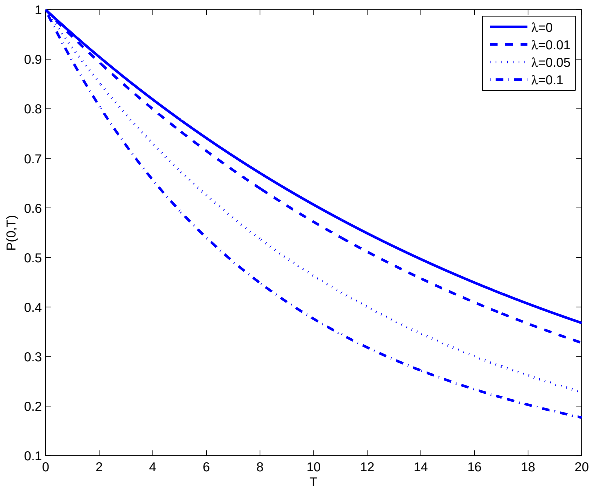
Figure 3 shows the mean of the price $P_e(t, 1) := \mathbb{E}[P(t, 1)]$ as a function of t. It is noted that the numerical illustration of the quantity $P(t, T)/B(t, T)$ discussed in Remark 5.5 is very similar to that of $P(t, T)$, since $B(t, T)$ here is a deterministic function $B(t, T) = e^{-r(T-t)}$ which is close to 1. We observe similar results as in the previous test: the counterparty jump risks in the density will decrease the bond prices.

Figure 3: $P_e(t, 1) := \mathbb{E}[P(t, 1)]$ as a function of t. Right hand side is the relative price $P_e(t, 1) := \mathbb{E}[P(t, 1)]/B(t, 1)$.



In the last graph, we show the quoted bond price at the initial time $t = 0$ as a function of the maturity time T for different values of intensities. Again we observe that the bond price is decreasing when there is larger default risks and for long term bonds.

Figure 4: $P(0, T)$ as a function of T.



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