Ordering policy rules with an unconditional welfare measure [*]

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Abstract

The unconditional expectation of social welfare is often used to assess alternative macroeconomic policy rules in applied quantitative research. It is shown that it is generally possible to derive a linear-quadratic problem that approximates the exact non-linear problem where the unconditional expectation of the objective is maximised and the steady-state is distorted. Thus, the measure of policy performance is a linear combination of second moments of economic variables which is relatively easy to compute numerically, and can be used to rank alternative policy rules. The approach is applied to a simple Calvo-type model under various monetary policy rules.

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1. Introduction

Taylor (1979) suggests that, in quantitative theoretical investigations under rational expectations, macroeconomic stabilization policies ought to optimize the unconditional expectation of the policymaker's objective function. That perspective on policy assessment has proven popular; some prominent examples include Whiteman (1986), Rotemberg and Woodford (1998), Woodford (1999), Clarida, Gali and Gertler (1999), Erceg, Henderson and Levin (2000), Kollman (2002), Kim and Henderson (2005) and Schmitt-Grohe and Uribe (2007).

The ordering of policies has conventionally been done by comparing losses calculated as a linear combination of the volatilities of output and inflation gaps. That criterion can be justified as a second-order approximation of the unconditional welfare of a representative agent around a non-distorted steady state. As is well known, the non-distorted steady state is the allocation which maximizes utility in an economy in the absence of constraints. To justify use of the non-distorted steady state as the approximation 'point', it is necessary to assume that lump-sum taxation is available.

However, in the more interesting case when the steady-state is distorted, it is not known whether the loss function can be expressed as a linear combination of quadratic terms. This paper devises a tractable LQ formulation to the unconditionally optimal (UO) policy problem when the steady-state is distorted. An advantage from doing this includes, as Benigno and Woodford (2007) note, the possibility of ranking alternative policies[1].

To design our algorithm we extend the methodology of Damjanovic, Damjanovic and Nolan (DDN) (2008), which derives the first-order necessary conditions for the policy optimizing the unconditional expectation of welfare[2]. Then, similar to Judd (1999) and Benigno and Woodford (2007), the linear-quadratic approximation is done around the optimal deterministic steady state;

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[1] See also Kim and Kim (2007).

[2] See also Blake (2001) and Jensen and McCallum (2002, 2010) and Whiteman (1986).

in our case around the unconditionally optimal steady-state and in Benigno and Woodford's case, the timeless-perspective (TP) steady-state.[3] In contrast to the timeless perspective, unconditional optimisation incorporates the effect of policy on the distribution of initial conditions. However, we show that accounting for these initial conditions does not preclude the possibility of LQ approximation.

The paper highlights important differences between UO and TP policies when the steady-state is distorted. Jensen and McCallum (2010) compare UO policies (what they call "optimal continuation" policies) with the TP, when the steady-state is efficient. In that case, it is shown that the form of the welfare function to be optimized is the same across policies. Here we show that the corresponding LQ problems can be significantly different when the steady-state is distorted. We find that UO and TP approaches imply different steady states, different arguments in the social welfare function and different dynamic constraints. Even the number of dynamic constraint may differ across the TP and UO policy problems.

Finally, we also develop a useful approach for constructing the unconditional welfare measure. Since this measure can be presented in the form of a linear combination of the second moments, one can apply the Anderson, McGrattan, Hansen and Sargent (1996) algorithm which has good convergence properties. Consequently, it is also straightforward numerically to analyze UO policies.

A specific application of the approach is provided employing the canonical New Keynesian model. A number of insights emerge. First, unconditionally optimal monetary policy is characterized by trend inflation. That trend in inflation complicates the linear-quadratification[4]. That explains a second insight: The second-order approximate loss function is no longer defined solely over terms in output and inflation as found in DDN for the non-distorted steady-state case. However, the loss function that one obtains is easily interpreted in light of the underlying distortions in the economy. The approximate loss function is used to evaluate and rank different simple rules for monetary policy (i.e., the nominal

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[3]See also Debortoli and Nunes (2006) and Levine, Pearlman and Pierse (2008).

[4]As shown in Ascari and Ropele (2008) and Damjanovic and Nolan (2010).

interest rate). The welfare implications of nominal income targeting versus inflation targeting are explored and our results are contrasted with some of those of Kim and Henderson (2005).

The rest of the paper is organized as follows. In section 2 the basic problem is set out in a general form. The problem is analyzed and it is shown that one can derive a purely quadratic approximation to the unconditional expectation of the objective function. Section 3 begins the application; first a canonical New Keynesian, Calvo-price-setting model is set up. Section 4 formalizes the policy problem and demonstrates the application of the various steps in the approach of section 2. There is then a brief discussion of the implications for optimal monetary policy when the steady state is distorted and the authorities are optimizing over the unconditional loss function. In Section 5 we use the unconditional welfare criterion to explore briefly the impact of different simple rules for monetary policy. Section 6 offers some conclusions. Appendices contain proofs and details of key derivations.

2. The general problem

Consider a discounted loss function of the form

$$L_t = (1 - \beta) E_t \sum_{j=0}^{\infty} \beta^j l(x_{t+j}), \qquad (2.1)$$

where E_t is the expectations operator *conditional on information up through date* t, β is the time discount factor, $l(x_{t+j})$ is the period loss function and x_t is a vector of target variables. Specifically, $x_t = [Z_t, z_t, i_t]$, where Z_t is a vector of predetermined endogenous variables (lags of variables that are included in z_t and i_t), z_t is a vector of non-predetermined endogenous variables (including 'jump' variables), the value of which will generally depend upon both policy actions and exogenous disturbances at date t, and i_t is a vector of policy instruments, the value of which is chosen in period t. Let μ_t denote a vector of exogenous disturbances. For simplicity, assume that μ_t is a function of primary i.i.d. shocks, $(e_i)_{-\infty}^t$.

Further, let the evolution of the endogenous variables z_t and Z_t be determined by a system of simultaneous equations,

$$E_t F\left(x_{t+1}, x_t, \mu_t\right) = 0. \tag{2.2}$$

Let us further assume, following Taylor (1979), that the policy maker seeks to minimize the unconditional expectation of the loss function (2.1), subject to constraints, $(2.2)^5$. That is, he or she searches for a policy rule

$$\varphi\left(E_t x_{t+1}, x_t, \mu_t\right) = 0 \tag{2.3}$$

such that

$$\varphi = \arg\min EL_t(\varphi), \tag{2.4}$$

where E is the unconditional expectations operator. We call such a policy "unconditionally optimal" and denote it 'UO-policy'.

2.1. Solution

The first step is to formulate the non-linear policy problem and identify the non-stochastic steady state around which approximation needs to take place. Next, the possibility of a second-order approximation to welfare is addressed; specifically the possibility of a loss function that is solely a function of quadratic terms. However, an alternative approach to analyzing (2.2)-(2.4) is to solve a non-linear problem and to analyze the linearized optimality conditions. So, finally in this section we establish the equivalence of the LQ approach (which is the central topic of this paper) with that alternative approach of "optimize then linearize".

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[5]Taylor's approach may be interpreted as a recommendation: Policymakers *ought* to seek to minimize the unconditional value of the loss function. This appears partly, perhaps largely, in response to the issue of time inconsistency. See Taylor (1979) for further discussion. McCallum (2005) is an interesting discussion of these, and related, issues.

2.1.1. Necessary conditions for an optimum

Consider the following Lagrangian function which derives from the above optimal policy problem:

$$\mathcal{L}\left(y_{t}, x_{t}, \mu_{t}\right) = E\left(l\left(x_{t}\right) + \xi_{t} F\left(y_{t}, x_{t}, \mu_{t}\right) + \rho_{t}\left(x_{t+1} - y_{t}\right)\right). \qquad (2.5)$$

DDN (2008) show that the necessary conditions for the optimality of policy, φ, is that it implies a path for the endogenous variables, x_{t} and y_{t}, and that there exists Lagrange multipliers, $\left(\xi_{t}, \rho_{t}\right)$, that together satisfy the first-order conditions (2.6), (2.7) and constraints (2.2)[6],

$$\frac{\partial H}{\partial x_{t}} = \frac{\partial l\left(x_{t}\right)}{\partial x} + \xi_{t}\frac{\partial F\left(y_{t}, x_{t}, \mu_{t}\right)}{\partial x} + \rho_{t-1} = 0; \qquad (2.6)$$

$$\frac{\partial H}{\partial y_{t}} = \xi_{t}\frac{\partial F\left(y_{t}, x_{t}, \mu_{t}\right)}{\partial y_{t}} - \rho_{t} = 0, \qquad (2.7)$$

where $H\left(y_{t}, x_{t}, \mu_{t}\right)$ is the Hamiltonian for (2.5), such that $\mathcal{L}\left(y_{t}, x_{t}, \mu_{t}\right) = E\left(H\left(y_{t}, x_{t}, \mu_{t}\right)\right).$

Judd (1999), Woodford (2002) and Benigno and Woodford (2005) demonstrate very clearly that the choice of the steady-state is crucial (along with the solution concept for forward-looking policy problems) in being able to obtain LQ approximations to general non-linear, forward-looking policy problems. To choose the deterministic steady state around which log-linearization takes place, one needs to solve the system of first-order conditions (2.6), (2.7) and constraints (2.2). The steady state (X, ξ) is defined by the system (2.8-2.9):

$$F\left(X, X, \mu\right) = 0; \qquad (2.8)$$

$$\frac{\partial l\left(X\right)}{\partial x_{t}} + \xi\frac{\partial F\left(X, X, \mu\right)}{\partial x} + \xi\frac{\partial F\left(X, X, \mu\right)}{\partial y} = 0, \qquad (2.9)$$

where X, ξ and μ indicate the vectors of steady state values of endogenous variables, Lagrange multipliers and the average value of shocks, respectively. We refer to (X, ξ) as the "unconditionally optimal steady state"[7].

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[6]The notation ξF is a shorthand for the tensor product, $\sum_{i=1}^{n} \xi_{i} F_{i}$.
[7]It is assumed throughout that system (2.8) has a unique solution.

In the absence of shocks, solution (2.8) shows that unconditionally optimal policy delivers the steady state with the highest level of steady state welfare. This is not the case for "timeless perspective"-optimal policy. It is worth emphasizing that the TP approach discussed in Woodford (2002) implies different first-order conditions and therefore a different center of approximation. That difference will be shown to lead to a different optimal monetary policy.[8]

2.2. The possibility of pure second-order approximation

The value of the loss function $El(x_t)$ should not change if combined with the unconditional expectation of the constraints $EF(y_t, x_t, \mu_t)$. Thus, the appendix demonstrates that the second-order approximation to this combination has a pure second-order form. That is,

$$
\begin{aligned}
El(x_t, \mu_t) &= E\left[l(x_t) + \xi F(y_t, x_t, \mu_t)\right] \\
&= EQ_l + \xi EQ_F + t.i.p + O3.
\end{aligned}
\tag{2.10}
$$

The notation $O3$ denotes third or higher-order terms. Q_l and Q_F are pure second-order terms of the log-approximation, around the unconditionally optimal steady state, to the loss function $l(x_t)$ and dynamic constraints $EF(x_{t+1}, x_t, \mu_t)$:

$$
\begin{aligned}
Q_l &= \frac{1}{2}\left(X^2 \frac{\partial^2 l}{\partial x^2}\widehat{x}_t\widehat{x}_t\right); \\
Q_F &= \frac{1}{2}X^2\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)\widehat{x}_t\widehat{x}_t \\
&\quad + XX\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_t\widehat{x}_{t+1} + X\mu\frac{\partial^2 F}{\partial x\partial \mu}\widehat{x}_t\widehat{\mu}_t + X\mu\frac{\partial^2 F}{\partial y\partial \mu}\widehat{x}_{t+1}\widehat{\mu}_t,
\end{aligned}
$$

where we use $\widehat{x}_t$ to denote a log deviation from steady state.

It is straightforward to show that the maximization of the unconditional objective (2.10) subject to the linearized analogues of equations (2.2) yields the same solution as log-linearization of the first-order conditions (2.6). This latter approach is proposed by Khan, King and Wolman (2004) in the context

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[8]Specifically, in the TP methodology, equation (2.6) is replaced by $\frac{\partial l(x_t)}{\partial x} + \xi_t\frac{\partial F(y_t, x_t, \mu_t)}{\partial x} + \beta^{-1}\rho_{t-1} = 0$, and therefore (2.9) becomes
$\frac{\partial l(X)}{\partial x_t} + \xi\frac{\partial F(X,X,\mu)}{\partial x} + \beta^{-1}\xi\frac{\partial F(X,X,\mu)}{\partial y} = 0.$

of conditional optimization, and is extended in DDN (2008) to unconditional optimization. See Appendix 7.2 for a confirmation of our assertion.

2.3. Substitution techniques for UO and TP policies

Although the method of pure second-order approximation, (2.10), is straightforward and quite efficient, it may be useful to show how one can replicate the same welfare analysis by substituting variables employing the dynamic constraints, (2.2). In particular, it demonstrates that even though UO policy cannot ignore initial conditions, that does not prevent one from using a substitution approach for UO policy analysis. Consider a second-order approximation to the dynamic constraint equations,

$$\widehat{x}_{t+1} = \alpha\widehat{x}_t + \widehat{y}_t + Q_t + O_2 \tag{2.11}$$

where Q_t is a pure quadratic form.

The TP methodology expresses the discounted sum of $\{\widehat{x}_{t+s}\}_{s=0}^{+\infty}$ as a function of $\{\widehat{y}_{t+s}\}_{s=0}^{+\infty}$. In that case, equation (2.11) is integrated forward to yield

$$\sum_{s=0}^{+\infty}\beta^s\widehat{x}_{t+1+s} = a\sum_{s=0}^{+\infty}\beta^s\widehat{x}_{t+s} + \sum_{s=0}^{+\infty}\beta^s\widehat{y}_{t+s} + \sum_{s=0}^{+\infty}\beta^s Q_{t+s} + O_2.$$

That expression can be simplified as

$$\left(\beta^{-1} - a\right)\sum_{s=0}^{+\infty}\beta^s\widehat{x}_{t+s} - \beta^{-1}\widehat{x}_t = \sum_{s=0}^{+\infty}\beta^s\widehat{y}_{t+s} + \sum_{s=0}^{+\infty}\beta^s Q_{t+s} + O_2$$

Then an initial value, $\widehat{x}_t$, is ignored as a "term independent of policy" and the final expression appears as

$$\sum_{s=0}^{+\infty}\beta^s\widehat{x}_{t+s} = \frac{1}{\beta^{-1} - a}\sum_{s=0}^{+\infty}\beta^s\widehat{y}_{t+s} + \frac{1}{\beta^{-1} - a}\sum_{s=0}^{+\infty}\beta^s Q_{t+s} + O_2.$$

This expression is then used to calculate approximate utility.

To deriver the analogous expression in the case of UO policy one applies the unconditional expectations operator to (2.11)

$$E\widehat{x}_{t+1} = E\alpha\widehat{x}_t + E\widehat{y}_t + EQ_t + O_2. \tag{2.12}$$

Then, one uses the fact that $E\widehat{x}_{t+1} = E\widehat{x}_t$, which transforms (2.12) into

$$E\widehat{x}_t = \frac{1}{1 - a}E\widehat{y}_t + \frac{1}{1 - a}EQ_t + O_2, \tag{2.13}$$

which is the desired expression.

3. Example: Calvo model with distorted steady state

A more or less canonical dynamic New Keynesian model is now developed and two issues in particular are pursued. First, which model variables appear in the approximate loss function? Second, some insight is sought into the nature of optimal monetary policy.

3.1. The Households

There is a large number of identical agents in this (closed) economy where the only input to production is labour. Each agent evaluates utility using the following criterion:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(Y_t, N_t(i)) = E_0 \sum_{t=0}^{\infty} \beta^t \left(\log(Y_t) - \frac{\lambda}{1+v} \left(\int_i N_t(i) di \right)^{1+v} \right). \quad (3.1)$$

E_t denotes the conditional expectations operator at time $t \geq 0$, β is the discount factor, Y_t is consumption and $N_t(i)$ is the quantity of labour supplied to industry i; labour is industry specific. $v \geq 0$ measures the labour supply elasticity while λ is a 'preference' parameter.

Consumption is defined over a Dixit-Stiglitz basket of goods

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}. \quad (3.2)$$

The average price-level, P_t, is known to be

$$P_t = \left[\int_0^1 p_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}}. \quad (3.3)$$

The demand for each good is given by

$$Y_t(i) = \left(\frac{p_t(i)}{P_t} \right)^{-\theta} Y_t^d, \quad (3.4)$$

where $p_t(i)$ is the nominal price of the final good produced in industry i and Y_t^d denotes aggregate demand.

Agents face the flow constraint

$$P_t Y_t + B_t = [1 + i_{t-1}] B_{t-1} + W_t N_t (1 - \tau) + \Pi_t. \tag{3.5}$$

As all agents are identical, the only financial assets traded in equilibrium will be those issued by the fiscal authority. Here B_t denotes the nominal value of government bond holdings, at the end of date t, $1 + i_t$ is the nominal interest rate on this 'riskless' one-period nominal asset, W_t is the nominal wage in period t (our assumptions mean that we do not need to index wages on i), and Π_t indicates any profits remitted to the individual. It is assumed that labour income is taxed at rate τ. The usual conditions are assumed to apply to the consumer's limiting net savings behavior. Hence, necessary conditions for an optimum include:

$$-\frac{U_N'(Y_t, N_t)}{U_Y'(Y_t, N_t)} = \lambda N_t^v Y_t = w_t (1 - \tau); \tag{3.6}$$

$$w_t = \frac{\lambda}{1 - \tau} N_t^v Y_t; \tag{3.7}$$

and

$$E_t \left\{ \frac{\beta U_Y'(Y_{t+1}, N_{t+1})}{U_Y'(Y_t, N_t)} \frac{P_t}{P_{t+1}} \right\} = \frac{1}{1 + i_t}. \tag{3.8}$$

Here w_t denotes the real wage. The complete markets assumption implies the existence of a unique stochastic discount factor,

$$Q_{t,t+k} = \beta \frac{Y_t P_t}{Y_{t+k} P_{t+k}}, \tag{3.9}$$

where

$$E_t \{Q_{t,t+k}\} = E_t \prod_{j=0}^{k} \frac{1}{1 + i_{t+j}}.$$

3.2. Representative firm: factor demand

As noted, labour is the only factor of production. Firms are monopolistic competitors who produce their distinctive goods according to the following technology

$$Y_t(i) = A_t [N_t(i)]^{1/\phi}, \tag{3.10}$$

where $N_t(i)$ denotes the amount of labour hired by firm i in period t, A_t is a stochastic productivity shock and $1 < \phi$.

The demand for output determines the demand for labour. Hence one finds that

$$N_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\theta\phi}\left(\frac{Y_t}{A_t}\right)^{\phi}. \tag{3.11}$$

There is an economy-wide labour market so that all firms pay the same wage for the same labour. As a result, as asserted above, one may write $w_t(i) = w_t, \quad \forall i$. All households provide the same share of labour to all firms. The total amount of labour will then be

$$N_t = \int N_t(i)di = \left(\frac{Y_t}{A_t}\right)^{\phi}\int\left(\frac{P_t(i)}{P_t}\right)^{-\theta\phi}di = \left(A_t^{-1}Y_t\right)^{\phi}\Delta_t, \tag{3.12}$$

where Δ_t is the measure of price dispersion:

$$\Delta_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\theta\phi}di. \tag{3.13}$$

3.3. Representative firm: price setting

As in Calvo (1983), each period a fixed proportion of firms are allowed to adjust prices. Those firms choose the nominal price which maximizes their expected profit given that they have to charge the same price in k periods time with probability α^k. As usual, we assume that firms are cost-takers. Let $p_t'(i)$ denote the choice of nominal price by a firm that is permitted to re-price in period t. As all firms who are permitted to reprice will choose the same price, optimal repricing implies

$$\left(\frac{p_t'}{P_t}\right)^{1+\theta(\phi-1)} = \frac{\left(\frac{\theta}{\theta-1}\right)\sum_{k=0}^{\infty}(\alpha\beta)^k Y_{t+k}^{-1}\left[\phi\mu_{t+k}w_{t+k}A_{t+k}^{-\phi}Y_{t+k}^{\phi}(P_t/P_{t+k})^{-\theta\phi}\right]}{\sum_{k=0}^{\infty}(\alpha\beta)^k(P_t/P_{t+k})^{1-\theta}}. \tag{3.14}$$

where μ_t is a cost-push shock. The price index then evolves according to the law of motion,

$$P_t = \left[(1-\alpha)p_t'^{1-\theta} + \alpha P_{t-1}^{1-\theta}\right]^{1/(1-\theta)}. \tag{3.15}$$

Because the relative prices of the firms that do not change their prices in period t fall by the rate of inflation, the law of motion for the measure of price dispersion is

$$\Delta_t = \alpha\Delta_{t-1}\pi_t^{\theta\phi} + (1-\alpha)\left(p_t'/P_t\right)^{-\theta\phi}. \tag{3.16}$$

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4. UO Monetary Policy

Proposition 4.1 sets out the relevant UO Ramsey problem.

Proposition 4.1. *The UO Ramsey plan is a choice of state contingent paths for the endogenous variables* $\{\pi_{t+k}, \Delta_{t+k}, p_{t+k}, u_{t+k}, X_{t+k}, Z_{t+k}\}_{k=0}^{\infty}$ *from date t onwards given* $\{E_t A_{t+k}, E_t \mu_{t+k}\}_{k=0}^{\infty}$, *so as to maximize social welfare function (4.1) subject to constraints (4.2)-(4.4):*

$$\max EE_t \sum_{k=0}^{\infty} \beta^k \left(\frac{\log u_{t+k}}{(1+v)\phi} - \frac{1}{\phi} \log \Delta_{t+k} - u_{t+k} \right); \tag{4.1}$$

subject to:

- *The Phillips block*

$$p_t^{\theta\phi-\theta+1} X_t = Z_t; \tag{4.2}$$

$$X_t = 1 + \alpha\beta E_t X_{t+1} \pi_{t+1}^{\theta-1};$$

$$Z_t = \frac{(1+v)\phi}{\Phi} \left(\frac{\mu_t}{\mu} \frac{u_t}{\Delta_t} \right) + \alpha\beta E_t Z_{t+1} \pi_{t+1}^{\theta\phi}.$$

- *The law of motion of prices*

$$\Delta_t = \alpha \Delta_{t-1} \pi_t^{\theta\phi} + (1-\alpha) p_t^{-\theta\phi}. \tag{4.3}$$

- *Prices*: p_t *is the relative price set by firms updating at time t,*

$$p_t = \left(\frac{1 - \alpha\pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{1}{1-\theta}}. \tag{4.4}$$

It is useful in formalizing this policy problem to define some variables as follows: Discounted marginal revenue is $X_t := E_t \sum_{k=0}^{\infty} (\beta\alpha)^k \left(\frac{P_t}{P_{t+k}} \right)^{1-\theta}$; discounted marginal cost is $Z_t := E_t \sum_{k=0}^{\infty} (\beta\alpha)^k \frac{\mu_{t+k}}{\mu} \frac{(1+v)\phi}{\Phi} \frac{u_{t+k}}{\Delta_t} \left(\frac{P_t}{P_{t+k}} \right)^{-\theta\phi}$; period marginal cost is $u_{t+k} := \frac{\lambda}{1+v} \Delta_{t+k}^{v+1} \left(A_{t+k}^{-1} Y_{t+k} \right)^{(v+1)\phi}$; and $\Phi := \frac{\theta-1}{\theta} \frac{1-\tau}{\mu} < 1$, indexes the steady state distortions in this economy.

One can set up the Hamiltonian for this problem, as proposed in section 2, as follows:

$$
\begin{aligned}
H &= \left(\frac{1}{(v+1)\phi} \log u_t - \frac{1}{\phi} \log \Delta_t - u_t \right) \\
&\quad + \rho_t \left(X_t - 1 - \beta \alpha \pi_{t+1}^{\theta-1} X_{t+1} \right) \\
&\quad + \varphi_t \left(Z_t - \frac{\mu_t}{\mu} \frac{(1+v)\phi}{\Phi} \frac{u_t}{\Delta_t} - \beta \alpha \pi_{t+1}^{\theta\phi} Z_{t+1} \right) \\
&\quad + \xi_t \left(Z_t - p_t^{\theta\phi-\theta+1} X_t \right) \\
&\quad + \eta_t \left(\Delta_t - \alpha \Delta_{t-1} \pi_t^{\theta\phi} - (1-\alpha) p_t^{-\theta\phi} \right) \\
&\quad + \delta_t \left(p_t - \left(\frac{1 - \alpha \pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{1}{1-\theta}} \right).
\end{aligned}
$$

The necessary conditions for an optimum include:

$$
\begin{aligned}
u_t \frac{\partial}{\partial u_t} H &= \frac{1}{(v+1)\phi} - u_t - \varphi_t \frac{\mu_t}{\mu} \frac{(1+v)\phi}{\Phi} \frac{u_t}{\Delta_t}; \\
\frac{\partial}{\partial \Delta_t} H &= \left(-\frac{1}{\phi \Delta_t} \right) + \varphi_t \left(\frac{\mu_t}{\mu} \frac{(1+v)\phi}{\Phi} \frac{u_t}{\Delta_t^2} \right) + \eta_t - E_t \alpha \eta_{t+1} \pi_{t+1}^{\theta\phi}; \\
\frac{\partial}{\partial X_t} H &= \rho_t - \rho_{t-1} \beta \alpha \pi_t^{\theta-1} - \xi_t p_t^{\theta\phi-\theta+1}; \\
\frac{\partial}{\partial Z_t} H &= \varphi_t - \varphi_{t-1} \beta \alpha \pi_t^{\theta\phi} + \xi_t; \\
\pi_t \frac{\partial}{\partial \pi_t} H &= -(\theta-1)\rho_{t-1}\beta\alpha\pi_t^{\theta-1} X_t - \varphi_{t-1}\beta\alpha\theta\phi\pi_t^{\theta\phi} Z_t \\
&\quad - \eta_t \alpha\theta\phi\Delta_{t-1}\pi_t^{\theta\phi} - \delta_t p_t^\theta \frac{\alpha\pi_t^{\theta-1}}{1-\alpha}; \\
p_t \frac{\partial}{\partial p_t} H &= -\xi_t (\theta\phi-\theta+1) X_t p_t^{1-\theta+\theta\phi} + \theta\phi\eta_t (1-\alpha) p_t^{-\theta\phi} + p_t \delta_t.
\end{aligned}
$$

$$(4.5)$$

To reduce a little on notation, denote

$$
c_t := \left(\frac{\mu_t}{\mu} \frac{(1+v)\phi}{\Phi} \frac{u_t}{\Delta_t} \right), \tag{4.6}
$$

which represents marginal production costs.

4.1. The steady state

As noted, unconditionally optimal policy is associated with the highest level of steady-state welfare, unlike TP optimal policy. It is well known (see Benigno and Woodford, 2005) that TP optimal policy requires price stability in the steady state.

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On the other hand, King and Wolman (1999) argue that a slightly positive inflation rate maximises steady-state welfare. We now turn in more detail to steady-state analysis.

The value of the endogenous variables in steady state should solve the system of constraints (4.2), (4.3), (4.4), (4.6) and the first-order conditions, (4.5). As a result one obtains the following steady state equations:

$$
\begin{array}{ll}
p = \left(\frac{1-\alpha\pi^{\theta-1}}{1-\alpha}\right)^{\frac{1}{1-\theta}}; & \varphi(v+1)\phi c = 1 - \Phi\Delta c; \\
\Delta = \left(\frac{1-\alpha}{1-\alpha\pi^{\theta\phi}}\right)p^{-\theta\phi}; & \eta\Delta\left(1-\alpha\pi^{\theta\phi}\right) = \left(\frac{1}{\phi} - \varphi c\right); \\
X = \frac{1}{1-\beta\alpha\pi^{\theta-1}}; & \xi = -\varphi\left(1-\alpha\beta\pi^{\theta\phi}\right); \\
Z = Xp^{\theta\phi-\theta+1}; & \rho = \xi Xp^{\theta\phi-\theta+1} = -\varphi\left(1-\alpha\beta\pi^{\theta\phi}\right)Xp^{\theta\phi-\theta+1}; \\
c = \left(1-\alpha\beta\pi^{\theta\phi}\right)Z; & \delta p = (\theta\phi-\theta+1)\rho - \theta\phi\eta(1-\alpha)p^{-\theta\phi}. \\
u = \Phi\frac{c\Delta}{(1+v)\phi};
\end{array}
\tag{4.7}
$$

Using these equations, one can derive the following expression

$$
(\theta-1)\rho\beta\alpha\pi^{\theta-1}X + \varphi\beta\alpha\theta\phi\pi^{\theta\phi}Z + \eta\alpha\theta\phi\Delta\pi^{\theta\phi} + \delta p^\theta\frac{\alpha\pi^{\theta-1}}{1-\alpha} = 0, \tag{4.8}
$$

which can be used to infer certain properties of the optimal steady-state inflation rate.

Proposition 4.2. *The steady state inflation is positive, $\pi \geq 1$. Price stability is only optimal if either $\beta = 1$ or if $\Phi = 1$ (which corresponds to the non-distorted steady state). Moreover π is unique and bounded: $\pi \leq \min(\beta^{1/(\theta-1-\phi\theta)}, \alpha^{-1/(\phi\theta)})$.*

Proof. See Appendix. ■

Proposition 4.3. *Steady state inflation increases with the distortion, $1 - \Phi$, and declines in the discount factor β and the labour elasticity, v.*

Proof. See Appendix. ■

Using parameter values typically found in the literature, expression (4.8) implies that optimal steady state inflation is of the order of 0.2% a year. As

discussed in King and Wolman (1999), this small positive trend in inflation reflects a number of conflicting effects. On the one hand, a small amount of inflation can boost demand, as it partially offsets the markup distortion. On the other hand, price dispersion, which is rising in inflation, acts rather like a cost shock on firms, for reasons analyzed in Damjanovic and Nolan (2010). Hence, one finds that optimal trend inflation has a U-shaped relation to price stickiness, α; it is increasing in α when initial price dispersion is relatively small, and declines once initial price dispersion is sufficiently large. Optimal inflation declines in the discount factor, β. As discussed in more detail in DDN (2008) and demonstrated in section 2.3, UO policy in contrast to timeless perspective policy, gives some weight to the distribution of initial conditions. In particular, it considers the distribution of the initial output gap. That is partly why some stimulation of output via inflation is desirable. So the smaller the discount factor, the higher is the relative weight on initial conditions and the higher the optimal inflation rate. Finally, we note that the nominal interest rate is positive in the UO steady state. That conclusion follows from the Euler equation (3.8) which yields $1/(1+i) = \beta/\pi < 1$.

4.2. The quadratic form

Having recovered the optimal steady state, one can obtain a quadratic loss function; that is, an equation of the form (2.10):

$$EU = EQ_l + \rho EQ_X + \varphi EQ_Z + \xi EQ_{ZX} + \eta EQ_\Delta + \delta EQ_p,$$

where

$$
\begin{aligned}
Q_l &= -\frac{1}{2} u \widehat{u}_t^2; \\
Q_X &= \frac{1}{2} X \left(\widehat{X}_t^2 \right) - \frac{1}{\beta \alpha \pi^{\theta-1}} X \frac{1}{2} \widehat{X}_t^2 = -\frac{1}{2} \frac{1}{\beta \alpha \pi^{\theta-1}} \widehat{X}_t^2; \\
Q_z &= \frac{1}{2} Z \widehat{Z}_t^2 - \frac{1}{2} c \widehat{c}_t^2 - \frac{1}{2} \beta \alpha \pi^{\theta\phi} Z \left(\widehat{Z}_{t+1} + \theta\phi \widehat{\pi}_{t+1} \right)^2; \\
Q_{xz} &= \frac{1}{2} Z \widehat{Z}_t^2 - Z \frac{1}{2} \left((\theta\phi - \theta + 1) \widehat{p}_t + \widehat{X}_t \right)^2 = 0; \\
Q_\Delta &= \frac{1}{2} \Delta \widehat{\Delta}_t^2 - (1-\alpha) p^{\theta\phi} \frac{1}{2} (\theta\phi \widehat{p}_t)^2 - \frac{1}{2} a \Delta \pi^{\theta\phi} \left(\widehat{\Delta}_{t-1} + \theta\phi \widehat{\pi}_t \right)^2; \\
Q_p &= \frac{1}{2} p \widehat{p}_t^2 - \frac{1}{2} p \frac{\alpha \pi^{\theta-1}}{1 - \alpha \pi^{\theta-1}} \left(\theta \frac{\alpha \pi^{\theta-1}}{1 - \alpha \pi^{\theta-1}} - 1 \right) \widehat{\pi}_t^2.
\end{aligned}
$$

The details of the derivation are set out in the appendix. One can simplify the above expression in a number of ways. Consider the following expression:[9]

$$EU = -\frac{1}{2}E\left[u\widehat{u}_t^2 + \varphi c\widehat{c}_t^2 + \Lambda_x \widehat{X}_t^2 + \Lambda_\Delta \widehat{\Delta}_t^2 + \Lambda_\pi \widehat{\pi}_t^2\right].$$ (4.9)

It is possible to write equation (4.9) in a way that relates it more clearly to the 'standard' loss function often employed which is simply defined over output and inflation. First, recall the definitions of $\widehat{u}_t$:

$$\widehat{u}_t = (v+1)\widehat{\Delta}_t + (v+1)\phi\left(\widehat{Y}_t - \widehat{A}_t\right).$$

Now note that $\widehat{c}_t$ can be represented as

$$\widehat{c}_t = \widehat{u}_t - g_t.$$

$\widehat{g}_t$ can be thought of as the 'labour wedge' of inefficiency (note the role price dispersion):

$$g_t := \frac{\partial U}{\partial N} \Big/ \frac{\partial F}{\partial N} = \frac{u_t}{c_t} = \left(\frac{\mu}{\mu_t}\frac{\Phi}{(1+v)\phi}\Delta_t\right),$$

which in log-linearized form is simply:

$$\widehat{g}_t = \widehat{\Delta}_t - \widehat{\mu}_t.$$

So one can further simplify (4.9) to

$$EU = -\frac{1}{2}E\left[\phi(1+v)\left(\widehat{Y}_t - \widehat{Y}_t^*\right)^2 + G\widehat{g}_t^2 + \Lambda_x \widehat{X}_t^2 + \Lambda_\Delta \widehat{\Delta}_t^2 + \Lambda_\pi \widehat{\pi}_t^2\right].$$ (4.10)

The term $\widehat{Y}_t^*$ represents the 'target' level of output $Y_t^* = \widehat{A}_t - \widehat{\mu}_t - v_\Delta \widehat{\Delta}_t$ (and where details concerning coefficients are again given in the Appendix). The 'target' rate is increasing in productivity and declining in the cost-push shock; it is also declining in price dispersion. The variable $\widehat{X}_t$ represents, in effect, the losses to the firm forced to charge suboptimal prices due to price stickiness and expected inflation, to which they may not be able to react.

This form of the loss function can easily be nested to familiar cases, either the non-distorted steady state where $\Phi = 1$, or where the steady state of the model economy remains distorted but where the social discount rate is equal to the private

---

[9]The coefficients of equation (4.9) are positive for reasonable parameterizations.

rate of discount, $\beta = 1$ (in which case the UO policy and the timeless perspective policies coincide). In both special cases optimal monetary policy corresponds to price stability and the loss function (4.10) reduces to a familiar form defined simply over inflation and output. Specifically, if the optimal steady state is characterized by price stability, then $\Lambda_x = 0$. Moreover one can easily show that price dispersion, $\widehat{\Delta}_t$, is a second-order term in that case. Lastly, the labour wedge $\widehat{g}_t$ is then simply a cost-push shock, $\widehat{\mu}_t$, and can be considered as a term independent of policy.

5. Application: Unconditional ordering of simple rules

The foregoing approach is easily used to evaluate simple rules for monetary policy and to highlight the potential significance for policy design of a distorted steady-state. First, write the model in vector autoregressive form as follows:

$$\widehat{p}_t - \frac{\alpha\pi^{\theta-1}}{1 - \alpha\pi^{\theta-1}}\widehat{\pi}_t = 0 \tag{5.1}$$

$$E_t\widehat{\pi}_{t+1} + E_t\widehat{Y}_{t+1} = \widehat{Y}_t + (1-\beta)\widehat{i}_t \tag{5.2}$$

$$\beta\alpha\pi^{\theta\phi}E_t\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right) = \widehat{Z}_t - \frac{c}{Z}\widehat{c}_t \tag{5.3}$$

$$\beta\alpha\pi^{\theta-1}E_t\left(\widehat{X}_{t+1} + (\theta-1)\widehat{\pi}_{t+1}\right) = \widehat{X}_t \tag{5.4}$$

$$\widehat{Z}_t - \left((\theta\phi - \theta + 1)\frac{\alpha\pi^{\theta-1}}{1 - \alpha\pi^{\theta-1}}\widehat{\pi}_t + \widehat{X}_t\right) = 0 \tag{5.5}$$

$$-Y_t^* + \widehat{A}_t - \widehat{\mu}_t - \left(u + \frac{v}{(v+1)\phi}\right)\widehat{\Delta}_t = 0 \tag{5.6}$$

$$-\widehat{g}_t + \widehat{\Delta}_t - \widehat{\mu}_t = 0 \tag{5.7}$$

$$-\widehat{c}_t + \widehat{u}_t - \widehat{g}_t = 0 \tag{5.8}$$

$$-\widehat{u}_t + (v+1)\widehat{\Delta}_t + (v+1)\phi(\widehat{Y}_t - \widehat{A}_t) = 0 \tag{5.9}$$

$$\widehat{\Delta}_{t+1} - a\pi^{\theta\phi}\theta\phi\widehat{\pi}_{t+1} + \frac{(1-\alpha)p^{-\theta\phi}\theta\phi}{\Delta}\widehat{p}_{t+1} = a\pi^{\theta\phi}\widehat{\Delta}_t \tag{5.10}$$

$$-\widehat{i}_t = \phi_\pi\widehat{\pi}_t + \phi_f\widehat{f}_t + m_t. \tag{5.11}$$

More compactly, one writes

$$V_{t+1} = AV_t + B\varepsilon_{t+1},$$

where V_t is the vector of endogenous variables and ε_{t+1} is the vector of exogenous shocks. In this form it is straightforward to construct the variance-covariance

matrix, $R \equiv EV_tV_t'$, using standard software such as Dynare. That is, R is recovered by solving the following matrix equation

$$R = ARA' + B\Upsilon B' \tag{5.12}$$

where $\Upsilon = E\varepsilon_t\varepsilon_t'$ is the unconditional variance-covariance matrix of the underlying shock processes. Equation (5.12) can be solved numerically using a doubling algorithm as described in Anderson, McGrattan, Hansen, and Sargent (1996) using an equivalent form

$$R = \sum_{j=0}^{+\infty} A^j B\Upsilon B' A'^j.$$

As demonstrated in section 2.2, the social welfare function is then a linear combination of the elements of matrix R.

In the above linearized system of equations the final equation (5.11) is the policy rule, where $\widehat{i}_t$ is the gross nominal interest rate, $\widehat{i}_t = \log\left(\frac{\beta}{\pi}(1 + i_t)\right)$, and $\widehat{f}_t$ represents a linear combination of policy feedback variables, while m_t is a policy shock.[10]

It is clear that steady-state distortions complicate the policy problem so far as the policymaker's objective function is concerned[11]. However, does it make any difference so far as the design of simple rules are concerned?[12].

First, a simple interest rate feedback rule is considered, where the interest rate responds to current and lagged inflation only. The feedback on current inflation is fixed at $\phi_\pi = 1.5$. Given this, the optimized weight on lagged inflation, $f = \widehat{\pi}_{t-1}$, is computed. In both the distorted and non-distorted case the optimal feedback is about 15 in the distorted case and 14 in the non-distorted steady state case.

---

[10]The following parameterization is used in the quantitative investigation: $\beta = 0.9$, $v = 1.1$, $\theta = 7$, $\alpha = 0.5$, and $\phi = 1.3$. It is assumed that shocks, A_t, μ_t and m_t follow $AR(1)$ processes with: $\rho_A = 0.98$, $\sigma_A = 0.008$, $\rho_m = 0.9$, $\sigma_m = 0.005$, and $\rho_\mu = 0.9$, $\sigma_\mu = 0.02$.

[11]That is, complicates it relative to the objective function in the non-distorted case.

[12]In the particular model developed above, the UO trend inflation is rather small and the policy ordering across distorted and non-distorted steady states is often the same for given simple rules. However, in simulations not reported, it was possible to find simple, plausible rules that result in welfare "reversals"; that is, where rule 1 welfare dominates rule 2 in the distorted economy, but where the ranking switched in the non-distorted economy.

However, the difference in welfare between responding and not responding to lagged inflation is quite substantial and may be up to 16 percentage points in terms of consumption equivalent units (see the top right hand graph in the panel below, ϕ_f is at its optimal value)). As in the TP approach, relative price distortion is very costly and the optimal simple rule may be very close to price stability ($\phi_\pi = +\infty$). However, if for any reason the policy reaction on current inflation is restricted, the economy may significantly benefit from a response to lagged inflation.

One can also show that the optimal feedback on output should be slightly *negative*, $\phi_f = -0.015$. Furthermore, inclusion of real output targeting leads to very modest welfare improvements, in the order of 10^{-3} compared with targeting inflation alone. This result is consistent with Schmitt-Grohe and Uribe (2007) who found that a positive feedback on real output did not increase welfare.

The results are summed up in Figure 1 (where the broken line is the non-distorted economy).

Figure 1: Suboptimal simple policies
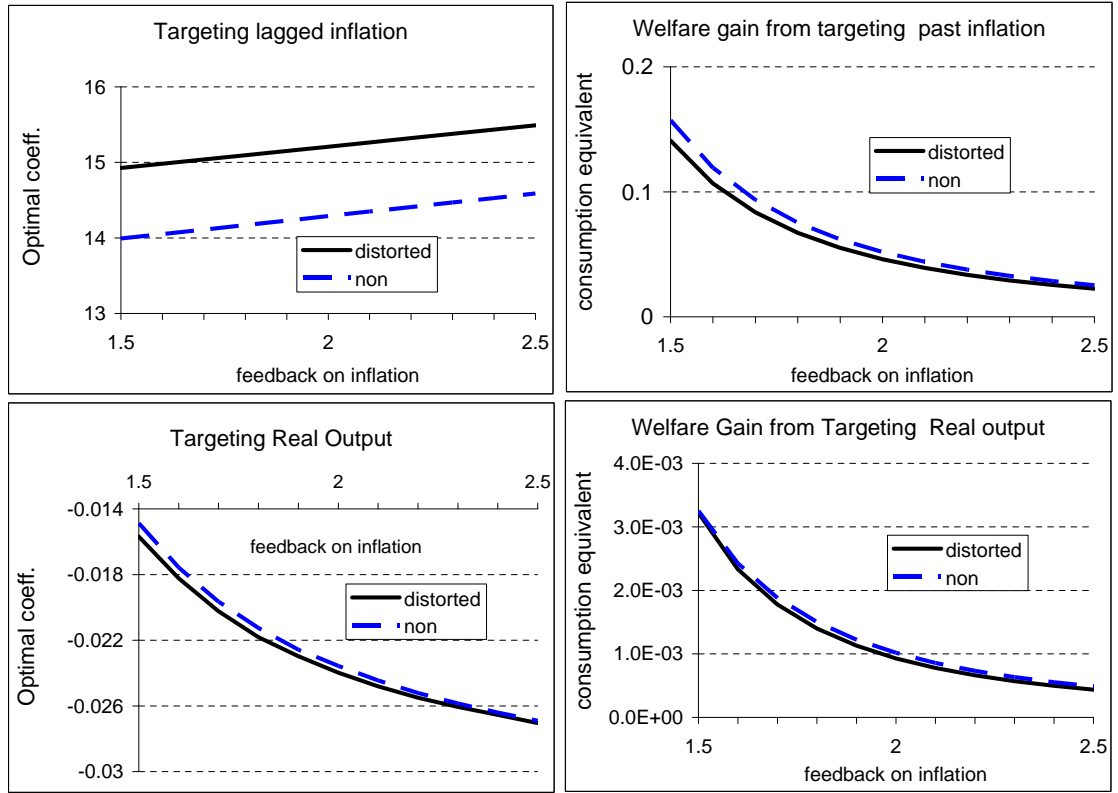
5.0.1. Targeting Nominal Income Growth

Finally, inflation targeting and nominal income targeting are compared under an UO policy criterion as in Kim and Henderson (2005). Kim and Henderson suggest, in a model with one-period price stickiness, that nominal income targeting may have superior welfare properties to inflation targeting. Two rules are compared:

$$\text{Nominal income growth targeting:} \quad i_t = 0.05\left(y_t - y_{t-1} + \pi_t\right) \quad (5.13)$$
$$+ \left(\phi_\pi - 0.05\right)\pi_t + m_t;$$
$$\text{Inflation targeting:} \quad i_t = \phi_\pi \pi_t + m_t. \quad (5.14)$$

In the case of a non-distorted steady state, and a "low" feedback on inflation the findings are similar to some of Kim and Henderson's findings. Specifically, in the case of a distorted steady-state model, the net welfare gain from targeting

nominal income growth over inflation targeting is positive. In the distorted case, inflation targeting is rarely dominated by nominal income targeting. In Figure 2 below, the relative welfare gain (over inflation targeting) in targeting nominal income growth is plotted against ϕ_π.

Figure 2: Relative welfare gain in targeting nominal income growth.
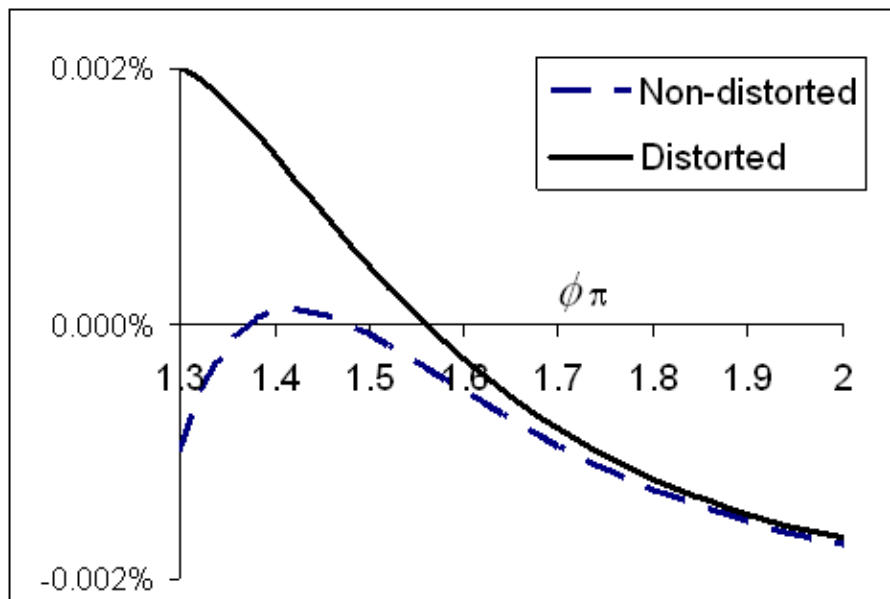
The precise position of these net welfare schedules is quite sensitive to parameterization of the model (in particular, the persistence of shocks) but in general one finds that as the feedback on inflation rises, inflation targeting is likely to dominate nominal income targeting.

6. Conclusion

The paper demonstrates that, in general, one is able to obtain a purely quadratic approximate unconditional loss function to a model economy with a distorted steady state. It develops a straightforward, efficient approach to implementing the UO algorithm. In an application, it is shown that the loss function may be somewhat more complex than in a model with no steady-state distortions; inflation and output are no longer the sole arguments in the loss function. However, the

loss function so obtained is easily interpreted in terms of the underlying distortions in the economy. Furthermore, optimal inflation and nominal interest rates are positive in the steady state. The implications for the ordering of simple rules is briefly explored.

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7. Appendices

7.1. The possibility of the second-order approximation

The first part of the appendix demonstrates the key result in Section 2.2, namely the existence of the quadratic form, (2.10). The first line of the following block of equations corresponds to the top line of (2.10), the subsequent lines being its quadratic approximation:

$$
\begin{aligned}
El\left(x_{t}\right) &= E\left[l\left(x_{t}\right)+\xi F\left(y_{t}, x_{t}, \mu_{t}\right)\right] = \\
&= E\left(l + X\frac{\partial l}{\partial x}\widehat{x}_{t} + \frac{1}{2}\left(X^{2}\frac{\partial^{2} l}{\partial x^{2}} + X\frac{\partial l}{\partial x}\right)\widehat{x}_{t}\widehat{x}_{t}\right) \\
&\quad + E\xi\left(F + X\frac{\partial F}{\partial x}\widehat{x}_{t} + X\frac{\partial F}{\partial y}\widehat{y}_{t} + \mu\frac{\partial F}{\partial \mu}\widehat{\mu}_{t}\right) \\
&\quad + \frac{1}{2}\xi\left(X\frac{\partial F}{\partial x} + X^{2}\frac{\partial^{2} F}{\partial x^{2}}\right)E\widehat{x}_{t}\widehat{x}_{t} + \frac{1}{2}\xi\left(X\frac{\partial F}{\partial y} + XX\frac{\partial F}{\partial y^{2}}\right)E\widehat{y}_{t}\widehat{y}_{t} \\
&\quad + \frac{1}{2}\xi\left(\mu\frac{\partial F}{\partial x} + \mu^{2}\frac{\partial^{2} F}{\partial x^{2}}\right)E\widehat{\mu}_{t}\widehat{\mu}_{t} \\
&\quad + \xi E\left(XX\frac{\partial F}{\partial x\partial y}\widehat{x}_{t}\widehat{y}_{t} + X\mu\frac{\partial^{2} F}{\partial x\partial \mu}\widehat{x}_{t}\widehat{\mu}_{t} + X\mu\frac{\partial F}{\partial y\partial \mu}\widehat{y}_{t}\widehat{\mu}_{t}\right) + O3.
\end{aligned}
$$

Using the constraints $E_{t}x_{t+1} = y_{t}$, and the property of unconditional expectations that $Ez_{t+1} = Ez_{t}$, this can be rewritten as

$$
\begin{aligned}
El\left(x_{t}\right) &= XE\widehat{x}_{t}\left(\frac{\partial l}{\partial x} + \xi\frac{\partial F}{\partial x} + \xi\frac{\partial F}{\partial y}\right) + \frac{1}{2}XE\widehat{x}_{t}\widehat{x}_{t}\left(\frac{\partial l}{\partial x} + \xi\frac{\partial F}{\partial x} + \xi\frac{\partial F}{\partial y}\right) \quad (7.1) \\
&\quad + EQ_{l} + \xi EQ_{F} \\
&\quad + l + \xi F + \xi\mu\frac{\partial F}{\partial \mu}E\widehat{\mu}_{t} + \frac{1}{2}\xi\left(\mu\frac{\partial F}{\partial x} + \mu^{2}\frac{\partial^{2} F}{\partial x^{2}}\right)E\widehat{\mu}_{t}\widehat{\mu}_{t} + O3. \quad (7.2)
\end{aligned}
$$

Here Q_{l} and Q_{F} are pure second-order terms:

$$
\begin{aligned}
Q_{l} &= \frac{1}{2}X^{2}\frac{\partial^{2} l}{\partial x^{2}}\widehat{x}_{t}\widehat{x}_{t}; \\
Q_{F} &= \frac{1}{2}X^{2}\left(\frac{\partial^{2} F}{\partial x^{2}} + \frac{\partial^{2} F}{\partial y^{2}}\right)\widehat{x}_{t}\widehat{x}_{t} + XX\frac{\partial^{2} F}{\partial x\partial y}\widehat{x}_{t}\widehat{x}_{t+1} + X\mu\frac{\partial^{2} F}{\partial x\partial \mu}\widehat{x}_{t}\widehat{\mu}_{t} + X\mu\frac{\partial^{2} F}{\partial y\partial \mu}\widehat{x}_{t+1}\widehat{\mu}_{t}.
\end{aligned}
$$

Furthermore, using the steady state conditions (2.8), one can show that the first line of expression (7.1) equals zero. Moreover, expression (7.2) consists of $l + \xi F = l$, the steady state value of the loss function and shocks. These are terms independent of policy (*t.i.p.*). Thus, it is proved that the loss function can be represented in a pure quadratic form.

$$
El\left(x_{t}\right) = EQ_{l} + \xi EQ_{F} + t.i.p + O3.
$$

7.2. Alternative approaches to recovering UO policy

The approach of some researchers is to solve non-linear problems and then linearize the resulting optimality conditions. For example, in the context of *conditionally optimal monetary policy*, that is the approach taken by Khan, King and Wolman (2003). This section demonstrates that this alternative approach also works in the case of unconditionally optimal policy. Specifically, the maximization of the unconditional objective (2.10) subject to the linearized analogues of equations (2.2) yields the same solution as log-linearization of the first-order conditions (2.6). The first-order conditions to the non-linear problem are written as

$$
\frac{\partial H}{\partial x_t} = \frac{\partial l\left(x_t\right)}{\partial x} + \xi_t \frac{\partial F\left(y_t, x_t, \mu_t\right)}{\partial x} + \rho_{t-1} = 0;
$$
$$
\frac{\partial H}{\partial y_t} = \xi_t \frac{\partial F\left(y_t, x_t, \mu_t\right)}{\partial y_t} - \rho_t.
$$

The log-linearized versions of these equations are:

$$
\frac{\partial H}{\partial x_t} = \frac{\partial l}{\partial x} + X \frac{\partial^2 l}{\partial x^2} \widehat{x}_t
$$
$$
+ \xi \frac{\partial F}{\partial x} + \xi \frac{\partial F}{\partial x} \widehat{\xi}_t + \xi X \frac{\partial^2 F}{\partial x^2} \widehat{x}_t + \xi X \frac{\partial^2 F}{\partial x \partial y} \widehat{y}_t + \xi \mu \frac{\partial^2 F}{\partial x \partial \mu} \widehat{\mu}_t
$$
$$
+ \rho + \rho \widehat{\rho}_{t-1} + O2; \tag{7.3}
$$
$$
\frac{\partial H}{\partial y_t} = \xi \frac{\partial F}{\partial y} + \xi \frac{\partial F}{\partial y} \widehat{\xi}_t + \xi X \frac{\partial^2 F}{\partial y^2} \widehat{y}_t + \xi X \frac{\partial^2 F}{\partial x \partial y} \widehat{x}_t + \xi X \frac{\partial^2 F}{\partial \mu \partial y} \widehat{\mu}_t
$$
$$
- \rho - \rho \widehat{\rho}_t + O2. \tag{7.4}
$$

These are simplified by plugging (7.4) into (7.3) and using the steady state conditions (2.8),

$$
\frac{\partial H}{\partial x_t} = X \frac{\partial^2 l}{\partial x^2} \widehat{x}_t
$$
$$
\xi \frac{\partial F}{\partial x} \widehat{\xi}_t + \xi X \frac{\partial^2 F}{\partial x^2} \widehat{x}_t + \xi X \frac{\partial^2 F}{\partial x \partial y} \widehat{y}_t + \xi X \frac{\partial^2 F}{\partial x \partial y} \widehat{y}_t + \xi \mu \frac{\partial^2 F}{\partial x \partial \mu} \widehat{\mu}_t
$$
$$
\xi \frac{\partial F}{\partial y} \widehat{\xi}_{t-1} + \xi X \frac{\partial^2 F}{\partial y^2} \widehat{y}_{t-1} + \xi X \frac{\partial^2 F}{\partial x \partial y} \widehat{x}_{t-1} + \xi \mu \frac{\partial^2 F}{\partial \mu \partial y} \widehat{\mu}_{t-1} = 0. \tag{7.5}
$$

Turning now to the LQ approach, utility is represented as (2.10). Hence, the relevant optimization problem is

$$
\max E l\left(\widehat{x}_t\right) = \max \frac{1}{2} E X^2 \frac{\partial^2 l}{\partial x^2} \widehat{x}_t \widehat{x}_t + \frac{1}{2} X^2 \xi E \left(\frac{\partial^2 F}{\partial x^2} \widehat{x}_t \widehat{x}_t + \frac{\partial^2 F}{\partial y^2} \widehat{y}_t \widehat{y}_t \right)
$$
$$
+ \xi X X \frac{\partial^2 F}{\partial x \partial y} E \widehat{x}_t \widehat{y}_t + \xi X \mu \frac{\partial^2 F}{\partial x \partial \mu} E \widehat{x}_t \widehat{\mu}_t + \xi X \mu \frac{\partial^2 F}{\partial y \partial \mu} E \widehat{y}_t \widehat{\mu}_t,
$$

subject to log-linearized constraints

$$F\left(E_t x_{t+1}, x_t, \mu_t\right) = X\frac{\partial F}{\partial x_t}\widehat{x}_t + X\frac{\partial F}{\partial y}\widehat{y}_t + \mu\frac{\partial F}{\partial \mu_t}\widehat{\mu}_t = 0; \tag{7.6}$$

$$\widehat{y}_t = E_t\widehat{x_{t+1}}. \tag{7.7}$$

The new Hamiltonian can be written as

$$
\begin{aligned}
\widetilde{H} &= \frac{1}{2}X^2\frac{\partial^2 l}{\partial x^2}\widehat{x}_t\widehat{x}_t + \frac{1}{2}X^2\xi\left(\frac{\partial^2 F}{\partial x^2}\widehat{x}_t\widehat{x}_t + \frac{\partial^2 F}{\partial y^2}\widehat{y}_t\widehat{y}_t\right) \\
&\quad +\xi XX\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_t\widehat{y}_t + \xi X\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{x}_t\widehat{\mu}_t + \xi X\mu\frac{\partial^2 F}{\partial y\partial\mu}\widehat{y}_t\widehat{\mu}_t \\
&\quad +s_t\left(X\frac{\partial F}{\partial x_t}\widehat{x}_t + X\frac{\partial F}{\partial y}\widehat{y}_t + \mu\frac{\partial F}{\partial \mu_t}\widehat{\mu}_t\right) + r_t\widehat{y}_t - r_{t-1}\widehat{x}_t,
\end{aligned}
$$

where s_t and r_t are the corresponding Lagrange multipliers attached to linearized constraints (7.6) and (7.7). The resulting first-order conditions are

$$
\begin{aligned}
\frac{\partial\widetilde{H}}{\partial\widehat{x}_t} &= X^2\frac{\partial^2 l}{\partial x^2}\widehat{x}_t + \xi X^2\frac{\partial^2 F}{\partial x^2}\widehat{x}_t \\
&\quad +\xi XX\frac{\partial^2 F}{\partial x\partial y}\widehat{y}_t + \xi X\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{\mu}_t + s_t X\frac{\partial F}{\partial x_t}\widehat{x}_t - \widehat{r}_{t-1},
\end{aligned}
$$

and

$$\frac{\partial\widetilde{H}}{\partial\widehat{y}_t} = \xi X^2\frac{\partial^2 F}{\partial y^2}\widehat{y}_t + \xi XX\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_t + \xi X\mu\frac{\partial^2 F}{\partial y\partial\mu}\widehat{\mu}_t + \widehat{s}_t X\frac{\partial F}{\partial y} + \rho\widehat{r}_t.$$

So, it follows that one may write

$$
\begin{aligned}
\frac{1}{X}\frac{\partial\widetilde{H}}{\partial\widehat{x}_t} &= X\frac{\partial^2 l}{\partial x^2}\widehat{x}_t + \xi X\frac{\partial^2 F}{\partial x^2}\widehat{x}_t \\
&\quad +\xi X\frac{\partial^2 F}{\partial x\partial y}\widehat{y}_t + \xi\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{\mu}_t + s_t\frac{\partial F}{\partial x_t} \\
&\quad \xi X\frac{\partial^2 F}{\partial y^2}\widehat{y}_{t-1} + \xi X\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_{t-1} + \xi\mu\frac{\partial^2 F}{\partial y\partial\mu}\widehat{\mu}_{t-1} + \widehat{s}_{t-1}\frac{\partial F}{\partial y}.
\end{aligned}
$$

This is identical to (7.5) with the following relations between Lagrange multipliers $s_t = \xi\widehat{\xi}_t$, $r_t = \rho\widehat{\rho}_t$.

7.3. Optimal steady state

7.3.1. Proof of Proposition 4.2: Existence

One may rewrite (4.8) as (7.8):

$$F(\pi) = \upsilon g(\pi) + [g(\pi) - f(\pi)] + \Phi h(\pi)f(\pi) = 0, \tag{7.8}$$

where $h(\pi) = \frac{1-\alpha\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta-1}}\frac{1-\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta\phi}} > 0$, and $g(\pi) = \frac{\theta\phi}{1-\alpha\pi^{\theta\phi}} - \frac{\theta\phi}{1-\alpha\pi^{\theta-1}}$; $f(\pi) = \left[\frac{\theta\phi}{1-\alpha\pi^{\theta\phi}} - \frac{\theta\phi}{1-\alpha\beta\pi^{\theta\phi}}\right] - \left[\frac{\theta-1}{1-\alpha\pi^{\theta-1}} - \frac{\theta-1}{1-\beta\alpha\pi^{\theta-1}}\right]$.

It is easy to see that $g(1) = 0$; $h(1) = 1$ and $f(1) = \frac{(\theta\phi-\theta+1)\alpha(1-\beta)}{(1-a)(1-\alpha\beta)} > 0$, which implies that $F(1) = -(1-\Phi)\frac{(\theta\phi-\theta+1)\alpha(1-\beta)}{(1-a)(1-\alpha\beta)} \leq 0$. The strict equality obtains in three cases only. First, when prices are flexible, $\alpha = 0$; second, when the future is not discounted by firms, $\beta = 1$; and finally when there are no distortions in steady state, $\Phi = 1$.

Define $\pi_h = \alpha^{-1/(\phi\theta)}$ and note that the functions g, f, and h are defined on an interval $[1, \pi_h)$. The difference $[g(\pi_h) - f(\pi_h)]$ is bounded while $g(\pi)$, $h(\pi)$ and $f(\pi)$ tends to positive infinity as π approaches π_h. Hence, $\lim_{\pi \longrightarrow \pi_h} F(\pi) = +\infty$. Since $F(\pi)$ is a continuous function, one can conclude that there is a solution to (7.8) on the interval $[1, \alpha^{-1/(\phi\theta)})$. One may easily show then that if $\pi_m = \beta^{1/(\theta-1-\phi\theta)}$ then it follows that $F(\pi_m) > 0$, since $g(\pi_m) - f(\pi_m) > 0$. Therefore, optimal inflation is smaller than π_m.

7.3.2. Proof of Proposition 4.2: Uniqueness

The proof is by contradiction. First it is proved that if $\beta < 1$, for any $\pi_1 < \pi_m$ such that $F(\pi_1) = 0$, it is necessary that $F'(\pi_1) > 0$. By direct differentiation it follows that

$$F'(\pi_1) = (v+1)\, g'(\pi_1) + (\Phi h(\pi_1) - 1)\, f'(\pi_1) + \Phi h'(\pi_1) f(\pi_1).$$

Moreover, since $F(\pi_1) = 0$, it follows that $(\Phi h(\pi_1) - 1) = -(v+1)\, g(\pi_1)/f(\pi_1)$. Therefore

$$F'(\pi_1) = \frac{(v+1)}{f(\pi_1)}\left[g'(\pi_1)f(\pi_1) - f'(\pi_1)g(\pi_1)\right] + \Phi h'(\pi_1)f(\pi_1),$$

and it is easy to show that for any $\pi_1 < \pi_m$, $g'(\pi_1)f(\pi_1) - f'(\pi_1)g(\pi_1) > 0$, and therefore, $F'(\pi_1)$ is positive.

Since F is continuously differentiable, if a solution of (7.8) is not unique, there will be at least one solution such that $F'(\pi_1) \leq 0$. It has been demonstrated that such a solution is impossible and the necessary contradiction is obtained.

7.3.3. Proof of Proposition 4.3

By the implicit function theorem one concludes that $\frac{d\pi}{d\Phi} = -\frac{\partial F}{\partial \pi}/\frac{\partial F}{\partial \Phi}$. From section 7.3.2, we know that $\frac{\partial F}{\partial \pi} > 0$, while $\frac{\partial F}{\partial \Phi} = h(\pi)f(\pi) > 0$. Therefore $\frac{d\pi}{d\Phi} < 0$, and equilibrium inflation increases with steady state distortions, measured as $1 - \Phi$.

Similarly $\frac{d\pi}{dv} = -\frac{\partial F}{\partial \pi}/\frac{\partial F}{\partial v}$, where $\frac{\partial F}{\partial v} = g(\pi) > 0$ for $\pi > 1$, therefore $\frac{d\pi}{dv} < 0$, and optimal inflation declines with the elasticity of labour.

Moreover $\frac{d\pi}{d\beta} = -\frac{\partial F}{\partial \pi}/\frac{\partial F}{\partial \beta}$, where $\frac{\partial F}{\partial \beta} = -\left(1 - \Phi h(\pi)\right)\frac{\partial f}{\partial \beta} + \Phi f(\pi)\frac{\partial h}{\partial \beta}$, and one may prove by direct differentiation that $\frac{\partial f}{\partial \beta} < 0$, $\frac{\partial \ln h}{\partial \beta} > 0$, and $\left(1 - \Phi h(\pi)\right) = \left(v + 1\right)g(\pi)/f(\pi) > 0$. Therefore $\frac{\partial F}{\partial \beta} > 0$, and $\frac{d\pi}{d\beta} < 0$.

Finally, it is worth noting that steady state inflation can both increase or decrease in price stickiness, since the sign of $\frac{\partial F}{\partial \alpha}$ may be positive or negative.

7.4. A2: The second-order approximation to unconditional welfare.

In Section 4.2 of the main text we asserted the existence of the following quadratic equation,

$$EU = E\left(Q_l + \rho Q_X + \varphi Q_Z + \xi Q_{ZX} + \eta Q_\Delta + \delta Q_p\right),$$

where Q_l is the second-order term of the loss function and Q_X, Q_Z, Q_{ZX}, Q_Δ, Q_p are the second-order terms of the log linear approximation to constraints (4.2)-(4.4). This section demonstrates how one derives that equation. The model can

be rewritten in the following linear-quadratic representation

$$(\frac{1}{(v+1)\,\phi}\log u_t - \frac{1}{\phi}\log \Delta_t - u_t) - O3$$

$$= \frac{1}{(v+1)\,\phi}\widehat{u}_t - \frac{1}{\phi}\widehat{\Delta}_t - u\left(\widehat{u}_t + \frac{1}{2}\widehat{u}_t^2\right) + tip;$$

$$\left(X_t - 1 - \beta\alpha\pi_{t+1}^{\theta-1}X_{t+1}\right) - O3$$

$$= X\left(\widehat{X}_t + \frac{1}{2}\widehat{X}_t^2\right) - \beta\alpha\pi^{\theta-1}X\left(\widehat{X}_{t+1} + (\theta-1)\,\widehat{\pi}_{t+1} + \frac{1}{2}\left(\widehat{X}_{t+1} + (\theta-1)\,\widehat{\pi}_{t+1}\right)^2\right);$$

$$\left(Z_t - c_t - \beta\alpha\pi_{t+1}^{\theta\phi}Z_{t+1}\right) - O3$$

$$= Z\left(\widehat{Z}_t + \frac{1}{2}\widehat{Z}_t^2\right) - c\left(\widehat{c}_t + \frac{1}{2}c_t^2\right) - \beta\alpha\pi^{\theta\phi}Z\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1} + \frac{1}{2}\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right)^2\right);$$

$$\left(Z_t - p_t^{\theta\phi-\theta+1}X_t\right) - O3$$

$$= Z\left(\widehat{Z}_t + \frac{1}{2}\widehat{Z}_t^2\right) - p^{\theta\phi-\theta+1}X\left((\theta\phi-\theta+1)\,\widehat{p}_t + \widehat{X}_t + \frac{1}{2}\left((\theta\phi-\theta+1)\,\widehat{p}_t + \widehat{X}_t\right)^2\right);$$

$$\Delta_t - \alpha\Delta_{t-1}\pi_t^{\theta\phi} - (1-\alpha)\,p_t^{-\theta\phi} - O3$$

$$= \Delta\left(\widehat{\Delta}_t + \frac{1}{2}\widehat{\Delta}_t^2\right) - a\Delta\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t + \frac{1}{2}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right)^2\right)$$

$$- (1-\alpha)\,p^{-\theta\phi}\left(-\theta\phi\widehat{p}_t + \frac{1}{2}(\theta\phi\widehat{p}_t)^2\right);$$

$$p_t - \left(\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right)^{\frac{1}{1-\theta}} - O3;$$

$$= p\left(\widehat{p}_t + \frac{1}{2}\widehat{p}_t^2\right) - p\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\widehat{\pi}_t - \frac{1}{2}p\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\left(\theta\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} - 1\right)\widehat{\pi}_t^2.$$

The linear relations are therefore,

$$\widehat{X}_t - \beta\alpha\pi^{\theta-1}\left(\widehat{X}_{t+1} + (\theta-1)\,\widehat{\pi}_{t+1}\right) = O2; \tag{7.9}$$

$$\widehat{Z}_t - \frac{c}{Z}\widehat{c}_t - \beta\alpha\pi^{\theta\phi}\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right) = O2; \tag{7.10}$$

$$\widehat{Z}_t - \left((\theta\phi-\theta+1)\,\widehat{p}_t + \widehat{X}_t\right) = O2; \tag{7.11}$$

$$\widehat{\Delta}_t - a\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right) + (1-\alpha)\frac{p^{-\theta\phi}}{\Delta}\theta\phi\widehat{p}_t = O2; \tag{7.12}$$

$$\widehat{p}_t - \frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\widehat{\pi}_t = O2, \tag{7.13}$$

and the following are the quadratic relations:

$$Q_l = -\frac{1}{2}u\widehat{u}_t^2;$$

$$Q_X = \frac{1}{2}X\left(\widehat{X}_t^2\right) - \frac{1}{\beta\alpha\pi^{\theta-1}}X\frac{1}{2}\widehat{X}_t^2 = -\frac{1}{2}\frac{1}{\beta\alpha\pi^{\theta-1}}\widehat{X}_t^2;$$

$$Q_z = \frac{1}{2}Z\widehat{Z}_t^2 - \frac{1}{2}c\widehat{c}_t^2 - \frac{1}{2}\beta\alpha\pi^{\theta\phi}Z\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right)^2; \tag{7.14}$$

$$Q_{xz} = \frac{1}{2}Z\widehat{Z}_t^2 - Z\frac{1}{2}\left((\theta\phi - \theta + 1)\widehat{p}_t + \widehat{X}_t\right)^2 = 0;$$

$$Q_\Delta = \frac{1}{2}\Delta\widehat{\Delta}_t^2 - (1-\alpha)p^{-\theta\phi}\frac{1}{2}(\theta\phi\widehat{p}_t)^2 - \frac{1}{2}a\Delta\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right)^{2;} \tag{7.15}$$

$$Q_p = \frac{1}{2}p\widehat{p}_t^2 - \frac{1}{2}p\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\left(\theta\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} - 1\right)\widehat{\pi}_t^2. \tag{7.16}$$

One can simplify these expressions as follows.

Simplification of Q_p :   Use (7.13) in (7.16) to find that

$$Q_p = -\frac{1}{2}\Pi\widehat{\pi}_t^2,$$

where we define

$$\Pi := \left(\frac{1-\alpha\pi^{\theta-1}}{1-\alpha}\right)^{\frac{1}{1-\theta}}\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\left(\frac{(\theta-1)\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} - 1\right).$$

Simplification of Q_z :   Use (7.10) in (7.14),

$$
\begin{aligned}
2\frac{EQz}{Z} &= \widehat{Z}_t^2 - \left(1-\alpha\beta\pi^{\theta\phi}\right)\widehat{c}_t^2 - \beta\alpha\pi^{\theta\phi}\left(\widehat{Z}_t + \theta\phi\widehat{\pi}_t\right)^2 \tag{7.17}\\
&= \left(1-\beta\alpha\pi^{\theta\phi}\right)\widehat{Z}_t^2 - \left(1-\alpha\beta\pi^{\theta\phi}\right)\widehat{c}_t^2 - \beta\alpha\pi^{\theta\phi}\left(\theta\phi\widehat{\pi}_t\right)\left(2\widehat{Z}_t + \theta\phi\widehat{\pi}_t\right)\\
&= \left((\theta\phi - \theta + 1)\widehat{p}_t + \widehat{X}_t\right)\left(\left(1-\beta\alpha\pi^{\theta\phi}\right)\left((\theta\phi - \theta + 1)\widehat{p}_t + \widehat{X}_t\right) - 2\beta\alpha\pi^{\theta\phi}(\theta\phi)\widehat{\pi}_t\right)\\
&\quad - \left(1-\alpha\beta\pi^{\theta\phi}\right)\widehat{c}_t^2 - \beta\alpha\pi^{\theta\phi}(\theta\phi)^2\widehat{\pi}_t^2\\
&= \widehat{X}_t^2\left(1-\alpha\beta\pi^{\theta\phi}\right) - \left(1-\alpha\beta\pi^{\theta\phi}\right)\widehat{c}_t^2\\
&\quad -\beta\alpha\pi^{\theta\phi}(\theta\phi)^2\widehat{\pi}_t^2 + \left(1-\beta\alpha\pi^{\theta\phi}\right)(\theta\phi - \theta + 1)^2\widehat{p}_t^2 - (\theta\phi - \theta + 1)2\beta\alpha\pi^{\theta\phi}(\theta\phi)\widehat{\pi}_t\widehat{p}_t\\
&\quad 2\left(\frac{\theta\phi\left(\alpha\pi^{\theta-1} - \beta\alpha\pi^{\theta\phi}\right) - (\theta-1)\alpha\pi^{\theta-1}\left(1-\alpha\beta\pi^{\theta\phi}\right)}{1-\alpha\pi^{\theta-1}}\right)\widehat{\pi}_t\widehat{X}_t.
\end{aligned}
$$

Furthermore, from (7.9) one can find an expression for $E2\widehat{\pi}_t\widehat{X}_t$

$$E\widehat{X}_t^2 = E\left(\beta\alpha\pi^{\theta-1}\right)^2\left(\widehat{X}_{t+1}^2 + 2\left(\theta-1\right)\widehat{\pi}_{t+1}\widehat{X}_{t+1} + \left(\theta-1\right)^2\widehat{\pi}_{t+1}^2\right),$$

which implies that

$$2E\widehat{\pi}_t\widehat{X}_t = \frac{1}{(\theta-1)}\left(\frac{1-\left(\beta\alpha\pi^{\theta-1}\right)^2}{\left(\beta\alpha\pi^{\theta-1}\right)^2}\right)\widehat{X}_t^2 - (\theta-1)\widehat{\pi}_t^2. \qquad (7.18)$$

Now, combine (7.18) with (7.17) to yield

$$EQz = -Z\frac{1}{2}E\left[\left(1-\alpha\beta\pi^{\theta\phi}\right)\widehat{c}_t^2 + Z_\pi\widehat{\pi}_t^{\,2} + Z_x\widehat{X}_t^2\right],$$

where

$$Z_\pi = \frac{(\theta\phi-\theta+1)\left[\theta\phi\left(\alpha\beta\pi^{\theta\phi}-\alpha\pi^{\theta-1}\alpha\pi^{\theta-1}\right)+(\theta-1)\alpha\pi^{\theta-1}\left(1-\beta\alpha\pi^{\theta\phi}\right)\right]}{\left(1-\alpha\pi^{\theta-1}\right)^2};$$

$$Z_x = \frac{1-\beta\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta-1}}\frac{1-\beta^2\alpha\pi^{\theta-1}}{\beta^2\alpha\pi^{\theta-1}} + \frac{\theta\phi}{\theta-1}\frac{1-\left(\beta\alpha\pi^{\theta-1}\right)^2}{\left(\beta\alpha\pi^{\theta-1}\right)^2}\frac{\beta\alpha\pi^{\theta\phi}-\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}.$$

Simplification of Q_Δ :

$$\frac{2}{\Delta}Q_\Delta = \widehat{\Delta}_t^2 - (1-\alpha)\frac{p^{-\theta\phi}}{\Delta}\left(\theta\phi\widehat{p}_t\right)^2 - a\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right)^2.$$

One can simplify (7.15) using (7.12)

$$
\begin{aligned}
a\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right)^2 &= \frac{1}{a\pi^{\theta\phi}}\left(\widehat{\Delta}_t + (1-\alpha)\frac{p^{-\theta\phi}}{\Delta}\frac{\theta\phi\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\widehat{\pi}_t\right)^2 \\
&= \frac{1}{a\pi^{\theta\phi}}\widehat{\Delta}_t^2 + \frac{1}{a\pi^{\theta\phi}}\left(\frac{\theta\phi\left(1-\alpha\pi^{\theta\phi}\right)\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\right)^2\widehat{\pi}_t^{\,2} \\
&\quad +2\theta\phi\frac{1-\alpha\pi^{\theta\phi}}{a\pi^{\theta\phi}}\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\widehat{\Delta}_t\widehat{\pi}_t.
\end{aligned}
$$

Next, using constraint (7.12), one finds $E2\widehat{\Delta}_t\widehat{\pi}_t$

$$a\pi^{\theta\phi}\widehat{\Delta}_{t-1} = \widehat{\Delta}_t + \theta\phi\left((1-\alpha)\frac{p^{-\theta\phi}}{\Delta}\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} - a\pi^{\theta\phi}\right)\widehat{\pi}_t.$$

Recall that $\Delta = \left(\frac{1-\alpha}{1-\alpha\pi^{\theta\phi}}\right)p^{-\theta\phi}$, so that

$$a\pi^{\theta\phi}\widehat{\Delta}_{t-1} = \widehat{\Delta}_t + \theta\phi\left(\frac{\alpha\pi^{\theta-1}-\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta-1}}\right)\widehat{\pi}_t + O2.$$

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This implies

$$E\left(a\pi^{\theta\phi}\widehat{\Delta}_{t-1}\right)^2 = E\widehat{\Delta}_t^2 + E\left(\theta\phi\frac{\alpha\pi^{\theta-1}-\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta-1}}\right)^2\widehat{\pi}_t^{\,2} + 2\theta\phi\left(\frac{\alpha\pi^{\theta-1}-\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta-1}}\right)E\widehat{\pi}_t\widehat{\Delta}_t.$$

One can simplify the final terms in the expression as follows

$$2\frac{\theta\phi}{1-\alpha\pi^{\theta-1}}E\widehat{\pi}_t\widehat{\Delta}_t \quad = \tag{7.19}$$

$$-\frac{1-\left(a\pi^{\theta\phi}\right)^2}{\alpha\pi^{\theta-1}-\alpha\pi^{\theta\phi}}E\widehat{\Delta}_t^2 - \frac{\theta\phi^2}{1-\alpha\pi^{\theta-1}}\frac{\alpha\pi^{\theta-1}-\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta-1}}E\widehat{\pi}_t^{\,2};$$

$$\frac{2}{\Delta\left(1-\alpha\pi^{\theta\phi}\right)}EQ_\Delta = -E\frac{1-a\pi^{\theta\phi}\alpha\pi^{\theta-1}}{\alpha\pi^{\theta\phi}-\alpha\pi^{\theta-1}}\widehat{\Delta}_t^2 - E\alpha\pi\left[\frac{\theta\phi}{1-\alpha\pi^{\theta-1}}\right]^2\widehat{\pi}_t^{\,2}.$$

Hence, using these simplifications, we return to the quadratic expression.

$$\begin{aligned}
EU &= E\left(Q_l + \rho Q_X + \varphi Q_Z + \xi Q_{ZX} + \eta Q_\Delta + \delta Q_p\right); \\
&= -\frac{1}{2}Eu\widehat{u}_t^2 - \frac{1}{2}\rho\frac{1}{\beta\alpha\pi^{\theta-1}}E\widehat{X}_t^2 - \frac{1}{2}\varphi ZE\left[\frac{c}{Z}\widehat{c}_t^2 + Z_\pi\widehat{\pi}_t^{\,2} + Z_x\widehat{X}_t^2\right] \\
&\quad -\frac{1}{2}\Delta\eta\left(1-\alpha\pi^{\theta\phi}\right)E(\alpha\pi^{\theta-1}\left[\frac{\theta\phi}{1-\alpha\pi^{\theta-1}}\right]^2\widehat{\pi}_t^{\,2} + \frac{1-a\pi^{\theta\phi}\alpha\pi^{\theta-1}}{\alpha\pi^{\theta\phi}-\alpha\pi^{\theta-1}}\widehat{\Delta}_t^2) \\
&\quad -\frac{1}{2}\Pi\delta E\widehat{\pi}_t^{\,2} \\
&= -\frac{1}{2}E\left(u\widehat{u}_t^2 + \varphi c\widehat{c}_t^2 + \Lambda_x\widehat{X}_t^2 + \Lambda_\pi\widehat{\pi}_t^{\,2} + \Lambda_\Delta\widehat{\Delta}_t^2\right), \tag{7.20}
\end{aligned}$$

where

$$\Lambda_x = \varphi Z\left(Z_x - \frac{1-\beta\alpha\pi^{\theta\phi}}{\beta\alpha\pi^{\theta-1}}\right);$$

$$\Lambda_\pi = \varphi Z Z_\pi + \Delta\eta D_\pi + \Pi\delta;$$

$$\Lambda_\Delta = \Delta\eta\left(1-\alpha\pi^{\theta\phi}\right)\frac{\left(1-a\pi^{\theta\phi}\alpha\pi^{\theta-1}\right)}{\alpha\pi^{\theta\phi}-\alpha\pi^{\theta-1}} > 0.$$

Further simplification
The log-linear expression of the marginal disutility from labour u_t is

$$\widehat{u}_t = (v+1)\widehat{\Delta}_t + (v+1)\phi\left(\widehat{Y}_t - \widehat{A}_t\right).$$

Marginal production costs are approximately written as

$$\widehat{c}_t = \widehat{u}_t - \widehat{g}_t,$$

where $\widehat{g}_t$ is the labour wedge defined as

$$g_t := \frac{\partial U}{\partial N}\Big/\frac{\partial F}{\partial N} = \frac{u_t}{c_t} = \frac{\mu}{\mu_t}\frac{\Phi}{(1+v)\phi}\Delta_t. \tag{7.21}$$

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The first two terms in the quadratic loss function (7.20) can be simplified as follows:

$$
\begin{aligned}
u\widehat{u}_t^2 + \varphi c \widehat{c}_t^2 &= \widehat{u}_t^2 + \varphi c \left(\widehat{u}_t - \widehat{g}_t\right)^2 \\
&= (u + \varphi c)\widehat{u}_t^2 + \varphi c \left(\widehat{g}_t\right)^2 - 2\varphi c \widehat{u}_t \widehat{g}_t \\
&= (u + \varphi c)\left(\widehat{u}_t - \frac{\varphi c}{(u + \varphi c)}\widehat{g}_t\right)^2 + \varphi c \left(1 - (u + \varphi c)\varphi c\right)\widehat{g}_t^2 \\
&= \frac{1}{(v+1)\phi}\left((v+1)\widehat{\Delta}_t + (v+1)\phi\left(\widehat{Y}_t - \widehat{A}_t\right) - \frac{\varphi c}{u + \varphi c}\left(\widehat{\Delta}_t - \widehat{\mu}_t\right)\right)^2 \\
&\quad + \varphi c\left(1 - \frac{\varphi c}{(v+1)\phi}\right)\widehat{g}_t^2; \\
&= (v+1)\phi\left(\frac{1}{\phi}\widehat{\Delta}_t + \widehat{Y}_t - \widehat{A}_t - \varphi c\left(\widehat{\Delta}_t - \widehat{\mu}_t\right)\right)^2 \\
&\quad + \frac{\varphi c}{(v+1)\phi}\left((v+1)\phi - \varphi c\right)\widehat{g}_t^2; \\
&= (v+1)\phi\left(\widehat{Y}_t - \left(\widehat{A}_t - \widehat{\mu}_t - \left(\frac{v}{(v+1)\phi} + u\right)\widehat{\Delta}_t\right)\right)^2 \\
&\quad + \frac{\varphi c}{(v+1)\phi}\left((v+1)\phi - \varphi c\right)\widehat{g}_t^2 \\
&= (v+1)\phi\left(\widehat{Y}_t - Y_t^*\right)^2 + G\widehat{g}_t^2,
\end{aligned}
$$

where we define Y_t^* and G as

$$
\begin{aligned}
Y_t^* \quad &: \quad = \widehat{A}_t - \widehat{\mu}_t - \left(\frac{v}{(v+1)\phi} + u\right)\widehat{\Delta}_t; \\
G \quad &: \quad = \frac{\varphi c}{(v+1)\phi}\left((v+1)\phi - \varphi c\right).
\end{aligned}
$$

To obtain this result, recall that the steady state value of the Lagrange multiplier φ satisfies the following equation:

$$
\varphi c + u = \frac{1}{(v+1)\phi}\left(1 - \Phi \Delta c\right) + u = \frac{1}{(v+1)\phi}.
$$

7.5. Linearized equations of the model

For completeness, details are provided of the linear approximate model, consisting of the first-order conditions (4.5) and a system of constraints (4.2), (4.3), (4.4),

(4.6). The linearized block of equations is thus:

$$u_t \frac{\partial}{\partial u_t} H = -u\widehat{u}_t - \varphi c\left(\widehat{\varphi}_t + \widehat{c}_t\right);$$

$$\Delta \frac{\partial}{\partial \Delta_t} H = -\frac{1}{\phi}\widehat{\Delta}_t + \varphi c\left(\widehat{\varphi}_t + \widehat{c}_t - \widehat{\Delta}_t\right) + \Delta\eta\widehat{\eta}_t - \alpha\Delta\eta\pi^{\theta\phi}E_t\left(\widehat{\eta}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right);$$

$$\frac{\partial}{\partial X_t} H = \rho\widehat{\rho}_t - \rho\beta\alpha\pi^{\theta-1}\left(\widehat{\rho}_{t-1} + (\theta-1)\widehat{\pi}_t\right) - \xi p^{\theta\phi-\theta+1}\left(\widehat{\xi}_t + (\theta\phi-\theta+1)\widehat{p}_t\right);$$

$$\frac{\partial}{\partial Z_t} H = \varphi\widehat{\varphi}_t - \varphi\beta\alpha\pi^{\theta\phi}\left(\widehat{\varphi}_{t-1} + \theta\phi\widehat{\pi}_t\right) + \xi\widehat{\xi}_t;$$

$$\pi_t \frac{\partial}{\partial \pi_t} H = -(\theta-1)\rho\beta\alpha\pi^{\theta-1}X\left(\widehat{\rho}_{t-1} + (\theta-1)\widehat{\pi}_t + \widehat{X}_t\right) - \varphi\beta\alpha\theta\phi\pi^{\theta\phi}Z\left(\widehat{\varphi}_{t-1} + \theta\phi\widehat{\pi}_t + \widehat{Z}_t\right);$$

$$-\eta\alpha\theta\phi\Delta\pi^{\theta\phi}\left(\widehat{\eta}_t + \widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right) - \delta p^{\theta}\frac{\alpha\pi^{\theta-1}}{1-\alpha}\left(\widehat{\delta}_t + \theta\widehat{p}_t + (\theta-1)\widehat{\pi}_t\right);$$

$$p_t \frac{\partial}{\partial p_t} H = -\xi(\theta\phi-\theta+1)Xp^{1-\theta+\theta\phi}\left(\widehat{\xi}_t + \widehat{X}_t + (1-\theta+\theta\phi)\widehat{p}_t\right);$$

$$+\theta\phi\eta p^{-\theta\phi}(1-\alpha)\left(\widehat{\eta}_t - \theta\phi\widehat{p}_t\right) + \delta p\left(\widehat{\delta}_t + \widehat{p}_t\right);$$

$$\widehat{X}_t - \beta\alpha\pi^{\theta-1}E_t\left(\widehat{X}_{t+1} + (\theta-1)\widehat{\pi}_{t+1}\right) = 0;$$

$$Z\widehat{Z}_t - c\widehat{c}_t - \beta\alpha\pi^{\theta\phi}ZE_t\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right) = 0;$$

$$\widehat{Z}_t - \left((\theta\phi-\theta+1)\widehat{p}_t + \widehat{X}_t\right) = 0;$$

$$\Delta\widehat{\Delta}_t - a\Delta\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right) + (1-\alpha)p^{-\theta\phi}\theta\phi\widehat{p}_t = 0;$$

$$\widehat{p}_t - \frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\widehat{\pi}_t = 0;$$

$$-\widehat{c}_t + \widehat{u}_t + \widehat{\mu}_t - \widehat{\Delta}_t = 0.$$