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The Deterministic Impulse Control Maximum Principle in Operations Research: Necessary and Sufficient Optimality Conditions

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Abstract

This paper considers a class of optimal control problems that allows jumps in the state variable. We present the necessary optimality conditions of the Impulse Control Maximum Principle based on the current value formulation. By reviewing the existing impulse control models in the literature, we point out that meaningful problems do not satisfy the sufficiency conditions. In particular, such problems either have a concave cost function, contain a fixed cost, or have a control-state interaction, which have in common that they each violate the concavity hypotheses used in the sufficiency theorem. The implication is that the corresponding problem in principle has multiple solutions that satisfy the necessary optimality conditions. Moreover, we argue that problems with fixed cost do not satisfy the conditions under which the necessary optimality conditions can be applied. However, we design a transformation, which ensures that the application of the Impulse Control Maximum Principle still provides the optimal solution. Finally, we show for the first time that for some existing models in the literature no optimal solution exists.

Key words: Impulse Control Maximum Principle, Optimal Control, discrete continuous system, state-jumps, present value formulation

JEL-codes: C61, D90

1 Introduction

For many problems in the area of economics and operations research it is realistic to allow for jumps in the state variable. This paper therefore considers optimal control models in which the time moment of these jumps as well as the size of the jumps are taken as (new) decision variables. An example is Blaquière (1979) that deals with optimal maintenance and life time of machines. Here the firm has to decide when a certain machine has to be repaired (impulse

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control variable), and it has to determine the rate of maintenance expenses (ordinary control variable), so that the profit is maximized over the planning period. Blaquière (1977a; 1977b; 1979; 1985) extends the standard theory on optimal control by deriving a Maximum Principle, the so called Impulse Control Maximum Principle, that gives necessary (and sufficient) optimality conditions for solving such problems. Like Blaquière (1977a; 1977b; 1979; 1985), we consider a framework where the number of jumps is not restricted. This distinguishes our approach from, e.g., Liu et al. (1998), Augustin (2002, pp. 71-81) and Wu and Teo (2006), where the number of jumps is fixed (i.e. is taken as given).

This contribution focuses on deterministic impulse control problems that are analyzed by using the Impulse Control Maximum Principle. This implies that we do not consider stochastic impulse control problems. This excludes the theory of real options (see Dixit and Pindyck (1994)) and also the theory of Quasi-Variational Inequalities (QVI) (see Bensoussan and Lions (1984)). It is well known that in a stochastic framework these methodologies are much more useful than the stochastic Maximum Principle. Other insightful QVI references include Bensoussan et al. (2006) on an inventory model employing an (s, S) policy and Øksendal and Sulem (2007).

The contribution of this paper is fourfold. *First*, we give a correct formulation of the necessary optimality conditions of the Impulse Control Maximum Principle based on the current value formulation. In this way we correct Feichtinger and Hartl (1986, appendix 6) and Kort (1989, pp. 62-70). *Second*, by reviewing the existing impulse control models in the literature, we point out that meaningful problems do not satisfy the sufficiency conditions. In particular, such problems either have a concave cost function, contain a fixed cost, or have a control-state interaction that each violate the concavity hypotheses used in the sufficiency theorem. The implication of not satisfying the sufficiency conditions is that the corresponding problem in principle has multiple solutions that satisfy the necessary optimality conditions. In many cases, these multiple solutions can be represented by a so called tree-structure (see, e.g., Luhmer (1986), Kort (1989), Chahim et al. (2011). *Third*, we show for the first time that several existing problems (Blaquière (1977a; 1977b; 1979), Kort (1989, pp. 62-70)) do not have an optimal solution. In particular, the solution of these problems contain an interval where a singular arc is approximated as much as possible by applying impulse chattering. *Fourth*, we observe that problems with a fixed cost have the property that the cost function is not a C^1 function i.e. it is not continuously differentiable. This implies that in principle, also the necessary optimality conditions do not hold, although they were applied in Luhmer (1986), Gaimon1985; 1986a; 1986b and Chahim et al. (2011) leading to correct solutions. This paper provides a transformation, which ensures that the Impulse Control Maximum Principle can still be applied to problems with a fixed cost

This paper is organized as follows. Section 2 gives the general formulation of an impulse control model with discounting and presents the correct Impulse Control Maximum Principle in current value formulation (i.e. the necessary optimality conditions). Further we give the sufficient conditions for optimality and provide the transformation which makes clear why the Impulse Control Maximum Principle can still be applied to problems with a fixed cost. In Section 3 we classify existing economic models involving impulse control, show why optimal solutions for some of them do not exist, and discuss the problems that arise with the sufficiency conditions. Section 4 contains our conclusion and further remarks.

2 Impulse control

The theory of optimal control has its origin in physics and engineering where discounting cash flows does not occur. For this reason Blaquière (1977a; 1977b; 1979; 1985) derived his Maximum Principle considering impulse control problems without using current value Hamiltonians. Instead, he presents his Maximum Principle in the present value Hamiltonian form.

Section 2.1 transforms Blaquière present value analysis to a current value one, whereas Section 2.2 presents sufficiency conditions. Section 2.3 considers a subclass of impulse control problems, where the cost function contains a fixed cost.

2.1 Necessary Conditions

In this section we derive necessary optimality conditions for impulse control in current value Hamiltonian form. In doing so, we correct the necessary optimality conditions for impulse control given in Feichtinger and Hartl (1986, appendix 6). Their theorem is based on the current value present value transformation. However, applying it here turns out to be not as straightforward as usual.

A general formulation of the impulse control problem with discounting is:

$$\max_{\boldsymbol{u},N,\tau_i,\boldsymbol{v^i}} \int_0^T e^{-rt}F(\boldsymbol{x}(t),\boldsymbol{u}(t),t)dt + \sum_{i=1}^{N} e^{-r\tau_i}G(\boldsymbol{x}(\tau_i^-),\boldsymbol{v}^i,\tau_i) + e^{-rT}S(\boldsymbol{x}(T^+)). \qquad \text{(IC)}$$

s.t.

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(\boldsymbol{x}(t),\boldsymbol{u}(t),t), \quad \text{for } t \notin \{\tau_1,\dots,\tau_N\}, \\ \boldsymbol{x}(\tau_i^+) - \boldsymbol{x}(\tau_i^-) &= \boldsymbol{g}(\boldsymbol{x}(\tau_i^-),\boldsymbol{v}^i,\tau_i), \quad \text{for } i \in \{1,\dots,N\}, \\ \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{u} &\in \Omega_{\boldsymbol{u}}, \quad \boldsymbol{v}^i \in \Omega_v, \quad \text{and} \quad x(0^-) = x_0, \quad \tau_i \in [0,T]. \end{aligned}$$

Here, $\boldsymbol{x}$ is the state variable, $\boldsymbol{u}$ is an ordinary control variable and $\boldsymbol{v}^i$ is the impulse control variable, where $\boldsymbol{x}$ and $\boldsymbol{u}$ are piecewise continuous functions of time[1]. Future cash flows are discounted at a constant rate r leading to the discount factor e^{-rt}. The number of jumps is denoted by N, τ_i is the time moment of the i-th jump, and τ_i^- and τ_i^+ represent the time moment just before and just after the jump, respectively (i.e. $\boldsymbol{x}(\tau_i^-)$ and $\boldsymbol{x}(\tau_i^+)$ represent the left-hand and right-hand limit of $\boldsymbol{x}$, respectively). The terminal time or horizon date of the system or process is denoted by $T > 0$, and T^+ stands for the time moment just after T. The profit of the system is given by $F(\boldsymbol{x}(t),\boldsymbol{u}(t),t)$, $G(\boldsymbol{x}(t),\boldsymbol{v}^i,t)$ is the profit function associated with the i-th jump, and $S(\boldsymbol{x}(T^+))$ is the salvage value, i.e. the total costs or profit associated with the system after time T. Finally, $\boldsymbol{f}(\boldsymbol{x}(t),\boldsymbol{u}(t),t)$ describes the continuous change of the state variable over time between the jump points and $\boldsymbol{g}(\boldsymbol{x}(t),\boldsymbol{v}^i,t)$ is a function that represents the instantaneous (finite) change of the state variable when there is an impulse or jump.

We assume that the domains, $\Omega_{\boldsymbol{u}}$ and Ω_v are bounded convex sets. Further we impose that F, $\boldsymbol{f}$, $\boldsymbol{g}$ and G are continuously differentiable in $\boldsymbol{x}$ on $\mathbb{R}^n$ and $\boldsymbol{v}^i$ on Ω_v, $S(\boldsymbol{x}(T^+))$ is continuously differentiable in $\boldsymbol{x}(T^+)$ on $\mathbb{R}^n$, and that $\boldsymbol{g}$ and G are continuous in τ. Finally, when there is no impulse or jump, i.e. $\boldsymbol{v}^i = 0$, we assume that

$$\boldsymbol{g}(\boldsymbol{x}(t),0,t) = 0,$$

[1] Note that the necessary conditions also hold for measurable controls. We restrict ourselves to piecewise continuous functions since this is needed for sufficiency. Applications typically have piecewise continuous functions.

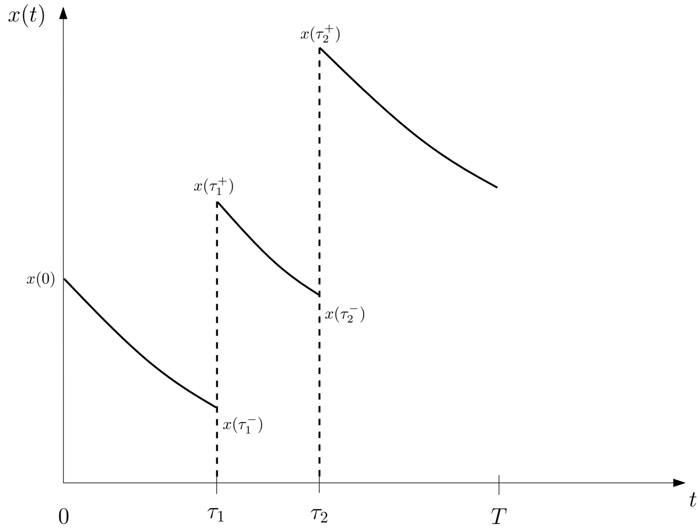


Figure 1: Solution of Impulse Control system

for all $\boldsymbol{x}$ and t. A typical solution for an Impulse Control problem is presented in Figure 1.

Let us define the present value Hamiltonian

$$\mathcal{H}am(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\mu}(t), t) = e^{-rt} F(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + \boldsymbol{\mu}(t)\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t),$$

and the present value Impulse Hamiltonian

$$\mathcal{IH}am(\boldsymbol{x}(t), \boldsymbol{v}^i, \boldsymbol{\mu}(t), t) = e^{-rt} G(\boldsymbol{x}(t), \boldsymbol{v}^i, t) + \boldsymbol{\mu}(t)\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{v}^i, t),$$

where $\boldsymbol{\mu}(t)$ denotes the present value costate variable. The following theorem presents the necessary optimality conditions associated with the impulse control problem defined in (IC).

Theorem 2.1 (Impulse Control Maximum Principle (present value)).
Let $(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), N, \tau_1^*, \ldots, \tau_k^*, v^{1*}, \ldots, v^{k*})$ be an optimal solution for the impulse control problem defined in (IC). Then there exists an adjoint variable $\boldsymbol{\mu}$ such that the following conditions hold:

$$\boldsymbol{u}^*(t) = \arg\max_{\boldsymbol{u}\in\Omega_{\boldsymbol{u}}} \mathcal{H}am(\boldsymbol{x}^*(t), \boldsymbol{u}, \boldsymbol{\mu}(t), t), \tag{1}$$

$$\dot{\boldsymbol{\mu}}(t) = -\frac{\partial \mathcal{H}am}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{\mu}(t), t). \tag{2}$$

At the impulse or jump points, it holds that

$$\frac{\partial \mathcal{IH}am}{\partial \boldsymbol{v}^i}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^i, \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*)(\boldsymbol{v}^i - \boldsymbol{v}^{i*}) \leq 0^1, \tag{3}$$

$$\boldsymbol{\mu}(\tau_i^{*+}) - \boldsymbol{\mu}(\tau_i^{*-}) = -\frac{\partial \mathcal{IH}am}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^{i*}, \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*), \tag{4}$$

$$\begin{aligned}&\mathcal{H}am(\boldsymbol{x}^*(\tau_i^{*+}), \boldsymbol{u}^*(\tau_i^{*+}), \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*)) - \mathcal{H}am(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{u}^*(\tau_i^{*-}), \boldsymbol{\mu}(\tau_i^{*-}), \tau_i^*)\\ &-\frac{\partial \mathcal{IH}am}{\partial \tau}(\boldsymbol{x}^*(\tau_i^{*-}), v^{i*}, \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*) \begin{cases} > 0 & \text{for } \tau_i^* = 0 \\ = 0 & \text{for } \tau_i^* \in (0, T) \\ < 0 & \text{for } \tau_i^* = T. \end{cases}\end{aligned} \tag{5}$$

For all points in time at which there is no jump, i.e. $t \neq \tau_i$ $(i = 1, \ldots k)$, it holds that

$$\frac{\partial \mathcal{IH}am}{\partial \boldsymbol{v}^i}(\boldsymbol{x}^*(t), \boldsymbol{0}, \boldsymbol{\mu}(t), t)\boldsymbol{v}^i \leq 0. \tag{6}$$

At the horizon date the transversality condition

$$\boldsymbol{\mu}(T^+) = e^{-rT}\frac{\partial S}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(T^+)) \tag{7}$$

holds.

Proof: see Blaquière (1977a; 1985) or Rempala and Zabczyk (1988) □

In Blaquière (1977a; 1985) it is assumed that the Impulse Hamiltonian is concave in v. In this case (3) and (6) are replaced by

$$v^{i*} = \arg\max_{v^i \in \Omega_v} \mathcal{IH}am(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^i, \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*),$$

$$\boldsymbol{0} = \arg\max_{v^i \in \Omega_v} \mathcal{IH}am(\boldsymbol{x}^*(t), \boldsymbol{0}, \boldsymbol{\mu}(t), t).$$

Next we determine the current value formulation of Theorem 2.1. By doing this we correct Feichtinger and Hartl (1986, appendix 6), in which the current value version of condition (5) is wrongly stated. First, we define the current value Hamiltonian

$$Ham(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t), t) = F(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + \boldsymbol{\lambda}(t)\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t),$$

and the current value Impulse Hamiltonian

$$IHam(\boldsymbol{x}(t), \boldsymbol{v^i}, \boldsymbol{\lambda}(t), t) = G(\boldsymbol{x}(t), \boldsymbol{v^i}, t) + \boldsymbol{\lambda}(t)\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{v^i}, t).$$

with $\boldsymbol{\lambda}(t)$ the current value costate variable. The following theorem presents necessary optimality conditions to solve the impulse control problem defined in (IC), based on the current value approach.

Theorem 2.2 (Impulse Control Maximum Principle (current value))**.**
Let $(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), N, \tau_1^*, \ldots, \tau_k^*, v^{1*}, \ldots, v^{k*})$ be an optimal solution for the impulse control problem defined in (IC). Then there exists an adjoint variable $\boldsymbol{\lambda}$ such that the following conditions hold:

$$\boldsymbol{u}^*(t) = \arg\max_{\boldsymbol{u} \in \Omega_{\boldsymbol{u}}} Ham(\boldsymbol{x}^*(t), \boldsymbol{u}, \boldsymbol{\lambda}, t), \tag{8}$$

$$\dot{\boldsymbol{\lambda}}(t) = r\boldsymbol{\lambda} - \frac{\partial Ham}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(t), \boldsymbol{u}, \boldsymbol{\lambda}, t). \tag{9}$$

At the impulse or jump points, it holds that:

$$\frac{\partial IHam}{\partial \boldsymbol{v}^i}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^i}, \boldsymbol{\lambda}(\tau_i^{*+}), \tau_i^*)(\boldsymbol{v}^i - \boldsymbol{v}^{i*}) \leq 0, \tag{10}$$

$$\boldsymbol{\lambda}(\tau_i^{*+}) - \boldsymbol{\lambda}(\tau_i^{*-}) = -\frac{\partial IHam}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(\tau_i^{*-}), v^{i*}, \boldsymbol{\lambda}(\tau_i^{*+}), \tau_i^*), \tag{11}$$

$$\begin{aligned} &Ham(\boldsymbol{x}^*(\tau_i^{*+}), \boldsymbol{u}^*(\tau_i^{*+}), \boldsymbol{\lambda}(\tau_i^{*+}), \tau_i^*)) - Ham(\boldsymbol{x}^*(\tau_i^{*-}), u^*(\tau_i^{*-}), \boldsymbol{\lambda}(\tau_i^{*-}), \tau_i^*) \\ &\quad - \left[\frac{\partial G}{\partial \tau}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^{i*}, \boldsymbol{\lambda}(\tau_i^{*+}), \tau_i^*) - rG(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^{i*}, \boldsymbol{\lambda}(\tau_i^{*+}), \tau_i^*)\right] \\ &\quad -\boldsymbol{\lambda}(\tau_i^+)\frac{\partial \boldsymbol{g}}{\partial \tau}(\boldsymbol{x}(\tau_i^-), \boldsymbol{v^i}, \tau_i) \begin{cases} > 0 & \textit{for } \tau_i^* = 0 \\ = 0 & \textit{for } \tau_i^* \in (0, T) \\ < 0 & \textit{for } \tau_i^* = T. \end{cases} \end{aligned} \tag{12}$$

For all points in time at which there is no jump, i.e. $t \neq \tau_i^*$ $(i = 1, \dots k)$*, it holds that:*

$$\frac{\partial IHam}{\partial \boldsymbol{v}^i}(\boldsymbol{x}^*(t), \mathbf{0}, \boldsymbol{\lambda}(t), t)\boldsymbol{v}^i \leq 0. \tag{13}$$

At the horizon date the transversality condition

$$\boldsymbol{\lambda}(T^+) = \frac{\partial S}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(T^+)) \tag{14}$$

holds.

Proof: The relation between present value and current value Hamiltonian, Impulse Hamiltonian and co-state variables is given by

$$\mathcal{H}am = e^{-rt} Ham,$$

$$\mathcal{IH}am = e^{-rt} IHam,$$

and

$$\boldsymbol{\mu}(t) = e^{-rt}\boldsymbol{\lambda}(t).$$

Under these transformations, conditions (8)-(11),(13) and (14) are equal to conditions (1)-(4),(6) and (7). In this proof we show that (12) is the current value equivalent of the analogous condition (5) derived by Blaquière. From the definitions of $IHam$ and $\mathcal{IH}am$ we obtain that

$$\begin{aligned} e^{-rt} IHam(\boldsymbol{x}(t), \boldsymbol{v^i}, \boldsymbol{\lambda}(t), t) &= e^{-rt} G(\boldsymbol{x}(t), \boldsymbol{v^i}, t) + e^{-rt}\boldsymbol{\lambda}(t)\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{v^i}, t) \\ &= e^{-rt} G(\boldsymbol{x}(t), \boldsymbol{v^i}, t) + \boldsymbol{\mu}(t)\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{v^i}, t) \\ &= \mathcal{IH}am(\boldsymbol{x}(t), \boldsymbol{v^i}, \boldsymbol{\mu}(t), t) \end{aligned}$$

Combining this with (5) we get for $\tau_i \in (0, T)$:

$$\mathcal{H}am^+ - \mathcal{H}am^- = e^{-rt}(\frac{\partial G(\boldsymbol{x}(t), \boldsymbol{v^i}, t)}{\partial t} - rG(\boldsymbol{x}(t), \boldsymbol{v^i}, t)) + \boldsymbol{\mu}(t)\frac{\partial \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{v^i}, t)}{\partial t},$$

which implies that

$$\begin{aligned} Ham^+ - Ham^- &= e^{rt}(e^{-rt}(\frac{\partial G(\boldsymbol{x}(t), \boldsymbol{v^i}, t)}{\partial t} - rG(\boldsymbol{x}(t), \boldsymbol{v^i}, t)) + e^{-rt}\boldsymbol{\lambda}(t)\frac{\partial \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{v^i}, t)}{\partial t}) \\ &= (\frac{\partial G(\boldsymbol{x}(t), \boldsymbol{v^i}, t)}{\partial t} - rG(\boldsymbol{x}(t), \boldsymbol{v^i}, t)) + \boldsymbol{\lambda}(t)\frac{\partial \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{v^i}, t)}{\partial t}. \end{aligned}$$

This is condition (12) for $\tau_i^* \in (0, T)$. The other two cases, $\tau_i^* = 0$ and $\tau_i^* = T$, follow the same steps. □

2.2 Sufficiency conditions

The following theorem can be found in Seierstad and Sydsæter (1987, pp. 198-199).

Theorem 2.3 (Sufficient Conditions for Impulse Control)**.** *Let there be a feasible solution,* $(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), N, \tau_1^*, \dots, \tau_k^*, v^{1*}, \dots, v^{k*})$*, for the impulse control problem (IC) and a piecewise continuous costate trajectory, so that the necessary optimality conditions of Theorem 2.2 hold. When the maximized Hamiltonian function* $Ham^0 = \max_{\boldsymbol{u}} Ham(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}, t)$ *is concave in* $\boldsymbol{x}$ *for all* $(\boldsymbol{\lambda}(t), t)$*, the* $IHam$*, concave in* $(\boldsymbol{x}, \boldsymbol{v})$ *for all* t *and* $S(\boldsymbol{x})$ *concave in* $\boldsymbol{x}$*, then that solution,* $(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), N, \tau_1^*, \dots, \tau_k^*, v^{1*}, \dots, v^{k*})$*, is optimal.*

For the proof of this theorem we refer to Theorem 1 in Seierstad (1981), which is equivalent to the theorem stated above. However, we will show in Section 3 that this result is not very useful since most (relevant) problems given in the literature do not fulfil these conditions.

2.3 Impulse control: G including constant term (i.e. a fixed cost)

When there is some *fixed cost* involved in the impulse cost function, the function G has a jump discontinuity at point $v^i = 0$. The implication is that G is not continuously differentiable. Consequently, strictly speaking the Impulse Control Maximum Principle cannot be applied. However, the Impulse Control Maximum Principle has been applied a few times while ignoring this continuity requirement (see, e.g., Luhmer (1986), Gaimon(1985; 1986a; 1986b) and Chahim et al. (2011)). In this section we show that by applying some transformation, a general fixed cost problem can be represented by a problem with continuous cost function so that still the necessary optimality conditions can be applied.

Reconsider the above general impulse control problem. For applying the Impulse Control Maximum Principle, zero needs to be in the set $\Omega_v = [0, \bar{v}]$ and $g(x(\tau_i^-), 0, \tau_i) = 0$ (see e.g., Blaquière (1977a; 1977b; 1979; 1985) and Seierstad and Sydsæter (1987)). Furthermore, the impulse cost function needs to be continuously differentiable. As said before, this is not the case in the specification where G is discontinuous because of a fixed cost term (for simplicity we delete the superscript i in v^i):

$$G(x, v, \tau) = \begin{cases} 0 & \text{for } v = 0 \\ K(\tau) + \alpha(v, \tau)v & \text{for } v > 0, \end{cases}$$

where $K(\tau) > 0$. Clearly G is lower semi-continuous.

The idea is to approximate the linear impulse cost function $K + \alpha v$ by a continuously differentiable one that assumes the same value for $v > \varepsilon$, where we let ε go to zero. A possible specification would be

$$G_\varepsilon(x, v, \tau) = \begin{cases} -\frac{K(\tau)}{\varepsilon^2}v^2 + (\frac{2K(\tau)}{\varepsilon} + \alpha(v, \tau))v & \text{for } v \in [0, \varepsilon] \\ K(\tau) + \alpha(v, \tau)v & \text{for } v > \varepsilon. \end{cases}$$

Letting ε go to zero it follows that G_ε approaches G. Other specifications of $G_\varepsilon(x, v, \tau)$ are also possible, but the common denominator is that $\lim_{\varepsilon \to 0} \frac{\partial}{\partial v} G_\varepsilon(x, 0, \tau) = \infty$. The argument is that the optimal solution of a problem with cost G will never have "very small" jumps because of the fixed costs. Then, for ε small enough, G_ε will always give the same cost as G and the optimal solutions will be the same. Hence, the necessary optimality conditions still hold for G with fixed cost. The following lemma and proposition formalize these statements.

Lemma 2.4. *Let $0 < \varepsilon_1 < \varepsilon_0$ and let $(x_\varepsilon, u_\varepsilon, v_\varepsilon)$ (for simplicity we omit τ and N) be an optimal solution of the problem with cost function G_ε, while (x^*, u^*, v^*) is an optimal solution of problem (IC). Furthermore, we denote by $J(x, u, v)$ the value of the objective function of the original problem evaluated at (x, u, v), and by $J_\varepsilon(x, u, v)$ the value of the objective function of the approximated problem with cost function G_ε evaluated at (x, u, v). Then*

$$J(x, u, v) \leq J_{\varepsilon_1}(x, u, v) \leq J_{\varepsilon_0}(x, u, v), \tag{15}$$

and also

$$J(x^*, u^*, v^*) \leq J_{\varepsilon_1}(x_{\varepsilon_1}, u_{\varepsilon_1}, v_{\varepsilon_1}) \leq J_{\varepsilon_0}(x_{\varepsilon_0}, u_{\varepsilon_0}, v_{\varepsilon_0}). \tag{16}$$

Proof: The first result (15) follows directly from $G_{\varepsilon_0} \leq G_{\varepsilon_1} \leq G$, whereas (16) follows from (15) and

$$J_{\varepsilon_1}(x_{\varepsilon_1}, u_{\varepsilon_1}, v_{\varepsilon_1}) \leq J_{\varepsilon_0}(x_{\varepsilon_1}, u_{\varepsilon_1}, v_{\varepsilon_1}) \leq J_{\varepsilon_0}(x_{\varepsilon_0}, u_{\varepsilon_0}, v_{\varepsilon_0}).$$

□

Proposition 2.5. *Let (x^*, u^*, v^*) (for simplicity we omit τ and N) be an optimal solution of problem (IC). Then the Impulse Control Maximum Principle provides necessary optimality conditions, even though the model function G is not continuous. More precisely, if the optimal solution is unique, it satisfies these necessary optimality conditions. Otherwise there is at least one optimal solution for which this holds.*

Proof: Let ε_0 be some small positive number and let $(x_{\varepsilon_0}, u_{\varepsilon_0}, v_{\varepsilon_0})$ be an optimal solution of the problem with cost function G_{ε_0}, which thus satisfies the necessary optimality conditions. Let further $v^i_{\varepsilon_0}$ be the smallest jump parameter in this optimal solution. If $v^i_{\varepsilon_0} \geq \varepsilon_0$, the proposition automatically holds. If $v^i_{\varepsilon_0} < \varepsilon_0$, choose a lower ε_0, and check again if $v^i_{\varepsilon_0} \geq \varepsilon_0$. If yes we are done, if not continue this procedure. □

3 Classification of existing operations research models involving impulse control

This section classifies existing operations research impulse control problems found in the literature. When considering impulse control problems in an operations research context, a common feature is discounting. The resulting general impulse control problem *(where for reasons of exposition both the state and impulse control are one dimensional)* can be represented by

$$\max_{u,v^i,\tau_i,N} \int_0^T e^{-rt} F(x(t), u(t), t)dt + \sum_{i=1}^{N} e^{-r\tau_i} G(x(\tau_i^-), v_i, \tau_i) + e^{-rT} S(x(T^+)), \qquad (17)$$

s.t.

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t), \quad \text{for } t \notin \{\tau_1, \ldots, \tau_N\}, \\ x(\tau_i^+) - x(\tau_i^-) &= g(x(\tau_i^-), v^i, \tau_i), \quad \text{for } i \in \{1, \ldots, N\}, \\ x \in \mathbb{R}, \quad u \in \Omega_u, \quad & v^i \in \Omega_v, \quad \text{and} \quad x(0^-) = x_0, \quad \tau_i \in [0, T]. \end{aligned}$$

For applying the Impulse Control Maximum Principle, zero needs to be in the set $\Omega_v = [0, \infty)$ and $g(x(\tau_i^-), 0, \tau_i) = 0$. The objective is typically to maximize profit or minimize cost. We distinguish between

- linear impulse control problem, i.e. a problem where the impulse control variable occurs linearly in the Impulse Hamiltonian, and *no* continuous control present (*Case A*)
- linear impulse control problem and continuous control present (*Case B*)
- *non*-linear impulse control problem and *no* continuous control present (*Case C*)
- *non*-linear impulse control problem and continuous control present (*Case D*)

In the *linear* impulse control case where *no continuous* control u is present (*Case A*), a typical solution would be to reach some kind of singular arc by applying impulse control, but, if the state equation contains some decay term (for instance $\delta K(t)$ with δ the depreciation rate and $K(t)$ the capital stock), then it might be formally impossible to stay there. One has to use some kind of impulse chattering, i.e. an infinitely large number of impulses of infinitely small size. We elaborate on this when discussing the model by Blaquière (1977a; 1977b) in Section 3.1.

In the *linear* impulse control case where *also a continuous* control u is present (*case B*) and both go into the same direction i.e. increase or decrease the state, the two controls (i.e. the ordinary and impulse control) are in some sense substitutes to each other. Then one can distinguish the cases

1. Continuous control u and impulse control v have the same monetary effect (e.g. cost or profit). An example is the model by Seierstad and Sydsæter (1987, pp. 199-202) where just the impulse control is used to sell the complete stock of the resource at the best point in time. It is a non-autonomous model were the two controls appear in the model in the same way and are substitutes. The jump occurs at one time instant and in that sense this model is comparable to a model that has the most rapid approach path (MRAP) property (see e.g. Hartl and Feichtinger (1987)), where the singular arc is reached by applying impulse control at one point of time (usually the initial time point), followed by a singular arc which is maintained using the continuous control. The same analysis hold for the model by Seierstad and Sydsæter (1987, pp. 202-206). Other existing optimal control models having this MRAP property are, e.g., Jorgenson(1963; 1967), and Sethi (1973). These kinds of models are not considered in this paper any further.

2. The impulse control has a higher cost. An example is the model by Blaquière (1979)(see Section 3.2), where, for suitable values of $x(0)$, only the continuous control is used to do preventive maintenance for the machine but no impulse control to repair or upgrade the machine. If $x(0)$ is very low an impulse jump occurs at the initial time (MRAP-property), after which preventive maintenance is applied.

3. The impulse control has a lower cost. An example would be the model by Blaquière (1979)(see Section 3.2), with modified parameters so that repair is more attractive than preventive maintenance. Then one would not do preventive maintenance but only repair during the planning period. This will lead to an impulse chattering solution. We demonstrate in Section 3.1 that in such cases no optimal solution exists.

In some sense, these results are trivial, i.e. there is no interesting combination of the two types of control. Such interesting cases occur when there is some *fixed cost* involved in the impulse cost function. In the *non-linear* impulse control case where *no continuous* control u is present (*Case C*) this fixed cost in the impulse cost function often occurs, examples are e.g. Luhmer (1986) and Chahim et al. (2011). In Kort (1989) a model is given that analyzes the behavior of a firm under a concave adjustment cost function where impulse control is applied. However, in Section 3.5 we demonstrate that an optimal impulse control solution does not exist!

In the literature no problems exist dealing with the non-linear impulse control case where the continuous control u is present (*Case D*). This is different in the literature on stochastic impulse control, where, e.g., Bensoussan and Lions (1984, Chapter 1, Section 4) discuss an inventory problem with continuous production and impulse ordering of goods. However, as said before, our paper restricts itself to a deterministic impulse control framework, and, since "*Case D* problems" do not occur in this literature, we will not consider this case any further.

In the next sections we will discuss several (relevant) problems, check whether the sufficiency conditions 2.3 hold, and describe how their solution looks like. In particular we prove that in the roadside inn problem (Section 3.1), in one scenario of the maintenance problem in Section 3.2, and in the investment problem of Section 3.5 no optimal solution exists. These problems have in common that "impulse chattering" occurs on a time interval with positive length. On the other hand, for problems in Section 3.3 (Luhmer (1986)), Section 3.4 (Gaimon (1985; 1986a; 1986b)) and Section 3.6 (Chahim et al. (2011)) an algorithm can be designed that employs the necessary optimality conditions to find all candidate solutions for optimality, as is shown in Luhmer (1986) (see also Kort (1989) and Chahim et al. (2011)). Out of these candidate solutions we can simply select the one with the highest objective value, which is then for sure the optimal solution.

3.1 Blaquière (1977a; 1977b): Maximizing the profit of a roadside inn (*Case A*)

In Blaquière (1977a; 1977b) an example is given that deals with maximizing the profit of the owner of a roadside inn. The owner attracts more customers if he repaints the inn. The following model is given:

$$\max_{v^i,N} \quad W(T) = A\int_0^T x(t)dt - \sum_{i=0}^{N} v^i C, \tag{18}$$

s.t.

$$\begin{aligned} \dot{x}(t) &= -kx(t), && \text{for } t \notin \{\tau_1, \dots, \tau_N\}, \\ x(\tau_i^+) - x(\tau_i^-) &= v^i(1 - x(\tau_i^-)), && \text{for } i \in \{1, \dots, N\}, \\ x \in \mathbb{R}, \quad v^i \in [0,1], &\quad \text{and} \quad x(0^-) = x_0, && \tau_i \in [0,T], \end{aligned}$$

where N is the number of times the inn is (re)painted, v^iC, $i = 0, ..., N$, the cost of each (re)paint job, and A a strictly positive constant. It is assumed that $0 \le x \le 1$, and each time the inn is repainted the index of appearances of the inn x undergoes an upward jump from its previous value $x(\tau_i^-)$. Between (re)painting x decays as given above, with the depreciation rate k being a positive constant. Furthermore, we assume that after the planning period the inn will not be used (i.e. the salvage value is set to zero). In Sethi and Thompson (2006, pp. 324-330) this problem has been reinterpreted as "The Oil Driller's Problem".

The Hamiltonian and Impulse Hamiltonian in short hand notation are

$$\mathcal{H}am(x, \mu) = Ax + \mu(-kx),$$

$$\mathcal{IH}am(x, v^i, \mu(\tau_i)) = v^i(-C) + \mu(\tau_i)v^i(1 - x) = v^i(-C + \mu(\tau_i)(1 - x)).$$

Both the impulse control variable and state variable are linear in $\mathcal{IH}am$ and $\mathcal{H}am$. Due to the interaction term between the impulse control variable and the state variable in the Impulse Hamiltonian, $\mathcal{IH}am$ is not concave in (x, v^i) jointly, so that the necessary optimality conditions are not sufficient.

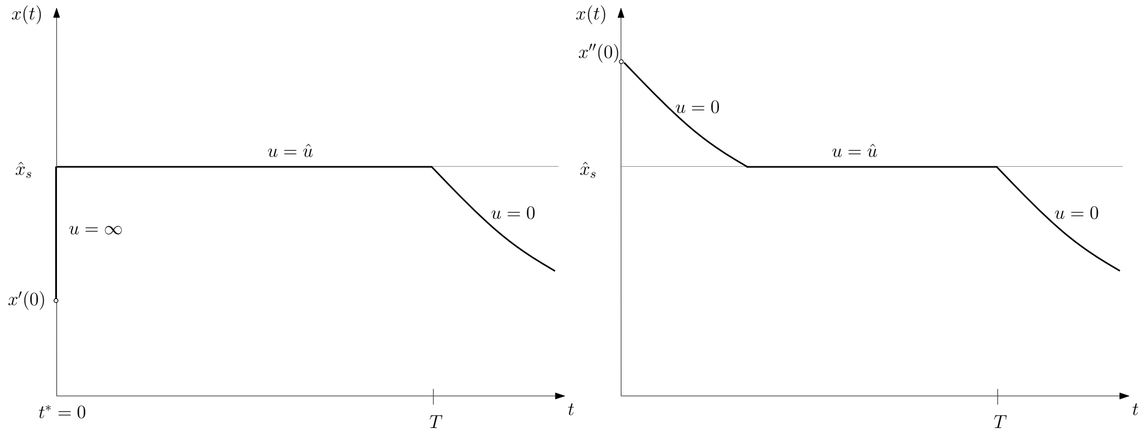
To solve the above stated model we first consider the continuous version of this problem (i.e. the problem where the impulse control v^i is replaced by a continuous control u):

$$\max_{u} \quad W(T) = \int_0^T Ax(t) - u(t)Cdt, \tag{19}$$

s.t.

$$\begin{aligned} \dot{x}(t) &= -kx(t) + u(1 - x(t)), \\ x \in \mathbb{R}, \quad u \in [0, \infty] &\quad \text{and} \quad x(0) = x_0. \end{aligned}$$

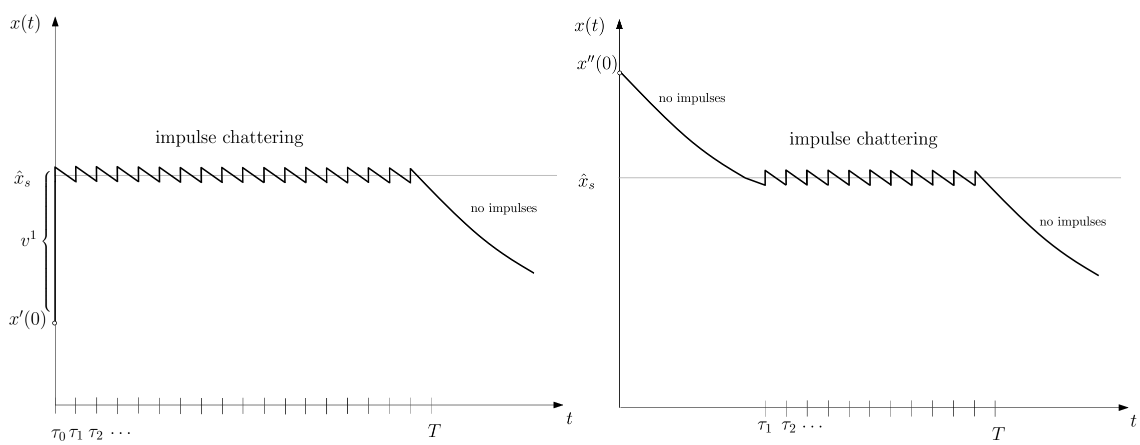
We can identify this model as the *Vidale-Wolfe* advertising model discussed in Sethi (1973). The solution for this model is given in Figure 2. If the initial value of $x(0)$ is lower than the singular arc value of $x(t)$ (i.e. $\hat{x}_s$) at t^*, we set the control $u = \infty$ so that the singular arc is reached immediately (MRAP property). If the initial value of $x(0)$ is higher than $\hat{x}_s$ the control $u = 0$ is applied until x has reached $\hat{x}_s$. At the singular arc the control is set at $u = \hat{u}_s = k\hat{x}_s/(1 - \hat{x}_s)$, so that $x(t)$ is kept constant at the level $\hat{x}_s$. At the final planning period the control is equal to zero, since the remaining time period is too short to defray the cost uC.



(a) $x'(0) < \hat{x}_s$ (b) $x''(0) > \hat{x}_s$

Figure 2: Vidale-Wolfe model solution

To solve the Blaquière (1977a; 1977b) impulse control model, we only need to approximate the continuous *Vidale-Wolfe* advertising model as much as possible. This is straightforward for the solution part where $u = 0$ (then simply put $v^i = 0$) or where $u = \infty$. In the latter case apply an initial impulse control jump, where $v^1 = \hat{x}_s - x'(0)$. On the singular arc we divide the interval $[t_{sa}, T]$ (with t_{sa} the time the singular arc is reached) in l parts of equal length and set within each interval first $v^i = \bar{v}$ (where $\bar{v}$ is such that $\tilde{x}+\bar{v}-\hat{x}_s = \hat{x}_s-\tilde{x}$ with $\tilde{x} = x(\tau_1^-) = \ldots = x(\tau_N^-)$) and then $v^i = 0$. In this way we create a "saw-toothed" shape around the singular arc. This control policy is shown in Figure 3 and is the impulse control equivalent of chattering control (see e.g. Feichtinger and Hartl (1986, pp. 78-81) or Kort (1989, pp. 62-70)). It is important to note that for each given "saw-toothed" solution, a better solution is available by increasing l and decreasing $\bar{v}$. We conclude that an optimal solution does not exist. This observation cannot be found in Blaquière (1977a; 1977b), or in Sethi and Thompson (2006, pp. 324-330).



(a) $x'(0) < \hat{x}_s$ (and $\tau_0 = 0$) (b) $x''(0) > \hat{x}_s$

Figure 3: Blaquière (1977) model solution with impulse chattering

3.2 Blaquière (1979): Optimal maintenance of machines (*Case B*)

The following problem is taken from Blaquière (1979) and is also extensively analyzed in Sethi and Thompson (2006, pp. 331-337). This example deals with the optimal maintenance of machines.

$$\max_{v^i,u,\tau_i,N} \quad W(T) = \int_0^T (Ax(t)-u(t))dt + \sum_{i=1}^N v^i(Kx(\tau_i^-)-C), \tag{20}$$

s.t.

$$\begin{aligned} \dot{x}(t) &= -kx(t)+mu, \quad \text{for } t \notin \{\tau_1,\ldots,\tau_N\}, \\ x(\tau_i^+)-x(\tau_i^-) &= v^i(1-x(\tau_i^-)), \quad \text{for } i \in \{1,\ldots,N\}, \\ x \in \mathbb{R}, \quad v^i \in [0,1], &\quad u \in [0,\bar{u}], \quad x(0^-) = x_0 \quad \text{and} \quad \tau_i \in [0,T], \end{aligned}$$

where N is the number of times the machines is repaired, $C - Kx(\tau_i)$, $i = 1,...,N$, the cost of each reparation, and A a strictly positive constant. It is assumed that $0 \le x \le 1$, and each time the machine is repaired (where full repair, i.e. $v^i = 1$ stands for replacing the machine with a new one) the index of appearances of the machine, x, undergoes an upward jump starting from its previous value $x(\tau_i^-)$. Between reparations x decays as given above, with k and m positive constant. The rate of maintenance expenses is denoted by u (i.e. the continues control). Moreover it is assumed that the cost of a reparation indexed by v^i is of the form $v^i(C - Kx_1)$, where C and K are strictly positive constants. Furthermore, we assume that after the planning period the machine will not be used (i.e. the salvage value is set to zero). The Hamiltonian and Impulse Hamiltonian in short hand notation are

$$\mathcal{H}am(x,\mu,u) = Ax - u + \mu(-kx+mu),$$

$$\mathcal{IH}am(x,v^i,\mu(\tau_i)) = v^i(Kx-C) + \mu(\tau_i)v^i(1-x) = v^i(Kx - C + \mu(\tau_i)(1-x)).$$

Both the impulse control variable and state variable are linear in $\mathcal{IH}am$ and $\mathcal{H}am$. Due to the interaction term between the impulse control variable and the state variable in the Impulse Hamiltonian the necessary optimality conditions are not sufficient, since $\mathcal{IH}am$ is not concave in (x, v^i). Because the sufficient conditions do not hold we know that multiple solutions can occur for this problem. Here we will distinguish between two cases:

- The impulse control has a higher cost. When $x(0)$ is sufficiently large, only the continuous control is used to do preventive maintenance for the machine, so no impulse control is applied to repair or upgrade the machine. In this case the coefficients satisfy $mK \le 1 < mC < \frac{mA}{k}$. When $x(0)$ is very low, besides preventive maintenance, an impulse jump occurs at the initial time and in that sense this model is comparable to a model that has the most rapid approach path (MRAP) property. For the analysis of this case we refer to Blaquière (1979).

- The impulse control has lower cost. Then one would not do preventive maintenance but repair during the planning period. This results in impulse chattering analogous to the Blaquière (1977a; 1977b) model in Section 3.1. Hence, for this case no optimal solution exists.

3.3 Luhmer (1986)): Minimizing inventory cost (*Case C*)

Luhmer (1986) applies the Impulse Control Maximum Principle to solve an inventory problem. The following model is presented:

$$C(T,v^i) = \min_{v^i,\tau_i,N} \int_0^T h(I(t),t)e^{-rt}dt + \sum_i^N \left(p(v^i,\tau_i)v^i + C(\tau_i) \right) e^{-r\tau_i} - S(I(T))e^{-rT}, \tag{21}$$

s.t.

$$\begin{aligned}
\dot{I}(t) &= -d(t) - g(I(t),t) \quad \text{for } t \notin \{\tau_1,\dots,\tau_N\},\\
I(\tau_i^+) - I(\tau_i^-) &= v^i > 0 \quad \text{for } i \in \{1,\dots,N\},\\
I \in \mathbb{R}, \quad v^i &\in [0,\infty) \quad ,I(0) = I_0, \quad I(T) = I_e \quad \text{and} \quad \tau_i \in [0,T],
\end{aligned}$$

where h denotes the holding or shortage cost and $I(t)$ the inventory level at time t. $I(t)$ decreases over time by the demand rate $d(t)$ and leakage losses $g(I(t),t)$. At a time instance τ_i the inventory is increased by a quantity v^i and the unit ordering costs are given by $p(v^i,\tau_i)$. An order of size v^i at time τ_i results in a variable cost of $(p(v^i,\tau_i)v^i$ plus a fixed ordering cost of size $C(\tau_i)$. At the end of the planning period a scrap value for inventory is left over, which is denoted by $S(I(T))$. Finally, r stands for the risk-free discount rate.

Due to the fixed cost, the model violates the requirement that the cost function should be continuously differentiable in the control in order for the Impulse Control Maximum Principle to be applicable. However, performing our transformation of Section 2.3 ensures that the Impulse Control Maximum Principle can still be applied. Moreover, the discontinuity in the cost function causes that the sufficient conditions do not hold, i.e. the Impulse Hamiltonian is not concave in (I,v^i) jointly. This implies that we can have multiple solutions satisfying the necessary optimality conditions. To solve this problem, Luhmer (1986) describes an algorithm that finds all these candidate solutions. Typically, this gives a tree structure in which the jumps of all candidate solutions are presented (cf. Section 3.6). The optimal solution is that candidate solution with the highest objective value.

3.4 Gaimon (1985; 1986a; 1986b): Optimal dynamic mix of manual and automatic output (*Case B*)

Gaimon(1985; 1986a) determines the optimal times of impulse acquisition of automation and the change for manual output. The objective is to minimize cost associated with deviation from a goal level of output. The purchase of automation is used to directly substitute for output resulting from manually operated equipment. Since automation is acquired at discrete times in the planning period the author solves the model using the impulse control maximum principle. The following model is given:

$$\begin{aligned}
\min_{h,s,v^i,\tau_i,N} \quad J(T) &= \int_0^T \{w[p(t)+q(t)-g(t)]^2 + c_1(t)h^2(t)\\
&+c_2(t)s^2(t) + f_1(t)p(t) + f_2(t)q(t)\}e^{-rt}dt,\\
&+\sum_{i=1}^{N} c_3(\tau_i)v^i e^{-r\tau_i} - \beta[p(T)+q(T)]e^{-rT}, \qquad (22)
\end{aligned}$$

s.t.

$$\begin{aligned}
\dot{p}(t) &= -d(t) + h(t) - s(t), \quad \text{for } t \notin \{\tau_1,\dots,\tau_N\},\\
q(\tau_i^+) - q(\tau_i^-) &= \mu v^i, \quad \text{for } i \in \{1,\dots,N\},\\
h(t) \in [0,H(t)], &\quad s(t) \in [0,S(t)], \quad p(0) = p_0,\\
q(0^-) = q_0, &\quad v^i \in \{0,1\} \quad \text{and} \quad \tau_i \in [0,T],
\end{aligned}$$

where N is the number of times automation equipment is acquired. $c_3(\tau_i)v^i$, $i=0,...,N$, the cost of acquiring the ith automation at time τ_i, where v^i denotes the ith technology purchase. The level of automation output and manual output are given by $q(t)$ and $p(t)$ respectively. The cost of producing output manually at time t is given by $f_1(t)$ and the cost of producing output automatically at time t is given by $f_2(t)$. The cost of increasing and reducing the level of manual output per unit squared at time t is represented by $c_1(t)h^2(t)$ and $c_2(t)s^2(t)$, respectively, where

$h(t)$ denotes the level of increase in manual output at time t, with $H(t)$ the available supply of labor and $s(t)$ denotes the level of reduction in manual output at time t, with $S(t)$ the maximum permitted level of reduction at time t. The level of reduction in manual output at time t in units of output is represented by $d(t)$, and $g(t)$ represents the goal level of output at time t also in units of output. Finally, w stands for the weight or cost of the squared deviation between the actual and the goal levels of output, μ the units of increase in output due to purchased automation, r is the discount rate, and β the value of the production per unit of output at the end of the planning period.

The difference with the other impulse control models is that the impulse control variable v^i can admit only two values: 0 or 1. It follows that the term $c_3(\tau_i)v^i$ works as a fixed cost. Hence, analoguous to the model in Section 3.3, sufficient conditions do not hold, so that in principle multiple solutions can satisfy the necessary optimality conditions. Furthermore our transformation of Section 2.3 is needed to apply the Impulse Control Maximum Principle. This is not mentioned in Gaimon (1985; 1986a). A similar reasoning holds for Gaimon and Thompson (1984).

Gaimon (1986b) determines the optimal times and levels of impulse acquisition of automation and the levels of change for manual output with a similar objective. The main difference is that in Gaimon (1986b) the magnitude of automation output can have different values. So Gaimon (1986b) not only determines the time of acquiring automation but also the size of this acquisition. The model is :

$$\begin{aligned} \min_{h,s,f_2,v^i,\tau_i,N} \quad J(T) &= \int_0^T \{w[p(t)+q(t)-g(t)]^2 + c_1(t)h^2(t) \\ &\quad + c_2(t)s^2(t) + f_1(t)p(t) + [F_2(t)+f_2(t)]q(t)\}e^{-rt}dt, \\ &\quad + \sum_{i=1}^{N} c_3(v^i,\tau_i)e^{-r\tau_i} - \beta[p(T)+q(T)]e^{-rT}, \qquad (23) \end{aligned}$$

s.t.

$$\begin{aligned} \dot{p}(t) &= -d(t) + h(t) - s(t), \quad && \text{for } t \notin \{\tau_1,\ldots,\tau_N\}, \\ q(\tau_i^+) - q(\tau_i^-) &= v^i, && \text{for } i \in \{1,\ldots,N\}, \end{aligned}$$

$$f_2(\tau_i^+) = f_2(\tau_i^-)[1-\alpha v^i],$$
$$h(t) \in [0,H(t)], \quad s(t) \in [0,S(t)], \quad p(0) = p_0, \quad p(t) \geq 0,$$
$$q(0^-) = q_0, \quad v^i \in [0,A(\tau_i)] \quad \text{and} \quad \tau_i \in [0,T],$$

where in addition to the notation also used in model (22), $F_2(t)$ is the component of the per unit cost of operating automatic equipment that is unaffected by the acquisition of automation at time t, $f_2(t)$ is the per unit cost of obtaining output automatically at time t, whereas α stands for the effectiveness of a unit acquisition of automation on reducing $f_2(\tau_i)$ at time τ_i $(0 \leq \alpha \leq 1/A(\tau_i))$.

All examples in Gaimon (1986b) have an impulse cost function of the form $c_3(v^i,\tau_i) = C_0 + C_1 {v^i}^2$. This again implies that the problem contains a fixed cost, and thus sufficiency conditions do not hold so that multiple solutions can satisfy the necessary optimality conditions.

3.5 Kort (1989, pp. 62-70): Firm behavior under a concave adjustment cost function (*Case C*)

In Kort (1989) a model is given that analyzes the behavior of a firm under a concave adjustment cost function. Kort (1989) applies impulse control because the concave cost function results in a Hamiltonian that is convex in the control. The following model is studied:

$$C(T, v^i) = \max_{v^i, \tau_i, N} \int_0^T S(K)e^{-rt}dt - \sum_i^N \left(v^i + A(v^i)\right) e^{-r\tau_i} + K(T)e^{-rT}, \tag{24}$$

s.t.

$$\begin{aligned} \dot{K}(t) &= -aK(t), \quad \text{for } t \notin \{\tau_1, \dots, \tau_N\}, \\ K(\tau_i^+) - K(\tau_i^-) &= v^i > 0, \quad \text{for } i \in \{1, \dots, N\}, \\ K \in \mathbb{R}_+, \quad v^i \in [0, \infty) \quad & K(0) = K_0 \quad \text{and} \quad \tau_i \in [0, T]. \end{aligned}$$

where v^i stands for the i-th investment impulse, and τ_i is the time of the i-th impulse. The adjustment costs of the i-th investment impulse are given by $A(v^i)$ (with $\frac{\partial A(v^i)}{\partial v^i} > 0$ and $\frac{\partial^2 A(v^i)}{\partial v^i 2} < 0$), $K(t)$ is the amount of capital goods at time t, and a is a constant depreciation rate. Like Feichtinger and Hartl (1986), Kort (1989) applies the incorrect current value Impulse Control Maximum Principle and designs an algorithm to find all candidate solutions that starts at time T and works backward in time (this is different from Luhmer (1986), whose algorithm starts at time zero). The Hamiltonian and Impulse Hamiltonian in short hand notation are

$$\mathcal{H}am(K, \lambda) = S(K) - \lambda a K,$$

$$\mathcal{IH}am(v^i, \lambda(\tau_i)) = -\big(v^i + A(v^i)\big) + \lambda(\tau_i)v^i.$$

Note that the Impulse Hamiltonian does not depend on K so here there is no state-control interaction. However the sufficient conditions do not hold due to the concave adjustment cost function which implies that the Impulse Hamiltonian is not concave in v^i. The continuous case of this problem is also described in Kort (1989, pp. 57-62) and consists of a chattering control solution. Consequently, the impulse control model has a "singular" arc with chattering too. Analogous to the Blaquière (1977a; 1977b) model in section 3.1, also here we have to conclude that no optimal solution exists. This was not noted in Kort (1989, pp. 62-70).

3.6 Chahim et al. (2011): Dike height optimization (*Case C*)

This section analyzes the problem of the optimal timing of heightening a dike. The cost-benefit-economic decision problem contains two types of cost, namely investment cost and cost due to damage (caused by failure of protection by the dikes). Clearly, there is a trade off between investment cost and damage cost. The model in Chahim et al. (2011) is as follows:

$$\min_{v^i, \tau_i, N} \Big(\int_0^T S(t)e^{-rt}dt + \sum_{i=1}^N I(v^i, H(\tau_i^-))e^{-r\tau_i} + e^{-rT}\frac{S(T)}{r} \Big), \tag{25}$$

s.t.

$$\begin{aligned} \dot{H}(t) &= 0, \quad \text{for } t \notin \{\tau_1, \dots, \tau_N\}, \\ H(\tau_i^+) - H(\tau_i^-) &= v^i > 0, \quad \text{for } i \in \{1, \dots, N\}, \\ H \in \mathbb{R}_+, \quad v^i \in [0, \infty) \quad & H_0 = 0 \quad \text{and} \quad \tau_i \in [0, T], \end{aligned}$$

where v^i stands for the i-th dike heightening, $H(t)$ is the height of the dike at time t relative to the initial situation, i.e. $H(0) = 0$ (cm), τ stands for the time of the dike update (years), and r is the risk-free discount rate. The objective (25) consist of two parts. The first part is the total (discounted) expected damage cost, which is given by

$$\int_0^T S(t)e^{-rt}dt + \frac{S(T)e^{-rT}}{r},$$

where $S(t)$ denotes the expected damage at time t, i.e. $S(t) = P(t)V(t)$. The flood probability $P(t)$ (1/year) in year t is defined as

$$P(t) = P_0 e^{\alpha\eta t} e^{-\alpha H(t)}, \tag{26}$$

where α (1/cm) stands for the parameter in the exponential distribution regarding the flood probability, η (cm/year) is the parameter that indicates the increase of the water level per year, and P_0 denotes the flood probability at $t = 0$. The damage of a flood V(t) (million €) is given by

$$V(t) = V_0 e^{\gamma t} e^{\zeta H(t)}, \tag{27}$$

in which γ (per year) is the parameter for economic growth, and ζ (1/cm) stands for the damage increase per cm dike height. V_0 (million €) denotes the loss by flooding at time $t = 0$. The second part of the objective is the total (discounted) investment cost

$$\sum_{i=1}^{N} I(v^i, H(\tau_i^-))e^{-r\tau_i},$$

where N is the number of dike heightenings and $H(\tau^-)$ the height of the dike (in cm) just before the dike update at time τ (left-limit of $H(t)$ at $t = \tau$). The investment cost is given by

$$I(v^i, H(\tau^-)) = \begin{cases} A_0(H(\tau^-) + v^i)^2 + b_0 v^i + c_0 & \text{for } v^i \neq 0 \\ 0 & \text{for } v^i = 0, \end{cases}$$

for suitably chosen constants A_0, b_0 and c_0. The current value Hamiltonian is

$$Ham(t, H(t)) = -S_0 e^{\beta t} e^{-\theta H(t)},$$

while the Impulse Hamiltonian (if jump $v^i > 0$) is given by

$$IHam(t, H(\tau^-), v^i, \lambda(t)) = -I(v^i, H(\tau_i^-)) + \lambda(t)v^i = -A_0(H(\tau^-) + v^i)^2 - b_0 v^i - c_0 + \lambda(t)v^i.$$

This problem is modeled as an impulse control problem due to the fixed cost, c_0, involved with each dike heightening v^i. As was the case for Luhmer (1986), due to this fixed cost a discontinuity arises in the cost function. The first implication is that the Impulse Control Maximum Principle cannot be straightforwardly applied (although our transformation in Section 2.3 makes up for this), and, second, the sufficiency conditions do not hold (i.e. the Impulse Hamiltonian is not concave in (H, v^i) jointly). Chahim et al. (2011) implement the backward algorithm designed by Kort (1989, pp. 62-70). This algorithm solves the above stated problem (25) for different values of $H(T)$. The optimal $H(T)$ could be found, because this one led to the lowest value of the objective function. In Figure 4 the tree for dike ring area 10 is presented. The tree shows all candidate solutions for (the optimal) $H(t) = 282.57$.

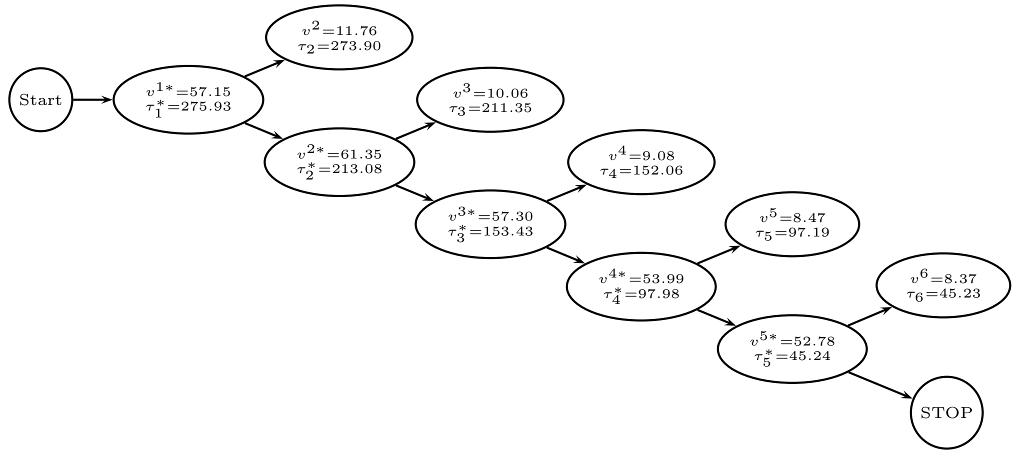


Figure 4: Example Tree, Dike ring area 10, $H(T) = 282.57$

Due to the fixed costs, small jumps cannot be optimal which is why one can cut away all the upper branches in Figure 4. Formally this can be proved by observing that a solution that contains such a small jump, is dominated by a solution where the small jump is deleted, while instead it is added to the previous jump. This implies that only the optimal solution is left over. In Table 1 this optimal solution (and corresponding cost) are presented.

No.	10	
	τ_i (years)	$v^i(cm)$
Updates($\tau_i : v^i$)	275.93	57.15
	213.08	61.35
	153.43	57.30
	97.98	53.99
	45.24	52.78
$H(T)(cm)$	282.57	
Investment cost (million €)	10.17	
Damage cost (million €)	29.96	
Total cost (million €)	40.13	

Table 1: Impulse control solutions for dike ring area 10 with quadratic investment cost

4 Conclusion and recommendations

This paper gives a correct formulation of the necessary optimality conditions of the Impulse Control Maximum Principle based on the current value formulation. In this way we correct Feichtinger and Hartl (1986, appendix 6) and Kort (1989, pp. 62-70). We review the existing impulse control models in the literature and show that all meaningful problems found in the literature do not satisfy the sufficiency conditions. We observe that these problems either have a concave cost function, contain a fixed cost, or have a control-state interaction, which all lead to non-concavities violating sufficiency. The implication of not satisfying the sufficiency conditions is that multiple solutions can arise and a so called tree-structure of jumps can be identified. We also show for the first time that for some problems no optimal solution exists since part of the trajectory consists of staying on the singular arc by applying some kind of impulse chattering. Finally, we provide a transformation, which makes clear why the Impulse Control Maximum Principle can still be applied to problems with a fixed cost despite the fact that this violates the continuous differentiability property of the model.

In this paper, we classify existing operations research models involving impulse control in four categories. In doing so we observe that *non*-linear deterministic impulse control problems in which a continuous control is present (*case D*) are missing in the literature. Some possibilities for future research arise here. A possibility is to extend Chahim et al. (2011) with continuous dike maintenance.

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