Stable syndicates of factor owners and distribution of social output: A Shapley value approach. *

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Abstract

The purpose of this paper is to examine the incentive of a player to join a syndicate in an environment of team production and payoff distribution according to Shapley value. We consider an economy in which a single output is produced by an increasing returns to scale production function using two inputs: labor and capital. By assuming that syndicates of factor owners can form, we are interested in their stability, i.e., the willingness of the members of the syndicate to stay in the syndicate. Our analysis, based on the Shapley value, allows us to find a fair imputation of the gains of cooperation and the conditions under which syndicates are stable.

KEYWORDS: Shapley value, Increasing returns to scale, Syndicate of factor owners.

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1 Introduction

It seems intuitively obvious that the formation of binding agreements between the members of a group of economic agents improves their bargaining power. Trade unions, syndicates of property owners or of consumers provide natural examples of this phenomenon. The purpose of this paper is to give a rationale for this in a cooperative game setting. The problem is the following: What is the influence of binding agreements on the outcome of bargaining processes? Under which conditions are such binding agreements stable?

Following Gabszewicz and Drèze [1971], we will term a syndicate a group of identical agents who delegate to a single decision unit the task of representing their economic interests. This leads to assimilate a syndicate to a single agent. As remarked by Guesnerie [1977], the study of syndicates so defined raises the question of their stability, i.e., the willingness of the members of the syndicate to stay in the syndicate. This stability will depend upon the advantages the syndicate gives to its members and different comparisons seem relevant for evaluating them. The first one is the comparison of the situation of the syndicate members in the presence, or absence, of the syndicate; this is the notion of total stability defined by Gabszewicz and Drèze [1971]. The second one is the comparison of the situation of syndicated and nonsyndicated agents of the same type when there is a syndicate; this is the notion of marginal stability defined by Gabszewicz and Drèze [1971]. We introduce a third and new comparison which consists in assuming that a syndicate of one type has already formed and we are interested in the reaction of the agents of the other type. This raises the following question: Will the formation of a syndicate of one type induce the formation of a syndicate of another type?

The stability of syndicates has been studied in the context of a productive economy using the core solution concept (Hansen and Gabszewicz [1972]). Hansen and Gabszewicz consider the case of a productive economy in which the production possibilities are characterized by constant returns to scale. Then, one may wonder what happens when returns are increasing. In that context, we consider economies of scale since production is cheaper when scale is larger. This is for example the case when we consider the economy of software engineering where some programmers can concentrate more on particular kinds of programming, and get better at them.

Gabszewicz [1996] recognized that core solution concept is rather limited when returns are increasing. First of all, the equal-treatment property does not hold in the core, i.e., agents of the same type can receive a different amount.[1] Therefore, the equal treatment property, which is satisfied in a

[1]Following Gabszewicz [1996], let us assume an economy in which a single output is produced by an increasing returns to scale production function using only one input, given by $F(z) = z^2$. By assuming that the economy consists of five factor owners and that each

productive economy characterized by constant returns to scale, is no longer satisfied when returns are increasing. Second, generally the larger the degree of increasing returns, the larger the core will be and the more the core solution will be unsuitable. Indeed, it is intuitively clear that the more pronounced is the degree of increasing returns, the more difficult it will be for proper subgroups to do better by their own means and therefore the more we could depart from the proportional imputation while remaining in the core.

So the core solution concept is not a very appropriate one, in a productive economy characterized by increasing returns to scale, due to the large core and hence the need for a deterministic allocation. Another solution seems more appropriate, namely the Shapley value. Indeed, the Shapley value is a solution concept which attempts to describe a reasonable, or "fair", way to divide the gains from cooperation and a powerful tool for evaluating the power structure in a coalitional game. The criterion of fairness to which value theory adheres is egalitarianism: the aim is to distribute the gains from cooperation equally. Moreover, given a game, the Shapley value assigns to it a single outcome - in contrast, the core solution assigns a set of outcomes. This is the reason why we investigate in this paper the stability of syndicates when the allocation of the gains of cooperation is governed by the Shapley value.

The paper is organized as follows. In section 2, we state our assumptions and we determine the Shapley values according to various states of competition. Section 3 contains the results. We show that a monopoly syndicate can increase the bargaining power of its members in terms of the Shapley value. Sufficient conditions are given for a monopoly to be stable.

2 The model

2.1 Assumptions

We consider an economy in which a single output is produced by an increasing returns to scale production function using two inputs or factors: labor and capital. Following Hansen and Gabszewicz [1972], we assume that these inputs are initially distributed among a given set of factor owners and that any coalition of owners has initial access to its aggregate input endowment.

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agent owns initially a single unit of the input, the total output which can be produced in the economy is then 25. Let x be an imputation defined by

$$x = (x_1, x_2, x_3, x_4, x_5) = (1, 3, 5, 7, 9)$$

We can easily check that this imputation, which assigns a different amount to identical agents, belongs to the core!

The economy consists of a set I_1 of workers - each owning initially a single unit of labor - and a set I_2 of capitalists - each owning initially a single unit of capital. There are two types of agents, namely:

$$I_1 = \{11, \ldots, 1j, \ldots, 1n\}$$

and
$$I_2 = \{21, \ldots, 2j, \ldots, 2n\},$$

where I_1 denotes the set of labor owners with $|I_1| = n$; I_2, the set of capital owners with $|I_2| = n$ and N, the set of all factor owners with $N = I_1 \cup I_2$.[2]

We assume that the production function of the economy is a Cobb-Douglas function:

$$F(z_1, z_2) = z_1^{\alpha} \cdot z_2^{\beta},$$

where $\alpha > 0$, $\beta > 0$, $\alpha + \beta > 1$ and z_i, the units of factors.

The total output which can be produced in the economy is:

$$F(n, n) = n^{\alpha+\beta}.$$

We are then interested in the imputation of this amount among the factor owners. In other words, the problem is the following: How can we impute the total output among the workers and the capitalists?

By definition, an imputation is a $2n$-tuple of numbers

$$(x_{11}, \ldots, x_{1j}, \ldots, x_{1n}; x_{21}, \ldots, x_{2j}, \ldots, x_{2n})$$

satisfying:

$$\sum_{j=1}^{n} x_{1j} + \sum_{j=1}^{n} x_{2j} = n^{\alpha+\beta},$$

where x_{1j} denotes the amount received by a worker j, and x_{2j} the amount obtained by a capitalist j.

Interesting imputations can be obtained through a collective decision mechanism. Indeed, some coalitions can form in order to improve their bargaining power. Consider a given coalition S composed of some workers (i.e., $s_1 = |S \cap I_1|$) and some capitalists (i.e., $s_2 = |S \cap I_2|$). The coalition S can produce, on its own means, an amount of output $F(s_1, s_2)$. This defines the characteristic function, v, by:

$$v(S) = F(s_1, s_2) = s_1^{\alpha} \cdot s_2^{\beta}, \text{ for all } S \subset N.$$

To obtain these imputations, we will define the Shapley value associated with this characteristic function.

[2] $|I_1| = |I_2|$ allows us to avoid size effects.

Following Gabszewicz and Drèze [1972] and Gabszewicz [1996], we first assume that any group of factor owners is free to form a coalition, and also to disband it. A factor owner is then free to join or leave any coalition.

What happens if some factor owners decide to act together as one unit relative to the rest of the agents? Let us introduce the notion of syndicate. We define a syndicate as a group of identical agents, i.e., a group of agents of the same type, who agree that no proper subset of them can act independently from the group itself. In this case, it is no longer true that the formation of any coalition is possible: any proper subset of these groups is no longer a feasible coalition, only those coalitions which either contain all the members of the group or contain none of them are admissible. The set of permissible coalitions is then restricted. Moreover, as members of the syndicate are identical players, it is natural to suppose that all members of the group will accept an equal treatment, i.e., an equal sharing of the gains of cooperation. Consider then the syndicates that factor owners can form: Let us call J_i a syndicate of agents of type i, where $J_i \subset I_i$, $i = 1, 2$.

By assuming that syndicates of factor owners can form, we are interested in their stability, i.e., the willingness of the members of the syndicate to stay in the syndicate. So, the question is: Under what conditions will stable syndicates exist? Different definitions of stability might explain the formation of syndicates. The first one compares the amount a syndicate member can obtain with what he would have received if the syndicates did not exist; this is the total stability of Aumann [1973], called A-stability.

Definition 1 *A syndicate J_i is A-stable if $x_{ij} > x'_{ij}$, $\forall ij \in J_i$, where x_{ij} is the amount received by a syndicate member ij when $J_i \neq \emptyset$, and x'_{ij} the amount received by a syndicate member ij when $J_i = \emptyset$ (every player ij is independent).*

The second definition consists in comparing the situations of a syndicated and nonsyndicated agent of the same type, when there is a syndicate. This is the marginal stability of Gabzsewicz and Drèze [1971], called B-stability.

Definition 2 *A syndicate J_i is B-stable if $x_{ij} > x_{ik}$, $\forall ij \in J_i$ and $\forall ik \in I_i \backslash J_i$, where x_{ij} is the amount received by a syndicate member ij and x_{ik} the amount received by a nonsyndicate member ik of the same type when $J_i \neq \emptyset$.*

Finally, we introduce an alternative definition of stability which we call C-stability. This definition rests on the assumption that a syndicate of some type is formed and we are interested in the reaction of the agents of the other type: Do they form a syndicate or decide to remain separate? In order to answer this question, we compare the situation of a syndicate member when the two types of syndicates exist, and when its own syndicate does not exist.

Definition 3 *A syndicate* J_1 *(resp.* J_2*) is C-stable if* $x_{1j} > x''_{1j}$, $\forall 1j \in J_1$, $\left(\textit{resp. } x_{2j} > x''_{2j}, \forall 2j \in J_2\right)$, *where* x_{1j} *(resp.* x_{2j}*) is the amount received by a syndicate member* $1j$ *(resp.* $2j$*) when* $J_1,\ J_2 \neq \emptyset$ *and* x''_{1j} $\left(\textit{resp. } x''_{2j}\right)$ *the amount received by a syndicate member* $1j$ *(resp.* $2j$*) when* $J_1 = \emptyset$ *(every player* $1j$ *is independent),* $J_2 \neq \emptyset$ *(resp.* $J_1 \neq \emptyset, J_2 = \emptyset$*).*

2.2 Payoff Distribution by the Shapley value

In order to study the incentive of a player to join a syndicate, it is necessary to determine the imputation each factor owner can obtain according to various states of competition. The solution concept we are dealing with is the Shapley value defined by Shapley [1953].

2.2.1 Shapley value with no syndicate

In an economy with no syndicate, agents of the same type are not constrained to behave identically. Then there are $2n$ independent players in the game.

The value of the game, to a given player, can be described intuitively as his average marginal worth over all possible coalitions. Hence, for a situation of perfect competition, we get the following expression for the Shapley value of factor owner j of type i:

$$\varphi_i = \sum_{\substack{S \subset N \\ ij \notin S}} \frac{s!\,(2n-s-1)!}{(2n)!} \left[v\left(S \cup \{ij\}\right) - v\left(S\right)\right], \text{ for all } S \subset N,$$

$\forall j = 1, \ldots, n$ and $i = 1, 2$ and where s denotes the number of players in S, $s = |S|$.

Theorem 1 *In a productive economy with increasing returns to scale, the Shapley value, denoted by* $\varphi = (\varphi_1, \varphi_2)$*, and the asymptotic Shapley value, denoted by* $\varphi(n) \sim (\varphi_1(n), \varphi_2(n))$*, are given by:*

(i) symmetric agents: $\alpha = \beta \Rightarrow \varphi = (\frac{1}{2}n^{2\alpha-1},\ \frac{1}{2}n^{2\alpha-1})$, $\forall n$.

(ii) asymmetric agents: $\alpha \neq \beta \Rightarrow \varphi \sim (\frac{\alpha}{\alpha+\beta}\, n^{\alpha+\beta-1},\ \frac{\beta}{\alpha+\beta}\, n^{\alpha+\beta-1})$, *for* n *large enough.*

Proof. With increasing returns to scale, production functions are quasi-concave but not concave. Thus, the Shapley value does not converge to the classical "competitive equilibrium" solution when the number of agents is increased[3].

(i) With symmetric agents, the proof is immediate. Since the Shapley value gives the same value to all the identical players, the total amount of output, $F(n,n)$, is equally shared between the $2n$ players, i.e.:

$$\varphi_1 = \varphi_2 = \frac{F(n,n)}{2n} \Leftrightarrow \varphi_1 = \varphi_2 = \tfrac{1}{2} n^{2\alpha-1}, \forall n.$$

(ii) With asymmetric agents, the proof is less immediate. Suppose that there is a large finite number n of players, each of whom is individually insignificant. We know from Aumann and Shapley [1974] that there exists another way of defining the value: The infinite-person game is approximated by games with finitely many players (with the aid of sequences of increasingly fine partitions of the players space) and it is proved that the value for finite games (called the "asymptotic value") coincides with the value for non-atomic games.

As a first step in the proof, we define the value for non-atomic games. Let x_1 be the fraction of workers owning a single unit of labor and zero unit of capital, i.e., $(1,0)$; and let x_2 be the fraction of capitalists owning a single unit of capital and zero unit of labor, i.e., $(0,1)$. Thus, the function $F(x_1,x_2) = x_1^{\alpha} \cdot x_2^{\beta}$ determines the output which can be produced in this normalized game. Then, the Shapley value is given by the following diagonal formula:

$$\varphi = \int_0^1 (\nabla F)_{t(1,1)}\, dt,$$

where (∇F) denotes the gradient of the function $F(x_1,x_2)$.

We obtain

$$\varphi = (\alpha, \beta) \frac{1}{\alpha+\beta}$$

And we deduce the asymptotic value for our finite game as

$$\begin{cases} \varphi_1(n) \sim \frac{\alpha}{\alpha+\beta}\, n^{\alpha+\beta-1}, \text{for } n \text{ large enough.} \\ \varphi_2(n) \sim \frac{\beta}{\alpha+\beta}\, n^{\alpha+\beta-1}, \text{for } n \text{ large enough.} \end{cases}$$

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We can note that the game associated with the characteristic function $v(S) = s_1^{\alpha} \cdot s_2^{\beta}$, for all $S \subset N$, is convex if $\alpha \geq 1$ and $\beta \geq 1$. Therefore, the Shapley value, φ_i, belongs to the core if the marginal productivities with respect to all inputs are increasing. Then, in a productive economy with no syndicate and characterized by increasing returns to scale, the Shapley

[3] Shapley [1964] proved that in a k-fold market with transferable utility and concave utility functions, the value converges to the competitive solution.

value is a "stable" imputation since it cannot be disrupted by the actions of any coalition of agents.

2.2.2 Shapley value with one syndicate

Let us assume that k agents of type 1, $k \leq n$, decide to establish a syndicate, denoted by J_1, $J_1 \subset I_1$. As a consequence, a new game is played which has formally (since the syndicate acts as a single agent) $(1+2n-k)$ players.

The Shapley value is represented by three numbers $(\hat{x}_{1s}, \hat{x}_{1ns}, \hat{x}_2)$ where $\hat{x}_{1s}$ denotes the value of a syndicated worker; $\hat{x}_{1ns}$ the value of a nonsyndicated worker and $\hat{x}_2$, the value of a capitalist. The purpose of this section is to determine this Shapley value.

First, by definition, the value of a syndicated worker, $\hat{x}_{1s}$, is given by:

$$\hat{x}_{1s} = \frac{1}{k} \sum_{\substack{S \subset N \\ J_1 \notin S}} \frac{s!\,(2n-k-s)!}{(1+2n-k)!} \left[v\left(S \cup \{J_1\}\right) - v\left(S\right) \right]$$

We can note that the marginal contribution of syndicate J_1 in a given coalition S will depend on the number of nonsyndicated workers in S i.e. $s_1 = \left| S \cap I_1 \backslash J_1 \right|$, and on the number of capitalists in S i.e. $s_2 = |S \cap I_2|$. Then

$$\hat{x}_{1s} = \frac{1}{k} \sum_{s_1=0}^{n-k} \sum_{s_2=0}^{n} \frac{(s_1+s_2)!\,(2n-k-s_1-s_2)!}{(1+2n-k)!} \binom{n-k}{s_1} \binom{n}{s_2} \left[s_2^{\beta} \left((s_1+k)^{\alpha} - s_1^{\alpha} \right) \right]$$

Next, we note that the value of a nonsyndicated worker, $\hat{x}_{1ns}$, is given by

$$\hat{x}_{1ns} = \sum_{\substack{S \subset N \\ 1j \notin S}} \frac{s!\,(2n-k-s)!}{(1+2n-k)!} \left[v\left(S \cup \{1j\}\right) - v\left(S\right) \right]$$

We shall distinguish between two cases: (1) the syndicate J_1 belongs to the coalition S (first term of the following expression), (2) the syndicate J_1 does not belong to the coalition S (second term). Then

$$\begin{aligned} \hat{x}_{1ns} &= \sum_{s_1=0}^{n-k-1} \sum_{s_2=0}^{n} \tfrac{(s_1+s_2+1)!(2n-k-s_1-s_2-1)!}{(1+2n-k)!} \binom{n-k-1}{s_1} \binom{n}{s_2} \left[s_2^{\beta} \left((s_1+k+1)^{\alpha} - (s_1+k)^{\alpha} \right) \right] \\ &\quad + \sum_{s_1=0}^{n-k-1} \sum_{s_2=0}^{n} \tfrac{(s_1+s_2)!(2n-k-s_1-s_2)!}{(1+2n-k)!} \binom{n-k-1}{s_1} \binom{n}{s_2} \left[s_2^{\beta} \left((s_1+1)^{\alpha} - s_1^{\alpha} \right) \right] \end{aligned}$$

Finally, we deduce the value of a capitalist, $\hat{x}_2$ from the property of efficiency of the Shapley value:

$$\hat{x}_2 = \frac{1}{n} \left[n^{\alpha+\beta} - k\hat{x}_{1s} - (n-k)\,\hat{x}_{1ns} \right]$$

Similarly, the symmetric case is obtained if we assume the formation of a syndicate of capitalists, J_2, $J_2 \subset I_2$, composed of k members. The Shapley value is then described by three numbers $(\hat{y}_1, \hat{y}_{2s}, \hat{y}_{2ns})$ where $\hat{y}_1$ is the value of a worker; $\hat{y}_{2s}$, the value of a syndicated capitalist and $\hat{y}_{2ns}$, the value of a nonsyndicated capitalist.

2.2.3 Shapley value with two syndicates

We now assume that k agents of type 1, $k \leq n$, decide to establish a syndicate, denoted by J_1, $J_1 \subset I_1$, and that k agents of type 2, $k \leq n$, decide to establish a syndicate, denoted by J_2, $J_2 \subset I_2$. As a consequence, a new game is played which has $2(n-k+1)$ players.

The Shapley value is then represented by four numbers $(\hat{z}_{1s}, \hat{z}_{1ns}, \hat{z}_{2s}, \hat{z}_{2ns})$ where $\hat{z}_{1s}$ denotes the value of a syndicated worker; $\hat{z}_{1ns}$, the value of a nonsyndicated worker; $\hat{z}_{2s}$, the value of a syndicated capitalist and $\hat{z}_{2ns}$, the value of a nonsyndicated capitalist. So, we want to determine successively these four values.

First, the value of a syndicated worker, $\hat{z}_{1s}$, is given by

$$\hat{z}_{1s} = \frac{1}{k} \sum_{\substack{S \subset N \\ J_1 \notin S}} \frac{s!\,(2(n-k+1)-s-1)!}{(2(n-k+1))!} \left[v(S \cup \{J_1\}) - v(S) \right]$$

Let s_1 be the number of nonsyndicated workers in a given coalition S; and let s_2 be the number of nonsyndicated capitalists in the coalition S. We shall again consider two cases depending on whether the syndicate J_2 belongs or not to the coalition S. Then we have

$$\begin{aligned} \hat{z}_{1s} \quad = \quad & \frac{1}{k} \sum_{s_1=0}^{n-k} \sum_{s_2=0}^{n-k} \tfrac{(s_1+s_2+1)!(2(n-k)-s_1-s_2)!}{(2(n-k+1))!} \tbinom{n-k}{s_1} \tbinom{n-k}{s_2} \left[(s_2+k)^{\beta} \left((s_1+k)^{\alpha} - s_1^{\alpha} \right) \right] \\ & + \frac{1}{k} \sum_{s_1=0}^{n-k} \sum_{s_2=0}^{n-k} \tfrac{(s_1+s_2)!(2(n-k)-s_1-s_2+1)!}{(2(n-k+1))!} \tbinom{n-k}{s_1} \tbinom{n-k}{s_2} \left[s_2^{\beta} \left((s_1+k)^{\alpha} - s_1^{\alpha} \right) \right] \end{aligned}$$

We now define the value of a nonsyndicated worker as

$$\hat{z}_{1ns} = \sum_{\substack{S \subset N \\ 1j \notin S}} \frac{s!\,(2(n-k+1)-s-1)!}{(2(n-k+1))!} \left[v(S \cup \{1j\}) - v(S) \right]$$

We shall distinguish between four cases: (1) the two syndicates belong to the coalition S (first term of the following expression), (2) the syndicate J_1 belongs to the coalition S (second term), (3) the syndicate J_2 belongs to the coalition S (third term), (4) the two syndicates do not belong to the coalition S (fourth term). Then

$$\begin{aligned}\hat{z}_{1ns} &= \sum_{s_1=0}^{n-k-1}\sum_{s_2=0}^{n-k} \tfrac{(s_1+s_2+2)!(2(n-k)-s_1-s_2-1)!}{(2(n-k+1))!}\tbinom{n-k-1}{s_1}\tbinom{n-k}{s_2}\left[(s_2+k)^{\beta}\left((s_1+k+1)^{\alpha}-(s_1+k)^{\alpha}\right)\right]\\ &+\sum_{s_1=0}^{n-k-1}\sum_{s_2=0}^{n-k} \tfrac{(s_1+s_2+1)!(2(n-k)-s_1-s_2)!}{(2(n-k+1))!}\tbinom{n-k-1}{s_1}\tbinom{n-k}{s_2}\left[s_2^{\beta}\left((s_1+k+1)^{\alpha}-(s_1+k)^{\alpha}\right)\right]\\ &+\sum_{s_1=0}^{n-k-1}\sum_{s_2=0}^{n-k} \tfrac{(s_1+s_2+1)!(2(n-k)-s_1-s_2)!}{(2(n-k+1))!}\tbinom{n-k-1}{s_1}\tbinom{n-k}{s_2}\left[(s_2+k)^{\beta}\left((s_1+1)^{\alpha}-s_1^{\alpha}\right)\right]\\ &+\sum_{s_1=0}^{n-k-1}\sum_{s_2=0}^{n-k} \tfrac{(s_1+s_2)!(2(n-k)-s_1-s_2+1)!}{(2(n-k+1))!}\tbinom{n-k-1}{s_1}\tbinom{n-k}{s_2}\left[s_2^{\beta}\left((s_1+1)^{\alpha}-s_1^{\alpha}\right)\right]\end{aligned}$$

Similarly, the value of a syndicated capitalist, $\hat{z}_{2s}$, and the value of a nonsyndicated capitalist, $\hat{z}_{2ns}$, are determined.

3 Results

We first assume that all agents of type 1 form a syndicate whereas the agents of the other type remain unorganized.

Assumption 1 $J_1 = I_1$ *and* $J_2 = \emptyset$.

Proposition 1 *Let $\hat{x}_{1s}(n)$ be the asymptotic Shapley value of a syndicated worker under assumption 1 and let $\varphi_1(n)$ be the asymptotic Shapley value of a worker in an economy without syndicate, then:*

$$\hat{x}_{1s}(n) > \varphi_1\,(n) \textit{if and only if } \alpha < 1, \forall \beta > 0,$$

where $\hat{x}_{1s}(n) \sim \dfrac{1}{1+\beta}\, n^{\alpha+\beta-1}$ *and* $\varphi_1(n) \sim \dfrac{\alpha}{\alpha+\beta}\, n^{\alpha+\beta-1}$, *for n large enough.*

Proof. Let $\hat{x}_{1s}$ be the Shapley value of a syndicated worker. Since $k = n$ and $s_1 = 0$, we have:

$$\hat{x}_{1s} = \frac{1}{n+1}\sum_{s_2=0}^{n} n^{\alpha-1} s_2^{\beta}$$

Therefore, we obtain the asymptotic Shapley value:

$$\hat{x}_{1s}(n) \sim \frac{1}{1+\beta}\, n^{\alpha+\beta-1}, \text{ for } n \text{ large enough.}$$

The stability of syndicate J_1 depends on the comparison of the members of the syndicate with the members of the unorganized coalition, thus we compare $\hat{x}_{1s}(n)$ to $\varphi_1(n)$. Since $\varphi_1(n) \sim \dfrac{\alpha}{\alpha+\beta}\, n^{\alpha+\beta-1}$, we obtain the desired result.

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Proposition 1 implies that, under assumption 1, the imputation $\hat{x}_{1s}(n)$ is strictly preferred by the members of the syndicate to the imputation $\varphi_1(n)$. Hence, the syndicate J_1 is stable (A-stable) when the parameter α is less than one and the syndicate J_1 is not stable when α is greater than one. The critical value, $\alpha = 1$, corresponds to the case where the marginal productivity of labor equals the average productivity of labor, i.e., $Pm_{z_1} = PM_{z_1}$. So, if $\alpha < 1$, the marginal productivity of labor is decreasing and we have $Pm_{z_1} < PM_{z_1}$ which means that a worker's average contribution is always greater than a worker's marginal contribution.

Therefore, as the number of agents is increased, a syndicate composed of all the agents of one type (the agents of the other type remaining unorganized) is A-stable if the decreasing marginal productivity property of the factor owned by this syndicate is satisfied, whatever be the marginal productivity of the other factor. In other words, a monopoly syndicate is stable when its marginal productivity is less than its average productivity.

We now assume that both agents of type 1 and type 2 form a monopoly syndicate.

Assumption 2 $J_1 = I_1$ *and* $J_2 = I_2$*.*

Proposition 2 *Let* $\hat{z} = (\hat{z}_{1s}, \hat{z}_{2s})$ *be the Shapley value under assumption 2 and let* $\varphi(n) \sim (\varphi_1(n), \varphi_2(n))$ *be the asymptotic Shapley value in an economy without syndicate, then:*

$$\begin{cases} \hat{z}_{1s} < \varphi_1(n) \\ \hat{z}_{2s} > \varphi_2(n) \end{cases} , \textit{ if and only if } \alpha > \beta,$$

where $\hat{z}_{1s} = \hat{z}_{2s} = \frac{1}{2}n^{\alpha+\beta-1}$, $\forall n$,
and $\varphi_1(n) \sim \dfrac{\alpha}{\alpha+\beta}\, n^{\alpha+\beta-1}$, $\varphi_2(n) \sim \dfrac{\beta}{\alpha+\beta}\, n^{\alpha+\beta-1}$, *for* n *large enough.*

Proof. Let $\hat{z}_{1s}$ be the Shapley value of a syndicated worker. Since $k = n$ and $s_1 = s_2 = 0$, we have:

$$\hat{z}_{1s} = \tfrac{1}{2}n^{\alpha+\beta-1}, \quad \forall \alpha, \beta > 0$$

Similarly, let $\hat{z}_{2s}$ be the Shapley value of a syndicated capitalist. Since $k = n$ and $s_1 = s_2 = 0$, we have:

$$\hat{z}_{2s} = \tfrac{1}{2}n^{\alpha+\beta-1}, \quad \forall \alpha, \beta > 0$$

In order to analyze the stability of these two syndicates, we compare $\hat{z}_{1s}$ to $\varphi_1(n)$ and $\hat{z}_{2s}$ to $\varphi_2(n)$, and then we obtain the desired result.

■

Proposition 2 shows that, under assumption 2, only the monopoly syndicate which has the weaker marginal productivity will be A-stable. In other words, since α and β measure the impact of a variation in inputs on the quantity of output, the syndicate J_2 will be A-stable if the impact on output from the variation of capital inputs is weaker than that of a variation in labor input, i.e. $\alpha > \beta$.

We will now consider another definition of stability: C-stability. Without loss of generality, we can compare the situation of a syndicated worker when two monopoly syndicates are present and when there is only a syndicate of capitalists, i.e., $\hat{z}_{1s}$ to $\hat{y}_1$.

Assumption 3 *$J_1 = \emptyset$ and $I_2 = J_2$.*

Proposition 3 *Let $\hat{z}_{1s}$ be the Shapley value of a syndicated worker under assumption 2 and let $\hat{y}_1(n)$ be the asymptotic Shapley value of a worker under assumption 3, then:*

$$\hat{z}_{1s} > \hat{y}_1(n) \text{ if and only if } \alpha < 1,\ \forall \beta > 0,$$

where $\hat{z}_{1s} = \frac{1}{2} n^{\alpha+\beta-1}$, $\forall n$, and $\hat{y}_1(n) \sim \dfrac{\alpha}{1+\alpha} n^{\alpha+\beta-1}$, for n large enough.

Proof. In Proposition 2, we have shown that $\hat{z}_{1s} = \frac{1}{2} n^{\alpha+\beta-1}$ when $k = n$ and $s_1 = s_2 = 0$.

Let us consider the formation of a monopoly syndicate of capitalists. Some easy computations give the asymptotic Shapley value of a worker:

$$\hat{y}_1(n) \sim \frac{\alpha}{1+\alpha} n^{\alpha+\beta-1}, \text{ for } n \text{ large enough.}$$

We compare $\hat{z}_{1s}$ to $\hat{y}_1(n)$ and we obtain the desired result.

■

Proposition 3 implies that a syndicate of workers is C-stable when the parameter α is less than one. In other words, this means that the formation of a syndicate of capitalists leads to the formation of the syndicate of workers if the marginal productivity of labor is decreasing. Hence, the formation of a syndicate of one type can lead to the formation of a syndicate of the other type.

We can summarize our results in the following bi-matrix[4]:

[4] Given n, the players will share the average productivity $\frac{y}{n} = n^{\alpha+\beta-1}$ according to the shares that are varying with the type of the market ((I_1, I_2) = bilateral oligopoly, $(I_1, \emptyset)$ = monopoly syndicate, $(\emptyset, I_2)$ = monopsony, $(\emptyset, \emptyset)$ = competitive economy).

	$J_2 = I_2$	$J_2 = \emptyset$
$J_1 = I_1$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{1+\beta}, \frac{\beta}{1+\beta}$
$J_1 = \emptyset$	$\frac{\alpha}{1+\alpha}, \frac{1}{1+\alpha}$	$\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}$

In the light of empirical studies, we are able to let $\alpha < 1$, $\beta < 1$ and $\alpha > \beta$ (and, where applicable, $\alpha+\beta$ slightly greater than 1) (see, for example, Pendharkar and al. [2008]). Which implies for the workers:

$$\frac{1}{1+\beta} > \frac{\alpha}{\alpha+\beta} > \frac{1}{2} > \frac{\alpha}{1+\alpha}$$

and for the preferences we get:

$$(I_1, \emptyset) \succ_{I_1} (\emptyset, \emptyset) \succ_{I_1} (I_1, I_2) \succ_{I_1} (\emptyset, I_2)$$

$$(\emptyset, I_2) \succ_{I_2} (I_1, I_2) \succ_{I_2} (\emptyset, \emptyset) \succ_{I_2} (I_1, \emptyset)$$

4 Conclusion

This paper analyses the stability of syndicates of factor owners in a productive economy characterized by increasing returns to scale. Our analysis, based on the Shapley value, allows us to find a fair allocation of the gains of cooperation and the conditions under which syndicates are stable.

We prove that the formation of a monopoly syndicate can increase the bargaining power of its members in terms of the Shapley value. We determine conditions for the monopoly to be stable: it turns out that the stability of a syndicate critically depends upon the marginal productivity of the factor that its members own.

Under the particular conditions where $\alpha, \beta < 1$, $\alpha > \beta$, in the presence of increasing returns to scale, we show that workers have never interest that capitalists form a syndicate; whereas capitalists still have interest to form it. Then, it remains to workers to form a syndicate so as to maximize their payoffs.

Under these circumstances it seems that the answer to the question to know whether the presence of a type of syndicate leads to the formation of a syndicate of the other type, is not trivial. Thus, it is true that workers have always interest in forming a syndicate if one exists in capitalist and vice versa. Alongside the capitalists still have an interest in forming a syndicate, in which case, the presence or absence of a labor syndicate cannot be considered as incentive. Finally, starting from a situation of bilateral monopoly, the workers are also willing not to form a syndicate if the capitalists do the same.

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