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Kurrild-Klitgaard, Peter
University of Copenhagen

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Election inversions, coalitions and proportional representation: Examples from Danish elections¹

PETER KURRILD-KLITGAARD

Dept. of Political Science, University of Copenhagen

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Abstract. When collective choices are made in more than one round and with different groups of decision-makers, so-called election inversions may take place, where each group have different majority outcomes. We identify two versions of such compound majority paradoxes specifically, but not exclusively, relevant for systems of proportional representation with governing coalitions: The “Threshold Paradox” and the “Federal Paradox”. The empirical relevance of the two paradoxes is illustrated with examples from three Danish elections (1971, 1990, 2011), where a majority of the voters voted for one bloc of parties but where a majority of the seats fell to another.

Keywords: Social choice; voting paradoxes; electoral systems; election inversions.

JEL-codes: D71; D72.

1. Introduction

The central, defining and legitimating characteristic of modern democracies, especially those aiming at proportional representation, is usually claimed to be that those governing in some meaningful sense represent a majority of the voters (cf. Dahl 1956; Dahl 1989; Riker 1982). Given this, it would certainly seem an odd kind of democracy if an election produced another government than what a majority of the voters had voted for.

However, for centuries—and especially since the 1950s—mathematicians and social scientists have been researching the possibilities of counterintuitive discrepancies between the

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preferences of individual decision-makers and the collective choices they make. Most famous is no doubt Condorcet's Paradox, i.e., the phenomenon where preferences may be "cyclical" so that there may not be any alternative that cannot be beaten by at least one other alternative (cf. Condorcet [1785] 1994; Condorcet [1789] 1994; Gehrlein 1983; Van Deemen 1998).² Most such studies have dealt with voting on legislative proposals or elections in plurality/majority systems (cf., e.g., Riker 1982; Nurmi 1987). However, recent work has also turned the attention to the types of voting paradoxes that may occur in systems of proportional representation (e.g., Van Deemen 1993; Van Deemen and Vergunst 1998; Härd 2000; Kurrild-Klitgaard 2008). Common for these studies is the identification of the potential discrepancy between the voters' preferences and the election outcomes. The present paper identifies two versions of a particular kind of voting paradoxes: so-called "compound majority paradoxes" that may occur in both first-past-the-post electoral systems (FPTP) and proportional representation systems (PR) but which are perhaps mostly overlooked in the latter. The two paradoxes are illustrated with data derived from Danish elections (1971, 1990) and election surveys (2011).

2. Voting paradoxes and election inversions

One broad type of voting paradoxes is what has been called "compound majority paradoxes" (e.g., Lagerspetz 1996; Nurmi 1997a; Nurmi 1997b; Nurmi 1999: Ch. 7). Such paradoxes deal with the way in which issues are being voted upon, specifically the agenda of voting (i.e., how issues are presented to the voters), and how outcomes may be sensitive to differences in the agendas (cf.

² Condorcet's Paradox has been particularly influential through its role in Kenneth Arrow's General Impossibility Theorem (Arrow [1951] 1963).

Nurmi 1999: 70; Laffond and Lainé 2008: 447). Specifically, various types of compound majority paradoxes may occur when decisions take place in two (or more) rounds (or tiers), and where there preferences of the relevant decision-makers in the one do not correspond to the preferences of the relevant decision-makers in the other round.

2.1. Some compound majority paradoxes

The most well-known compound majority paradox is Ostrogorski's Paradox. This shows that for a given set of voter preferences it is possible for one party to be the majority winner in a two-party contest while the other party is preferred by a majority of the voters on a majority of the issues (Nurmi 1999: 70ff). A closely related paradox is Anscombe's Paradox, where a majority of the voters may be frustrated on a majority of issues (e.g., a group of voters faced with choice of voting "yes" or "no" to a number of issues), and where a majority of the group may lose on a majority of the issues considered (Nurmi 1999: 73ff). Related is also the so-called Paradox of Multiple Elections, where an outcome consisting of several issues may receive unanimous support from a group of voters when compared with an alternative, but where the outcome simultaneously would coincide with no voter's opinions (Brams, Kilgour and Zwicker 1998; Nurmi 1999: 74ff).

Finally, the so-called Referendum Paradox deals with the fact that in many democratic systems direct and indirect forms of voting are intertwined, e.g., where voters first elect one group of representatives who then proceeds to make other types of choices, and where there may be a discrepancy between the latter and the majority view of the voters (Nurmi 1999: 76ff; Nurmi 1997b; Nurmi 1997b). An illustration could be a consultative referendum, where a majority of the voters vote in favor of an alternative, but where the parliament subsequently rejects it. In itself this is not necessarily paradoxical but it is so if each MP acts as he does because he is

following the opinions of a majority of his voters. Nurmi has used a simple illustration of the paradox, where 3 million voters are to elect 200 MPs, and where we for simplicity assume that each MP represents exactly 15,000 voters, and that there is a simply binary choice between “yes” and “no”, as in Table 1.

Table 1. Referendum Paradox: Hypothetical example.

Referendum stand	MP 1	...	MP 167	MP 168	...	MP 200
Yes	7,000	...	7,000	15,000	...	15,000
No	8,000	...	8,000	0	...	0

Source: Nurmi 1997b; Nurmi 1999.

If the preferences and votes are as in the example, the proposal will pass a referendum with 55 pct. of the 3 million votes cast in favor and 45 pct. opposed. However, if the MPs follow the opinions of their constituents no less than 5/6 of them (167 of 200) will vote “No”. This paradoxical outcome is in reality related to the well-known, so-called “ecological fallacy”, i.e., where erroneous conclusions are reached about micro-properties based in observations about macro-properties (cf. Nurmi 1999: 78f). It is also similar to the phenomenon known in medicine, statistics and probability theory as Simpson’s Paradox, and which may be transferred to voting (cf. Nurmi 1999: 78f): A correlation present in different groups may be reversed when the groups are combined.

In the following we shall identify two different but related compound majority paradoxes which bear similarities to, e.g., the Referendum Paradox but which may be particularly relevant for systems with proportional representation and parliamentary government. In such cases there may be a discrepancy between (1) what type of governing coalition a majority of the voters vote for, and (2) what type of governing coalition a majority of the electors prefer.

The phenomenon is an example of what is often called “election inversions”, i.e., cases “when the candidate (or party) that wins the most votes from the nationwide electorate fails to win the most electoral votes (or parliamentary seats) and therefore loses the election.” (Miller 2011a: 1). Election inversions may occur, e.g., in the US presidential system, where it has happened on occasion that the (plurality) winner of the popular vote has not been the same as the one winning a majority of the electors’ votes in the Electoral College (e.g., 1876, 1888, 2000 and arguably also 1960) (Riker 1982; Longley and Peirce 1999; Feix et al. 2004; Miller 2011a; Miller 2011b; cf. Gaines 2001). Election inversions have also occurred in parliamentary elections and are familiar from political systems with first-past-the-post electoral systems (FPTP), e.g., UK (1929, 1951, 1974), New Zealand (1978, 1981), Canada (1979) and France (cf. Feix et al. 2004; Miller 2011a).

2.2. Some sources of election inversions

In practice there are several possible sources for the occurrence of election inversions.

Faithless decision-makers: Most obviously, there is the risk that electors/parliamentarians may behave differently at a later round than they had promised at the earlier round, or that the promises they made were not able to be considered. Examples are, e.g., so-called “faithless electors” in US presidential elections, where individuals who have been pledged to vote for a particular candidate for president subsequently have votes for other candidates (Longley and Peirce 1999). This may also occur in parliamentary or quasi-parliamentary systems, where individual politicians jump ship and vote for someone else than their party’s candidate for, say, prime minister or mayor. However, strictly speaking these are not “paradoxical” results, since they do not deal with the logic of the decision-making process and accordingly will be disregarded here.

Wasted votes: In FPTP systems many votes are often “wasted” and election inversions can occur easily—in the sense that many voters’ intentions may not be reflected in the allocation of seats because votes cast for losing candidates are not transferred to the party’s candidates elsewhere. However, “wasted votes” may also occur in PR systems, e.g., if the electoral system includes a formal threshold and one or more parties running in the election fail to pass that threshold. Even if there is no formal threshold the phenomenon may occur due to parties (or candidates) failing to qualify for the requirements for a single seat or due to “rounding effects” in the proportional allocation of seats.

Asymmetric representation: Many electoral systems have aspects that make some seats relatively “cheaper” (or more “costly”) to get than others, measured in terms of the votes needed, and this is found in both FPTP and PR systems. Such asymmetries may be due to, e.g., a more or less deliberate overrepresentation of smaller “units” (as is the case with small states in the US Electoral College or in the US Senate). It can also be due to a deliberate overrepresentation of geographically particular parts of the country (e.g., rural areas in the Norwegian *Storting*, some countries in the EU’s allocation of votes in the Council of Ministers and seats in the European Parliament, or the semi-autonomous Greenland and Faroe Islands in the Danish *Folketing*). Finally, it may be due to a failure to redraw electoral district borders and reallocate seats proportionally to fit demographic changes (e.g., UK where the seats/voter ratio for years has been to the advantage of the Labour Party).

2.3. Coalitions and compound majorities

Most studies of election inversions and compound majority paradoxes have focused on the potential discrepancy between (a) the preferences of the voters in a majority of electoral districts and (b) the preferences of a majority of the voters at a national level. It is, as indicated, easy to identify such examples in FPTP systems but

less so in PR systems which by their very nature have it as their purpose to establish an allocation of seats at a national level that is proportional. Nonetheless, we shall demonstrate here that some types of election inversions may occur under PR systems too, specifically if elections become a choice between parties forming governing coalitions but where the choice confronting the voters on the ballot is not between competing coalitions but competing parties, and where every vote cast may not weigh equally heavy in the collective decision.

In PR systems it is rare that a single party obtains enough votes to form a majority government alone, and accordingly governments tend to be coalition governments based in the seats allocated to two or more parties. Furthermore, many proportional systems tend to go hand in hand with parliamentary systems, whereby a government is formed on the basis of the parliamentary strength of alternative coalitions. Given such circumstances it is easy to construct a hypothetical situation where there is a difference between what government is preferred by a majority of the voters and what government is preferred (or formed) by a majority of the members of parliament.

Table 2. Threshold Paradox: Hypothetical example.

	Votes, parties (pct.)	Votes, coalitions (pct.)	Seats, parties	Seats, coalitions
Party A (coalition X)	47%	<u>51%</u>	49	49
Party B (coalition X)	4%		0	
Party C (coalition Y)	49%	49%	52	<u>52</u>
Total	100%	100%	101	101

Note: The formal threshold for representation is set a 5%. Majority winners are emphasized.

Let us first look at what may happen when votes are “wasted”, e.g., due to formal thresholds. Consider Table 1, which posits an election

for a parliament with three parties, *A*, *B* and *C*, competing for 101 seats allocated proportionally and with a 5 pct. formal threshold, and where *A* and *B* are collaborating in one coalition, *X*, whereas party *C* represents another government alternative, *Y*: In this hypothetical scenario the two parties in coalition *X*, *A* and *B*, together obtain 51 pct. of the votes cast, while their opponent, *C*, receives 49 pct. However, given that party *B* only receives 4 pct., and hence fails to pass the formal threshold, party *C* receives an absolute majority of the seats on the basis of a simple plurality of votes. Accordingly, *C* can form a government with a majority of the seats, even though a majority of the voters voted for an alternative coalition. The obvious but counterintuitive paradox thus is that a majority of the voters wanted one thing but got the opposite.

Table 3. Federal Paradox: Hypothetical example.

	Votes, parties	Votes, parties (pct. at regional level)	Votes, parties (pct. at national level)	Seats, parties (regiona l level)	Vote share, parties (national level)	Seats, parties (national level)
<i>Region I (96 seats)</i>						
Party A	975,000	51.32%	49.97%	49	<u>A:</u> <u>976,000</u> (50.02%)	A: 49 <u>B: 52</u>
Party B	925,000	48.68%	47.41%	47	B: 975,000 (49.97%)	
<i>Region II (5 seats)</i>						
Party A	1,000	1.96%	0.05%	0		
Party B	50,000	98.04%	2.56%	5		
Total	1,951,000	-	100%	101	1,951,000 (100%)	101

Note: Majority winners are emphasized.

Now, let us consider an alternative scenario relating to the issue of asymmetric representation. Specifically, let us hypothesize a country with a political system composed of two electorally different types of electoral districts: One region that is

overrepresented in terms of the seats/voters ratio, and another that is underrepresented. Table 3 gives a hypothetical example of what we for lack of a better description may call the Federal Paradox—“Federal” because it involves the representation of different units at a higher, supra-regional level. In the example two parties, *A* and *B*, compete for 101 seats in a parliament, but where 96 seats are allocated proportionally in one part of the country, while five seats are allocated proportionally in another region.

Party *A* receives a majority of the grand total of 1,951,000 votes cast (50.02 pct.), but because of the asymmetry in the seats/voter ratio between the two regions, *A* only wins 49 of the 101 seats. Party *B*, in other words, controls a majority of the seats (52) although only supported by a minority of the voters (49.97 pct.). The paradox thus, again, is the counterintuitive result that one coalition receives most votes but another coalition wins most seats.

Both paradoxes identified here may be seen as variations of the Referendum Paradox. They share the election inversion as a feature: That there is a discrepancy between two different “tiers”—one where the voters’ intentions are imputed into the process and one where the MPs form a governing majority coalition. The difference vis-à-vis the Referendum Paradox is that the cause of the potentially paradoxical outcomes are the institutional arrangements (threshold leading to wasted votes; uneven representation of geographical areas).

When seen in this perspective—votes for parties as being votes for governing coalitions—the two versions also connect to some more general examples of voting paradoxes found specifically in systems of proportional representation: The More-Preferred-Less-Seats Paradox (where a party *A* is preferred by a majority to a party *B*, but where *B* wins more seats than *A*); the Condorcet-Winner-Turns-Loser Paradox (where a party *A* is preferred by a majority to all other parties when compared in pairwise contests, but where another party wins more seats than *A*); the Condorcet-Loser-Turns-Winner Paradox (where all other parties are preferred by a majority to party *A* when compared in pairwise contests, but where *A* wins

more seats than at least one other party) (cf. Van Deemen 1993; Kurrild-Klitgaard 2008). In this case, we are just focusing on government coalitions as alternatives rather than individual parties.³

3. Empirical illustrations from Danish elections

The two paradoxes identified here may occur in a number of political systems, and in the case of the Danish political system it is certainly the case that they may both occur, even simultaneously.⁴

The Danish *Folketing* is composed of a total of 179 seats, where the 175 seats are allocated proportionally in a two-tier system and with a 2 pct. formal threshold being the primary hurdle that needs to be passed in order to obtain representation.⁵ These 175 seats are allocated (using Hare's Quota, i.e., the Largest Remainder) with: a) 135 seats at the one tier in multimember constituencies, and b) 40 seats at the other (national) tier as so-called adjustment (or compensatory) seats. The former are re-allocated between the electoral districts on the basis of population every five years. Together these features produce one of the electoral systems in the world with the smallest amount of disproportionality between votes and seats (Laakso and Taagepera 1978; Farrell 2001: 157; Grofman and Lijphart 2002; cf. Benoit 2000). The remaining four seats are,

³ We do not have the preference orderings of the voters over all parties, not even over the coalitions. We only have their demonstrated preferences, in the form of the actual votes, and then with an assumption that a vote for a party taking part in a specific parliamentary coalition equals a first preference of that coalition over the alternative.

⁴ For other social choice theoretic studies of voting paradoxes in Danish politics, see, e.g., Kurrild-Klitgaard 1999; Kurrild-Klitgaard 2001; Kurrild-Klitgaard 2005.

⁵ For introductions to the Danish electoral system, including changes, see Elklit 1993; Elklit 2002; Elklit 2006; Elklit 2008. On the procedures and empirics of government formation in Denmark, see Skjæveland 2003.

independently of the underlying population size, allocated to the two autonomous regions within the Kingdom of Denmark, Greenland and the Faroe Islands, with two each (allocated using d'Hondt's Formula). As is the case with, e.g., US senators, there is a resulting unevenness in the "cost" of a seat in the *Folketing*, depending on which one of the three "parts" of the Kingdom of Denmark is considered, with one of the four seats from the North Atlantic costing only about half of one of the 175 "continental" ones.⁶

A government is formed under a system of "negative parliamentarism", where the right to form a government is given to the party leader with most members of parliament supporting and as long as a majority does not express its lack of confidence in the government. Given these features it is obvious that the Danish political system is potentially vulnerable to the occurrence of both paradoxes, but have they actually manifested themselves?

It is fairly easy to identify the presence or absence of the Threshold Paradox. All that is needed is information about what parties supported what coalitions at the time of government formation, their vote shares, and the number of seats awarded to them. The former is the most difficult, given that coalitions to some extent are endogenous to government formation (cf. Laver and Schofield [1998] 1990; Laver and Shepsle 1996; Skjæveland 2003),⁷ perhaps especially given that Denmark practices "negative

⁶ At the 2011 General Election the price of a seat in the *Folketing* was, based in the number of votes cast: Denmark: $3,545,368/175 = 20,259.2$ votes; Greenland: $22,301/2 = 11,150.5$ votes; Faroe Islands: $20,644/2 = 10,322$ votes. If measured as number of eligible voters relative to seats, the difference is less dramatic but still with the same logic: Denmark: $4,082,518/175 = 23,328.7$ votes; Greenland: $40,930/2 = 20,465$ votes; Faroe Islands: $35,047/2 = 17,523.5$ votes. Cf. the official election results at the website of the Ministry of the Interior: <http://www.dst.dk/valg/Valg1204271/valgopg/valgopgHL.htm>

⁷ On the location of the parties in the Danish political "space", see the comparison of alternative data and methods by Kurrild-Klitgaard, Klemmensen and Pedersen 2008.

parliamentarism". The latter fact has occasionally meant lengthy coalition negotiations with the Radicals, as the median party, trying to broker governments other than to the left or to the right (e.g. in 1975 and 1988).

The Federal Paradox is harder to detect in the real world given that the political parties running in Greenland and on the Faroe Islands are not the same (and that parties did not run at all in Greenland before 1977), and neither are they identical to the parties standing for elections in "continental" Denmark. For that reason judgments need to be made as to what parliamentary coalitions the four North Atlantic seats would likely caucus or collaborate with. Similarly, it has often been stated by the parties of Greenland and the Faroe Islands that they do not want to interfere in "domestic" Danish politics and usually claim to not to seek to be kingmakers. Indeed, until the 1960s the North Atlantic MPs usually did not intervene to save the lives of Danish governments (Skjæveland 2003: 114-24). For the present purposes the historical cases have been examined, and when it doubt it has been assumed that they will support the government coalition of the party they usually caucus with or would seem to be ideologically aligned with.

Given these preliminaries we may consider some general elections where one or both of the paradoxes considered here may have occurred. Let us first consider the 1971 General Election, where a total of 2,882,981 voters in "continental" Denmark cast their votes (cf. Table 4). Of these a narrow majority, 1,453,162 (50.4%), cast their votes for parties that supported (or would have supported if they had been represented) the incumbent center-right government led by Hilmar Baunsgaard, while 1,423,076 voters (49.4%) voted for socialist parties supporting a government led by former Prime Minister Jens Otto Krag. The result was 88 seats for the center-right and 87 seats for the left, and hence there was no instance of the Threshold Paradox.

However, an election inversion took place relative to how the Danes had cast their votes: Krag came to form a government because he obtained support from both MPs from Greenland (with

one of them being made a cabinet minister in return) and one of the two MPs from the Faroe Islands. That gave Krag a government with 90 of the 179 MPs supporting him (Skjæveland 2003: 116f).

Table 4. Danish General Election result 21.IX.1971 (votes excl. the Faroe Islands and Greenland).

Parties	Votes	Vote share	Seats
Social Democrats (A)	1,074,777	37.3%	70
Radicals (B)	413,620	14.4%	27
Conservatives (C)	481,335	16.7%	31
Justice Party (E)	50,231	1.7%	0
Socialist People's Party (F)	262,756	9.1%	17
Danish Communist Party (K)	39,564	1.4%	0
Christian People's Party (Q)	57,072	1.9%	0
German Minority Party (P)	6,743	0.2%	0
Liberal Party (V)	450,904	15.6%	30
Left Socialists (Y)	45,679	1.4%	0
Total, left (A + F + K + Y)	1,423,076	49.4%	87
Total, center-right (B + C + E + Q + V)	<u>1,453,162</u>	<u>50.4%</u>	<u>88</u>
Others (P)	6,743	0.2%	0

Source: The Danish Parliament (www.ft.dk). Note: Majority winners are emphasized.

In other words: A majority of the Danish voters voted for a center-right government but ended up with a socialist government. That raises the question of how the North Atlantic votes added up? In the Faroe Islands 5,897 voted for parties aligned with the Danish center-right, while 4,170 voted for parties that supported the socialists, and 3,058 voted for parties or candidates that were not represented and whose favored governments seem unclear. The same calculation cannot be made for the votes from Greenland, because MPs before 1977 were not elected for political parties but as individual candidates. If the votes from the Faroe Islands are added

to the results of Table 4 a Baunsgaard government was supported by parties collecting a total of 1,459,059 voters (50.4% of the combined Danish and Faroe Island votes), while a Krag government was supported by parties receiving 1,427,246 votes (49.3%). A total of 9,801 voters (0.3%) from Denmark and the Faroe Islands cannot be counted for the one or the other coalition. Given that voter turnout in Greenland in the 1970s never was above 20,000 votes, and often closer to 10,000 votes, there is no way in which candidates supporting Krag could have delivered enough votes for him to surpass a Baunsgaard government in the popular vote. In other words, the 1971 election is an example of the Federal Paradox.

Another case is the 1990 election where there is clear evidence that the Threshold Paradox occurred, cf. the vote shares and seats for the political parties given in Table 5. Although the election saw solid gains for the Social Democrats and a majority of the popular vote for the socialists (1,630,219 votes), too many leftist parties failed to clear the 2 pct. threshold, and the total “wasted votes” due to this for just two, the populist Common Course and the Communist Unity List, amounted to 3.5 pct. of the votes, corresponding to six seats. The center-right parties, which supported the incumbent Conservative Prime Minister Poul Schlüter, only collected a total of 1,609,219 (49.7%) of the popular vote but won a majority (91) of the 175 seats. The four North Atlantic seats split equally between left and right and did not affect the outcome, neither in seats or votes.⁸ In other

⁸ In Greenland the parties of the left, *Siumut* and *Inuit Ataqatigiit*, received 8,128 and 3,281 votes respectively. The parties of the right, *Atassut* and *Issittup Partiia*, received 7,087 votes and 366 votes respectively, while 333 voted for other candidates. In the Faroe Islands, the parties of the left, *Javnaðarflokkurin* (4,835) and *Sjálvstýrisflokkurin* (1,240), while the parties of the right were *Sambandsflokkurin* (4,558), *Kristiligi Fólkaflokkurin* (285), *Fólkaflokkurin* (4,582). In total that would be 17,484 votes for the left, 16,878 for the right, and (ca.) 333 for other candidates. In other words, Faroese and Greenlandic votes for the right were insufficient to give the Danish center-right parliamentary majority a majority of the total popular vote; if anything the left’s vote margin increased.

words, the Threshold Paradox manifested itself, but not the Federal Paradox.

Table 5. Danish General Election result 12.XII.1990 (votes excl. the Faroe Islands and Greenland).

Parties	Votes	Vote share	Seats
Social Democrats (A)	1,221,121	37.4%	69
Radicals (B)	114,888	3.5%	7
Conservatives (C)	517,293	16.0%	30
Center-Democrats (D)	165,556	5.1%	9
Justice Party (E)	17,181	0.5%	0
Socialist People's Party (F)	268,759	8.3%	15
Greens (G)	27,642	0.9%	0
Humanist Party (H)	763	0.0%	0
Common Course (P)	57,896	1.8%	0
Christian People's Party (Q)	74,174	2.3%	4
Liberal Party (V)	511,643	15.8%	29
Progress Party (Z)	208,484	6.4%	12
Unity List (Ø)	54,038	1.7%	0
Total, left (A + F + G + H + P + Ø)	<u>1,630,219</u>	<u>50.3%</u>	84
Total, center-right (B + C + D + E + Q + V + Z)	1,609,219	49.7%	<u>91</u>

Source: The Danish Parliament (www.ft.dk). Note: Majority winners are emphasized.

A third example stems from the aftermath of the 2011 General Election. The election itself was free of the voting paradoxes identified here but subsequent polling produced results, which illustrate the paradoxes very well. Table 6 summarizes the results of five surveys taken within a month after the General Election 15.IX.2011. A center-left coalition led by the Social Democrats' Helle Thorning-Schmidt had won a surprisingly narrow victory (89-86) over the incumbent Liberal-Conservative government headed by Lars Løkke Rasmussen, after having held comfortable leads in the polls for a year and a half, and from the very first polls published

after the election—while the negotiations were still going on—the public mood swung back giving a lead to the center-right again. Table 6 summarizes the results of five polls from September-October 2011 that all gave a popular vote majority to the coalition of parties that supported a government headed by Rasmussen. However, three of these polls simultaneously—due to rounding effects—gave a majority of the 175 seats elected in “continental” Denmark to Thorning-Schmidt’s “red bloc” (88-87), while it gave Rasmussen’s “blue bloc” an equally narrow majority of the seats in two other polls.

The results show that in three of the five polls considered here the Threshold Paradox was present in the votes and seats relating to Denmark itself: In two polls “blue bloc” had a majority of the votes and the closest possible majority of (non-North Atlantic) seats (88-87), but in the three others “blue bloc” had a majority of the votes while suffering an election inversion in terms of seats (88-87 in favor of “red bloc”).

Furthermore, the Federal Paradox seems to occur in all five polls considered, at least if we simply add the four North-Atlantic seats with their 3-1 split to the two Danish parliamentary “blocs”. Doing so, “blue bloc” had a majority of the Danish votes cast (ranging from 50.0% to 50.4%) but would only win 88-89 of the 179 seats, i.e., just one or two short of a majority.

Table 6. Five Danish opinion polls, September-October 2011.

<i>Parties, Denmark</i>	Voxmeter, 28.IX.2011		Gallup, 6.X.2011		Greens, 7.X.2011		Voxmeter, 9.X.2011		Rambøll, 13.X.2011	
	Vote share	Seats	Vote share	Seats	Vote share	Seats	Vote share	Seats	Vote share	Seats
Social Democrats (A)	23.8%	42	23.0%	40	24.4%	43	22.9%	41	23.0%	41
Radicals (B)	9.1%	16	10.7%	19	10.1%	18	10.3%	18	10.0%	18
Conservatives (C)	4.0%	7	3.5%	6	3.3%	6	4.0%	7	3.6%	6
Socialist People's Party (F)	8.7%	15	8.9%	16	7.9%	14	8.6%	15	9.4%	17
Liberal Alliance (I)	5.1%	9	5.1%	9	4.6%	8	4.2%	8	4.2%	7
Christian Democrats (K)	0.5%	0	0.6%	0	0.3%	0	0.5%	0	0.6%	0
Danish People's Party (O)	11.7%	21	12.2%	21	11.3%	20	11.5%	20	12.3%	22
Liberal Party (V)	28.8%	51	28.9%	51	30.5%	55	30.2%	53	29.7%	52
Unity List (Ø)	8.1%	14	7.1%	13	7.5%	13	7.5%	13	6.9%	12
Others	0.2%	0	0.0%	0	0.1%	0	0.3%	0	0.3%	0
Total, "Red bloc" (A + B + F + Ø)	49.7%	87	49.7%	88	49.9%	88	49.3%	88	49.3%	87
Total, "Blue bloc" (C + I + K + O + V)	50.1%	88	50.3%	87	50.0%	87	50.4%	87	50.4%	88
Total, "Red bloc" + 2 (Greenland) + 1 (Faroe Islands)	-	90	-	91	-	91	-	91	-	91
Total, "Blue bloc" + 1 (Faroe Islands)	-	89	-	88	-	88	-	88	-	88

Source: Berlingske Barometer (www.b.dk). Note: Majority winners are emphasized.

That these are not simply “fluke” results derived from some cherry picked polls may be seen by comparing them to other polls from the same period and seeing these in total. This is done in Appendix A which includes the average from all opinion polls taken in the first month after the 2011 General Election. Taking the simple arithmetic average of all polls the “blue bloc” would receive 50.3 pct. of the votes, while the “red bloc” would receive 49.0 pct. Splitting the 175 seats between them would not produce an election inversion: The “blue” would win 88 versus 87 for the “red”. However, adding the four North Atlantic seats with a 3-1 split would then result in an 89 versus 90 seats win for the left and thereby something that seems like a case of the Federal Paradox.⁹

However, it might be argued that this is a slightly uneven tabulation of the vote shares since we are thus considering seats from the entire realm but only vote shares from Denmark. In order to properly test whether the Federal Paradox occurs when the entire Danish-Greenland-Faroe Islands electorates are considered it is necessary to make some simplifications and assumptions. However, we are confronted with uneven data: We have information about preferences in surveys when it comes to Danish voters—but not the exact number of voters that would vote as such. In contrast, we do not have opinion polls for Greenland and the Faroe Islands matching those in Denmark—where we only have the exact votes from the 15.IX.2011 election to rely on.

In order to simulate what the “Greater-Denmark” results would have been in the aftermath of the election, one possibility is to assume the number of valid votes cast in the 2011 election in “continental” Denmark as given (3,545,368) and then “translate” the vote shares from the individual polls back into simulated votes for each party in each poll. To this we then add the valid votes from the

⁹ A second average of the polls was calculated, weighing each poll by the number of respondents. However, given that all polls had relatively similar sample sizes (roughly between 1,000 and 1,500), the end result was the same for all practical purposes.

actual election results from Greenland and the Faroe Islands from the 2011 election and assume—for lack of better data—that these are identical for all the polling times. Doing so we obtain a total number of shares for the parties, from which we may recalculate the total vote shares. These figures, given in Appendix B, should, of course, be taken with considerable caution; they are artificial. Nonetheless, the end-results largely mirror those of Table 6: In the four of the five cases considered the “blue bloc”-parties and their North Atlantic collaborators would receive either a plurality (one) of the votes or an outright majority (three), but in all five a majority of the seats (91-88 or 90-89) would go to the “red bloc”.

In other words, the polls considered give us reason to believe that if the 2011 General Election had been held a month later, or if a new election had been called, the outcome might have been one, where one or even both compound majority paradoxes identified here might have manifested themselves: That one political wing (the center-right), with parties preferring one government, would have received a majority of the votes cast, while the parties of the other wing (the center-left) would have received a majority of the seats, either because of disproportionality due to rounding problems caused by the formal threshold, or because of the built-in asymmetry when it comes to seats/votes ratios.

4. Conclusion

The present article has identified two types of compound majority voting paradoxes that may occur in PR systems: The Threshold Paradox and the Federal Paradox, which both identify situations where a majority of the voters vote for parties supporting one government alternative but where a majority of the seats are allocated to parties supporting another. As such they demonstrate that political systems with proportional representation and parliamentarism, such as those found in much of the West, are not

immune to the voting paradoxes, neither in theory nor in practice. Indeed, they have been manifest in at least two Danish elections over the last 40 years (1971 and 1990), and they seem to have been abundantly present in opinion polls of the Danish voters in the aftermath of the 2011 election.

In what William Riker called the “populist” interpretation of democracy the central aspect of democracy as the method of making authoritative choices is that it in some meaningful sense reflects “the will of the people” (Riker 1982). The Danish political system is frequently praised in such “populist” terms as being one of the most “fair” in the world, due to a high degree of proportionality between votes cast and seats won. However, the present analysis has demonstrated that even the Danish system may suffer from the defect that while a majority of the voters may vote for one outcome, the seats allocated may produce the exact opposite outcome.

Appendix A.

Average, all Danish published opinion polls, 15th September – 15th October 2011

Party	Vote shares	Seats
Social Democrats (A)	23.3%	41
Radicals (B)	10.1%	18
Conservatives (C)	3.9%	7
Socialist People's Party (F)	8.4%	15
Liberal Alliance (I)	4.8%	8
Christian Democrats (K)	0.6%	0
Danish People's Party (O)	12.0%	21
Liberal Party (V)	29.7%	52
Unity List (Ø)	7.3%	13
Others	0.2%	0
Total, "Red bloc" (A + B + F + Ø)	49.0%	87
Total, "Blue bloc" (C + I + K + O + V)	50.3%	88
Total, "Red bloc" + 2 (Greenland) + 1 (Faroe Islands)	-	90
Total, "Blue bloc" + 1 (Faroe Islands)	-	89

Sources: Voxmeter/Ritzau 27.IX.2011, 9.X.2011; Gallup/Berlingske 8.X.2011; Greens/Børsen 7.X.2011, 14.X.2011; Megafon/TV 2 & Politiken 7.X.2011; Rambøll/Jyllands-Posten 15.X.2011; YouGov/MetroXpress 12.X.2011.

Note: Majority winners are emphasized.

Appendix B.

Simulation of Danish election outcomes, based in 2011 General Election and five polls, September-October 2011

Polis, Denmark, October-September 2011	Voxmeter, 28.IX.2011		Gallup, 6.X.2011		Greens, 7.X.2011		Voxmeter, 9.X.2011		Rambøll, 13.X.2011	
<i>Parties, Denmark:</i>	Vote share	Simulated votes*	Vote share	Simulated votes*	Vote share	Simulated votes*	Vote share	Simulated votes*	Vote share	Simulated votes*
Social Democrats (A)	23.8%	843,798	23.0%	815,435	24.4%	865,070	22.9%	811,889	23.0%	815,435
Radicals (B)	9.1%	322,628	10.7%	379,354	10.1%	358,082	10.3%	365,173	10.0%	354,537
Conservatives (C)	4.0%	141,815	3.5%	124,088	3.3%	116,997	4.0%	141,815	3.6%	127,633
Socialist People's Party (F)	8.7%	308,447	8.9%	315,538	7.9%	280,084	8.6%	304,902	9.4%	333,265
Liberal Alliance (I)	5.1%	180,814	5.1%	180,814	4.6%	163,087	4.2%	148,905	4.2%	148,905
Christian Democrats (K)	0.5%	17,727	0.6%	21,272	0.3%	10,636	0.5%	17,727	0.6%	21,272
Danish People's Party (O)	11.7%	414,808	12.2%	432,535	11.3%	400,627	11.5%	407,717	12.3%	436,080
Liberal Party (V)	28.8%	1,021,066	28.9%	1,024,611	30.5%	1,081,337	30.2%	1,070,701	29.7%	1,052,974
Unity List (Ø)	8.1%	287,175	7.1%	251,721	7.5%	265,903	7.5%	265,903	6.9%	244,630
Others	0.2%	7,091	0.0%	-	0.1%	3,545	0.3%	10,636	0.3%	10,636
Total, "Red bloc" (A + B + F + Ø)	49.7%	1,762,048	49.7%	1,762,048	49.9%	1,769,139	49.3%	1,747,867	49.3%	1,747,867
Total, "Blue bloc" (C + I + K + O + V)	50.1%	1,776,230	50.3%	1,783,320	50.0%	1,772,684	50.4%	1,786,865	50.4%	1,786,864
Others	0.2%	7,091	0.0%	-	0.1%	3,545	0.3%	10,636	0.3%	10,636
North Atlantic election results, 15.IX.2011	Vote share	Votes	Vote share	Votes	Vote share	Votes	Vote share	Votes	Vote share	Votes
<i>Parties, Faroe Islands:</i>										
Sambandsflokkurin#	30.8%	6,361	30.8%	6,361	30.8%	6,361	30.8%	6,361	30.8%	6,361
Javnaðarflokkurin§	21.0%	4,328	21.0%	4,328	21.0%	4,328	21.0%	4,328	21.0%	4,328
Tjóðveldi§	19.4%	3,998	19.4%	3,998	19.4%	3,998	19.4%	3,998	19.4%	3,998
Fólkaflokkurin#	19.0%	3,932	19.0%	3,932	19.0%	3,932	19.0%	3,932	19.0%	3,932
Miðflokkurin	4.2%	872	4.2%	872	4.2%	872	4.2%	872	4.2%	872
Sjálvstýrisflokkurin	2.3%	481	2.3%	481	2.3%	481	2.3%	481	2.3%	481
Others	3.3%	672	3.3%	672	3.3%	672	3.3%	672	3.3%	672
<i>Parties, Greenland:</i>										
Inuit Ataqatigiit§	42.7%	9,780	42.7%	9,780	42.7%	9,780	42.7%	9,780	42.7%	9,780
Siumut§	37.1%	8,499	37.1%	8,499	37.1%	8,499	37.1%	8,499	37.1%	8,499
Demokraatit#	12.6%	2,882	12.6%	2,882	12.6%	2,882	12.6%	2,882	12.6%	2,882
Atassut#	7.6%	1,728	7.6%	1,728	7.6%	1,728	7.6%	1,728	7.6%	1,728
Others	0.1%	24	0.1%	24	0.1%	24	0.1%	24	0.1%	24
Real plus simulated results:										
"Red bloc"	49.8%	1,788,653	49.8%	1,788,653	50.0%	1,795,744	49.4%	1,774,471	49.4%	1,774,471
"Blue bloc"	49.9%	1,791,132	50.1%	1,798,223	49.8%	1,787,587	50.2%	1,801,768	50.2%	1,801,768
Others	0.3%	9,140	0.1%	2,049	0.2%	5,594	0.4%	12,685	0.4%	12,685
Total	100.0%	3,588,925	100.0%	3,588,925	100.0%	3,588,925	100.0%	3,588,925	100.0%	3,588,925

Sources: The Danish Parliament (www.ft.dk); Berlingske Barometer (www.b.dk). Notes: Majority winners are emphasized. Total number of valid votes cast in 2011 General Election in Denmark (excl. the Faroe Islands and Greenland): 3,545,368. * Rounded number of votes. § Presumed to caucus with Danish "red bloc" parties; # Presumed to caucus with Danish "blue bloc" parties.

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