# Price dynamics in a Markovian limit order market

Rama CONT & Adrien de LARRARD

We propose and study a simple stochastic model for the dynamics of a limit order book, in which arrivals of market order, limit orders and order cancelations are described in terms of a Markovian queueing system. Through its analytical tractability, the model allows to obtain analytical expressions for various quantities of interest such as the distribution of the duration between price changes, the distribution and autocorrelation of price changes, and the probability of an upward move in the price, *conditional* on the state of the order book. We study the diffusion limit of the price process and express the volatility of price changes in terms of parameters describing the arrival rates of buy and sell orders and cancelations. These analytical results provide some insight into the relation between order flow and price dynamics in order-driven markets.

*Key words*: limit order book, market microstructure, queueing, diffusion limit, high-frequency data, liquidity, duration analysis, point process.

## Contents

## 1. Introduction

An increasing number of stocks are traded in electronic, order-driven markets, in which orders to buy and sell are centralized in a *limit order book* available to market participants and market orders are executed against the best available offers in the limit order book. The dynamics of prices in such markets are not only interesting from the viewpoint of market participants –for trading and order execution (Alfonsi et al. (2010), Predoiu et al. (2011))– but also from a fundamental perspective, since they provide a rare glimpse into the dynamics of supply and demand and their role in price formation.

Equilibrium models of price formation in limit order markets (Parlour (1998), Rosu (2009)) have shown that the evolution of the price in such markets is rather complex and depends on the state of the order book. On the other hand, empirical studies on limit order books (Bouchaud et al. (2008), Farmer et al. (2004), Gourieroux et al. (1999), Hollifield et al. (2004), Smith et al. (2003)) provide an extensive list of statistical features of order book dynamics that are challenging to incorporate in a single model. While most of these studies have focused on unconditional/steady–state distributions of various features of the order book, empirical studies ( Harris and Panchapagesan (2005), Cont et al. (2010a)) show that the state of the order book contains information on short-term price movements so it is of interest to provide forecasts of various quantities *conditional* on the state of the order book. Providing analytically tractable models which enable to compute and/or reproduce *conditional* quantities which are relevant for trading and intraday risk management has proven to be challenging, given the complex relation between order book dynamics and price behavior.

The search for tractable models of limit order markets has led to the development of stochastic models which aim to retain the main statistical features of limit order books while remaining computationally manageable. Stochastic models also serve to illustrate how far one can go in reproducing the dynamic properties of a limit order book without resorting to detailed behavioral assumptions about market participants or introducing unobservable parameters describing agent preferences, as in more detailed market microstructure models.

Starting from a description of order arrivals and cancelations as point processes, the dynamics of a limit order book is naturally described in the language of queueing theory. Engle and Lunde (2003) formulate a bivariate point process to jointly analyze trade and quote arrivals. Cont et al. (2010b) model the dynamics of a limit order book as a tractable multiclass queueing system and compute various transition probabilities of the price conditional on the state of the order book, using Laplace transform methods.

### 1.1. Summary

We propose a Markovian model of a limit order market, which captures some salient features of the dynamics of market orders and limit orders and their influence on price dynamics, yet is even simpler than the model of Cont et al. (2010b) and enables a wide range of properties of the price process to be computed analytically.

Our approach is motivated by the observation that, if one is primarily interested in the dynamics of the price, it is sufficient to focus on the dynamics of the (best) bid and ask queues. Indeed, empirical evidence shows that most of the order flow is directed at the best bid and ask prices (Biais et al. (1995)) and the imbalance between the order flow at the bid and at the ask appears to be the main driver of price changes (Cont et al. (2010a)).

Motivated by this remark, we propose a parsimonious model in which the limit order book is represented by the number of limit orders $(q_t^b, q_t^a)$ sitting at the bid and the ask, represented as a system of two interacting queues. The remaining levels of the order book are treated as a 'reservoir' of limit orders represented by the distribution of the size of the queues at the 'next-to-best' price levels.

Through its analytical tractability, the Markovian version of our model allows to obtain analytical expressions for various quantities of interest such as the distribution of the duration until the next price change, the distribution and autocorrelation of price changes, and the probability of an upward move in the price, *conditional* on the state of the order book. Compared with econometric models of high frequency data Engle and Russell (1998), Engle and Lunde (2003) where the link between durations and price changes is specified exogenously, our model links these quantities in an endogenous manner, and provides a first step towards joint 'structural' modeling of high frequency dynamics of prices and order flow.

A second important observation is that order arrivals and cancelations are very frequent and occur at millisecond time scale, whereas, in many applications such as order execution, the metric of success is the volume-weighted average price (VWAP) so one is interested in the dynamics of order flow over a large time scale, typically tens of seconds or minutes. As shown in Table 1.1, thousands of order book events may occur over such time scales. This observation enables us to use asymptotic methods to study the link between price volatility and order flow in this model by studying the *diffusion limit* of the price process. In particular, we prove a functional central limit theorem for the price process and express the volatility of price changes in terms of parameters describing the arrival rates of buy and sell orders and cancelations. For example, we show (Theorem 1) that the variance of intraday price changes in a 'balanced' limit order market is given by the following simple relation:

$$\sigma^2 = \frac{\pi \delta^2 \lambda}{D(f)}$$

where $\delta$ is the 'tick size', $\lambda$ is the intensity of order arrivals and $D(f)$ is a measure of market depth. These analytical results provide insights into the relation between order flow and price dynamics in order-driven markets. Comparison of these results with empirical data shows the main insights of the model to be correct.

| | Average no. of orders in 10s | Price changes in 1 day |
|---|---|---|
| Citigroup | 4469 | 12499 |
| General Electric | 2356 | 7862 |
| General Motors | 1275 | 9016 |

**Table 1**    Average number of orders in 10 seconds and number of price changes (June 26th, 2008).

### 1.2. Outline

The chapter is organized as follows. Section 2 introduces a reduced-form representation of a limit order book and presents a Markovian model in which limit orders, market orders and cancelations occur according to Poisson processes. Section 3 presents various analytical results for this model: we compute the distribution of the duration until the next price change (section 3.1), the probability of upward move in the price (section 3.2) and the dynamics of the price (section 3.4). In Section 4, we show that the price exhibits diffusive behavior at longer time scales and express the variance of price changes in terms of the parameters describing the order flow, thus establishing a link between volatility and order flow statistics.

## 2. A Markov model of limit order book dynamics

### 2.1. A stylized representation of a limit order book

Empirical studies of limit order markets suggest that the major component of the order flow occurs at the (best) bid and ask price levels (see e.g. Biais et al. (1995)). All electronic trading venues also allow to place limit orders *pegged* to the best available price (National Best Bid Offer, or NBBO); market makers used these *pegged* orders to liquidate their inventories. Furthermore, studies on the price impact of order book events show that the net effect of orders on the bid and ask queue sizes is the main factor driving price variations (Cont et al. (2010a)). These observations, together with the fact that queue sizes at the best bid and ask of the consolidated order book are more easily obtainable (from records on trades and quotes) than information on deeper levels of the order book, motivate a reduced-form modeling approach in which we represent the state of the limit order book by

- the bid price $s_t^b$ and the ask price $s_t^a$
- the size of the bid queue $q_t^b$ representing the outstanding limit buy orders at the bid, and
- the size of the ask queue $q_t^a$ representing the outstanding limit sell orders at the ask

Figure 1 summarizes this representation.

If the stock is traded in several venues, the quantities $q^b$ and $q^a$ represent the best bids and offers in the *consolidated* order book, obtained by aggregating over all (visible) trading venues. At every time $t$, $q_t^b$ (resp. $q_t^a$) corresponds to all visible orders available at the bid price $s_t^b$ (resp. $s_t^a$) across all exchanges.
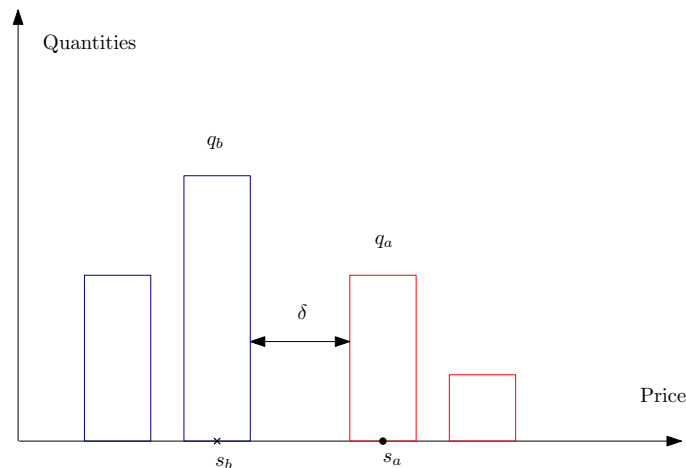


**Figure 1**    Simplified representation of a limit order book.

The state of the order book is modified by *order book events*: limit orders (at the bid or ask), market orders and cancelations (see Cont et al. (2010b,a), Smith et al. (2003)). A limit buy (resp. sell) order of size $x$ increases the size of the bid (resp. ask) queue by $x$, while a market buy (resp. sell) order decreases the corresponding queue size by $x$. Cancelation of $x$ orders in a given queue reduces the queue size by $x$. Given that we are interested in the queue sizes at the best bid/ask levels, market orders and cancelations have the same effect on the queue sizes $(q_t^b, q_t^a)$.

The bid and ask prices are multiples of the tick size $\delta$. When either the bid or ask queue is depleted by market orders and cancelations, the price moves up or down to the next level of the order book. The price processes $s_t^b, s_t^a$ is thus a piecewise constant process whose transitions correspond to hitting times of the axes $\{(0, y), y \in \mathbb{N}\} \cup \{(x, 0), x \in \mathbb{N}\}$ by the process $q_t = (q_t^b, q_t^a)$.

If the order book contains no 'gaps' (empty levels), these price increments are equal to one tick:

- when the bid queue is depleted, the (bid) price decreases by one tick.
- when the ask queue is depleted, the (ask) price increases by one tick.

If there are gaps in the order book, this results in 'jumps' (i.e. variations of more than one tick) in the price dynamics. We will ignore this feature in what follows but it is not hard to generalize our results to include it.

The quantity $s_t^a - s_t^b$ is the *bid-ask spread*, which may be one or several ticks. As shown in Table 2, for liquid stocks the bid-ask spread is equal to one tick for more than 98% of observations.

| Bid-ask spread | 1 tick | 2 tick | $\geq$ 3 tick |
|---|---|---|---|
| Citigroup | 98.82 | 1.18 | 0 |
| General Electric | 98.80 | 1.18 | 0.02 |
| General Motors | 98.71 | 1.15 | 0.14 |

**Table 2**    Percentage of observations with a given bid-ask spread (June 26th, 2008).

When either the bid or ask queue is depleted, the bid-ask spread widens immediately to more than one tick. Once the spread has increased, a flow of limit sell (resp. buy) orders quickly fills the gap and the spread reduces again to one tick. When a limit order is placed inside the spread, all the limit orders pegged to the NBBO price move in less than a millisecond to the price level corresponding to this new order. Once this happens, both the bid price and the ask price have increased (resp. decreased) by one tick.

The histograms in Figure 2 show that this 'closing' of the spread takes place very quickly: as shown in Figure 2 (left) the lifetime of a spread larger than one tick is of the order of a couple of milliseconds, which is negligible compared to the lifetime of a spread equal to one tick (Figure 2, right). In our model we assume that the second step occurs *infinitely fast*: once the bid-ask spread widens, it returns immediately to one tick. For the example of Dow Jones stocks (Figure 2), this is a reasonable assumption since the widening of the spread lasts only a few milliseconds. This simply means that we are not trying to describe/model how the orders flow inside the bid-ask spread at the millisecond time scale and, when we describe the state of the order book *after* a price change we have in mind the state of the order book *after* the bid-ask spread has returned to one tick.
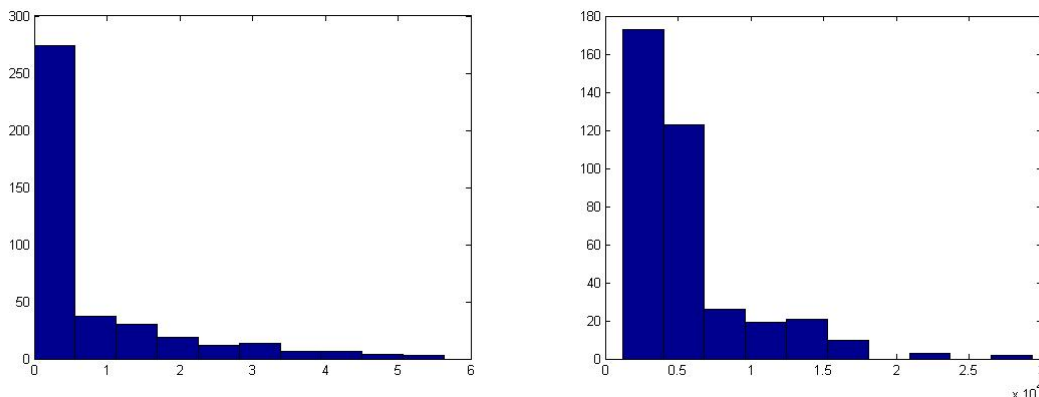


**Figure 2**    Left: Average lifetime, in milliseconds of a spread larger than one tick for Dow Jones stocks. Right: Average lifetime, in milliseconds of a spread equal to one tick.

Under this assumption, each time *one* of the queues is depleted, *both* the bid queue and the ask queues move to a new position and the bid-ask spread remains equal to one tick after the

price change. Thus, under our assumptions the bid-ask spread is equal to one tick, i.e. $s_t^a = s_t^b + \delta$, resulting in a further reduction of dimension in the model.

Once either the bid or the ask queue are depleted, the bid and ask queues assume new values. Instead of keeping track of arrival, cancelation and execution of orders at *all* price levels (as in Cont et al. (2010b), Smith et al. (2003)), we treat the queue sizes after a price change as stationary variables drawn from a certain distribution $f$ on $\mathbb{N}^2$ which represents, in a statistical sense, the *depth* of the order book after a price change: $f(x,y)$ represents the probability of observing $(q_t^b, q_t^a) = (x,y)$ right after a price increase. Similarly, we denote $\tilde{f}(x,y)$ the probability of observing $(q_t^b, q_t^a) = (x,y)$ right after a price decrease. More precisely, denoting by $\mathcal{F}_t$ the history of prices and order book events on $[0,t]$,

- if $q_{t-}^a = 0$ then $(q_t^b, q_t^a)$ is a random variable with distribution $f$, independent from $\mathcal{F}_{t-}$.
- if $q_{t-}^b = 0$ then $(q_t^b, q_t^a)$ is a random variable with distribution $\tilde{f}$, independent from $\mathcal{F}_{t-}$.

The distributions $f$ and $\tilde{f}$ summarize the interaction of the queues at the best bid/ask levels with the rest of the order book, viewed here as a 'reservoir' of limit orders. Figure 3 shows the (joint) empirical distribution of bid and ask queue sizes after a price move for Citigroup stock on June 26th 2008.

In summary, state of the limit order book is thus described by a continuous-time process $X_t = (s_t^b, q_t^b, q_t^a)$ which takes values in the discrete state space $\delta\mathbb{Z} \times \mathbb{N}^2$, with piecewise constant sample paths whose transitions correspond to the order book events. Denoting by $(T_i^a, i \geq 1)$ (resp. $T_i^b$) the durations between two consecutive orders arriving at the ask (resp. the bid) and $V_i^a$ (resp. $V_i^b$) the size of the associated change in queue size, the above assumptions translate into the following dynamics for $X_t = (s_t^b, q_t^b, q_t^a)$:

- If an order or cancelation arrives on the ask side at time $T$:

$$(s_T^b, q_T^b, q_T^a) = (s_{T-}^b, q_{T-}^b, q_{T-}^a + V_i^a)1_{q_{T-}^a > -V_i^a} + (s_{T-}^b + \delta, R_i^b, R_i^a)1_{q_{T-}^a \leq -V_i^a},$$

- If an order or cancelation arrives on the bid side i.e. $T \in \{T_i^b, i \geq 1\}$:

$$(s_T^b, q_T^b, q_T^a) = (s_{T-}^b, q_{T-}^b + V_i^b, q_{T-}^a)1_{q_{T-}^b > -V_i^b} + (s_{T-}^b - \delta, \tilde{R}_i^b, \tilde{R}_i^a)1_{q_{T-}^b \leq -V_i^b},$$

and $(R_i)_{i \geq 1} = (R_i^b, R_i^a)_{i \geq 1}$ is a sequence of IID variables with (joint) distribution $f$, and $(\tilde{R}_i)_{i \geq 1} = (\tilde{R}_i^b, \tilde{R}_i^a)_{i \geq 1}$ is a sequence of IID variables with (joint) distribution $\tilde{f}$.

## 2.2. A Markov model for order book dynamics

To give a complete statistical description of the dynamics of the limit order book, we need to describe the distributional properties of the sequences $T_i^a, T_i^b, V_i^a, V_i^b$ describing the timing and size of order book events.

We assume that these events occur according to independent Poisson processes:

- Market buy (resp. sell) orders arrive at independent, exponential times with rate $\mu$,
- Limit buy (resp. sell) orders at the (best) bid (resp. ask) arrive at independent, exponential times with rate $\lambda$,
- Cancelations occur at independent, exponential times with rate $\theta$.
- These events are mutually independent.
- All orders sizes are equal (assumed to be 1 without loss of generality).

Denoting by $(T_i^a, i \geq 1)$ (resp. $T_i^b$) the durations between two consecutive queue changes at the ask (resp. the bid) and $V_i^a$ (resp. $V_i^b$) the size of the associated change in queue size, the above assumptions translate into the following properties for the sequences $T_i^a, T_i^b, V_i^a, V_i^b$:

(i) $(T_i^a)_{i \geq 0}$ is a sequence of independent random variables with exponential distribution with parameter $\lambda + \theta + \mu$,

(ii) $(T_i^b)_{i \geq 0}$ is a sequence of independent random variables with exponential distribution with parameter $\lambda + \theta + \mu$,

(iii) $(V_i^a)_{i \geq 0}$ is a sequence of independent random variables with

$$\mathbb{P}[V_i^a = 1] = \frac{\lambda}{\lambda + \mu + \theta} \quad and \quad \mathbb{P}[V_i^a = -1] = \frac{\mu + \theta}{\lambda + \mu + \theta}, \tag{1}$$

(iv) $(V_i^b)_{i \geq 0}$ is a sequence of independent random variables with

$$\mathbb{P}[V_i^b = 1] = \frac{\lambda}{\lambda + \mu + \theta} \quad and \quad \mathbb{P}[V_i^b = -1] = \frac{\mu + \theta}{\lambda + \mu + \theta} \tag{2}$$
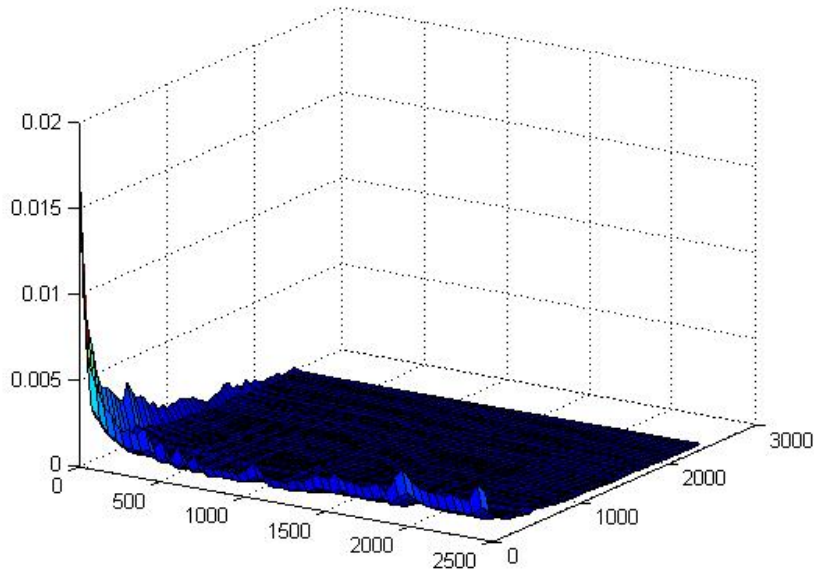
• These sequences are independent.



**Figure 3**    Joint density of bid and ask queue sizes after a price move (Citigroup, June 26th 2008).

Under these assumptions $q_t = (q_t^b, q_t^a)$ is thus a Markov process, taking values in $\mathbb{N}^2$, whose transitions correspond to the order book events $\{T_i^a, i \geq 1\} \cup \{T_i^b, i \geq 1\}$:

• At the arrival of a new limit buy (resp. sell) order the bid (resp. ask) queue increases by one unit. This occurs at rate $\lambda$.

• At each cancelation or market order, which occurs at rate $\theta + \mu$, either:

  (a) the corresponding queue decreases by one unit if it is $> 1$, or

  (b) if the ask queue is depleted then $q_t$ is a random variable with distribution $f$.

  (c) if the bid queue is depleted then $q_t$ is a random variable with distribution $\tilde{f}$.

The values of $\lambda$ and $\mu + \theta$ are readily estimated from high-frequency records of trades and quotes (see Cont et al. (2010b) for a description of the estimation procedure). Table 3 gives examples of such parameter estimates for the stocks mentioned above. We note that in all cases $\lambda < \mu + \theta$ but that the difference is small: $|(\mu + \theta) - \lambda| \ll \lambda$.

REMARK 1 (INDEPENDENCE ASSUMPTIONS). The IID assumption for the sequences $(R_n), (\tilde{R}_n)$ is only used in Section 4. The results of Section 3 do not depend on this assumption.

|  | $\hat{\lambda}$ | $\hat{\mu} + \hat{\theta}$ | $\frac{\hat{\mu}+\hat{\theta}-\hat{\lambda}}{\hat{\lambda}}$ |
|---|---|---|---|
| Citigroup | 2204 | 2331 | 0.0576 |
| General Electric | 317 | 325 | 0.0252 |
| General Motors | 102 | 104 | 0.0196 |

**Table 3**　　Estimates for the intensity of limit orders and market orders+cancellations, in number of batches per second (each batch representing 100 shares) on June 26th, 2008).

### 2.3. Quantities of interest

In applications, one is interested in computing various quantities that intervene in high frequency trading such as:
- the conditional distribution of the duration between price moves, given the state of the order book (Section 3.1),
- the probability of a price increase, given the state of the order book (Section 3.2),
- the dynamics of the price autocorrelations and distribution and autocorrelations of price changes (section 3.4), and
- the volatility of the price (section 4).

We will show that all these quantities may be characterized analytically in this model, in terms of order flow statistics.

## 3. Analytical results

The high-frequency dynamics of the price may be described in terms of *durations* between successive price changes and the magnitude of these price changes. It is of interest to examine what information the current state of the (consolidated) order book gives about the dynamics of the price. We now proceed to show how the model presented above may be used to compute the conditional distributions of durations and price changes, given the current state of the order book, in terms of the arrival rates of market orders, limit orders and cancelations. The results of this section do not depend on the assumptions on the sequences $(R_n), (\tilde{R}_n)$.

### 3.1. Duration until the next price change

We consider first the distribution of the duration until the next price change, starting from a given configuration $(x, y)$ of the order book. We define
- $\sigma_a$ the first time when the ask queue $(q_t^a, t \geq 0)$ is depleted,
- $\sigma_b$ the first time when the bid queue $(q_t^b, t \geq 0)$ is depleted

Since the queue sizes are constant between events, one can express these stopping times as:

$$\sigma_a = \inf\{T_1^a + ... + T_i^a, \ q_{T_1^a+...+T_i^a-}^a + V_i^a = 0\} \qquad \sigma_b = \inf\{T_1^b + ... + T_i^b, \ q_{T_1^b+...+T_i^b-}^b + V_i^b = 0\}$$

The price $(s_t, t \geq 0)$ moves when the queue $q_t = (q_t^b, q_t^a)$ hits one of the axes: the duration until the next price move is thus

$$\tau = \sigma_a \wedge \sigma_b.$$

The following theorem gives the distribution of the duration $\tau$, conditional on the initial queue sizes:

PROPOSITION 1 **(Distribution of duration until next price move)**. *The distribution of $\tau$ conditioned on the state of the order book is given by:*

$$\mathbb{P}[\tau > t | q_0^b = x, \ q_0^a = y] = \sqrt{(\frac{\mu + \theta}{\lambda})^{x+y}} \psi_{x,\lambda,\theta+\mu}(t) \psi_{y,\lambda,\theta+\mu}(t) \qquad (3)$$

$$\text{where} \qquad \psi_{n,\lambda,\theta+\mu}(t) = \int_t^\infty \frac{n}{u} I_n(2\sqrt{\lambda(\theta+\mu)}u)e^{-u(\lambda+\theta+\mu)}du \qquad (4)$$

and $I_n$ is the modified Bessel function of the first kind. The conditional law of $\tau$ has a regularly varying tail

- with tail exponent 2 if $\lambda < \mu + \theta$
- with tail exponent 1 if $\lambda = \mu + \theta$. In particular, if $\lambda = \mu + \theta$, $E[\tau|q_0^b = x, q_0^a = y] = \infty$ whenever $x > 0, y > 0$.

Proof. Since $(q_t^a, t \geq 0)$ follows a birth and death process with birth rate $\lambda$ and death rate $\mu + \theta$, $\mathcal{L}(s,x) := \mathbb{E}[e^{-s\sigma_a}|q_0^a = x]$ satisfies:

$$\mathcal{L}(s,x) = \frac{\lambda \mathcal{L}(s,x+1) + (\mu+\theta)\mathcal{L}(s,x-1)}{\lambda + \mu + \theta + s}.$$

We can find the roots of the polynomial: $\lambda X^2 - (\lambda + \mu + \theta + s)X + \mu + \theta$; one root is $> 1$, the other is $< 1$; since $\mathcal{L}(s,0) = 1$ and $\lim_{x\to\infty} \mathcal{L}(s,x) = 0$,

$$\mathcal{L}(s,x) = \left(\frac{(\lambda+\mu+\theta+s) - \sqrt{((\lambda+\mu+\theta+s))^2 - 4\lambda(\mu+\theta)}}{2\lambda}\right)^x.$$

Moreover if we use the relation $\mathbb{P}[\tau > t|q_0^b = x, q_0^a = y] = \mathbb{P}[\sigma_b > t|q_0^b = x]\mathbb{P}[\sigma_a > t|q_0^a = y]$,

$$\mathbb{P}[\tau > t|q_0^b = x, q_0^a = y] = \int_t^\infty \hat{\mathcal{L}}(u,x)du \int_t^\infty \hat{\mathcal{L}}(u,y)du.$$

This Laplace transform may be inverted (see (Feller 1971, XIV.7)) and the inversion yields

$$\hat{\mathcal{L}}(t,x) = \frac{x}{t}\sqrt{\left(\frac{\mu+\theta}{\lambda}\right)^x} \quad I_x(2\sqrt{\lambda(\theta+\mu)}t)e^{-t(\lambda+\theta+\mu)},$$

which gives us the expected result.

The above result allows in particular to study the *tail* behavior of the conditional distribution of $\tau$:

- If $\lambda < \mu + \theta$:

$$\mathcal{L}(s,x) = \alpha(s)^x \underset{s\to0}{\sim} 1 - \frac{x(\lambda+\mu+\theta)}{2\lambda(\mu+\theta-\lambda)}s,$$

so Karamata's Tauberian theorem (Feller 1971, XIII.5) yields

$$\mathbb{P}[\sigma_a > t|q_0^a = x] \underset{t\to\infty}{\sim} \frac{x(\lambda+\mu+\theta)}{2\lambda(\mu+\theta-\lambda)}\frac{1}{t};$$

therefore the conditional law of the duration $\tau$ is a regularly varying with tail index 2

$$\mathbb{P}[\tau > t|q_0^b = x, q_0^a = y] \underset{t\to\infty}{\sim} \frac{xy(\lambda+\mu+\theta)^2}{\lambda^2(\mu+\theta-\lambda)^2}\frac{1}{4t^2}. \qquad (5)$$

- If the order flow is balanced i.e. $\lambda = \mu + \theta$ then

$$\mathcal{L}(s,x) = \alpha(s)^x \underset{s\to0}{\sim} 1 - \frac{x}{\sqrt{\lambda}}\sqrt{s},$$

the law of $\sigma_a$ is regularly-varying with tail index $1/2$ and

$$\mathbb{P}[\sigma_a > t|q_0^a = x] \underset{t\to\infty}{\sim} \frac{x}{\sqrt{\pi\lambda}}\frac{1}{\sqrt{t}}.$$

The duration then follows a heavy-tailed distribution with infinite first moment:

$$\mathbb{P}[\tau > t|q_0^b = x, q_0^a = y] \underset{t\to\infty}{\sim} \frac{xy}{\pi\lambda}\frac{1}{t}; \qquad (6)$$
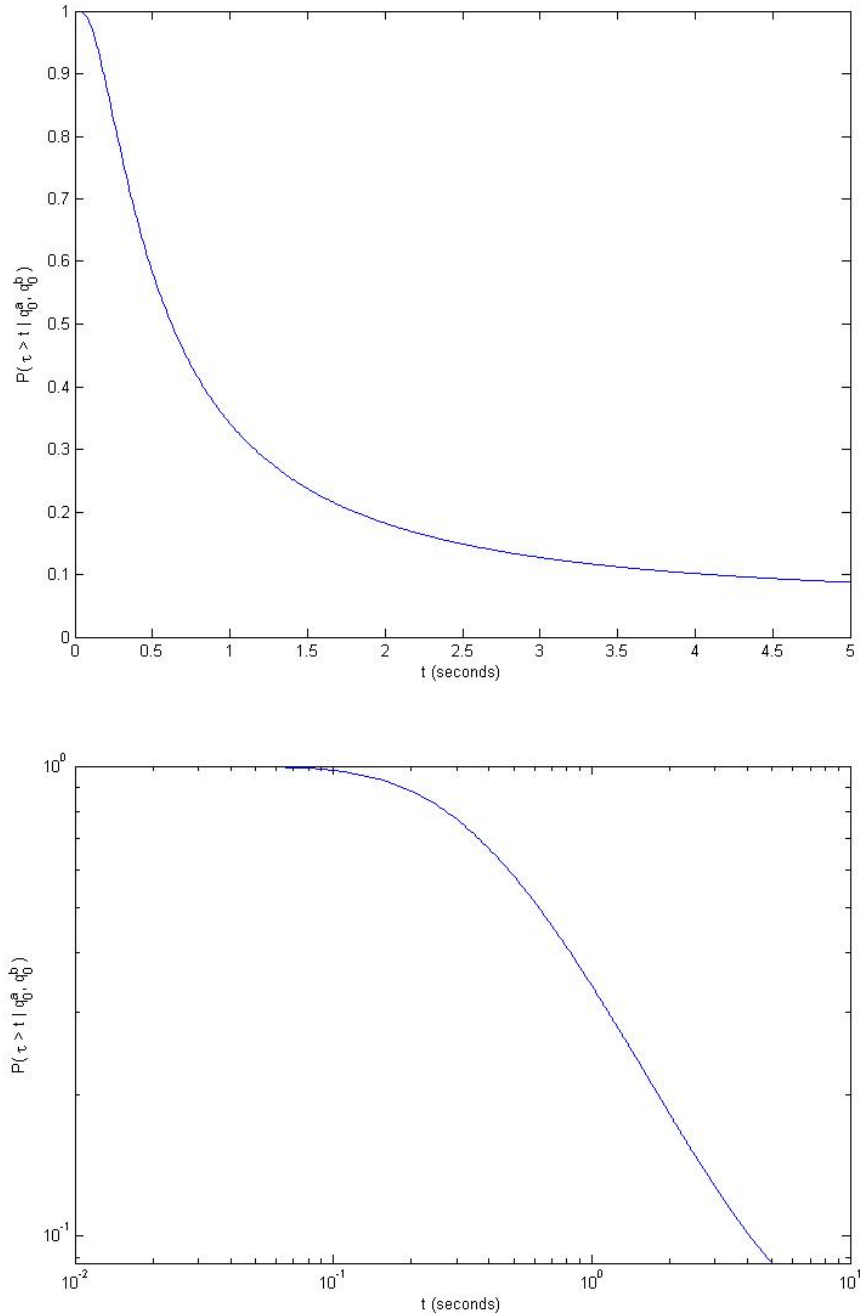
**Figure 4**    Above: $P(\tau > t | q_0^b = 4, q_0^a = 5)$ as a function of $t$ for $\lambda = 12, \mu + \theta = 13$. Below: same figure in log-log coordinates. Note the Pareto tail which decays as $t^{-2}$.

The expression given in (3) is easily computed by discretizing the integral in (4). Plotting (3) for a fine grid of values of $t$ typically takes less than a second on a laptop. Figure 3 gives a numerical example, with $\lambda = 12$ sec$^{-1}, \mu + \theta = 13$ sec$^{-1}, q_0^a = 4, q_0^b = 5$ (queue sizes are given in multiples of average batch size).

## 3.2. Probability of price increase: balanced limit order book

Starting from a given configuration of the limit order book, the probability that the next price move is an increase is given by the probability that the process $(q_t^b, q_t^a)$ hits the x-axis before the y-axis When $\lambda = \mu + \theta$, i.e. when the flow of limit orders is balanced by the flow of market orders and cancelations, this probability can be computed analytically in terms of hitting time distributions of a random walk in the orthant:

PROPOSITION 2. *For $(n, p) \in \mathbb{N}^2$, the probability $p_1^{up}(n, p)$ that the next price move is an increase, conditioned on having the n orders on the bid side and p orders on the ask side is:*

$$p_1^{up}(n, p) = \frac{1}{\pi} \int_0^\pi (2 - \cos(t) - \sqrt{(2 - \cos(t))^2 - 1})^p \frac{\sin(nt) \cos(\frac{t}{2})}{\sin(\frac{t}{2})} dt. \tag{7}$$

Proof.   One can notice that

$$\forall t \leq \tau, \quad q_t = M_{N_{2\lambda t}}$$

where $(M_n, n \geq 0)$ is a symmetric random walk in the positive orthant $\mathbb{Z}_+^2$ killed at the boundary and $(N_{2\lambda t}, t \geq 0)$ is a Poisson process with parameter $2\lambda$. Hence the probability of an upward move in the price starting from a configuration $q_t^b = n, q_t^a = p$ for the order book is equal to the probability that the random walk $M$ starting from $(n, p)$ hits the x-axis before the y-axis. The generator of the bivariate random walk $(M_n, n \geq 1)$ is the discrete Laplacian so $p_1^{up}(n, p) = \mathbb{P}[\sigma_a < \sigma_b | q_{0-}^b = n, q_{0-}^a = p]$ satisfies, for all $n \geq 1$ and $p \geq 1$,

$$4p_1^{up}(n, p) = p_1^{up}(n+1, p) + p_1^{up}(n-1, p) + p_1^{up}(n, p+1) + p_1^{up}(n, p-1), \tag{8}$$

with the boundary conditions: $p_1^{up}(0, p) = 0$ for all $p \geq 1$ and $p_1^{up}(n, 0) = 1$ for all $n \geq 1$. This problem is known as the *discrete Dirichlet* problem; solutions of (8) are called discrete harmonic functions. (Lawler and Limic 2010, Ch. 8) show that for all $t \geq 0$, the functions

$$f_t(x, y) = e^{xr(t)} \sin(yt), \quad \text{and} \quad \tilde{f}_t(x, y) = e^{-xr(t)} \sin(yt) \quad \text{with} \quad r(t) = \cosh^{-1}(2 - \cos t)$$

are solutions of (8). In (Lawler and Limic 2010, Corollary 8.1.8) it is shown that the probability that a simple random walk $(M_k, k \geq 1)$ starting at $(n, p) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ reaches the axes at $(x, 0)$ is

$$\frac{2}{\pi} \int_0^\pi e^{-r(t)p} \sin(nt) \sin(tx) dt,$$

therefore

$$p_1^{up}(n, p) = \sum_{k=1}^\infty \frac{2}{\pi} \int_0^\pi e^{-r(t)p} \sin(tn) \sin(tk) dt.$$

Since

$$\sum_{k=1}^m \sin(kt) = \frac{\sin(\frac{mt}{2}) \sin(\frac{(m+1)t}{2})}{\sin(t/2)} = \frac{\cos(\frac{t}{2}) - \cos((m+\frac{1}{2})t)}{2\sin(t/2)},$$

using integration by parts we see that the second term leads to the integral:

$$\int_0^\pi \underbrace{\frac{e^{-r(t)p} \sin(nt)}{\sin(t/2)}}_{g(t)} \cos((m+1/2)t) dt = -\frac{1}{m + \frac{1}{2}} \int_0^\pi g'(t) \sin((m+\frac{1}{2})t) dt \underset{m \to \infty}{\to} 0.$$

since $g'$ is bounded. So finally:

$$p_1^{up}(n, p) = \frac{1}{\pi} \int_0^\pi e^{-r(t)p} \sin(tn) \frac{\cos(\frac{t}{2})}{\sin(\frac{t}{2})} dt.$$

Noting that $e^{-r(t)} = (2 - \cos(t) - \sqrt{(2 - \cos(t))^2 - 1})$ we obtain the result.
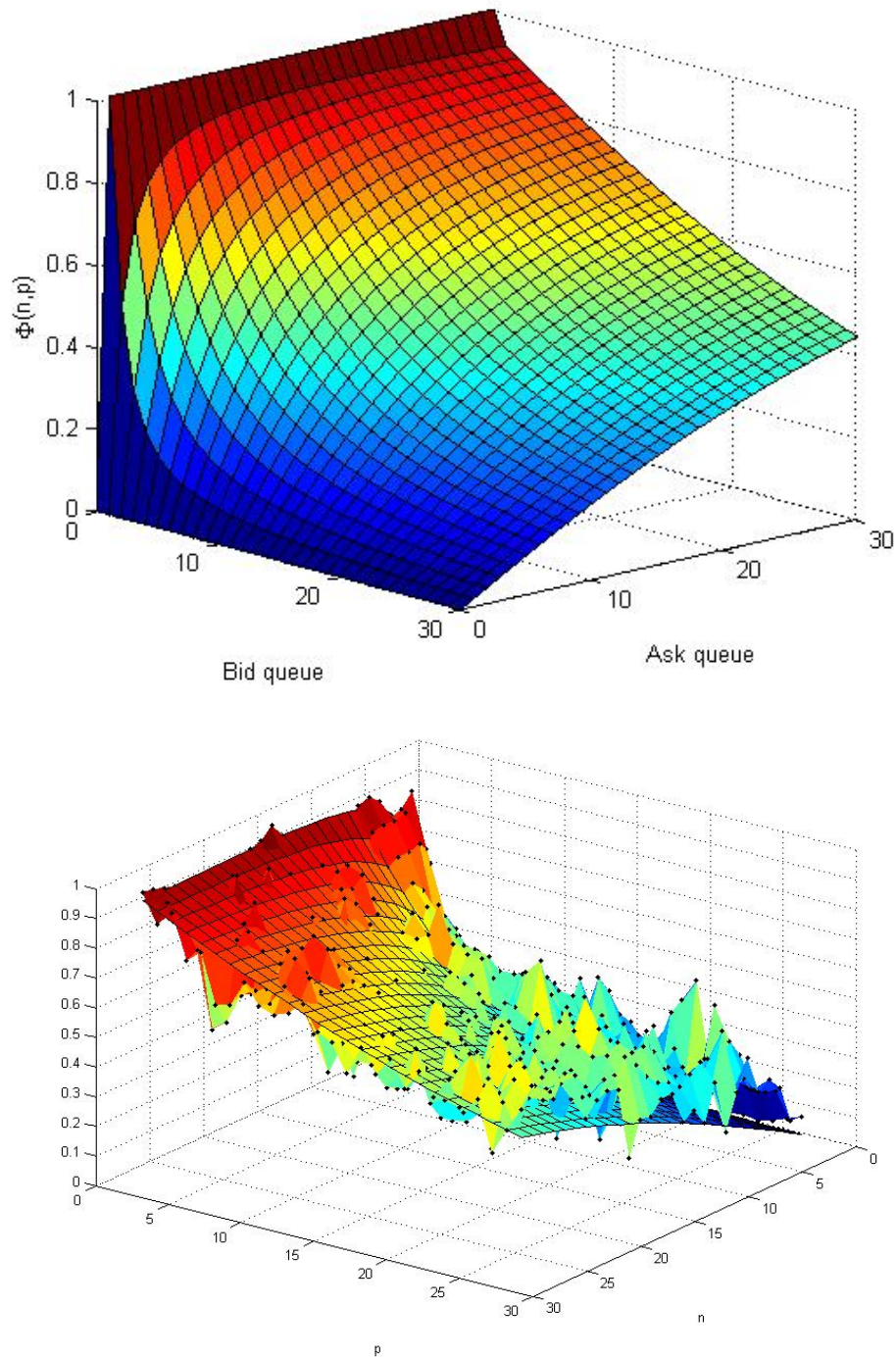
**Figure 5**    Above: Conditional probability of a price increase, as a function of the bid and ask queue size. Below: comparison with empirical transition frequencies for CitiGroup stock price tick-by-tick data on June 26, 2008.

Note that the conditional probabilities (7) are, in the case of a balanced order book, independent of the parameters describing the order flow.

The expression (7) is easily computed numerically: Figure 3.2 displays the shape of the function $p_1^{up}$. Comparison with empirical data for CitiGroup stock (June 2008) shows good agreement between the theoretical value (7) and the empirical transition frequencies of the price conditional on the state of the consolidated order book.

### 3.3. Probability of price increase: asymmetric order flow

In this subsection we relax the symmetry assumptions above and allow the intensity of limit and market orders at the bid and the ask to be different; more precisely we assume that:

• Limit orders at the ask arrive at independent, exponential time with parameter $\lambda^a$

• Market orders and cancelations at the ask arrive at independent, exponential time with parameter $\mu^a + \theta^a$

• Limit orders at the bid arrive at independent, exponential time with parameter $\lambda^b$

• Market orders and cancelations at the bid arrive at independent, exponential time with parameter $\mu^b + \theta^b$

and that these Poisson processes are independent. The dynamics of bid and ask queues may be then represented as

$$q_t = M_{N_{\Lambda t}} \quad \text{for} \quad \Lambda = \lambda^a + \mu^a + \theta^a + \lambda^b + \mu^b + \theta^b,$$

where $N_{\Lambda t}$ is a Poisson process with intensity $\Lambda$ and $(M_n, n \geq 0)$ is a random walk on $\mathbb{Z}^2$ killed when it hits either the x-axis or the y-axis whose the transition probabilities are:

$$p_{0,1} = \frac{\lambda^a}{\Lambda} \quad p_{1,0} = \frac{\lambda^b}{\Lambda} \quad p_{0,-1} = \frac{\mu^a + \theta^a}{\Lambda} \quad p_{-1,0} = \frac{\mu^b + \theta^b}{\Lambda}. \tag{9}$$

The following result generalizes Proposition 2 for an asymmetric order flow.

PROPOSITION 3. *Given* $(q^b, q^a) = (n, p)$, *the probability* $p_1^{up}(n, p)$ *that the next price move in an increase in:*

$$p_1^{up}(n,p) = 1 - \frac{1}{\pi} \left( \frac{\mu^a + \theta^a}{\lambda^a} \right)^p \frac{2[\lambda^a(\mu^a + \theta^a)]^{1/2}}{\mu^a + \theta^a + \lambda^a} \int_0^\pi Z(t)^n \sin(pt) \sin(t) \times$$

$$\times \frac{2\lambda^b Z_t - (\Lambda - 2[\lambda^a(\mu^a + \theta^a)]^{1/2}\cos(t))}{\Lambda \frac{2[(\mu^a+\theta^a)\lambda^a]^{1/2}}{\lambda^b + \mu^b + \theta^b}\cos(t) - \Lambda} \frac{\Lambda dt}{\sqrt{(\Lambda - 2[(\mu^a + \theta^a)\lambda^a]^{1/2}\cos(t))^2 - 4(\mu^b + \theta^b)\lambda^b}},$$

*where* $(Z_t, t \geq 0)$ *is the function defined by:*

$$\forall t \geq 0, \quad Z_t = \frac{\Lambda - 2[(\mu^a + \theta^a)\lambda^a]^{1/2}\cos(t) - \sqrt{(\Lambda - 2[(\mu^a + \theta^a)\lambda^a]^{1/2}\cos(t))^2 - 4\lambda^b(\theta^b + \mu^b)}}{2\lambda^a}.$$

Proof.  Using results from Kurkova and Raschel (2011), it is shown in Raschel (2012) that the probability that $M$ starting from $(n, p)$ hits the x-axis before the y-axis is given by

$$1 - \frac{1}{\pi} \left( \frac{p_{0,-1}}{p_{0,1}} \right)^p \frac{2[p_{0,1}p_{0,-1}]^{1/2}}{p_{0,1} + p_{0,-1}} \int_0^\pi Z(t)^n \sin(pt) \sin(t) \times$$

$$\times \frac{2p_{1,0}Z_t - (1 - 2[p_{0,1}p_{0,-1}]^{1/2}\cos(t))}{\frac{2[p_{0,1}p_{0,-1}]^{1/2}}{p_{0,1}+p_{0,-1}}\cos(t) - 1} \frac{dt}{\sqrt{(1 - 2[p_{0,1}p_{0,-1}]^{1/2}\cos(t))^2 - 4p_{1,0}p_{-1,0}}},$$

where $(Z_t, t \geq 0)$ is the function defined by:

$$\forall t \geq 0, \quad Z_t = \frac{1 - 2[p_{0,1}p_{0,-1}]^{1/2}\cos(t) - \sqrt{(1 - 2[p_{0,1}p_{0,-1}]^{1/2}\cos(t))^2 - 4p_{1,0}p_{-1,0}}}{2p_{1,0}}.$$

Applying this formula to the random walk whose transition probabilities are given in Equation (9) we obtain that the probability of a price increase at the next price move, starting from a configuration $q_t^b = n, q_t^a = p$ of the queues is equal to the probability that the random walk $M$ starting from $(n,p)$ hits the x-axis before the y-axis; thus, it is given by equation (3).

### 3.4. Dynamics of the price

The high-frequency dynamics of the price is described by a piecewise constant, right continuous process $(s_t, t \geq 0)$ whose jumps times correspond to times when the order book process $(q_t, t \geq 0)$ hits one of the axes. Denote by $(\tau_1, \tau_2, ...)$ the successive durations between price changes. The number of price changes that occur during $[0, t]$ is given by

$$N_t := \max\{ n \geq 0, \quad \tau_1 + ... + \tau_n \leq t \}$$

At $t = \tau_i$, $s_{\tau_i} = s_{\tau_{i-}} + 1$ if $q_{\tau_{i-}^a} = 0$ and $s_{\tau_i} = s_{\tau_{i-}} - 1$ if $q_{\tau_{i-}^b} = 0$. $(X_1, X_2, X_3, ..., X_n, ...)$ are the successive moves in the price which, in general, are not independent random variables. We have

$$s_t = Z_{N_t} \quad \text{where} \quad Z_n = \sum_{i=1}^{n} X_i$$

is the price after $n$ price changes have occurred. Hence, for all $t \geq 0$, $s_t = Z_{N_t}$. We are interested in the $n$-step ahead distribution of the price change:

$$p_n^{up}(x, y) = \mathbb{P}[X_n = +\delta \quad | \quad (q_0^b, q_0^a) = (x, y)] \tag{10}$$

For $n = 1$ this corresponds to the probability $p_1^{up}(x, y) = p_1(x, y)$ of an upward price move, computed in Theorem 2. To simplify the analysis we use, in this Section and the next one, the following symmetry assumption:

ASSUMPTION 1 (**Bid-ask symmetry**). $\tilde{f}(x, y) = f(y, x)$.

A key quantity for studying the dynamics of the price is the probability of two successive price changes in the same direction,

$$p_{cont} = \mathbb{P}[X_{k+1} = \delta | X_k = \delta] = \mathbb{P}[X_{k+1} = -\delta | X_k = -\delta] \tag{11}$$

which may be expressed in terms of the distribution of queue sizes $f$ after a price change:

PROPOSITION 4. *Let* $p_{cont} = \mathbb{P}[X_2 = \delta | X_1 = \delta] = \mathbb{P}[X_2 = -\delta | X_1 = -\delta]$ *be the probability of two successive price moves in the same direction.*
- $p_{cont} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(i, j) p_1^{up}(i, j)$.
- $\forall k \geq 1, \quad \mathbb{E}[X_k | q_0^b = x, q_0^a = y] = (2p_1^{up}(x, y) - 1)(2p_{cont} - 1)^{k-1}$.
- $cov(X_1, X_2 | q_0^b = x, q_0^a = y) = \delta^2(2p_{cont} - 1)(1 - (2p_1^{up}(x, y) - 1)^2)$.
- *Conditional on the current state of the limit order book, the n-step ahead distribution of the price change is given by:*

$$p_n^{up}(x, y) := \mathbb{P}[X_n = \delta | q_0^b = x, \ q_0^a = y] = \frac{1 + (2p_{cont} - 1)^{n-1}(2p_1^{up}(x, y) - 1)}{2}. \tag{12}$$

Proof. First, let us prove that $\mathbb{P}[X_2 = \delta | X_1 = \delta] = \mathbb{P}[X_2 = -\delta | X_1 = -\delta]$:

$$\mathbb{P}[X_2 = \delta | X_1 = \delta] = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(i,j) p_1^{up}(i,j),$$

where $p_1^{up}$ is given in 7, by symmetry of the bid and the ask, for all $(n,p) \in \mathbb{N}^2$, $p_1^{up}(n,p) = 1 - p_1^{up}(p,n)$. By assumption 1, for all $(i,j) \in \mathbb{N}^2$, $f(i,j) = \tilde{f}(j,i)$. Therefore,

$$\mathbb{P}[X_2 = \delta | X_1 = \delta] = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(i,j)(1 - p_1^{up}(j,i)),$$

$$\mathbb{P}[X_2 = \delta | X_1 = \delta] = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tilde{f}(j,i)(1 - p_1^{up}(j,i)),$$

$$\mathbb{P}[X_2 = \delta | X_1 = \delta] = \mathbb{P}[X_2 = -\delta | X_1 = -\delta].$$

$p_n^{up} = p_n^{up}(x,y)$ defined by (12) is then characterized by the following relation:

$$\begin{pmatrix} p_n^{up} \\ 1 - p_n^{up} \end{pmatrix} = \begin{pmatrix} p_{cont} & 1 - p_{cont} \\ 1 - p_{cont} & p_{cont} \end{pmatrix} \begin{pmatrix} p_{n-1} \\ 1 - p_{n-1} \end{pmatrix},$$

hence

$$\begin{pmatrix} p_n^{up} \\ 1 - p_n^{up} \end{pmatrix} = \begin{pmatrix} p_{cont} & 1 - p_{cont} \\ 1 - p_{cont} & p_{cont} \end{pmatrix}^{n-1} \begin{pmatrix} p_1 \\ 1 - p_1 \end{pmatrix}.$$

The eigenvalues of this matrix are 1 and $2p_{cont} - 1$:

$$\begin{pmatrix} p_{cont} & 1 - p_{cont} \\ 1 - p_{cont} & p_{cont} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2p_{cont} - 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

Therefore

$$p_n^{up}(x,y) = \frac{1 + (2p_{cont} - 1)^{n-1}(2p_1(x,y) - 1)}{2}.$$

Moreover for all $n \geq 2$,

$$\mathbb{E}[X_n | q_0^b = x, q_0^a = y] = (2p_n^{up}(x,y) - 1) = (2p_{cont} - 1)^{n-1}(2p_1(x,y) - 1).$$

and the correlation between two consecutive price moves is given by:

$$\begin{aligned}
\mathrm{cov}(X_1, X_2 | q_0^b = x, q_0^a = y) &= \mathbb{E}[X_1 X_2 | q_0^b = x, q_0^a = y] - \mathbb{E}[X_1 | q_0^b = x, q_0^a = y]\mathbb{E}[X_2 | q_0^b = x, q_0^a = y] \\
&= \delta^2(2p_{cont} - 1) - \delta^2(2p_1(x,y) - 1)(2p_2(x,y) - 1),
\end{aligned}$$

$$\mathrm{cov}(X_1, X_2 | q_0^b = x, q_0^a = y) = \delta^2(2p_{cont} - 1) - (2p_1(x,y) - 1)^2(2p_{cont} - 1) = (2p_{cont} - 1)(1 - (2p_1(x,y) - 1)^2). \tag{13}$$

REMARK 2 (NEGATIVE AUTOCORRELATION OF PRICE CHANGES AT FIRST LAG). It is empirically observed that high frequency price movements have a negative autocorrelation at the first lag; this observation is often attributed to the 'bid-ask' bounce of transaction prices, but in fact it also holds for the time series of bid or ask prices Cont (2001). Our model links the value of this autocorrelation at first lag to the properties of the distribution $f$ of order book depth. As observed from (13), the sign of the $\mathrm{cov}(X_1, X_2 | q_0^b, q_0^a)$ does not depend on the initial configuration $(q_0^b, q_0^a)$ of the bid/ask queues, so $\mathrm{cov}(X_k, X_{k+1}) = \mathrm{cov}(X_1, X_2) < 0$ if and only if

$$p_{cont} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(i,j) p_1^{up}(i,j) < 1/2$$

where $f$ is the joint distribution of queue sizes after a price increase. This condition is satisfied for most high-frequency data sets of Dow Jones stocks we have examined. For example, for CitiGroup stock we find

$$p_{cont} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(i,j) p_1^{up}(i,j) \simeq 0.35$$

This asymmetry of $f$ corresponds to the fact that, after an upward price move, the new bid queue is generally smaller than the ask queue since the ask queue corresponds to the limit order previously sitting at second best ask level, while the bid queue results from the accumulation of orders over the very short period since the last price move. Under this condition, high frequency increments of the price are negatively correlated: an increase in the price is more likely to be followed by a decrease in the price.

REMARK 3. The sequence of price increments $(X_1, X_2, ...)$ is uncorrelated if and only if

$$p_{cont} = \sum_{i,j \geq 1} f(i,j) p_1^{up}(i,j) = 1/2$$

### 3.5. "Efficient" price

Various authors (see e.g. Robert and Rosenbaum (2011)) have considered models in which the evolution of transaction prices is based on a non-observed (semi)martingale $\hat{s}$, sometimes called the "efficient" price: the observed price is then either a noisy version of $\hat{s}$ or the value of $\hat{s}$ rounded to the nearest tick.

Given the probability $p_1^{up}(q_t^b, q_t^a)$ that the next price move is an "uptick" (Equation (7)), we can construct an auxiliary process $\hat{s}$ whose value $\hat{s}_t$ represents the expected value of the price after its next move:

$$\forall t \geq 0, \ \ \hat{s}_t = (s_t + \delta) p_1^{up}(q_t^b, q_t^a) + (s_t - \delta) \left(1 - p_1^{up}(q_t^b, q_t^a)\right),$$
$$\hat{s}_t = s_t + \delta(2 p_1^{up}(q_t^b, q_t^a) - 1),$$

$(\hat{s}_t, t \geq 0)$ is a continuous-time stochastic process with values between $s_t - \delta$ and $s_t + \delta$:

$$\forall t \geq 0, \ \ s_t - \delta \leq \hat{s}_t \leq s_t + \delta.$$

The process $\hat{s}$ incorporates the information on the price $s_t$ and the state of the order book $(q^b, q^a)$ insofar as it affects the next price move. The following result shows

PROPOSITION 5. *If $p_{\mathrm{cont}} = 1/2$ then $(\hat{s}_t, t \geq 0)$ is a martingale.*

REMARK 4. This condition is verified in particular if $\forall (i,j) \in \mathbb{N}^2$, $f(i,j) = f(j,i)$ but more generally if

$$\sum_{i,j \geq 1} p_1^{up}(i,j) f(i,j) = \frac{1}{2}$$

Proof. Let $(\tau_1, \tau_2, ..., \tau_k)$ the sequence of times when the price $s$ moves and $(X_1, ..., X_n)$ the sequence of consecutive price moves. Since $p_{cont} = 1/2$, $(X_1, ..., X_k, ...)$ is a sequence of I.I.D bernoulli random variables with parameter $1/2$. Therefore we have the following property:

$$\forall (i,j) \in \mathbb{N}^2, \ i < j, \ \mathbb{E}[s_{\tau_j} | \mathcal{F}_{\tau_i}] = s_{\tau_i}.$$

The function $p_1^{up}$, from equation (7), satisfies the equation $L p_1^{up} = 0$, where $L$ is the generator of the process $(q^b, q^a)$. Hence $p_1^{up}$ is an harmonic function for the process $(q^b, q^a)$, and the process $(p_1^{up}(q_t^b, q_t^a), t \geq 0)$ is a martingale. We proved that

$$\forall \ s \leq t < \tau_1, \ \ \mathbb{E}[\hat{s}_t | \mathcal{F}_s] = \hat{s}_s.$$

By recurrence on $k$, one can easily notice that

$$\forall \ k \ \geq 1, \ \forall \tau_k \leq t < \tau_{k+1}, \quad \mathbb{E}[\hat{s}_t | \mathcal{F}_{\tau_1}] = \hat{s}_{\tau_1}.$$

Assuming $s \leq \tau_1 \leq t$,

$$
\begin{aligned}
\mathbb{E}[\hat{s}_t | \mathcal{F}_s] &= \mathbb{E}[\hat{s}_t | \mathcal{F}_s, X_1 = 1] \mathbb{P}[X_1 = 1 | \mathcal{F}_s] + \mathbb{E}[\hat{s}_t | \mathcal{F}_s, X_1 = -1] \mathbb{P}[X_1 = -1 | \mathcal{F}_s], \\
&= \mathbb{E}[\hat{s}_t | \mathcal{F}_s, X_1 = 1] p_1^{up}(q_t^b, q_t^a) + \mathbb{E}[\hat{s}_t | \mathcal{F}_s, X_1 = -1](1 - p_1^{up}(q_t^b, q_t^a)), \\
&= (s_s + \delta) p_1^{up}(q_s^b, q_s^a) + s_s(1 - p_1^{up}(q_s^b, q_s^a)), \\
&= s_s + \delta(2 p_1^{up}(q_s^b, q_s^a) - 1) = \hat{s}_s,
\end{aligned}
$$

which completes the proof.

Contrarily to the 'latent price' models alluded to above, here $\hat{s}$ is a function of the state variables $(s_t, q_t^b, q_t^a)$ and thus is observable, provided one observes trades and quotes.

REMARK 5. When $p_{cont} \neq 1/2$, the process $(\hat{s}_t, t \geq 0)$ fails to possess the martingale property. When $p_{cont} < 1/2$, the jump of $\hat{s}$ is negative after a price increase and positive after a price decrease.

## 4. Diffusion limit of the price process

As discussed in Section 3.4, the high frequency dynamics of the price is described by a piecewise constant stochastic process $s_t = Z_{N_t}$ where

$$Z_n = X_1 + ... + X_n \quad and \quad N_t = \sup\{k; \ \tau_1 + ... + \tau_k \leq t\}$$

is the number of price changes during $[0, t]$.

However, over time scales much larger than the interval between individual order book events, prices are observed to have diffusive dynamics and modeled as such. To establish the link between the high frequency dynamics and the diffusive behavior at longer time scales, we shall consider a time scale $t_n$ over which the average number of order book events is of order $n$ and exhibit conditions under which the scaled price process

$$(s_t^n := \frac{s_{t_n}}{\sqrt{n}}, t \geq 0)_{n \geq 1}$$

satisfies a functional central limit theorem i.e. converges in distribution to a non-degenerate process $(p_t, t \geq 0)$ as $n \to \infty$. The choice of the time scale $t_n$ is such that

$$\frac{\tau_1 + ... + \tau_n}{t_n}$$

has a well-defined limit: it is imposed by the distributional properties of the durations which, as observed in Section 3.1, are heavy tailed. In this section, we show that, under a symmetry condition, this limit can be identified as a diffusion process whose diffusion coefficient may be computed from the statistics of the order flow driving the limit order book.

Assume $\lambda + \theta \leq \mu$ and that the joint distribution $f$ of the queue sizes after a price move satisfies:

$$D(f) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} ij f(i, j) < \infty \tag{14}$$

The quantity $D(f)$ represents a measure of market depth: more precisely, $\sqrt{D(f)}$ is the geometric average of the size of the bid queue and the size of the ask queue after a price change.

In this section we assume that $f$ is symmetric: $\forall i, j \geq 0$, $f(i,j) = f(j,i)$. Under this assumption, price increments form a sequence $(X_i, i \geq 0)$ of independent random variables with distribution:

$$\mathbb{P}[X_1 = \delta] = \mathbb{P}[X_1 = -\delta] = \frac{1}{2}.$$

We will show that the limit $p$ is then a diffusion process which describes the dynamics of the price at lower frequencies. In particular, we will compute the volatility of this *diffusion limit* and relate it to the properties of the order flow.

In the following $\mathcal{D}$ denotes the space of right continuous paths $\omega : [0, \infty) \to \mathbb{R}^2$ with left limits, equipped with the Skorokhod topology $J_1$, and $\Rightarrow$ will designate weak convergence on $(\mathcal{D}, J_1)$ (see Billingsley (1968), Whitt (2002) for a discussion).

### 4.1. Balanced order book

We first consider the case of a *balanced* order flow for which the intensity of market orders and cancelations is equal to the intensity of limit orders. The study of high-frequency quote data indicates that this is an empirically relevant case for many liquid stocks: as shown in Table 3, the imbalance between arrival of limit orders on one hand and market orders/ cancelations on the other hand is around 5% or less for these stocks.

For balanced order flow, we proved in Section 3.1 that the distribution of price duration $\tau$ conditioned on observing $i$ shares at the bid and $j$ shares at the ask at $t = 0$ has a tail index 1:

$$\mathbb{P}[\tau > t | q_0^b = i, q_0^a = j] \sim \frac{ij}{\pi \lambda t}; \tag{15}$$

The unconditional distribution of price durations keeps a tail index of one:

$$\mathbb{P}[\tau > t] \overset{t \to \infty}{\sim} \frac{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} ij f(i,j)}{\pi \lambda t} = \frac{D(f)}{\pi \lambda t}. \tag{16}$$

The sequence of durations between consecutive move of the price consecutive $(\tau_1, \tau_2, \tau_3, ...)$ is a sequence of IID random variables with tail index 1. The following lemma 1 gives a central limit theorem for this sequence of durations.

LEMMA 1. *The sequence of durations $(\tau_1, \tau_2, \tau_3, ...)$ satisfies*

$$\frac{\tau_1 + \tau_2 + ... + \tau_k}{k \log k} \overset{k \to \infty}{\Rightarrow} \frac{D(f)}{\pi \lambda}.$$

Proof. The sequence $(\tau_1, \tau_2, \tau_3, ...)$ is a sequence of i.i.d, regularly varying random variables, with tail index equal to 1. Let $\mathcal{L}(s)$ be the Laplace transform of the distribution of $\tau_2$:

$$\mathcal{L}(s) = 1 - s \int_0^{\infty} \exp(-st)\mathbb{P}[\tau > t]dt.$$

Since $$\frac{1}{\log(n)} \int_0^n \exp(-st/n\log(n))\mathbb{P}[\tau > t]dt \to \frac{D(f)}{\pi \lambda}$$

using an integration by parts we obtain

$$\frac{1}{\log(n)} \int_n^{\infty} \exp(-st/n\log(n))\mathbb{P}[\tau > t]dt \to 0.$$

Therefore

$$\frac{s}{n\log(n)} \int_0^{\infty} \exp(-\frac{st}{n\log n})\mathbb{P}[\tau > t]dt = \frac{D(f)}{\pi \lambda}\frac{s}{n} + o(\frac{1}{n})$$

which implies

$$(\mathcal{L}(s/p_1^{up}(n)))^n = (1 + \frac{D(f)}{\pi\lambda} + o(\frac{1}{n}))^n \to \exp(-\frac{D(f)}{\pi\lambda}).$$

So one can conclude that

$$\frac{\tau_1 + \tau_2 + ... + \tau_n}{n \log n} \Rightarrow^{\mathcal{D}} \frac{D(f)}{\pi\lambda}.$$

Using the rescaling $t \mapsto t_n = tn \log n$, the process $N_{t_n}$ counting the number of price change in the interval $[0, n \log n]$ satisfies the implicit equation:

$$\frac{t_n}{N_{t_n} \log N_{t_n}} \overset{n\to\infty}{\sim} \frac{D(f)}{\pi\lambda}. \tag{17}$$

Hence if one defines $\rho : (1, \infty) \to (1, \infty)$ the inverse function of $t \mapsto t \log t$:

$$\forall t > 1, \quad \rho(t) \log \rho(t) = t, \tag{18}$$

and knowing that $\rho(t) \sim_{t\to\infty} \frac{t}{\log(t)}$, one can notice that the asymptotic number of price change occurring during $[0, t_n]$ is approximately:

$$N_{t_n} \sim \rho\left(\frac{\pi\lambda t_n}{D(f)}\right) \sim \frac{\pi\lambda t_n}{D(f) \log(\pi\lambda t_n/D(f))} \quad as \quad n \to \infty, \tag{19}$$

$$N_{t_n} \sim \frac{\pi\lambda tn \log n}{D(f) \log(n \log n)} \sim \frac{\pi\lambda t}{D(f)} n \quad as \quad n \to \infty. \tag{20}$$

Equation (20) shows that the number of price move occurring during $[0, tn \log n]$ is an order of $n$ for large $n$, and proportional to the ratio of the order intensity $\lambda$ to the order book depth $D(f)$. Since each price change is $\pm\delta$, this factor $\frac{\pi\lambda}{D(f)}$ also shows up in the expression of price volatility:

THEOREM 1. *If $\lambda = \mu + \theta$,*

$$\left(\frac{s_{tn \log n}}{\sqrt{n}}, t \geq 0\right) \overset{n\to\infty}{\Rightarrow} \left(\delta\sqrt{\frac{\pi\lambda}{D(f)}} W_t, t \geq 0\right)$$

*where $\delta$ is the tick size, $D(f)$ is given by (14) and $W$ is a standard Brownian motion.*

Proof. Let $t_n = tn \log n$. One can decompose the process $(\frac{s_{tn \log n}}{\sqrt{n}}, t \geq 0)$ as

$$\frac{s_{tn \log n}}{\sqrt{n}} = \frac{Z([tn\pi\lambda/D(f)])\delta}{\sqrt{n}} + \left(\frac{Z(N_{t_n})\delta}{\sqrt{n}} - \frac{Z([tn\pi\lambda/D(f)])\delta}{\sqrt{n}}\right) \tag{21}$$

Since $(X_1, X_2, ...)$ is a sequence of IID random variables with mean zero, one can apply the Donsker's invariance principle to the sequence of processes

$$\left(\frac{Z([t\pi\lambda n/D(f)])\delta}{\sqrt{n}}, t \geq 0\right)_{n\geq 1} \Rightarrow \left(\delta\sqrt{\frac{\pi\lambda}{D(f)}} W_t, t \geq 0\right) \quad as \quad n \to \infty, \tag{22}$$

where the convergence is in distribution on the Skorokhod space. As shown in (20),

$$N_{tn \log n} \overset{n\to\infty}{\sim} \frac{nt\pi\lambda}{D(f)},$$

therefore for $t \geq 0$,

$$\left(\frac{Z(N_{t_n})\delta}{\sqrt{n}} - \frac{Z([tn\pi\lambda/D(f)])\delta}{\sqrt{n}}\right) \overset{n\to\infty}{\Rightarrow} 0. \tag{23}$$

Hence the finite dimensional distributions of the sequence of processes

$$\left( \frac{Z(N_{t_n})\delta}{\sqrt{n}} - \frac{Z(t\pi\lambda/D(f))\delta}{\sqrt{n}}, t \geq 0 \right)_{n \geq 1} \tag{24}$$

converge to a point mass at zero. Since this sequence of processes is tight on $(\mathcal{D}, J_1)$, this sequence of processes converges weakly to zero on $(\mathcal{D}, J_1)$ (see Whitt (2002)). So finally

$$\left( \frac{s_{tn \log n}}{\sqrt{n}}, t \geq 0 \right) \overset{n \to \infty}{\Rightarrow} \delta \sqrt{\frac{\pi\lambda}{D(f)}} W. \tag{25}$$

### 4.2. Empirical test using high-frequency data

Theorem 1 relates the 'coarse-grained' volatility of intraday returns at lower frequencies to the high-frequency arrival rates of orders. Denote by $\tau_0 = 1/\lambda$ the typical time scale separating order book events. Typically $\tau_0$ is of the order of milliseconds. In plain terms, Theorem 1 states that, observed over a time scale $\tau_2 >> \tau_0$ (say, 10 minutes), the price has a diffusive behavior with a diffusion coefficient given by

$$\sigma_n = \delta \sqrt{\frac{n\pi\lambda}{D(f)}} \tag{26}$$

where $\delta$ is the tick size, $n$ is an integer verifying $n \ln n \quad \tau_0 = \tau_2$ which represents the average number of orders during an interval $\tau_2$ and $\sqrt{D(f)}$, the geometric mean of the size of the bid queue and the size of the ask queue after a price change, is a measure of market depth.

Formula (26) links properties of the price to the properties of the order flow. the left hand side represents the variance of price changes, whereas the right hand side only involves the tick size and quantities: it yields an estimator for price volatility which may be computed *without observing* the price!

The relation (26) has an intuitive interpretation. It shows that, in two 'balanced' limit order markets with the same tick size and same rate of arrival of orders at the next bid/ask, the market with higher depth of the next-to-best queues will lead to lower price volatility.

More precisely, this formula shows that the microstructure of order flow affects price volatility through the ratio $\lambda/D(f)$ where $\lambda$ is the rate of execution/cancelation of limit orders and $D(f)$, given by (14), is a measure of market depth: in fact, our model predicts a proportionality between the variance of price increments and this ratio. This is an empirically testable prediction.

Figure 4.2 compares, for stocks in the Dow Jones index, the standard deviation of 10-minute price increments with $\sqrt{\lambda/D(f)}$. We observe that, indeed, stocks with a higher value of the ratio $\lambda/D(f)$ have a higher variance, and standard deviation of price increments increases roughly proportionally to $\sqrt{\lambda/D(f)}$.

REMARK 6. When the intensity of *all orders coming in the limit order book* is multiplied by the same factor $x$,
- The intensity of limit orders becomes $\lambda x$
- The intensity of market orders and cancelations becomes $(\mu + \theta)x$
- The depth of the limit order book increases by a factor $x$, so $D(f)$ increases by a factor $x^2$.

Substituting in the above formula, we then see that price volatility is decreased by a factor $\sqrt{\frac{1}{x}}$. Interestingly, Rosu (2009) shows the same dependence in $1/\sqrt{x}$ of price volatility using an equilibrium approach.
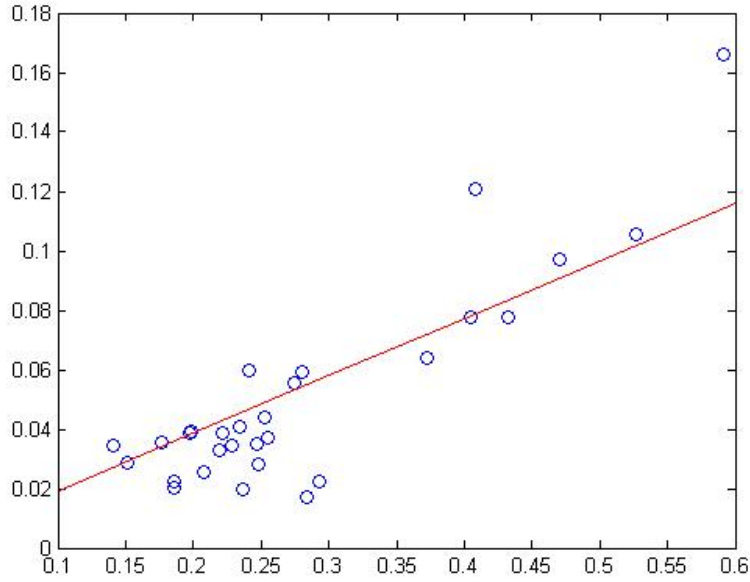
**Figure 6**     $\sqrt{\lambda/D(f)}$, estimated from tick-by-tick order flow (vertical axis) vs standard deviation of 10-minute price increments (horizontal axis) for stocks in the Dow Jones Index, estimated from high frequency data on June 26, 2008. Each point represents one stock. Red line indicates the best linear approximation.

### 4.3. Case when market orders and cancelations dominate

We now consider the case in which the flow of market orders and cancelations dominates that of limit orders: $\lambda < \theta + \mu$. In this case, price changes are more frequent since the order queues are depleted at a faster rate than they are replenished by limit orders. We also obtain a diffusion limit though with a different scaling:

THEOREM 2. *Let $\lambda < \theta + \mu$ and assume that the probability distribution $f$ satisfies*

$$m(\lambda, \theta + \mu, f) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} m(\lambda, \theta + \mu, i, j) f(i, j) < \infty,$$

*where for all $(x, y) \in (\mathbb{N}^*)^2$,*

$$m(\lambda, \theta + \mu, x, y) = \int_0^{\infty} dt \int_t^{\infty} \psi_{x,\lambda,\mu+\theta}(u) du \int_t^{\infty} \psi_{y,\lambda,\mu+\theta}(u) du$$

*where $\psi_{x,\lambda,\mu+\theta}$ is given by* (4). *Then*

$$\left( \frac{s_{nt}}{\sqrt{n}}, t \geq 0 \right) \overset{n \to \infty}{\Rightarrow} \left( \sqrt{\frac{1}{m(\lambda, \theta + \mu, f)}} \delta W_t, t \geq 0 \right)$$

*where $W$ is a standard Brownian motion.*

Proof.    The sequence $(\tau_2, \tau_3, ...)$ is a sequence of i.i.d random variables with finite mean equal to $m(\lambda, \theta + \mu, f)$. We apply the law of large numbers:

$$\frac{\tau_1 + \tau_2 + ... + \tau_n}{n} \to m(\lambda, \theta + \mu, f) \quad as \quad n \to \infty.$$

Therefore,

$$\forall t \geq 0, \qquad N_{tn} \sim [\frac{tn}{m(\lambda, \theta + \mu, f)}] \ \ as \ \ n \to \infty.$$

The rest of the proof follows the lines of the proof of theorem 1. We start to decompose for all $n \geq 1$ the process $(\frac{s_{[nt]}}{\sqrt{n}}, t \geq 0)$ in two terms:

$$\frac{s_{[nt]}}{\sqrt{n}} = \frac{\delta Z_{[nt/m(\lambda, \theta+\mu, f)]}}{\sqrt{n}} + (\frac{\delta Z_{N_{nt}}}{\sqrt{n}} - \frac{\delta Z_{[nt/m(\lambda, \theta+\mu, f)]}}{\sqrt{n}}) \tag{27}$$

By Donsker theorem,

$$\left(\frac{\delta Z_{[nt/m(\lambda, \theta+\mu, f)]}}{\sqrt{n}}, t \geq 0\right) \Rightarrow \left(\frac{1}{\sqrt{m(\lambda, \theta + \mu, f)}} \delta W_t, t \geq 0\right) \ \ as \ \ n \to \infty. \tag{28}$$

The second term of the decomposition converges to zero:

$$(\frac{\delta Z_{N_{nt}}}{\sqrt{n}} - \frac{\delta Z_{[nt/m(\lambda, \theta+\mu, f)]}}{\sqrt{n}}, t \geq 0) \overset{n \to \infty}{\Rightarrow} 0, \tag{29}$$

which concludes the proof.

*Variance of price change at intermediate frequency* Theorem 2 leads to an expression of the variance of the price at a time scale $\tau \gg \tau_0$, where $\tau_0(\sim \text{ms})$ is the average interval between order book events:

$$\sigma^2 = \frac{\tau}{\tau_0} \frac{1}{m(\lambda, \theta + \mu, f)} \delta^2 \tag{30}$$

Here, $m(\lambda, \theta + \mu, f)$ represents the expected hitting time of the axes by the Markovian queuing system $q$ with parameters $(\lambda, \theta + \mu)$ and random initial condition with distribution $f$.

This equation relates the variance of price changes (over a time scale $\tau_2$) to the tick size and the statistical properties of the order flow.

## 4.4. Conclusion

We have exhibited a simple model of a limit order market in which order book events are described in terms of a Markovian queueing system. The analytical tractability of our model allow us to compute various quantities of interest such as

- the distribution of the duration until the next price change,
- the distribution of price changes, and
- the diffusion limit of the price process and its volatility.

in terms of parameters describing the order flow. These results provide analytical insights into the relation between price dynamics and order flow, in particular the relation between liquidity and volatility, in a limit order market.

We view this stylized model as a first step in the analytical study of realistic stochastic models of order book dynamics. Yet, comparison with empirical data shows that even our simple modeling setup is capable of yielding useful analytical insights into the relation between volatility and order flow, worthy of being further pursued. Moreover, the connection with two-dimensional queueing systems allow us to use the rich analytical theory developed for these systems (see Cohen and Boxma (1983)) to compute many other quantities. We hope to pursue further some of these ramifications in future work.

A relevant question is to examine which of the above results are robust to departures from the model assumptions and whether the intuitions conveyed by our model remain valid in a more general context where one or more of these assumptions are dropped. This issue is further studied in Cont and de Larrard (2011) where we explore a more general queueing model relaxing some of the assumptions above.

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