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VAR for VaR: Measuring Systemic Risk Using Multivariate Regression Quantiles*

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Abstract

This paper proposes methods for estimation and inference in multivariate, multi-quantile models. The theory can simultaneously accommodate models with multiple random variables, multiple confidence levels, and multiple lags of the associated quantiles. The proposed framework can be conveniently thought of as a vector autoregressive (VAR) extension to quantile models. We estimate a simple version of the model using market returns data to analyse spillovers in the values at risk (VaR) of different financial institutions. We construct impulse-response functions for the quantile processes of a sample of 230 financial institutions around the world and study how financial institution-specific and system-wide shocks are absorbed by the system.

Keywords: Quantile impulse-responses, spillover, codependence, CAViaR
JEL classification: C13, C14, C32.

1 Introduction

The recent financial crisis has brought to the forefront the importance of having sound measures of financial spillover. In the current debate, great emphasis has been put on how to measure whether an institution is of systemic importance. In particular, it has been argued that since the failure of a systemically important financial institutions could produce severe negative externalities on the whole financial system, the supervision of financial institutions should, among other things, take into account the spillover of risks within the system. The regulatory constraints imposed on firms should therefore also reflect their overall systemic impact.

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A popular measure to assess the systemic importance of a financial institution is to look at the sensitivity of its Value at Risk (VaR) to shocks to the whole financial system (see, for instance, Adrian and Brunnermeier 2009, Acharya et al. 2009, Engle and Brownlees 2010). This paper proposes a novel method to estimate the sensitivity of VaR of a given financial institution to system-wide shocks and, vice versa, the sensitivity of the financial system VaR to shocks to individual financial institutions. We develop the econometric framework to estimate and make inferences in a "VAR for VaR" model, that is, a vector autoregressive (VAR) model where the dependent variables are the VaR of financial institutions, which depend on (lagged) VaR and past shocks. This allows us to study the spillover and feedback effects among the variables of the system. In addition, from the estimated parameters, we can compute the long run VaR equilibria, as well as impulse-response functions.

To illustrate our approach and its usefulness, consider a simple set-up with two financial institutions. Let Y_{1t} and Y_{2t} denote the returns at time t for the two institutions. All information available in both markets at time t is represented by the information set $\mathcal{F}_t$. As is standard, define VaR as the worst monetary loss over a relevant holding period and with a given level of confidence $\theta \in (0,1)$. Assuming that the total money value of each market is \$1, VaR_{it} for market $i = 1,2$ at time t is

$$\Pr[Y_{it} \leq -VaR_{it} | \mathcal{F}_{t-1}] = \theta. \tag{1}$$

Hence, $-VaR_{it}$ is the θ^{th} quantile of Y_{it} conditional on $\mathcal{F}_{t-1}$; we will find it convenient to denote this as $q^*_{i,t}$ in the analysis to follow.

A simple version of our proposed structure relates Value at Risk in two country-wide markets according to

$$\begin{aligned} VaR_{1t} &= X_t'\beta_1^* + b_{11}^* VaR_{1t-1} + b_{12}^* VaR_{2t-1} \\ VaR_{2t} &= X_t'\beta_2^* + b_{21}^* VaR_{1t-1} + b_{22}^* VaR_{2t-1}, \end{aligned}$$

where X_t represents predictors belonging to $\mathcal{F}_{t-1}$. The codependence between the two markets is measured by the off-diagonal coefficients b_{12}^* and b_{21}^*, and the hypothesis of no codependence can be tested by testing $H_0 : b_{12}^* = b_{21}^* = 0$. The direction of risk spread also can be detected by examining these two coefficients. For example, if $b_{12}^* = 0$ and $b_{21}^* \neq 0$, then the direction of risk spread is from country 1 to country 2, not the other way around. Our fully general model, explained in the next sections, is much richer than the above in that, among other things: (i) we can accommodate more than two markets; (ii) multiple lags of VaR_{it} can be included; and (iii) we can simultaneously consider multiple confidence levels, say $(\theta_1, ..., \theta_p)$.

In our empirical analysis, we estimate the VAR for VaR model using returns of individual financial institutions from around the world and a global financial sector index. By constructing the impulse-response functions, we can rank the banks by their resilience to shocks to the overall index and by the impact they have on the VaR of the financial sector index.

The plan of the paper is as follows. In Section 2, we set forth the multivariate multi-quantile CAViaR (MVMQ-CAViaR) framework, a generalization of White, Kim, and Manganelli's (2008) MQ-CAViaR extension of Engle and Manganelli's original CAViaR (2004) framework. Section 3 provides conditions ensuring the consistency and asymptotic normality of the MVMQ-CAViaR estimator, as well as results providing a consistent asymptotic covariance matrix estimator. Section 5 contains our empirical study. Section 5 provides a summary and concluding remarks. The appendix contains the remaining regularity conditions and the proofs of the theorems in the text.

2 The MVMQ-CAViaR Process and Model

We consider data generated as a realization of the following stochastic process.

Assumption 1 The sequence $\{(Y_t', X_t') : t = 0, \pm 1, \pm 2, ..., \}$ is a stationary and ergodic stochastic process on the complete probability space $(\Omega, \mathcal{F}, P_0)$, where Y_t is a finitely dimensioned $n \times 1$ vector and X_t is a countably dimensioned vector whose first element is one.

Let $\mathcal{F}_{t-1}$ be the σ-algebra generated by $Z^{t-1} := \{X_t, (Y_{t-1}, X_{t-1}), ...\}$, i.e. $\mathcal{F}_{t-1} := \sigma(Z^{t-1})$. For $i = 1, ..., n$, we let $F_{it}(y) := P_0[Y_{it} < y \mid \mathcal{F}_{t-1}]$ define the cumulative distribution function (CDF) of Y_{it} conditional on $\mathcal{F}_{t-1}$.

Let $0 < \theta_{i1} < ... < \theta_{ip} < 1$. For $j = 1, ..., p$, the $\theta_{ij}th$ quantile of Y_{it} conditional on $\mathcal{F}_{t-1}$, denoted $q^*_{i,j,t}$, is

$$q^*_{i,j,t} := \inf\{y : F_{it}(y) = \theta_{ij}\}, \tag{2}$$

and if F_{it} is strictly increasing,

$$q^*_{i,j,t} = F_{it}^{-1}(\theta_{ij}).$$

Alternatively, $q^*_{i,j,t}$ can be represented as

$$\int_{-\infty}^{q^*_{i,j,t}} dF_{it}(y) = E[1_{[Y_{it} \leq q^*_{i,j,t}]} \mid \mathcal{F}_{t-1}] = \theta_{ij}, \tag{3}$$

where $dF_{it}(\cdot)$ is the Lebesgue-Stieltjes probability density function (PDF) of Y_{it} conditional on $\mathcal{F}_{t-1}$, corresponding to F_{it}. Note that we specify the same number (p) of quantile indexes for each $i = 1, ..., n$; however, this is just for notational simplicity. Our theory easily applies to the case in which the number of quantile indexes differs with i; i.e., p can be replaced with p_i.

Our objective is to jointly estimate the conditional quantile functions $q^*_{i,j,t}$, $j = 1, 2, ..., p$, $i = 1, ..., n$. For this we write $q^*_t := (q^{*\prime}_{1,t}, q^{*\prime}_{2,t}, ..., q^{*\prime}_{n,t})'$ with $q^*_{i,t} := (q^*_{i,1,t}, q^*_{i,2,t}, ..., q^*_{i,p,t})'$ and impose additional appropriate structure.

First, we ensure that the conditional distributions of Y_{it} are everywhere continuous, with positive density at each conditional quantile of interest, $q^*_{i,j,t}$. We let f_{it} denote the conditional probability density function (PDF) corresponding to F_{it}. In stating our next condition (and where helpful elsewhere), we make explicit the dependence of the conditional CDF F_{it} on $\omega \in \Omega$ by writing $F_{it}(\omega, y)$ in place of $F_{it}(y)$. Similarly, we may write $f_{i,t}(\omega, y)$ in place of $f_{i,t}(y)$. Realized values of the conditional quantiles are correspondingly denoted $q^*_{i,j,t}(\omega)$.

Our next assumption ensures the desired continuity and imposes specific structure on the quantiles of interest.

Assumption 2 For $i = 1, ..., n$, (i) Y_{it} is continuously distributed such that for each $\omega \in \Omega$, $F_{it}(\omega, \cdot)$ and $f_{it}(\omega, \cdot)$ are continuous on $\mathbb{R}$, $t = 1, 2, ...$; (ii) For given $0 < \theta_{i1} < ... < \theta_{ip} < 1$ and $\{q^*_{i,j,t}\}$ as defined above, suppose: (a) For each i, j, t, and ω, $f_{it}(\omega, q^*_{i,j,t}(\omega)) > 0$; and (b) For given finite integers k and m, there exist a stationary ergodic sequence of random $k \times 1$ vectors $\{\Psi_t\}$, with Ψ_t measurable$-\mathcal{F}_{t-1}$, and real vectors $\beta^*_{ij} := (\beta^*_{i,j,1}, ..., \beta^*_{i,j,k})'$ and $\gamma^*_{i,j,\tau} := (\gamma^{*\prime}_{i,j,\tau,1}, ..., \gamma^{*\prime}_{i,j,\tau,n})'$, where each $\gamma^*_{i,j,\tau,k}$ is a $p \times 1$ vector, such that for $j = 1, ..., p$, $i = 1, ..., n$, and all t,

$$q^*_{i,j,t} = \Psi'_t \beta^*_{ij} + \sum_{\tau=1}^{m} q^{*\prime}_{t-\tau} \gamma^*_{i,j,\tau} \ . \tag{4}$$

The structure of eq. (4) is a multivariate version of the MQ-CAViaR process of White, Kim, and Manganelli (2008), itself a multi-quantile version of the CAViaR process introduced by Engle and Manganelli (2004). Under suitable restrictions on the $\gamma^*_{i,j,\tau}$'s, we obtain as special cases (1) separate MQ-CAViaR processes for each element of Y_t; (2) standard (single quantile) CAViaR processes for each element of Y_t; or (3) multivariate CAViaR processes, in which a single quantile of each element of Y_t is related dynamically to single quantiles of the (lags of) other elements of Y_t. Thus, we call processes satisfying our structure "Multivariate MQ-CAViaR" (MVMQ-CAViaR) processes.

For MVMQ-CAViaR, the number of relevant lags can differ across the elements of Y_t and the conditional quantiles; this is reflected in the possibility that for given j, elements of $\gamma^*_{i,j,\tau}$ may be zero for values of τ greater than some given integer. For notational simplicity, we do not represent m as depending on i or j. Nevertheless, by convention, for no $\tau \leq m$ does $\gamma^*_{i,j,\tau}$ equal the zero vector for all i and j.

The finitely dimensioned random vectors Ψ_t may contain lagged values of Y_t, as well as measurable functions of X_t and lagged X_t or Y_t. In particular, Ψ_t may contain Stinchcombe and White's (1998) GCR transformations, as discussed in White (2006).

For a particular quantile, say θ_{ij}, the coefficients to be estimated are β^*_{ij} and $\gamma^*_{ij} := (\gamma^{*\prime}_{i,j,1}, ..., \gamma^{*\prime}_{i,j,m})'$. Let $\alpha^{*\prime}_{ij} := (\beta^{*\prime}_{ij}, \gamma^{*\prime}_{ij})$, and write $\alpha^* = (\alpha^{*\prime}_{11}, ..., \alpha^{*\prime}_{1p}, ..., \alpha^{*\prime}_{n1}, ..., \alpha^{*\prime}_{np})'$, an $\ell \times 1$ vector, where $\ell := np(k + npm)$. We call α^* the "MVMQ-CAViaR coefficient vector." We estimate α^* using a correctly specified model of the MVMQ-CAViaR process.

First, we specify our model.

Assumption 3 Let $\mathbb{A}$ be a compact subset of $\mathbb{R}^{\ell}$. For $i = 1, ..., n$, and $j = 1, ..., p$, (i) the sequence of functions $\{q_{i,j,t} : \Omega \times \mathbb{A} \to \mathbb{R}^{p_i}\}$ is such that for each t and each $\alpha \in \mathbb{A}$, $q_{i,j,t}(\cdot, \alpha)$ is measurable$-\mathcal{F}_{t-1}$; for each t and each $\omega \in \Omega$, $q_{i,j,t}(\omega, \cdot)$ is continuous on $\mathbb{A}$; and for each i, j, and t,

$$q_{i,j,t}(\cdot, \alpha) = \Psi_t' \beta_{ij} + \sum_{\tau=1}^{m} q_{t-\tau}(\cdot, \alpha)' \gamma_{i,j,\tau}.$$

Next, we impose correct specification and an identification condition. Assumption 4(i.a) delivers correct specification by ensuring that the MVMQ-CAViaR coefficient vector α^* belongs to the parameter space, $\mathbb{A}$. This ensures that α^* optimizes the estimation objective function asymptotically. Assumption 4(i.b) delivers identification by ensuring that α^* is the only such optimizer. In stating the identification condition, we define $\delta_{i,j,t}(\alpha, \alpha^*) := q_{i,j,t}(\cdot, \alpha) - q_{i,j,t}(\cdot, \alpha^*)$ and use the norm $||\alpha|| := \max_{s=1,\ldots,\ell} |\alpha_s|$, where for convenience we also write $\alpha = (\alpha_1, ..., \alpha_\ell)'$.

Assumption 4 (i)(a) There exists $\alpha^* \in \mathbb{A}$ such that for all i, j, t

$$q_{i,j,t}(\cdot, \alpha^*) = q^*_{i,j,t}; \tag{5}$$

(b) There is a non-empty index set $\mathcal{I} \subseteq \{(1,1), ..., (1,p), ..., (n,1), ..., (n,p)\}$ such that for each $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for all $\alpha \in \mathbb{A}$ with $||\alpha - \alpha^*|| > \epsilon$,

$$P[\cup_{(i,j)\in\mathcal{I}}\{|\delta_{i,j,t}(\alpha, \alpha^*)| > \delta_\epsilon\}] > 0.$$

Among other things, this identification condition ensures that there is sufficient variation in the shape of the conditional distribution to support estimation of a sufficient number ($\#\mathcal{I}$) of variation-free conditional quantiles. As in the case of MQ-CAViaR, distributions that depend on a given finite number of variation-free parameters, say r, will generally be able to support r variation-free quantiles. For example, the quantiles of the $N(\mu, 1)$ distribution all depend on μ alone, so there is only one "degree of freedom" for the quantile variation. Similarly the quantiles of scaled and shifted $t-$distributions depend on three parameters (location, scale, and kurtosis), so there are only three "degrees of freedom" for the quantile variation.

3 MVMQ-CAViaR: Asymptotic Theory

We estimate α^* by the method of quasi-maximum likelihood. Specifically, we construct a quasi-maximum likelihood estimator (QMLE) $\hat{\alpha}_T$ as the solution to the optimization problem

$$\min_{\alpha\in\mathbb{A}} \bar{S}_T(\alpha) := T^{-1} \sum_{t=1}^{T} \{\sum_{i=1}^{n} \sum_{j=1}^{p} \rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha))\}, \tag{6}$$

where $\rho_\theta(e) = e\psi_\theta(e)$ is the standard "check function," defined using the usual quantile step function, $\psi_\theta(e) = \theta - 1_{[e\leq 0]}$.

We thus view

$$S_t(\alpha) := -\{\sum_{i=1}^{n}\sum_{j=1}^{p}\rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha))\}$$

as the quasi log-likelihood for observation t. In particular, $S_t(\alpha)$ is the log-likelihood of a vector of np independent asymmetric double exponential random variables (see White, 1994, ch. 5.3; Kim and White, 2003; Komunjer, 2005). Because $Y_{it} - q_{i,j,t}(\cdot,\alpha)$ does not need to actually have this distribution, the method is *quasi* maximum likelihood.

We establish consistency and asymptotic normality for $\hat{\alpha}_T$ by methods analogous to those of White, Kim, and Manganelli (2008). For conciseness, we place the remaining regularity conditions and technical discussion in the appendix.

Theorem 1 *Suppose that Assumptions* $1, 2(i, ii), 3(i), 4(i),$ *and* $5(i, ii)$ *hold. Then* $\hat{\alpha}_T \overset{a.s.}{\to} \alpha^*$.

With Q^* and V^* as given below, the asymptotic normality result is

Theorem 2 *Suppose that Assumptions 1-6 hold. Then*

$$T^{1/2}(\hat{\alpha}_T - \alpha^*) \overset{d}{\to} N(0, Q^{*-1}V^*Q^{*-1}).$$

To test restrictions on α^* or to obtain confidence intervals, we require a consistent estimator of the asymptotic covariance matrix $C^* := Q^{*-1}V^*Q^{*-1}$. First, we provide a consistent estimator $\hat{V}_T$ for V^*; then we give a consistent estimator $\hat{Q}_T$ for Q^*. It follows that $\hat{C}_T := \hat{Q}_T^{-1}\hat{V}_T\hat{Q}_T^{-1}$ is a consistent estimator for C^*.

We have $V^* := E(\eta_t^*\eta_t^{*\prime})$ with $\eta_t^* := \sum_{i=1}^{n}\sum_{j=1}^{p}\nabla q_{i,j,t}(\cdot,\alpha^*)\psi_{\theta_{ij}}(\varepsilon_{i,j,t})$, where $\psi_{\theta_{ij}}(\varepsilon_{i,j,t})$ is a generalized residual. A straightforward plug-in estimator of V^* is

$$\begin{aligned}
\hat{V}_T &:= T^{-1}\sum_{t=1}^{T}\hat{\eta}_t\hat{\eta}_t', \qquad \text{with} \\
\hat{\eta}_t &:= \sum_{i=1}^{n}\sum_{j=1}^{p}\nabla q_{i,j,t}(\cdot,\hat{\alpha}_T)\psi_{\theta_{ij}}(\hat{\varepsilon}_{i,j,t}) \\
\hat{\varepsilon}_{i,j,t} &:= Y_{it} - q_{i,j,t}(\cdot,\hat{\alpha}_T).
\end{aligned}$$

The next result establishes the consistency of $\hat{V}_T$ for V^*.

Theorem 3 *Suppose that Assumptions 1-6 hold. Then* $\hat{V}_T \overset{p}{\to} V^*$.

Next, we provide a consistent estimator of

$$Q^* := \sum_{i=1}^{n}\sum_{j=1}^{p} E[f_{i,j,t}(0)\nabla q_{i,j,t}(\cdot,\alpha^*)\nabla' q_{i,j,t}(\cdot,\alpha^*)].$$

We follow Powell's (1984) suggestion of estimating $f_{i,j,t}(0)$ with $1_{[-\hat{c}_T \leq \hat{\varepsilon}_{i,j,t} \leq \hat{c}_T]}/2\hat{c}_T$ for a suitably chosen sequence $\{\hat{c}_T\}$. This is also the approach taken in Kim and White (2003), Engle and Manganelli (2004), and White, Kim, and Manganelli (2008). Accordingly, our proposed estimator is

$$\hat{Q}_T = (2\hat{c}_T T)^{-1} \sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{j=1}^{p} 1_{[-\hat{c}_T \leq \hat{\varepsilon}_{i,j,t} \leq \hat{c}_T]} \nabla q_{i,j,t}(\cdot,\hat{\alpha}_T)\nabla' q_{i,j,t}(\cdot,\hat{\alpha}_T).$$

Theorem 4 *Suppose that Assumptions 1-7 hold. Then* $\hat{Q}_T \xrightarrow{p} Q^*$.

There is no guarantee that $\hat{\alpha}_T$ is asymptotically efficient. There is now a considerable literature investigating efficient estimation in quantile models; see, for example, Newey and Powell (1990), Otsu (2003), Komunjer and Vuong (2006, 2007a, 2007b). So far, this literature has only considered single quantile models. It is not obvious how the results for single quantile models extend to multivariate multi-quantile models. Nevertheless, Komunjer and Vuong (2007a) show that the class of QML estimators is not large enough to include an efficient estimator, and that the class of M-estimators, which strictly includes the QMLE class, yields an estimator that attains the efficiency bound. Specifically, when $p = n = 1$, they show that replacing the usual quantile check function $\rho_{\theta_{ij}}(\cdot)$ in eq.(6) with

$$\rho^*_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha)) = (\theta_{ij} - 1_{[Y_{it}-q_{i,j,t}(\cdot,\alpha)\leq 0]})(F_{it}(Y_{it}) - F_{it}(q_{i,j,t}(\cdot,\alpha)))$$

will deliver an asymptotically efficient quantile estimator. We conjecture that replacing $\rho_{\theta_{ij}}$ with $\rho^*_{\theta_{ij}}$ in eq.(6) will improve estimator efficiency for p and/or n not equal to 1. We leave the study of the asymptotically efficient multivariate multi-quantile estimator for future work.

4 Assessing the Systemic Importance of Financial Institutions

We apply the model developed in the previous sections to study spillovers in the returns quantiles of a sample of 230 financial institutions from around the world. In this section we first show how to compute impulse-response functions within the multivariate, multiquantile framework. We choose a particular quantile specification for our empirical analysis, linking it to the DGP of more familiar multivariate mean-variance models. We next describe the data and the optimization strategy. Finally, we present the results of an empirical application to market returns of financial institutions.

4.1 Impulse-response Functions for Multivariate CAViaR

Suppose data are structurally generated as:

$$Y_t = L_t u_t,$$

where $L_t := L_t(Z^{t-1})$ is an $\mathcal{F}_{t-1}$−measurable lower triangular matrix and the elements of u_t are mutually independent with $\{u_t|\mathcal{F}_{t-1}\}$ a martingale difference sequence. By convention, we let the first element of Y_t denote the per-period return on a financial sector index and the second element the per-period return on a specific bank. The identification assumption behind this decomposition is that shocks to the financial sector index are allowed to have a direct impact on the return of the specific bank, but shocks to the specific bank do not have a direct impact on the financial sector index. Here we limit ourselves to a bivariate system, as we are interested in the interaction between a financial sector index and an individual bank. The theoretical framework of this paper can accommodate higher dimensional models, although at the cost of a rapidly increasing computational burden.

For suitable choices of L_t, the conditional quantiles of Y_t obey (4). For our purposes here, suppose

$$q_{i,t} = c + A|Y_{t-1}| + Bq_{i,t-1} \tag{7}$$

where $q_{i,t}$, Y_{t-1}, and c are 2-dimensional vectors, and A, B are (2,2)-matrices. See Appendix 1 for an example of how this representation corresponds to a bivariate GARCH model.

Let the long run matrix L be defined as:

$$L := \lim_{t\to\infty} L_t(Z^{t-1})|_{Z^{t-1}=0}.$$

If we set $y_{t+n} = 0$ for $n > 0$, the system converges to $\bar{q} = (I-B)^{-1}c$. Assuming the system is at its long run equilibrium, if we denote by ι^1 a one standard deviation shock to the first element of the orthogonal error u at time t, such a shock implies the following quantile response:

$$\begin{aligned} q_{1,t+1} &\equiv c + A|L\iota^1| + B\bar{q} \\ q_{1,t+n} &\equiv c + Bq_{1,t+n-1} \qquad n > 1 \end{aligned}$$

The impulse-responses for a shock to the second element of u can be computed analogously.

We can compute four types of impulse-responses:

1. $\partial q_{1,t+n}/\partial u_{1,t}$ is the reaction of the system's risk to a system shock;

2. $\partial q_{1,t+n}/\partial u_{2,t}$ is the reaction of the system's risk to an individual bank's shock;

3. $\partial q_{2,t+n}/\partial u_{1,t}$ is the reaction of the bank's risk to a system shock;

4. $\partial q_{2,t+n}/\partial u_{2,t}$ is the reaction of the bank's risk to its own shock.

4.2 Data and Optimization Strategy

The data were downloaded from Datastream. We considered three main global sub-indices: banks, financial services, and insurances. The sample includes daily closing prices from 1 January 2000 to 6 August 2010. We eliminated all the stocks whose times series started later than 1 January 2000. At the end of this process, we were left with 230 stocks.

Table 1 reports the breakdown of the stocks by sector and by geographic area. There are twice as many financial institutions classified as banks in our sample as there are those classified as financial services or insurances. The distribution across geographic areas is more balanced, with a greater number of EU financial institutions and slightly lower Asian representation.

To cope with asynchronicity issues due to differing time zones, the data were transformed to weekly frequency. Weekly returns were computed as the log difference of weekly closing prices and expressed in percentage terms. The proxy for the overall index used in each bivariate quantile estimation is the World Financials price index, as provided by Datastream.

We estimated 230 bivariate 1% quantile models between the index and each of the 230 financial institutions in our sample. Each model is estimated using as starting values for optimization the univariate CAViaR estimates and initializing the remaining parameters at zero. We also generated 40 additional initial conditions by adding a normally distributed noise to this vector. For each of these 41 initial conditions, we minimized the regression quantile objective function (6) using the *fminsearch* optimization function in Matlab, which is based on the Nelder-Mead simplex algorithm. Finally, among the resulting 41 vectors of parameter estimates, we chose the vector yielding the lowest value for the function (6). We adopt this strategy because we find that parameter estimates are sensitive to the choice of initial conditions (possibly due to a very flat likelihood near the optimum). Such an optimization strategy is more time consuming, but delivers more reliable results. Still, for some time series, either we did not get convergence or the parameter estimates were associated with an explosive impulse-response function. This happened in about 20% of the sample. In this case, we restricted to zero the coefficients associated with the second lagged quantile (i.e. the quantile associated with the single financial institutions) in the process (7).

In calculating the standard errors, we have set the bandwidth to 1.

4.3 Results

Table 2 reports as an example the estimation results for the Sumitomo Mitsui Financial Group, the second largest bank in Japan by market value (as of November 2009). Notice that the non-diagonal coefficients for the B matrix are significantly different from zero, illustrating how the multivariate quantile model can uncover dynamics that cannot be detected by estimating univariate quantile models. In general, we reject the joint null hypothesis that all off-diagonal coefficients of the matrices A and B are equal to zero at the .05 level [****HW: correct level?**] for about 85% of the firms in our sample. The resulting estimated 1% quantiles for the index and the Sumitomo Mitsui Financial Group are reported in Figure 1. The quantile of the global index is generally much smaller in absolute terms that the quantile of Sumitomo, reflecting the portfolio diversification effect of the index. Only around the Lehman bankruptcy (September 2008) is the situation briefly inverted, with the estimates indicating a higher risk associated with the global index than with the Sumitomo bank.

Figures 2-5 plot the average impulse-response functions $\partial q_{1,t+n}/\partial u_{2,t}$ and $\partial q_{2,t+n}/\partial u_{1,t}$ measuring the impact of a two standard deviation individual bank shock on the index and the impact of a two standard deviation shock to the index on the individual bank's risk. In Figures 2 and 3, the average is taken with respect to the geographical distribution. That is, the average impulse-response for Europe, say, is obtained by averaging all the impulse-response functions for European financial institutions. We notice two things. First, the impact of a shock to the index is much stronger than the impact of a shock to the individual financial institution. This result is partly driven by our identification assumption that shocks to the index have a contemporaneous impact on the return of the single financial institutions, while the institution's specific shocks have only a lagged impact on the global financial index. Second, we notice that the risk of Asian financial institutions appears to be on average much less sensitive to global shocks than their European and North American counterparts.

Figures 4 and 5 plot the average impulse-response functions for the financial institutions grouped by line of business, i.e. banks, financial services, and insurances. We see that a shock to the index has a stronger initial impact on the group of insurance companies. Regarding the impact of shocks to the individual financial institutions on the risk of the global index, banks have on average a lower initial impact, but the shock appears to be more persistent than for financial services and insurance companies.

To rank the financial institutions according to their impact on risk, we integrated out all the individual impulse-responses and sorted the financial institutions according to this metric. The resulting ranking is reported in Table 3. The first two columns rank the 20 financial institutions whose risk is most and least sensitive to a shock to the index. Among the most sensitive institutions are household names such as Barclays, Unicredit, Citigroup, and Royal Bank of Scotland. Goldman Sachs, on the other hand, belongs to the group of financial institutions that are least affected by global shocks. The last two columns contain the financial institutions with the largest and smallest impact on the risk of the global index. These lists contain smaller and less well known financial institutions. To get an idea of the orders of magnitude involved, Figure 6 plots the average impulse-responses corresponding to the four lists of 20 financial institutions contained in Table 3. It is clear that the shocks to the index have an impact of an order of magnitude greater than the shocks to the individual financial institutions. A two standard deviation shock to the index produces an average initial increase in the VaR of the most sensitive financial institutions of more than 20%. The shock is also quite persistent, as it is not yet completely absorbed after 15 weeks. On the other hand, for the least sensitive financial institutions, a shock to the index produces an average immediate increase in the VaR of less than 3%, which is then entirely absorbed after the second week. The figure also shows that the shocks to the individual financial institutions have a significantly lower impact on the risk of the index, in line with the results shown in Figures 3 and 5.

5 Conclusion

We have developed theory ensuring the consistency and asymptotic normality of multivariate multi-quantile models. Our theory is general enough to comprehensively cover models with multiple random variables, multiple confidence levels and multiple lags of the quantiles.

We conduct an empirical analysis in which we estimate a VAR for VaR model using returns of individual financial institutions from around the world and a global financial sector index. By examining the impulse-response functions, we can rank the banks by their resilience to shocks to the overall index and by the impact they have on the VaR of the financial sector index. We find that the risk of Asian financial institutions tends to be less sensitive to systemic shocks, whereas insurance companies exhibit a greater sensitivity to global shocks. Ranking financial institutions by how they react to shocks, we uncover wide differences among them. Among the top 20 financial institutions that appear to be more vulnerable to system-wide shocks we find well-known names such as Barclays, Unicredit, Citigroup, and Royal Bank of Scotland. These findings are quite striking, as they are obtained without weighting the financial institutions by their market capitalization.

The methods developed in this paper can be applied to many other contexts. For instance, many stress-test models are built from vector autoregressive models on credit risk indicators and macroeconomic variables. Starting from the estimated mean and adding assumptions on the multivariate distribution of the error terms, one can deduce the impact of a macro shock on the quantile of the credit risk variables. Our methodology provides a convenient alternative for stress testing, by allowing researchers to estimate vector autoregressive processes directly on the quantiles of interest, rather than on the mean.

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6 Appendix 1 - Multi-quantile representation of a bivariate GARCH process with zero mean

Consider the following data generating process:

$$\begin{aligned}
\begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix} &= \begin{bmatrix} \alpha_t & 0 \\ \beta_t & \gamma_t \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \qquad\qquad \varepsilon_t \backsim N(0, I) \\
\sigma_t(Y_{1t}) &= \alpha_t \\
&\equiv \sigma_{1t} \\
&= c_1 + a_{11}|Y_{1t-1}| + a_{12}|Y_{2t-1}| + b_{11}\sigma_{1t-1} + b_{12}\sigma_{2t-1} \\
\sigma_t(Y_{2t}) &= \sqrt{\beta_t^2 + \gamma_t^2} \\
&\equiv \sigma_{2t} \\
&= c_2 + a_{21}|Y_{1t-1}| + a_{22}|Y_{2t-1}| + b_{21}\sigma_{1t-1} + b_{22}\sigma_{2t-1}.
\end{aligned}$$

The respective quantile processes are:

$$\begin{aligned}
q_{1t} &= kc_1 + ka_{11}|Y_{1t-1}| + ka_{12}|Y_{2t-1}| + b_{11}q_{1t-1} + b_{12}q_{2t-1} \\
q_{2t} &= kc_2 + ka_{21}|Y_{1t-1}| + ka_{22}|Y_{2t-1}| + b_{21}q_{1t-1} + b_{22}q_{2t-1},
\end{aligned}$$

where k is the α-quantile of the standard normal distribution. In this case, setting the shocks equal to zero is equivalent to setting $Y_t = 0$. The long-run quantiles are:

$$\bar{q} = \begin{bmatrix} 1 - b_{11} & b_{12} \\ b_{21} & 1 - b_{22} \end{bmatrix}^{-1} \begin{bmatrix} kc_1 \\ kc_2 \end{bmatrix}$$

7 Appendix 2 - Proofs

The proofs that follow are straightforward modifications of those in White, Kim, and Manganelli (2008).

We establish the consistency of $\hat{\alpha}_T$ by applying results of White (1994). For this we impose the following moment and domination conditions. In stating this next condition and where convenient elsewhere, we exploit stationarity to omit explicit reference to all values of t.

Assumption 5 (i) For $i = 1, ..., n$, $E|Y_{it}| < \infty$; (ii) let $D_{0,t} := \max_{i=1,...,n} \max_{j=1,...,p} \sup_{\alpha \in \mathbb{A}} |q_{i,j,t}(\cdot, \alpha)|$. Then $E(D_{0,t}) < \infty$.

Proof of Theorem 1: We verify the conditions of corollary 5.11 of White (1994), which delivers $\hat{\alpha}_T \to \alpha^*$, where

$$\hat{\alpha}_T := \arg\max_{\alpha \in \mathbb{A}} T^{-1} \sum_{t=1}^{T} \varphi_t(Y_t, q_t(\cdot, \alpha)),$$

and $\varphi_t(Y_t, q_t(\cdot, \alpha)) := -\{\sum_{i=1}^n \sum_{j=1}^p \rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha))\}$. Assumption 1 ensures White's Assumption 2.1. Assumption 3(i) ensures White's Assumption 5.1. Our choice of $\rho_{\theta_{ij}}$ satisfies White's Assumption 5.4. To verify White's Assumption 3.1, it suffices that $\varphi_t(Y_t, q_t(\cdot, \alpha))$ is dominated on $\mathbb{A}$ by an integrable function (ensuring White's Assumption 3.1(a,b)) and that for each α in $\mathbb{A}$, $\{\varphi_t(Y_t, q_t(\cdot, \alpha))\}$ is stationary and ergodic (ensuring White's Assumption 3.1(c), the strong uniform law of large numbers (ULLN)). Stationarity and ergodicity is ensured by Assumptions 1 and 3(i). To show domination, we write

$$\begin{aligned} |\varphi_t(Y_t, q_t(\cdot, \alpha))| &\leq \sum_{i=1}^{n} \sum_{j=1}^{p} |\rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha))| \\ &= \sum_{i=1}^{n} \sum_{j=1}^{p} |(Y_{it} - q_{i,j,t}(\cdot, \alpha))(\theta_{ij} - 1_{[Y_{it} - q_{i,j,t}(\cdot,\alpha) \leq 0]})| \\ &\leq 2 \sum_{i=1}^{n} \sum_{j=1}^{p} (|Y_{it}| + |q_{i,j,t}(\cdot, \alpha)|) \\ &\leq 2p \sum_{i=1}^{n} |Y_{it}| + 2np|D_{0,t}|, \end{aligned}$$

so that

$$\sup_{\alpha \in \mathbb{A}} |\varphi_t(Y_t, q_t(\cdot, \alpha))| \leq 2p \sum_{i=1}^{n} |Y_{it}| + 2np|D_{0,t}|.$$

Thus, $2p \sum_{i=1}^n |Y_{it}| + 2np|D_{0,t}|$ dominates $|\varphi_t(Y_t, q_t(\cdot, \alpha))|$; this has finite expectation by Assumption 5(i,ii).

It remains to verify White's Assumption 3.2; here this is the condition that α^* is the unique maximizer of $E(\varphi_t(Y_t, q_t(\cdot, \alpha))$. Given Assumptions 2(ii.b) and

4(i), it follows by argument directly parallel to that in the proof of White (1994, corollary 5.11) that for all $\alpha \in \mathbb{A}$,

$$E(\varphi_t(Y_t, q_t(\cdot, \alpha)) \leq E(\varphi_t(Y_t, q_t(\cdot, \alpha^*)).$$

Thus, it suffices to show that the above inequality is strict for $\alpha \neq \alpha^*$. Consider $\alpha \neq \alpha^*$ such that $||\alpha - \alpha^*|| > \epsilon$ and let $\Delta(\alpha) := \sum_{i=1}^n \sum_{j=1}^p E(\Delta_{i,j,t}(\alpha))$ with $\Delta_{i,j,t}(\alpha) := \rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha)) - \rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha^*))$. It will suffice to show that $\Delta(\alpha) > 0$. First, we define the "error" $\varepsilon_{i,j,t} := Y_{it} - q_{i,j,t}(\cdot, \alpha^*)$ and let $f_{i,j,t}(\cdot)$ be the density of $\varepsilon_{i,j,t}$ conditional on $\mathcal{F}_{t-1}$. Noting that $\delta_{i,j,t}(\alpha, \alpha^*) := q_{i,j,t}(\cdot, \alpha) - q_{i,j,t}(\cdot, \alpha^*)$, we next can show by some algebra and Assumption 2(ii.a) that

$$\begin{aligned}
E(\Delta_{i,j,t}(\alpha)) &= E[\int_0^{\delta_{i,j,t}(\alpha,\alpha^*)} (\delta_{i,j,t}(\alpha, \alpha^*) - s) \ f_{i,j,t}(s)ds] \\
&\geq E[\frac{1}{2}\delta_\epsilon^2 1_{[|\delta_{i,j,t}(\alpha,\alpha^*)|>\delta_\epsilon]} + \frac{1}{2}\delta_{i,j,t}(\alpha, \alpha^*)^2 1_{[|\delta_{i,j,t}(\alpha,\alpha^*)|\leq\delta_\epsilon]})] \\
&\geq \frac{1}{2}\delta_\epsilon^2 E[1_{[|\delta_{i,j,t}(\alpha,\alpha^*)|>\delta_\epsilon]}].
\end{aligned}$$

The first inequality above comes from the fact that Assumption 2(ii.a) implies that for any $\delta > 0$ sufficiently small, we have $f_{i,j,t}(s) > \delta$ for $|s| < \delta$. Thus,

$$\begin{aligned}
\Delta(\alpha) \quad &:= \sum_{i=1}^n \sum_{j=1}^p E(\Delta_{i,j,t}(\alpha)) \geq \frac{1}{2}\delta_\epsilon^2 \sum_{i=1}^n \sum_{j=1}^p E[1_{[|\delta_{i,j,t}(\alpha,\alpha^*)|>\delta_\epsilon]}] \\
&= \frac{1}{2}\delta_\epsilon^2 \sum_{i=1}^n \sum_{j=1}^p P[|\delta_{i,j,t}(\alpha, \alpha^*)| > \delta_\epsilon] \geq \frac{1}{2}\delta_\epsilon^2 \sum_{(i,j)\in\mathcal{I}} P[|\delta_{i,j,t}(\alpha, \alpha^*)| > \delta_\epsilon] \\
&\geq \frac{1}{2}\delta_\epsilon^2 P[\cup_{(i,j)\in\mathcal{I}}\{|\delta_{i,j,t}(\alpha, \alpha^*)| > \delta_\epsilon\}] > 0,
\end{aligned}$$

where the final inequality follows from Assumption 4(i.b). As α is arbitrary, the result follows. ■

Next, we establish the asymptotic normality of $T^{1/2}(\hat{\alpha}_T - \alpha^*)$. We use a method originally proposed by Huber (1967) and later extended by Weiss (1991). We first sketch the method before providing formal conditions and the proof.

Huber's method applies to our estimator $\hat{\alpha}_T$, provided that $\hat{\alpha}_T$ satisfies the asymptotic first order conditions

$$T^{-1}\sum_{t=1}^T \{\sum_{i=1}^n \sum_{j=1}^p \nabla q_{i,j,t}(\cdot, \hat{\alpha}_T) \ \psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \hat{\alpha}_T))\} = o_p(T^{1/2}), \tag{8}$$

where $\nabla q_{i,j,t}(\cdot, \alpha)$ is the $\ell \times 1$ gradient vector with elements $(\partial/\partial\alpha_s)q_{i,j,t}(\cdot, \alpha)$, $s = 1, ..., \ell$, and $\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \hat{\alpha}_T))$ is a generalized residual. Our first task is thus to ensure that eq. (8) holds.

Next, we define

$$\lambda(\alpha) := \sum_{i=1}^n \sum_{j=1}^p E[\nabla q_{i,j,t}(\cdot, \alpha)\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha))].$$

With $\lambda(\alpha)$ continuously differentiable at α^* interior to $\mathbb{A}$, we can apply the mean value theorem to obtain

$$\lambda(\alpha) = \lambda(\alpha^*) + Q_0(\alpha - \alpha^*), \tag{9}$$

where Q_0 is an $\ell \times \ell$ matrix with $(1 \times \ell)$ rows $Q_{0,s} = \nabla' \lambda(\bar{\alpha}_{(s)})$, where $\bar{\alpha}_{(s)}$ is a mean value (different for each s) lying on the segment connecting α and α^*, $s = 1, ..., \ell$. It is straightforward to show that correct specification ensures that $\lambda(\alpha^*)$ is zero. We will also show that

$$Q_0 = -Q^* + O(||\alpha - \alpha^*||), \tag{10}$$

where $Q^* := \sum_{i=1}^{n} \sum_{j=1}^{p} E[f_{i,j,t}(0) \nabla q_{i,j,t}(\cdot, \alpha^*) \nabla' q_{i,j,t}(\cdot, \alpha^*)]$ with $f_{i,j,t}(0)$ the value at zero of the density $f_{i,j,t}$ of $\varepsilon_{i,j,t} := Y_{it} - q_{i,j,t}(\cdot, \alpha^*)$, conditional on $\mathcal{F}_{t-1}$. Combining eqs. (9) and (10) and putting $\lambda(\alpha^*) = 0$, we obtain

$$\lambda(\alpha) = -Q^*(\alpha - \alpha^*) + O(||\alpha - \alpha^*||^2). \tag{11}$$

The next step is to show that

$$T^{1/2} \lambda(\hat{\alpha}_T) + H_T = o_p(1), \tag{12}$$

where $H_T := T^{-1/2} \sum_{t=1}^{T} \eta_t^*$, with $\eta_t^* := \sum_{i=1}^{n} \sum_{j=1}^{p} \nabla q_{i,j,t}(\cdot, \alpha^*) \psi_{\theta_{ij}}(\varepsilon_{i,j,t})$. Eqs. (11) and (12) then yield the following asymptotic representation of our estimator $\hat{\alpha}_T$:

$$T^{1/2}(\hat{\alpha}_T - \alpha^*) = Q^{*-1} T^{-1/2} \sum_{t=1}^{T} \eta_t^* + o_p(1). \tag{13}$$

As we impose conditions sufficient to ensure that $\{\eta_t^*, \mathcal{F}_t\}$ is a martingale difference sequence (MDS), a suitable central limit theorem (e.g., theorem 5.24 in White, 2001) applies to eq. (13) to yield the desired asymptotic normality of $\hat{\alpha}_T$:

$$T^{1/2}(\hat{\alpha}_T - \alpha^*) \xrightarrow{d} N(0, Q^{*-1} V^* Q^{*-1}), \tag{14}$$

where $V^* := E(\eta_t^* \eta_t^{*\prime})$.

We now strengthen the conditions above to ensure that each step of the above argument is valid.

Assumption 2 (iii) (a) There exists a finite positive constant f_0 such that for each i and t, each $\omega \in \Omega$, and each $y \in \mathbb{R}$, $f_{it}(\omega, y) \leq f_0 < \infty$; (b) There exists a finite positive constant L_0 such that for each i and t, each $\omega \in \Omega$, and each $y_1, y_2 \in \mathbb{R}$, $|f_{it}(\omega, y_1) - f_{it}(\omega, y_2)| \leq L_0 |y_1 - y_2|$.

Next we impose sufficient differentiability of q_t with respect to α.

Assumption 3 (ii) For each t and each $\omega \in \Omega$, $q_t(\omega, \cdot)$ is continuously differentiable on $\mathbb{A}$; (iii) For each t and each $\omega \in \Omega$, $q_t(\omega, \cdot)$ is twice continuously differentiable on $\mathbb{A}$;

To exploit the mean value theorem, we require that α^* belongs to $int(\mathbb{A})$, the interior of $\mathbb{A}$.

Assumption 4 (ii) $\alpha^* \in int(\mathbb{A})$.

Next, we place domination conditions on the derivatives of q_t.

Assumption 5 (iii) Let $D_{1,t} := \max_{i=1,\dots,n} \max_{j=1,\dots,p} \max_{s=1,\dots,\ell} \sup_{\alpha\in\mathbb{A}} |(\partial/\partial\alpha_s)\, q_{i,j,t}(\cdot,\alpha)|$. Then (a) $E(D_{1,t}) < \infty$; (b) $E(D_{1,t}^2) < \infty$; (iv) Let $D_{2,t} := \max_{i=1,\dots,n} \max_{j=1,\dots,p} \max_{s=1,\dots,\ell} \max_{h=1,\dots,\ell} \sup_{\alpha\in\mathbb{A}} |(\partial^2/\partial\alpha_s\partial\alpha_h) q_{i,j,t}(\cdot,\alpha)|$. Then (a) $E(D_{2,t}) < \infty$; (b) $E(D_{2,t}^2) < \infty$.

Assumption 6 (i) $Q^* := \sum_{i=1}^n \sum_{j=1}^p E[f_{i,j,t}(0)\nabla q_{i,j,t}(\cdot,\alpha^*)\nabla' q_{i,j,t}(\cdot,\alpha^*)]$ is positive definite; (ii) $V^* := E(\eta_t^* \eta_t^{*\prime})$ is positive definite.

Assumptions 3(ii) and 5(iii.a) are additional assumptions helping to ensure that eq. (8) holds. Further imposing Assumptions 2(iii), 3(iii.a), 4(ii), and 5(iv.a) suffices to ensure that eq. (11) holds. The additional regularity provided by Assumptions 5(iii.b), 5(iv.b), and 6(i) ensures that eq. (12) holds. Assumptions 5(iii.b) and 6(ii) help ensure the availability of the MDS central limit theorem.

We now have conditions sufficient to prove asymptotic normality of our MVMQ-CAViaR estimator.

Proof of Theorem 2: As outlined above, we first prove

$$T^{-1}\sum_{t=1}^{T}\{\sum_{i=1}^{n}\sum_{j=1}^{p}\nabla q_{i,j,t}(\cdot,\hat{\alpha}_T)\ \psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T))\} = o_p(1). \tag{15}$$

The existence of $\nabla q_{i,j,t}$ is ensured by Assumption 3(ii). Let e_i be the $\ell \times 1$ unit vector with i^{th} element equal to one and the rest zero, and let

$$G_s(c) := T^{-1/2}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}\rho_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T+ce_s)),$$

for any real number c. Then by the definition of $\hat{\alpha}_T$, $G_s(c)$ is minimized at $c = 0$. Let $H_s(c)$ be the derivative of $G_s(c)$ with respect to c from the right. Then

$$H_s(c) = -T^{-1/2}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}\nabla_s q_{i,j,t}(\cdot,\hat{\alpha}_T+ce_s)\ \psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T+ce_s)),$$

where $\nabla_s q_{i,j,t}(\cdot,\hat{\alpha}_T+ce_s)$ is the s^{th} element of $\nabla q_{i,j,t}(\cdot,\hat{\alpha}_T+ce_s)$. Using the facts that (i) $H_s(c)$ is non-decreasing in c and (ii) for any $\epsilon > 0$, $H_s(-\epsilon) \le 0$ and $H_s(\epsilon) \ge 0$, we have

$$\begin{aligned}
|H_s(0)| &\le H_s(\epsilon) - H_s(-\epsilon) \\
&\le T^{-1/2}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}|\nabla_s q_{i,j,t}(\cdot,\hat{\alpha}_T)|1_{[Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T)=0]} \\
&\le T^{-1/2}\max_{1\le t\le T} D_{1,t}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}1_{[Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T)=0]},
\end{aligned}$$

where the last inequality follows by the domination condition imposed in Assumption 5(iii.a). Because $D_{1,t}$ is stationary, $T^{-1/2}\max_{1\leq t\leq T} D_{1,t} = o_p(1)$. The second term is bounded in probability: $\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p} 1_{[Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T)=0]} = O_p(1)$ given Assumption 2(i,ii.a) (see Koenker and Bassett, 1978, for details). Since $H_s(0)$ is the s^{th} element of $T^{-1/2}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}\nabla q_{i,j,t}(\cdot,\hat{\alpha}_T)\ \psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T))$, the claim in (15) is proved.

Next, for each $\alpha \in \mathbb{A}$, Assumptions 3(ii) and 5(iii.a) ensure the existence and finiteness of the $\ell \times 1$ vector

$$\begin{aligned}\lambda(\alpha) \quad &:= \sum_{i=1}^{n}\sum_{j=1}^{p} E[\nabla q_{i,j,t}(\cdot,\alpha)\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\alpha))] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{p} E[\nabla q_{i,j,t}(\cdot,\alpha)\int_{\delta_{i,j,t}(\alpha,\alpha^*)}^{0} f_{i,j,t}(s)ds],\end{aligned}$$

where $\delta_{i,j,t}(\alpha,\alpha^*) := q_{i,j,t}(\cdot,\alpha) - q_{i,j,t}(\cdot,\alpha^*)$ and $f_{i,j,t}(s) = (d/ds)F_{it}(s+q_{i,j,t}(\cdot,\alpha^*))$ represents the conditional density of $\varepsilon_{i,j,t} := Y_{it} - q_{i,j,t}(\cdot,\alpha^*)$ with respect to Lebesgue measure. The differentiability and domination conditions provided by Assumptions 3(iii) and 5(iv.a) ensure (e.g., by Bartle, 1966, corollary 5.9) the continuous differentiability of $\lambda(\alpha)$ on $\mathbb{A}$, with

$$\nabla\lambda(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{p} E[\nabla\{\nabla' q_{i,j,t}(\cdot,\alpha)\int_{\delta_{i,j,t}(\alpha,\alpha^*)}^{0} f_{i,j,t}(s)ds\}].$$

Since α^* is interior to $\mathbb{A}$ by Assumption 4(ii), the mean value theorem applies to each element of $\lambda(\alpha)$ to yield

$$\lambda(\alpha) = \lambda(\alpha^*) + Q_0(\alpha - \alpha^*), \tag{16}$$

for α in a convex compact neighborhood of α^* where Q_0 is an $\ell \times \ell$ matrix with $(1 \times \ell)$ rows $Q_s(\bar{\alpha}_{(s)}) = \nabla'\lambda(\bar{\alpha}_{(s)})$, where $\bar{\alpha}_{(s)}$ is a mean value (different for each s) lying on the segment connecting α and α^* with $s = 1, ..., \ell$. The chain rule and an application of the Leibniz rule to $\int_{\delta_{i,j,t}(\alpha,\alpha^*)}^{0} f_{i,j,t}(s)ds$ then give

$$Q_s(\alpha) = A_s(\alpha) - B_s(\alpha),$$

where

$$\begin{aligned}A_s(\alpha) \quad &:= \sum_{i=1}^{n}\sum_{j=1}^{p} E[\nabla_s\nabla' q_{i,j,t}(\cdot,\alpha)\int_{\delta_{i,j,t}(\alpha,\alpha^*)}^{0} f_{i,j,t}(s)ds] \\ B_s(\alpha) \quad &:= \sum_{i=1}^{n}\sum_{j=1}^{p} E[f_{i,j,t}(\delta_{i,j,t}(\alpha,\alpha^*))\nabla_s q_{i,j,t}(\cdot,\alpha)\nabla' q_{i,j,t}(\cdot,\alpha)].\end{aligned}$$

Assumption 2(iii) and the other domination conditions (those of Assumption 5) then ensure that

$$\begin{aligned}A_s(\bar{\alpha}_{(s)}) \quad &= \quad O(||\alpha - \alpha^*||) \\ B_s(\bar{\alpha}_{(s)}) \quad &= \quad Q_s^* + O(||\alpha - \alpha^*||),\end{aligned}$$

where $Q_s^* := \sum_{i=1}^n \sum_{j=1}^p E[f_{i,j,t}(0)\nabla_s q_{i,j,t}(\cdot,\alpha^*)\nabla' q_{i,j,t}(\cdot,\alpha^*)]$. Letting $Q^* := \sum_{i=1}^n \sum_{j=1}^p E[f_{i,j,t}(0)\nabla q_{i,j,t}(\cdot,\alpha^*)\nabla' q_{i,j,t}(\cdot,\alpha^*)]$, we obtain

$$Q_0 = -Q^* + O(||\alpha - \alpha^*||). \tag{17}$$

Next, we have that $\lambda(\alpha^*) = 0$. To show this, we write

$$\begin{aligned}
\lambda(\alpha^*) &= \sum_{i=1}^n \sum_{j=1}^p E[\nabla q_{i,j,t}(\cdot,\alpha^*)\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha^*))] \\
&= \sum_{i=1}^n \sum_{j=1}^p E(E[\nabla q_{i,j,t}(\cdot,\alpha^*)\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha^*)) \mid \mathcal{F}_{t-1}])
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \sum_{j=1}^p E(\nabla q_{i,j,t}(\cdot,\alpha^*)E[\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha^*)) \mid \mathcal{F}_{t-1}]) \\
&= \sum_{i=1}^n \sum_{j=1}^p E(\nabla q_{i,j,t}(\cdot,\alpha^*)E[\psi_{\theta_{ij}}(\varepsilon_{i,j,t}) \mid \mathcal{F}_{t-1}]) \\
&= 0,
\end{aligned}$$

as $E[\psi_{\theta_{ij}}(\varepsilon_{i,j,t}) \mid \mathcal{F}_{t-1}] = \theta_{ij} - E[1_{[Y_{it} \le q^*_{i,j,t}]} \mid \mathcal{F}_{t-1}] = 0$, by definition of $q^*_{i,j,t}$ for $i = 1,...,n$ and $j = 1,...,p$ (see eq. (3)). Combining $\lambda(\alpha^*) = 0$ with eqs. (16) and (17), we obtain

$$\lambda(\alpha) = -Q^*(\alpha - \alpha^*) + O(||\alpha - \alpha^*||^2). \tag{18}$$

The next step is to show that

$$T^{1/2}\lambda(\hat{\alpha}_T) + H_T = o_p(1) \tag{19}$$

where $H_T := T^{-1/2}\sum_{t=1}^T \eta_t^*$, with $\eta_t^* := \eta_t(\alpha^*)$ and $\eta_t(\alpha) := \sum_{i=1}^n \sum_{j=1}^p \nabla q_{i,j,t}(\cdot,\alpha)$ $\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha))$. Let $u_t(\alpha, d) := \sup_{\{\tau:||\tau-\alpha||\le d\}} ||\eta_t(\tau) - \eta_t(\alpha)||$. By the results of Huber (1967) and Weiss (1991), to prove (19) it suffices to show the following: (i) there exist $a > 0$ and $d_0 > 0$ such that $||\lambda(\alpha)|| \ge a||\alpha - \alpha^*||$ for $||\alpha - \alpha^*|| \le d_0$; (ii) there exist $b > 0$, $d_0 > 0$, and $d \ge 0$ such that $E[u_t(\alpha, d)] \le bd$ for $||\alpha - \alpha^*|| + d \le d_0$; and (iii) there exist $c > 0, d_0 > 0$, and $d \ge 0$ such that $E[u_t(\alpha, d)^2] \le cd$ for $||\alpha - \alpha^*|| + d \le d_0$.

The condition that Q^* is positive-definite in Assumption 6(i) is sufficient for (i). For (ii), we have that for given (small) $d > 0$

$$
\begin{aligned}
& u_t(\alpha, d) \\
\leq & \sup_{\{\tau:||\tau-\alpha||\leq d\}} \sum_{i=1}^{n}\sum_{j=1}^{p} ||\nabla q_{i,j,t}(\cdot,\tau)\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\tau))-\nabla q_{i,j,t}(\cdot,\alpha)\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\alpha))|| \\
\leq & \sum_{i=1}^{n}\sum_{j=1}^{p} \sup_{\{\tau:||\tau-\alpha||\leq d\}} ||\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\tau))|| \times \sup_{\{\tau:||\tau-\alpha||\leq d\}} ||\nabla q_{i,j,t}(\cdot,\tau)-\nabla q_{i,j,t}(\cdot,\alpha)|| \\
& + \sum_{i=1}^{n}\sum_{j=1}^{p} \sup_{\{\tau:||\tau-\alpha||\leq d\}} ||\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha)) - \psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\tau))|| \\
& \qquad \times \sup_{\{\tau:||\tau-\alpha||\leq d\}} ||\nabla q_{i,j,t}(\cdot,\alpha)|| \\
& \leq npD_{2,t}d + D_{1,t} \sum_{i=1}^{n}\sum_{j=1}^{p} 1_{[|Y_{it}-q_{i,j,t}(\cdot,\alpha)|<D_{1,t}d]}
\end{aligned}
$$

using the following: (i) $||\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\tau))|| \leq 1$; (ii) $||\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha)) - \psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\tau))|| \leq 1_{[|Y_{it}-q_{i,j,t}(\cdot,\alpha)|<|q_{i,j,t}(\cdot,\tau)-q_{i,j,t}(\cdot,\alpha)|]}$; and (iii) the mean value theorem applied to $\nabla q_{i,j,t}(\cdot,\tau)$ and $q_{i,j,t}(\cdot,\alpha)$. Hence, we have

$$E[u_t(\alpha, d)] \leq npC_0 d + 2npC_1 f_0 d$$

for some constants C_0 and C_1, given Assumptions 2(iii.a), 5(iii.a), and 5(iv.a). Hence, (ii) holds for $b = npC_0 + 2npC_1 f_0$ and $d_0 = 2d$. The last condition (iii) can be similarly verified by applying the c_r-inequality to eq. (**??**) with $d < 1$ (so that $d^2 < d$) and using Assumptions 2(iii.a), 5(iii.b), and 5(iv.b). Thus, eq. (19) is verified.

Combining eqs. (18) and (19) thus yields

$$Q^* T^{1/2}(\hat{\alpha}_T - \alpha^*) = T^{-1/2}\sum_{t=1}^{T} \eta_t^* + o_p(1).$$

But $\{\eta_t^*, \mathcal{F}_t\}$ is a stationary ergodic martingale difference sequence (MDS). In particular, η_t^* is measurable$-\mathcal{F}_t$, and $E(\eta_t^*|\mathcal{F}_{t-1}) = E(\sum_{i=1}^{n}\sum_{j=1}^{p} \nabla q_{i,j,t}(\cdot,\alpha^*)\psi_{\theta_{ij}}(\varepsilon_{i,j,t}) \mid \mathcal{F}_{t-1}) = \sum_{i=1}^{n}\sum_{j=1}^{p} \nabla q_{i,j,t}(\cdot,\alpha^*)E(\psi_{\theta_{ij}}(\varepsilon_{i,j,t}) \mid \mathcal{F}_{t-1}) = 0$, as $E[\psi_{\theta_{ij}}(\varepsilon_{i,j,t}) \mid \mathcal{F}_{t-1}] = 0$ for all $i = 1, ..., n$ and $j = 1, ..., p$. Assumption 5(iii.b) ensures that $V^* := E(\eta_t^* \eta_t^{*\prime})$ is finite. The MDS central limit theorem (e.g., theorem 5.24 of White, 2001) applies, provided V^* is positive definite (as ensured by Assumption 6(ii)) and that $T^{-1}\sum_{t=1}^{T} \eta_t^* \eta_t^{*\prime} = V^* + o_p(1)$, which is ensured by the ergodic theorem. The standard argument now gives

$$V^{*-1/2} Q^* T^{1/2}(\hat{\alpha}_T - \alpha^*) \xrightarrow{d} N(0, I),$$

which completes the proof. ■

Proof of Theorem 3: We have

$$\hat{V}_T - V^* = (T^{-1}\sum_{t=1}^{T}\hat{\eta}_t\hat{\eta}_t' - T^{-1}\sum_{t=1}^{T}\eta_t^*\eta_t^{*\prime}) + (T^{-1}\sum_{t=1}^{T}\eta_t^*\eta_t^{*\prime} - E[\eta_t^*\eta_t^{*\prime}]),$$

where $\hat{\eta}_t := \sum_{i=1}^{n}\sum_{j=1}^{p}\nabla\hat{q}_{i,j,t}\hat{\psi}_{i,j,t}$ and $\eta_t^* := \sum_{i=1}^{n}\sum_{j=1}^{p}\nabla q_{i,j,t}^*\psi_{i,j,t}^*$, with $\nabla\hat{q}_{i,j,t} := \nabla q_{i,j,t}(\cdot,\hat{\alpha}_T), \hat{\psi}_{i,j,t} := \psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\hat{\alpha}_T)), \nabla q_{i,j,t}^* := \nabla q_{i,j,t}(\cdot,\alpha^*)$, and $\psi_{i,j,t}^* := \psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha^*))$. Assumptions 1 and 2(i,ii) ensure that $\{\eta_t^*\eta_t^{*\prime}\}$ is a stationary ergodic sequence. Assumptions 3(i,ii), 4(i.a), and 5(iii) ensure that $E[\eta_t^*\eta_t^{*\prime}] < \infty$. It follows by the ergodic theorem that $T^{-1}\sum_{t=1}^{T}\eta_t^*\eta_t^{*\prime} - E[\eta_t^*\eta_t^{*\prime}] = o_p(1)$. Thus, it suffices to prove $T^{-1}\sum_{t=1}^{T}\hat{\eta}_t\hat{\eta}_t' - T^{-1}\sum_{t=1}^{T}\eta_t^*\eta_t^{*\prime} = o_p(1)$.

The (h,s) element of $T^{-1}\sum_{t=1}^{T}\hat{\eta}_t\hat{\eta}_t' - T^{-1}\sum_{t=1}^{T}\eta_t^*\eta_t^{*\prime}$ is

$$T^{-1}\sum_{t=1}^{T}\{\sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{l=1}^{n}\sum_{k=1}^{p}(\hat{\psi}_{i,j,t}\hat{\psi}_{l,k,t}\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{l,k,t} - \psi_{i,j,t}^*\psi_{l,k,t}^*\nabla_h q_{i,j,t}^*\nabla_s q_{l,k,t}^*)\}.$$

Thus, it will suffice to show that for each (h,s) and (i,j,l,k),

$$T^{-1}\sum_{t=1}^{T}\{\hat{\psi}_{i,j,t}\hat{\psi}_{l,k,t}\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{l,k,t} - \psi_{i,j,t}^*\psi_{l,k,t}^*\nabla_h q_{i,j,t}^*\nabla_s q_{l,k,t}^*\} = o_p(1).$$

By the triangle inequality,

$$|T^{-1}\sum_{t=1}^{T}\{\hat{\psi}_{i,j,t}\hat{\psi}_{l,k,t}\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{l,k,t} - \psi_{i,j,t}^*\psi_{l,k,t}^*\nabla_h q_{i,j,t}^*\nabla_s q_{l,k,t}^*\}| \le A_T + B_T,$$

where

$$\begin{aligned}
A_T &= T^{-1}\sum_{t=1}^{T}|\hat{\psi}_{i,j,t}\hat{\psi}_{l,k,t}\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{l,k,t} - \psi_{i,j,t}^*\psi_{l,k,t}^*\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{l,k,t}| \\
B_T &= T^{-1}\sum_{t=1}^{T}|\psi_{i,j,t}^*\psi_{l,k,t}^*\nabla_h q_{i,j,t}^*\nabla_s q_{l,k,t}^* - \psi_{i,j,t}^*\psi_{l,k,t}^*\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{l,k,t}|.
\end{aligned}$$

We now show that $A_T = o_p(1)$ and $B_T = o_p(1)$, delivering the desired result. For A_T, the triangle inequality gives

$$A_T \le A_{1T} + A_{2T} + A_{3T},$$

where

$$\begin{aligned}
A_{1T} &= T^{-1}\sum_{t=1}^{T}\theta_{ij}|1_{[\varepsilon_{i,j,t}\le 0]} - 1_{[\hat{\varepsilon}_{i,j,t}\le 0]}||\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{l,k,t}| \\
A_{2T} &= T^{-1}\sum_{t=1}^{T}\theta_{lk}|1_{[\varepsilon_{l,k,t}\le 0]} - 1_{[\hat{\varepsilon}_{l,k,t}\le 0]}||\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{l,k,t}| \\
A_{3T} &= T^{-1}\sum_{t=1}^{T}|1_{[\varepsilon_{i,j,t}\le 0]}1_{[\varepsilon_{l,k,t}\le 0]} - 1_{[\hat{\varepsilon}_{i,j,t}\le 0]}1_{[\hat{\varepsilon}_{l,k,t}\le 0]}||\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{l,k,t}|.
\end{aligned}$$

Theorem 2, ensured by Assumptions $1-6$, implies that $T^{1/2}||\hat{\alpha}_T - \alpha^*|| = O_p(1)$. This, together with Assumptions 2(iii,iv) and 5(iii.b), enables us to apply the same techniques used in Kim and White (2003) to show $A_{1T} = o_p(1)$, $A_{2T} = o_p(1)$, and $A_{3T} = o_p(1)$, implying $A_T = o_p(1)$.

It remains to show $B_T = o_p(1)$. By the triangle inequality,

$$B_T \leq B_{1T} + B_{2T},$$

where

$$\begin{aligned} B_{1T} &= T^{-1}\sum_{t=1}^{T}|\psi^*_{i,j,t}\psi^*_{l,k,t}\nabla_h q^*_{i,j,t}\nabla_s q^*_{l,k,t} - E[\psi^*_{i,j,t}\psi^*_{l,k,t}\nabla_h q^*_{i,j,t}\nabla_s q^*_{l,k,t}]| \\ B_{2T} &= T^{-1}\sum_{t=1}^{T}|\psi^*_{i,j,t}\psi^*_{l,k,t}\nabla_h \hat{q}_{i,j,t}\nabla_s \hat{q}_{l,k,t} - E[\psi^*_{i,j,t}\psi^*_{l,k,t}\nabla_h q^*_{i,j,t}\nabla_s q^*_{l,k,t}]|. \end{aligned}$$

Assumptions 1, 2(i,ii), 3(i,ii), 4(i.a), and 5(iii) ensure that the ergodic theorem applies to $\{\psi^*_{i,j,t}\psi^*_{l,k,t}\nabla_h q^*_{i,j,t}\nabla_s q^*_{l,k,t}\}$, so $B_{1T} = o_p(1)$. Next, Assumptions 1, 3(i,ii), and 5(iii) ensure that the stationary ergodic ULLN applies to $\{\psi^*_{i,j,t}\psi^*_{l,k,t}\nabla_h q_{i,j,t}(\cdot,\alpha)$ $\nabla_s q_{l,k,t}(\cdot,\alpha)\}$. This and the result of Theorem 1 ($\hat{\alpha}_T - \alpha^* = o_p(1)$) ensure that $B_{2T} = o_p(1)$ by e.g., White (1994, corollary 3.8), and the proof is complete. ■

To establish consistency of $\hat{Q}_T$, we strengthen the domination condition on $\nabla q_{i,j,t}$ and impose conditions on $\{\hat{c}_T\}$.

Assumption 5 (iii.c) $E(D^3_{1,t}) < \infty$.

Assumption 7 $\{\hat{c}_T\}$ is a stochastic sequence and $\{c_T\}$ is a non-stochastic sequence such that (i) $\hat{c}_T/c_T \xrightarrow{p} 1$; (ii) $c_T = o(1)$; and (iii) $c_T^{-1} = o(T^{1/2})$.

Proof of Theorem 4: We begin by sketching the proof. We first define

$$Q_T := (2c_T T)^{-1}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p} 1_{[-c_T\leq\varepsilon_{i,j,t}\leq c_T]}\nabla q^*_{i,j,t}\nabla' q^*_{i,j,t},$$

and then we will show the following:

$$Q^* - E(Q_T) \xrightarrow{p} 0, \tag{20}$$

$$E(Q_T) - Q_T \xrightarrow{p} 0, \tag{21}$$

$$Q_T - \hat{Q}_T \xrightarrow{p} 0. \tag{22}$$

Combining the results above will deliver the desired outcome: $\hat{Q}_T - Q^* \xrightarrow{p} 0$.

For (20), one can show by applying the mean value theorem to $F_{i,j,t}(c_T) - F_{i,j,t}(-c_T)$, where $F_{i,j,t}(c) := \int 1_{\{s\leq c\}} f_{i,j,t}(s)ds$, that

$$E(Q_T) = T^{-1}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p} E[f_{i,j,t}(\xi_{i,j,T})\nabla q^*_{i,j,t}\nabla' q^*_{i,j,t}] = \sum_{i=1}^{n}\sum_{j=1}^{p} E[f_{i,j,t}(\xi_{i,j,T})\nabla q^*_{i,j,t}\nabla' q^*_{i,j,t}],$$

where $\xi_{i,j,T}$ is a mean value lying between $-c_T$ and c_T, and the second equality follows by stationarity. Therefore, the (h,s) element of $|E(Q_T) - Q^*|$ satisfies

$$\begin{aligned}
& |\sum_{i=1}^{n}\sum_{j=1}^{p} E\{f_{i,j,t}(\xi_{i,j,T}) - f_{i,j,t}(0)\nabla_h q^*_{i,j,t}\nabla_s q^*_{i,j,t}\}| \\
\leq & \sum_{i=1}^{n}\sum_{j=1}^{p} E\{|f_{i,j,t}(\xi_{i,j,T}) - f_{i,j,t}(0)||\nabla_h q^*_{i,j,t}\nabla_s q^*_{i,j,t}|\} \\
\leq & \sum_{i=1}^{n}\sum_{j=1}^{p} L_0 E\{|\xi_{i,j,T}||\nabla_h q^*_{i,j,t}\nabla_s q^*_{i,j,t}|\} \\
\leq & \; npL_0 c_T E[D^2_{1,t}],
\end{aligned}$$

which converges to zero as $c_T \to 0$. The second inequality follows by Assumption 2(iii.b), and the last inequality follows by Assumption 5(iii.b). Therefore, we have the result in eq.(20).

To show (21), it suffices simply to apply a LLN for double arrays, e.g. theorem 2 in Andrews (1988).

Finally, for (22), we consider the (h,s) element of $|\hat{Q}_T - Q_T|$, given by

$$\begin{aligned}
& |\frac{1}{2\hat{c}_T T}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p} 1_{[-\hat{c}_T \leq \hat{\varepsilon}_{i,j,t} \leq \hat{c}_T]}\nabla_h \hat{q}_{i,j,t}\nabla_s \hat{q}_{i,j,t} \\
& \qquad - \frac{1}{2c_T T}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p} 1_{[-c_T \leq \varepsilon_{i,j,t} \leq c_T]}\nabla_h q^*_{i,j,t}\nabla_s q^*_{i,j,t}| \\
& = \frac{c_T}{\hat{c}_T} \times |\frac{1}{2c_T T}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}(1_{[-\hat{c}_T \leq \hat{\varepsilon}_{i,j,t} \leq \hat{c}_T]} - 1_{[-c_T \leq \varepsilon_{i,j,t} \leq c_T]})\nabla_h \hat{q}_{i,j,t}\nabla_s \hat{q}_{i,j,t} \\
& \quad + \frac{1}{2c_T T}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p} 1_{[-c_T \leq \varepsilon_{i,j,t} \leq c_T]}(\nabla_h \hat{q}_{i,j,t} - \nabla_h q^*_{i,j,t})\nabla_s \hat{q}_{i,j,t} \\
& \quad + \frac{1}{2c_T T}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p} 1_{[-c_T \leq \varepsilon_{i,j,t} \leq c_T]}\nabla_h q^*_{i,j,t}(\nabla_s \hat{q}_{i,j,t} - \nabla_s q^*_{i,j,t}) \\
& \quad + \frac{1}{2c_T T}(1 - \frac{\hat{c}_T}{c_T})\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p} 1_{[-c_T \leq \varepsilon_{i,j,t} \leq c_T]}\nabla_h q^*_{i,j,t}\nabla_s q^*_{i,j,t}| \\
& \leq \frac{c_T}{\hat{c}_T}[A_{1T} + A_{2T} + A_{3T} + (1 - \frac{\hat{c}_T}{c_T})A_{4T}],
\end{aligned}$$

where

$$
\begin{aligned}
A_{1T} &:= \frac{1}{2c_T T}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}|1_{[-\hat{c}_T\leq\hat{\varepsilon}_{i,j,t}\leq\hat{c}_T]}-1_{[-c_T\leq\varepsilon_{i,j,t}\leq c_T]}|\times|\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{i,j,t}| \\
A_{2T} &:= \frac{1}{2c_T T}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}1_{[-c_T\leq\varepsilon_{i,j,t}\leq c_T]}|\nabla_h\hat{q}_{i,j,t}-\nabla_h q^*_{i,j,t}|\times|\nabla_s\hat{q}_{i,j,t}| \\
A_{3T} &:= \frac{1}{2c_T T}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}1_{[-c_T\leq\varepsilon_{i,j,t}\leq c_T]}|\nabla_h q^*_{i,j,t}|\times|\nabla_s\hat{q}_{i,j,t}-\nabla_s q^*_{i,j,t}| \\
A_{4T} &:= \frac{1}{2c_T T}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}1_{[-c_T\leq\varepsilon_{i,j,t}\leq c_T]}|\nabla_h q^*_{i,j,t}\nabla_s q^*_{i,j,t}|.
\end{aligned}
$$

It will suffice to show that $A_{1T}=o_p(1), A_{2T}=o_p(1), A_{3T}=o_p(1)$, and $A_{4T}=O_p(1)$. Then, because $\hat{c}_T/c_T\xrightarrow{p}1$, we obtain the desired result: $\hat{Q}_T-Q^*\xrightarrow{p}0$.

We first show $A_{1T}=o_p(1)$. It will suffice to show that for each i and j,

$$\frac{1}{2c_T T}\sum_{t=1}^{T}|1_{[-\hat{c}_T\leq\hat{\varepsilon}_{i,j,t}\leq\hat{c}_T]}-1_{[-c_T\leq\varepsilon_{i,j,t}\leq c_T]}|\times|\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{i,j,t}|=o_p(1).$$

Let α_T lie between $\hat{\alpha}_T$ and α^*, and put $d_{i,j,t,T}:=||\nabla q_{i,j,t}(\cdot,\alpha_T)||\times||\hat{\alpha}_T-\alpha^*||+|\hat{c}_T-c_T|$. Then

$$(2c_T T)^{-1}\sum_{t=1}^{T}|1_{[-\hat{c}_T\leq\hat{\varepsilon}_{i,j,t}\leq\hat{c}_T]}-1_{[-c_T\leq\varepsilon_{i,j,t}\leq c_T]}|\times|\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{i,j,t}|\ \leq U_T+V_T,$$

where

$$
\begin{aligned}
U_T &:= (2c_T T)^{-1}\sum_{t=1}^{T}1_{[|\varepsilon_{i,j,t}-c_T|<d_{i,j,t,T}]}|\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{i,j,t}| \\
V_T &:= (2c_T T)^{-1}\sum_{t=1}^{T}1_{[|\varepsilon_{i,j,t}+c_T|<d_{i,j,t,T}]}|\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{i,j,t}|.
\end{aligned}
$$

It will suffice to show that $U_T\xrightarrow{p}0$ and $V_T\xrightarrow{p}0$. Let $\eta>0$ and let z be an arbitrary positive number. Then, using reasoning similar to that of Kim and White (2003, lemma 5), one can show that for any $\eta>0$,

$$
\begin{aligned}
P(U_T &> \eta)\leq P((2c_T T)^{-1}\sum_{t=1}^{T}1_{[|\varepsilon_{i,j,t}-c_T|<(||\nabla q_{i,j,t}(\cdot,\alpha_T)||+1)zc_T]})|\nabla_h\hat{q}_{i,j,t}\nabla_s\hat{q}_{i,j,t}|>\eta) \\
&\leq \frac{zf_0}{\eta T}\sum_{t=1}^{T}E\left\{(||\nabla q_{i,j,t}(\cdot,\alpha_T)||+1)|\nabla_h\hat{q}_{j,t}\nabla_s\hat{q}_{j,t}|\right\} \\
&\leq zf_0\{E|D^3_{1,t}|+E|D^2_{1,t}|\}/\eta,
\end{aligned}
$$

where the second inequality is due to the Markov inequality and Assumption 2(iii.a), and the third is due to Assumption 5(iii.c). As z can be chosen arbitrarily small and the remaining terms are finite by assumption, we have $U_T \xrightarrow{p} 0$. The same argument is used to show $V_T \xrightarrow{p} 0$. Hence, $A_{1T} = o_p(1)$ is proved.

Next, we show $A_{2T} = o_p(1)$. For this, it suffices to show $A_{2T,i,j} := \frac{1}{2c_T T} \sum_{t=1}^{T} 1_{[-c_T \leq \varepsilon_{i,j,t} \leq c_T]} |\nabla_h \hat{q}_{i,j,t} - \nabla_h q^*_{i,j,t}| \times |\nabla_s \hat{q}_{i,j,t}| = o_p(1)$ for each i and j. Note that

$$
\begin{aligned}
A_{2T,i,j} &\leq \frac{1}{2c_T T} \sum_{t=1}^{T} |\nabla_h \hat{q}_{i,j,t} - \nabla_h q^*_{i,j,t}| \times |\nabla_s \hat{q}_{i,j,t}| \\
&\leq \frac{1}{2c_T T} \sum_{t=1}^{T} ||\nabla_h^2 q_{i,j,t}(\cdot, \tilde{\alpha})|| \times ||\hat{\alpha}_T - \alpha^*|| \times |\nabla_s \hat{q}_{i,j,t}| \\
&\leq \frac{1}{2c_T} ||\hat{\alpha}_T - \alpha^*|| \frac{1}{T} \sum_{t=1}^{T} D_{2,t} D_{1,t} \\
&= \frac{1}{2c_T T^{1/2}} T^{1/2} ||\hat{\alpha}_T - \alpha^*|| \frac{1}{T} \sum_{t=1}^{T} D_{2,t} D_{1,t},
\end{aligned}
$$

where $\tilde{\alpha}$ is between $\hat{\alpha}_T$ and α^*, and $\nabla_h^2 q_{j,t}(\cdot, \tilde{\alpha})$ is the first derivative of $\nabla_h \hat{q}_{j,t}$ with respect to α, which is evaluated at $\tilde{\alpha}$. The last expression above is $o_p(1)$ because: (i) $T^{1/2}||\hat{\alpha}_T - \alpha^*|| = O_p(1)$ by Theorem 2; (ii) $T^{-1} \sum_{t=1}^{T} D_{2,t} D_{1,t} = O_p(1)$ by the ergodic theorem; and (iii) $1/(c_T T^{1/2}) = o_p(1)$ by Assumption 7(iii). Hence, $A_{2T} = o_p(1)$. The other claims $A_{3T} = o_p(1)$ and $A_{4T} = O_p(1)$ can be analogously and more easily proven. Hence, they are omitted. Therefore, we finally have $Q_T - \hat{Q}_T \xrightarrow{p} 0$, which, together with (20) and (21), implies that $\hat{Q}_T - Q^* \xrightarrow{p} 0$. The proof is complete. ■

Table 1 – Breakdown of financial institutions by sector and by geographic area

	Banks	*Financial Services*	*Insurances*	
EU	47	22	27	**96**
North America	25	17	28	**70**
Asia	47	14	3	**64**
	119	**53**	**58**	**230**

Note: Classification as provided by Datastream. Swiss and Norwegian financial institutions have been classified as EU. Asia includes Australian financial institutions.

Table 2 – Estimates and standard errors for the Sumitomo Mitsui Financial Group

	c_1	a_{11}	a_{12}	b_{11}	b_{12}
	-2.71	**-0.89**	-0.13	**0.61**	**-0.11**
s.e.	*1.48*	*0.37*	*0.13*	*0.12*	*0.06*
	c_2	a_{21}	a_{22}	b_{21}	b_{22}
	-1.05	0.07	**-0.12**	**-0.08**	**0.93**
s.e.	*0.48*	*0.10*	*0.05*	*0.04*	*0.03*

Note: Coefficients significant at the 5% level formatted in bold.

Table 3 – Ranking of financial institutions according to impulse responses

	Financial institutions most sensitive to index's shocks	*Financial institutions least sensitive to index's shocks*	*Financial institutions with greatest impact on index VaR*	*Financial institutions with lowest impact on index's risk*
1	ING GROEP	WESTPAC BANKING	WESTPAC BANKING	SWISS RE 'R'
2	BARCLAYS	CHINA EVERBRIGHT	SLM	UNICREDIT
3	UNICREDIT	CREDITO VALTELLINES	CREDITO VALTELLINES	ING GROEP
4	STATE STREET	BANCA PPO.EMILIA ROMAGNA	BANCA PPO.EMILIA ROMAGNA	SUMITOMO TRUST & BANKING
5	CITIGROUP	PROVIDENT FINANCIAL	SUNCORP-METWAY	SUMITOMO MITSUI FINL.GP.
6	KBC GROUP	HIROSHIMA BANK	NATIONAL AUS.BANK	HYAKUJUSHI BANK
7	XL GROUP	BANCO ESPIRITO SANTO	DAISHI BANK	DBS GROUP HOLDINGS
8	BANK OF AMERICA	SUMITOMO MITSUI FINL.GP.	AGEAS (EX-FORTIS)	SURUGA BANK
9	ROYAL BANK OF SCTL.GP.	AWA BANK	BANCO POPULAR ESPANOL	BBV.ARGENTARIA
10	SWISS RE 'R'	NATIONAL AUS.BANK	MACQUARIE GROUP	COMPUTERSHARE
11	HARTFORD FINL.SVS.GP.	DAISHI BANK	ICAP	MIZUHO TST.& BKG.
12	AGEAS (EX-FORTIS)	HYAKUGO BANK	AUS.AND NZ.BANKING GP.	NANTO BANK
13	SLM	FAIRFAX FINL.HDG.	BANCO ESPIRITO SANTO	MEDIOBANCA
14	COMMERZBANK (XET)	CLOSE BROTHERS GROUP	AMP	BALOISE-HOLDING AG
15	ABERDEEN ASSET MAN.	AMLIN	CHINA EVERBRIGHT	INVESTOR 'B'
16	ORIX	ALPHA BANK	MAN GROUP	BANK OF KYOTO
17	BANK OF IRELAND	GOLDMAN SACHS GP.	FIFTH THIRD BANCORP	ABERDEEN ASSET MAN.
18	LINCOLN NAT.	HACHIJUNI BANK	BANK OF IRELAND	UNITED OVERSEAS BANK
19	AEGON	RENAISSANCERE HDG.	BANKINTER 'R'	IYO BANK
20	KEYCORP	VALIANT 'R'	SAMPO 'A'	KINNEVIK 'B'

Note: The ranking is obtained by first integrating out the impulse response functions and then sorting the financial institutions by the resulting number. The first and second columns report the top 20 financial institutions whose VaR reacts most and least strongly to a shock to the index. The third and fourth columns contain the 20 financial institutions whose shocks have the largest and smallest impact on the index VaR.

Figure 1 – 1% quantile for the overall index and Sumitomo Mitsui Financial Group

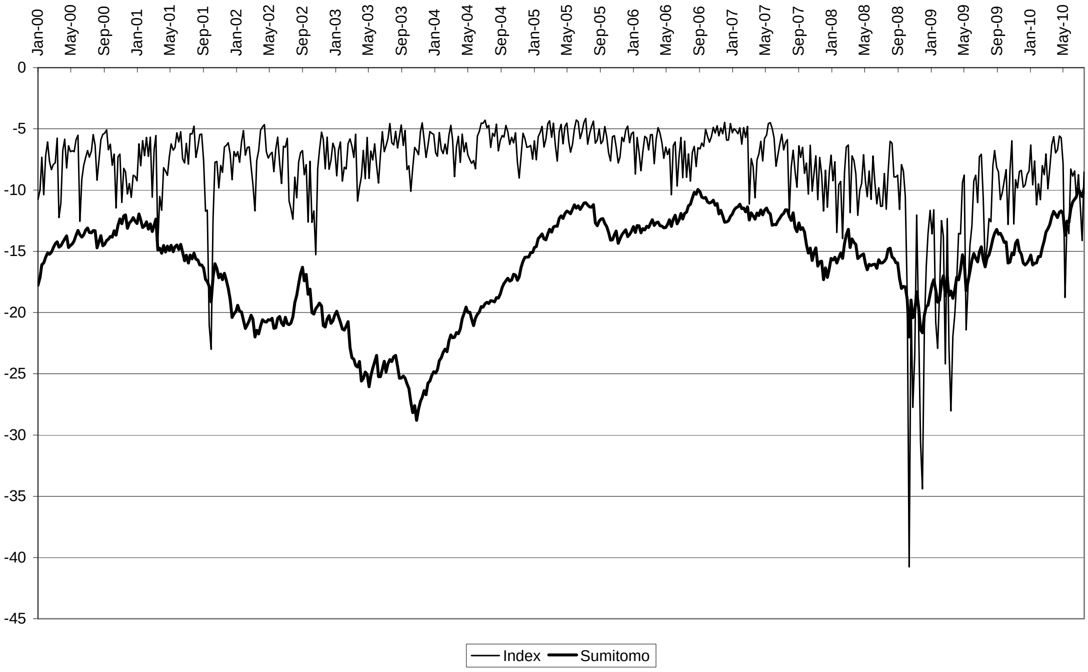
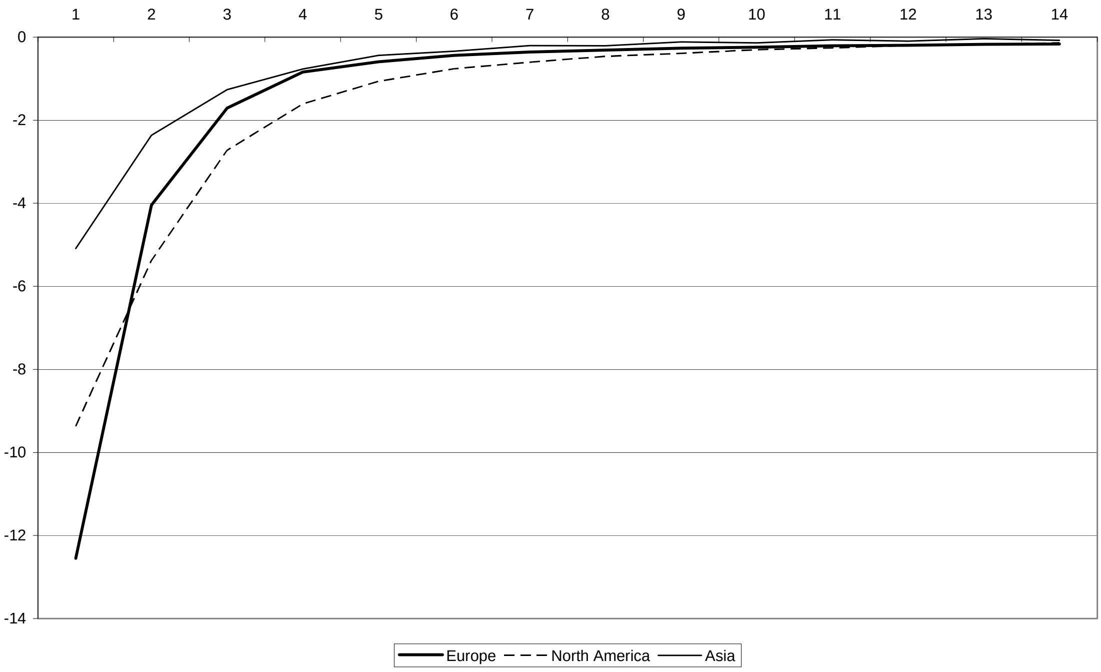
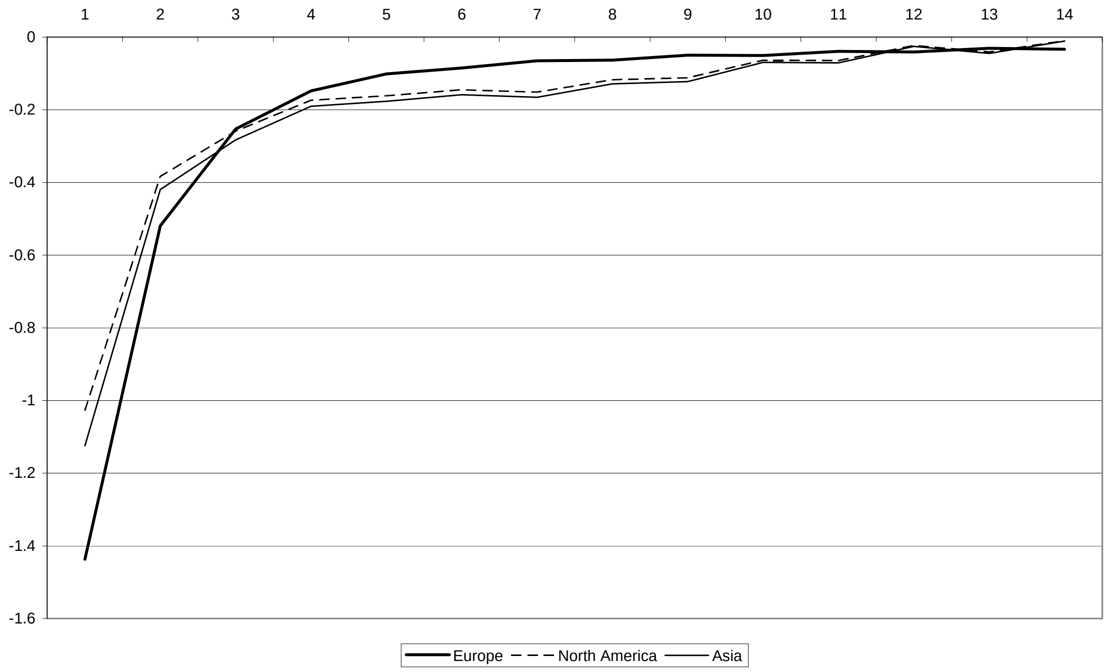


Figure 2 – Geographical breakdown of VaR impulse-responses: <u>shock to the index</u>



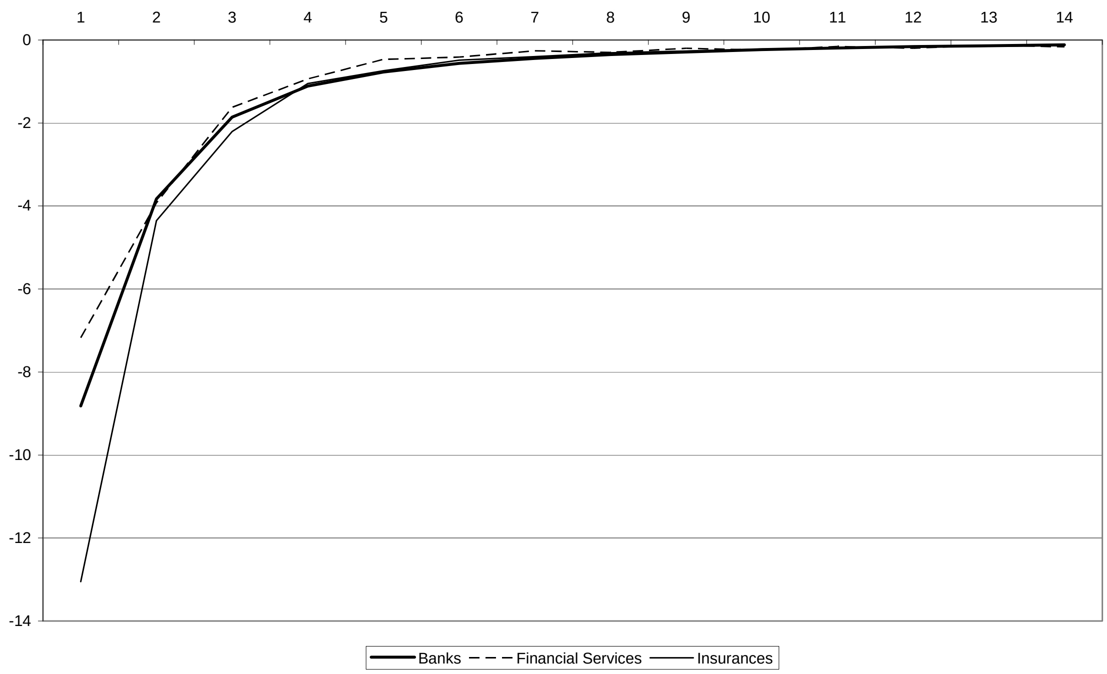
Note: The figure reports the average impulse-response function of European, North American and Asian financial institutions to a shock to the index.

Figure 3 – Geographical breakdown of VaR impulse-responses: <u>shock to the financial institution</u>



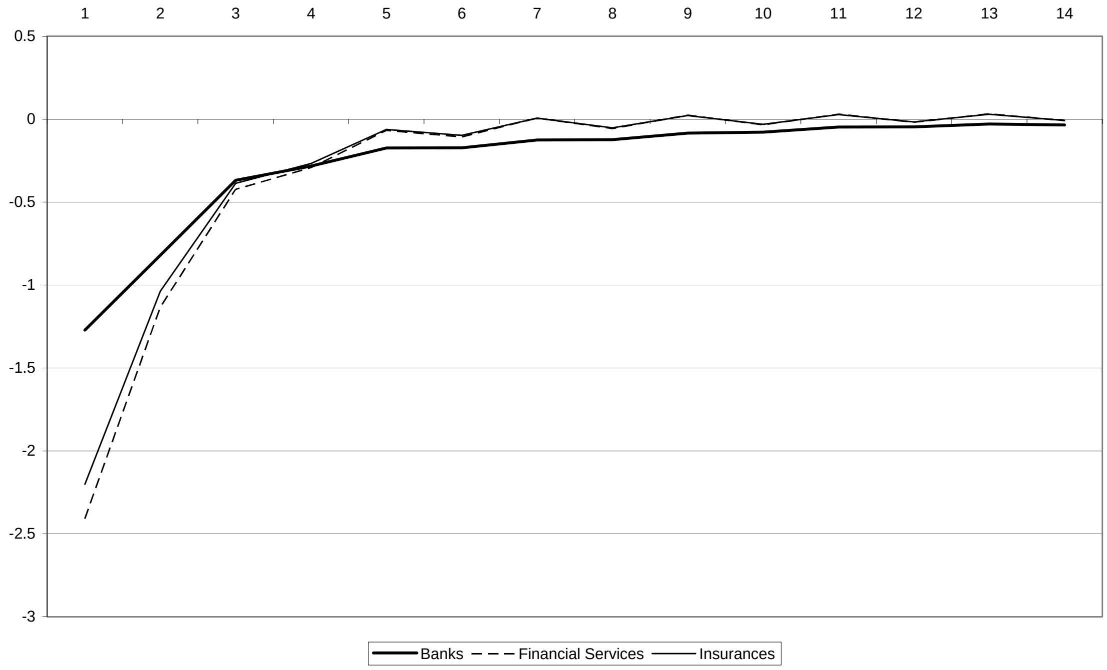
Note: The figure reports the average impulse-response function of the index to a shock to European, North American and Asian financial institutions.

Figure 4 – Breakdown by institution of VaR impulse-responses: <u>shock to the index</u>



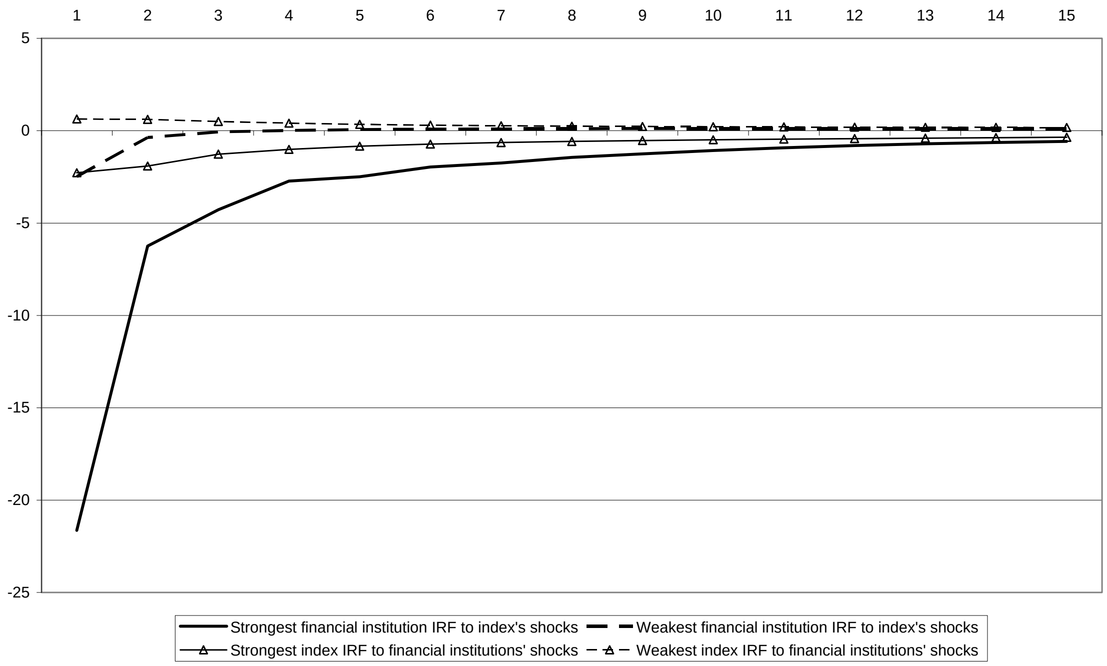
Note: The figure reports the average impulse-response function by sector to a shock to the index.

Figure 5 – Breakdown by institution of VaR impulse-responses: <u>shock to the bank</u>



Note: The figure reports the average impulse-response function of the index to a shock to the financial institutions classified by sector.

Figure 6 – Strongest and weakest VaR impulse-responses



Note: The figure reports the average impulse-response function of the 20 financial institutions with the strongest and weakest impact.