A partial differential equation to express a business cycle :an implication for Japan's law interest policy

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A Partial Differential Equation to Express a Business Cycle: Implications of Japan's Low Interest Rate Policy

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Abstract

This study presents an equation of income derived from the Keynesian IS curve and the consumption Euler equation that explains the business cycle. Drawing on multi-period data from Japan, the model confirms the conventional wisdom that the appropriate policy response to an inflationary gap is to increase the interest rate when economic growth accelerates and decrease it when growth decelerates. However, the model indicates that to stabilize a deflationary gap, policymakers should decrease the interest rate when growth accelerates and increase it when growth decelerates. This prescription defies generations of conventional wisdom but fits the historical data remarkably well.

(JEL Nos. E32, E52)

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1. Consumption Euler equation
2. Business Cycle
3. Capital
4. Interest rate
5. Elasticity of intertemporal substitution
6. Dynamic optimization problem
7. Variable separation method
8. Income elasticity of capital
9. Japan
10. Financial policy

Introduction

Thirty years ago, businesspeople and policymakers worldwide might have been reading a book to answer the question *How can we become like Japan?* Presently, they are more likely to ask, “How can we avoid becoming like Japan?” This study answers the latter question and argues that Japan’s adoption of a low interest rate policy following the burst of the bubble economy has been the major reason for its inability to escape deflation.

Policymakers’ doctrinaire response to a deflationary gap is to reduce interest rates in hopes of stimulating investment and escaping stagnation. Arguably, this response is based on a fundamental misconception. During a period of deflation, an economy inherently must suffer from productive overcapacity. If so, should you not discourage investment? When investment brings a higher interest rate, I presume that you achieve a higher growth rate.

What is needed is a reconsideration of previous equations of national income and theories of the business cycle. This paper proposes the following equation and discusses its implications:

$$\frac{\frac{d^2Y}{dt^2}}{\frac{dr}{dt}} = \alpha \quad \text{(constant) } Y \text{ (income), } r \text{ (interest rate), } t \text{ (time)} \qquad (1\text{-}1)$$

Section 2 explains how this equation is derived. Section 3 organizes assumptions of this theory. Section 4 solves the equation, and Section 5 calculates the value of α. Section 6 provides statistical proof, and Section 7 considers implications of the theory for financial policy. Section 8 presents the conclusion.

2. Derivation of the Equation

This section presents two ways of deriving Eq. (1-1): using the Old Keynesian IS curve and the consumption Euler equation familiar to New Keynesians. We begin with the Old Keynesian IS curve.

First, assume that the macro-economy model of Japan resembles the following, which includes credible figures for ease of illustration:

(trillion yen, interest rate is expressed in percent):

Consumption function $C = 30 + 0.8Y \ldots$ (2-1)

Investment function $I = 75 - 3r \ldots$ (2-2)

Income balance equation $Y = C + I \ \ldots$ (2-3)

We derive the IS curve from the following assumptions:

Substituting (2-1) and (2-2) into (2-3):

$Y = 30 + 0.8Y + 75 - 3r$

$0.2Y = 105 - 3r$

$Y = 525 - 15r \ldots$ (2-4)

Eq. (2-4) is the Old Keynesian IS curve. The IS curve can be expressed as a deferential equation as below. Differentiating both sides of (2-4) with respect to *r*,

$$\frac{dY}{dr} = -15$$

when $r = 0, Y_0 = 525$.

It is more generally expressed as the following equation:

$$\frac{dY}{dr} = \chi \text{ (constant)} \ldots \tag{2-5}$$

This means that IS curves can be rewritten universally as equations of the type expressed in (2-5).

We differentiate (2-5) with respect to *t* and express it as α (constant) and find $\alpha = 0$ in this case. In other words:

$$\frac{d^2Y}{drdt} = \chi' = \alpha \text{ (constant)} \ldots \tag{2-6}$$

Solving (2-6) as a partial differential equation and calculating the value of α yields $\alpha = 0$.

Rewriting the equation for ease of solution:

$$\frac{\frac{d^2Y}{dt^2}}{\frac{dr}{dt}} = \alpha \text{ (constant)} \qquad \ldots \tag{2-7}$$

Next, we consider the derivation of Eq. (1-1) based on micro-economic theory consistent with [i] New Keynesian ideas.

Household utility is defined below. First,

$$U(c_t, m_t) \equiv \frac{c_t^{1-\theta}}{1-\theta} + \frac{m_t^{1-\gamma}}{1-\gamma}, \tag{2-8}$$

where U is household utility, c_t is consumption in *t*, and m_t is the quantity of money held by households. Then utility is $\frac{m_t^{1-\gamma}}{1-\gamma}$.

There are two parameters, θ and γ. θ expresses relative risk aversion[ii] and is defined in (2-9)

$$\theta \equiv -\frac{U_{cc}C_t}{U_c} > 0 \ldots \tag{2-9}$$

U_c and U_{cc} are the first- and second-order derivatives, respectively, of U with respect to C.

The reciprocal is the elasticity of intertemporal substitution:

$$\frac{1}{\theta} \equiv -\frac{U_c}{U_{cc}\mathrm{C_t}} > 0 \ldots \tag{2-10}$$

This definition plays an important role later.

The limiting condition is

$$(1+\pi_t)(m_t + \mathrm{B}_t + C_t) = (1+i_{t-1})\mathrm{B}_{t-1} + m_{t-1}$$

$$\mathrm{B}_{-1}, m_{-1}$$

given that π_t is the inflation rate in *t*, B_t is a bond with a one-year maturity at *t*, and *i* is the interest rate.

Then,

$a_t \equiv \mathrm{B}_t + m_t$,

where a_t is the real net asset in *t*

The limiting conditions become

$$(a_t + C_t) = \frac{1}{1+\pi_t}((1+i_{t-1})a_{t-1} - (1+i_{t-1})m_{t-1} + m_{t-1})$$

and

$$(a_t + C_t) = \frac{1}{1+\pi_t}((1+i_{t-1})a_{t-1} - i_{t-1}m_{t-1}) .$$

This is a dynamic optimization problem.

$$\max : \mathrm{E}_0 \sum_{\mathrm{t}=0}^{\infty} \beta^{\mathrm{t}} U(C_t, m_t) = E_0 \sum_{t=o}^{\infty} \beta^{\mathrm{t}} \left(\frac{C_t^{1-\theta}}{1-\theta} + \frac{m_t^{1-\gamma}}{1-\gamma} \right)$$

Then the Lagrangian is

$$\Gamma_o = E_0 \sum_{t=o}^{\infty} \beta^{\mathrm{t}} \left(\frac{C_t^{1-\theta}}{1-\theta} + \frac{m_t^{1-\gamma}}{1-\gamma} - \phi_t \big((a_t + C_t) - \frac{1}{1+\pi_t} \big((1+i)a_{t-1} - i_{t-1}m_{t-1}) \big) \right).$$

ϕ_t is a Lagrange coefficient.

The first-order optimization conditions are

$$\mathrm{C}_t^{-\theta} = \phi_t$$

$$\mathrm{m}_t^{-\gamma} = E_t \frac{\beta i_t}{1+\pi_{t+1}} \phi_{t+1}$$

$$\phi_t = E_t \frac{\beta(1+i_t)}{1+\pi_{t+1}} \phi_{t+1}$$

Thus, we obtain the following equation:

$$C_t^{-\theta} = E_t \frac{\beta(1+i_t)}{1+\pi_{t+1}} C_{t+1}^{-\theta} \cdots \qquad \text{(2-11)}$$

This is the consumption Euler equation, which shows the relationship between the current and the following years' consumption.

Let us assume perfect foresight and that the subjective discount factor is 1. Then

$$C_t^{-\theta} = \frac{(1+i_t)}{1+\pi_{t+1}} C_{t+!}^{-\theta} .$$

Taking log of both sides, we get

$$inC_t - inC_{t+1} = -\frac{1}{\theta}(\mathrm{in}(1+\mathrm{i}_t) - in(1+\pi_{t+1}))\,.$$

So we assume the following relationships:

$i_t \cong \mathrm{in}(1+\mathrm{i}_t)$

$\pi_{t+1} \cong \mathrm{in}(1+\pi_{t+1})$

Then

$$inC_t - inC_{t+1} = -\frac{1}{\theta}(i_t - \pi_{t+1})$$

$$\frac{dC_t/dt}{C_t} = \frac{1}{\theta}(i_t - \pi_{t+1}) \ldots \qquad (2\text{-}12)$$

New Keynesians construct their models on the bold assumption that Y = C. Because their models do not include capital, their assumption does not fit my model. Therefore, I assume propensity to consume is constant (*c*).

$$\frac{C_t}{Y_t} = c$$

If c is constant, by differentiating c with respect to t , we find that the value of c' is 0.

$$c' = \frac{Y_t\, dC_t/dt - C_t\, dY_t/dt}{Y_t^{\,2}} = 0\,.$$

Then

$$\frac{dY_t/dt}{Y_t} = \frac{dC_t/dt}{C_t} \ldots \qquad (2\text{-}13)$$

Substitute (2-13) into (2-12)

$$\frac{dY_t/dt}{Y_t} = \frac{1}{\theta}(i_t - \pi_{t+1}) \ldots \qquad (2\text{-}14)$$

Then we substitute (2-10) into θ:

$$\frac{dY_t/dt}{Y_t} = -\frac{U_c}{U_{cc}C_t}(i_t - \pi_{t+1})$$

$$\frac{dY_t}{dt} = -\frac{U_c Y_t}{U_{cc} C_t}(i_t - \pi_{t+1}) = -\frac{U_c}{U_{cc} c}(i_t - \pi_{t+1})$$

We define the real interest rate as $r_t = i_t - \pi_{t+1}$. Then

$$\frac{dY_t}{dt} = -\frac{U_c}{U_{cc} c} r_t \ldots \qquad (2\text{-}15)$$

Differentiate both sides of (2-15) with respect to *r*:

$$\frac{d^2 Y_t}{dt dr_t} = -\frac{U_c}{U_{cc} c} = \alpha \ldots \qquad (2\text{-}16)$$

We assume α to be constant. Then we rewrite (2-16) as

$$\frac{\frac{d^2 Y}{dt^2}}{\frac{dr}{dt}} = \alpha \quad \ldots \qquad (2\text{-}17)$$

These are the two derivations of the equation shown.

3. Method of Analysis

Let us confirm the assumptions before solving the equation. The most important assumption is that the interest rate equals the marginal product of capital. Other assumptions that support this fact include perfect competition and the profit maximization principle.

We also assume Y is differentiable twice and that *r* is differentiable once with respect to *t* and that capital elasticity of real income is constant. The final assumption is that the income function is the product of the function of *t* and the function of *k*. The equation has only two variables, *t* (time) and *k* (capital stock). Although it may be said that a two-variable model is too simple to explain business cycles, I suggest that the process by which the equation is derived confirms its explanatory power.

I believe it is unique to explain business cycles by only one parameter, but the explanation is robust because the parameters that explain economic phenomena influence each other, and these parameters can be aggregated into only one parameter (see statistical proof in Section 6).

4. Solving the Equation

When we solve this partial differential equation using a variable separation method, we automatically hypothesize the following function:

Y = Y (t, k) = P (t) Q (k)… (4-1)

Therefore, Y is a function of t (time) and k (capital), which is a product of P (t) (price) and Q (k) (quantity).

First, let us differentiate Y with respect to t once,

$$\frac{dY}{dt} = P'(t)Q(k) + P(t)Q'(k).$$

Then we differentiate Y a second time.

$$\frac{d^2Y}{dt^2} = P''(t)Q(k) + P'(t)Q'(k) + P'(t)Q'(k) + P(t)Q''(k)$$

$$= P''(t)Q(k) + 2P'(t)Q'(k) + P(t)Q''(k)\ldots \qquad (4\text{-}2)$$

$$= \frac{\partial^2 P}{\partial t^2}Q(k) + 2\frac{\partial P}{\partial t}\frac{\partial Q}{\partial k}\frac{dk}{dt} + P(t)\frac{\partial^2 Q}{\partial k^2}\left(\frac{dk}{dt}\right)^2 \ldots \qquad (4\text{-}3)$$

The interest rate is equal to the marginal product of capital ($r = \frac{d\ Y}{dk} = P(t)\frac{\partial Q}{\partial k}$).

Therefore,

$$\frac{dr}{dk} = d\,\frac{dY}{dkdk} = \frac{d^2Y}{dk^2} = P(t)\frac{\partial^2 Q}{\partial k^2}\ldots \qquad (4\text{-}4)$$

When we differentiate $r = \frac{d\ Y}{dk} = P(t)\frac{\partial Q}{\partial k}$ with respect to t, we get the denominator of (1-1) as follows:

$$\frac{d\ r}{dt} = \frac{\partial P}{\partial t}\frac{\partial Q}{\partial k} + P(t)\frac{\partial^2 Q}{\partial k^2}\frac{dk}{dt}\ldots. \qquad (4\text{-}5)$$

From (4-3) and (4-5), (1-1) becomes

$$\frac{\partial^2 P}{\partial t^2}Q(k) + 2\frac{\partial P}{\partial t}\frac{\partial Q}{\partial k}\frac{dk}{dt} + P(t)\frac{\partial^2 Q}{\partial k^2}\left(\frac{dk}{dt}\right)^2 = \alpha\left(\frac{\partial P}{\partial t}\frac{\partial Q}{\partial k} + P(t)\frac{\partial^2 Q}{\partial k^2}\frac{dk}{dt}\right)$$

$$\frac{\partial^2 P}{\partial t^2}Q(k) + 2\frac{\partial P}{\partial t}\frac{\partial Q}{\partial k}\frac{dk}{dt} - \alpha\frac{\partial P}{\partial t}\frac{\partial Q}{\partial k} = \alpha P(t)\frac{\partial^2 Q}{\partial k^2}\frac{dk}{dt} - P(t)\frac{\partial^2 Q}{\partial k^2}\left(\frac{dk}{dt}\right)^2$$

$$\frac{\partial^2 P}{\partial t^2}Q(k) + \frac{\partial P}{\partial t}\frac{\partial Q}{\partial k}\left(2\frac{dk}{dt} - \alpha\right) = P(t)\frac{\partial^2 Q}{\partial k^2}\frac{dk}{dt}\left(\alpha - \frac{dk}{dt}\right).$$

If $P(t) = P_o e^{\lambda t}$, so $\frac{\partial^2 P}{\partial t^2} = \lambda\ \frac{\partial P}{\partial t}$.

Furthermore, we substitute $\frac{dk}{dt}$ for b .

Then

$$\frac{\partial P}{\partial t}\left(\lambda Q(k)+\frac{\partial Q}{\partial k}(2b-\alpha)\right)=P(t)\frac{\partial^2 Q}{\partial k^2}b(\alpha-b)$$

$$\frac{\frac{\partial P}{\partial t}}{P(t)}=\frac{\frac{\partial^2 Q}{\partial k^2}b(\alpha-b)}{\frac{\partial Q}{\partial k}(2b-\alpha)+\lambda Q(k)}=\lambda \text{ (Constant)}\ldots \qquad \text{(4-6)}$$

Therefore,

$$\frac{\partial\ P}{\partial t}=\lambda P(t)\ \ldots. \qquad \text{(4-7)}$$

$$\frac{\partial^2 Q}{\partial k^2}b(\alpha-b)=\lambda\frac{\partial Q}{\partial k}(2b-\alpha)+\lambda^2 Q(k)\ \ldots \qquad \text{(4-8)}$$

From (4-7) we derive

$$P(t)=P_o e^{\lambda t}\ \ldots \qquad \text{(4-9)}$$

From (4-8) we derive

$$b(\alpha-b)\frac{\partial^2 Q}{\partial k^2}-\lambda(2b-\alpha)\frac{\partial Q}{\partial k}-\lambda^2 Q(k)=0\,.$$

Therefore,

$$b(\alpha-b)Q''-\lambda(2b-\alpha)Q'-\lambda^2 Q=0\,.$$

This is a second-order linear differential equation. The characteristic equation is therefore

$$b(\alpha-b)\zeta^2-\lambda(2b-\alpha)\zeta-\lambda^2=0\,.$$

The basic solution is determined by

$$\zeta_{+-}=\frac{\lambda(2b-\alpha)\pm\sqrt{\lambda^2(2b-\alpha)^2+4\lambda^2 b(\alpha-b)}}{2b(\alpha-b)}=\frac{\lambda(2b-\alpha)\pm\lambda\alpha}{2b(\alpha-b)}\ldots \qquad \text{(4-10)}$$

$$\zeta_{+}=\frac{\lambda}{\alpha-b}\ldots \qquad \text{(4-11)}$$

$$\zeta_{-}=-\frac{\lambda}{b}\ldots \qquad \text{(4-12)}$$

The content of the radical sign is

$$\lambda^2(2b-\alpha)^2+4\lambda^2 b(\alpha-b)=\lambda^2(4b^2-4\alpha b+\alpha^2+4\alpha b-4b^2)\ =\alpha^2\lambda^2$$

The general solution follows.

If D = $\alpha\lambda > 0$

$$Q(k) = C_1 \exp(\frac{\lambda}{\alpha - b} k) + C_2 \exp(-\frac{\lambda}{b} k) \ldots \tag{4-13}$$

If D = $\alpha\lambda = 0$

$$Q(k) = (C_1 + C_2 k) \exp(\frac{\lambda(2b-\alpha)}{2b(\alpha - b)} k) \ldots \tag{4-14}$$

If D = $\alpha\lambda < 0$,

$$Q(k) = \exp(\frac{\lambda(2b-\alpha)}{2b(\alpha - b)} k)(C_1 \cos(\frac{\alpha\lambda}{2b(\alpha - b)} k) + C_2 \sin(\frac{\alpha\lambda}{2b(\alpha - b)} k) \ldots \tag{4-15}$$

From (4-1), (4-9), (4-13), (4-14), and (4-15) we find the following:

When D = $\alpha\lambda > 0$

$$Y(t,k) = C_1 \exp(\lambda t + \frac{\lambda}{\alpha - b} k) \ldots \tag{4-16}$$

or

$$Y(t,k) = C_1 \exp(\lambda t + \frac{-\lambda}{b} k) \ldots \tag{4-17}$$

When D = $\alpha\lambda = 0$, it means that either $\alpha = 0$ or $\lambda = 0$.

Then if $\alpha = 0$,

$$Y(t,k) = (C_1 + C_2 k) \exp(\frac{\lambda(2b-\alpha)}{2b(\alpha - b)} k) \ldots \tag{4-18}$$

$$Y(t,k) = (C_1 + C_2 k) \exp(\lambda t + \frac{-\lambda}{b} k) \ldots \tag{4-19}$$

If $\lambda = 0$, from (4-9) and (4-19), we find

Y (t,k)= $P_1(C_2 + C_3 k)$.

Thus, *Y* is a primary function of *k*.

If D = $\alpha\lambda < 0$,

$$Y(t,k) = P_1 \exp(\lambda t + \frac{\lambda(2b-\alpha)}{2b(\alpha-b)}k)(C_2 \cos(\frac{\alpha\lambda}{2b(\alpha-b)}k) + C_3 \sin(\frac{\alpha\lambda}{2b(\alpha-b)}k) \ldots \qquad (4\text{-}20)$$

5. Value of α

In this section, we consider the value of α.

From (4-1),

$$Y(t,\mathrm{k}) = \mathrm{P(t)Q(k)} \ldots \qquad (4\text{-}1)$$

So from (4-9) and (4-10), we can divide (4-1) into two functions, (5-1) and (5-2), as below.

$$P(t) = P_o e^{\lambda t} \ldots \qquad (5\text{-}1)$$

$$Q(k) = B\exp(\frac{\lambda(2b-\alpha) \pm \alpha\lambda}{2b(\alpha-b)}k) = Be^{\mu k} \ \ldots \qquad (5\text{-}2)$$

$$: \ \mu = \frac{\lambda(2b-\alpha) \pm \alpha\lambda}{2b(\alpha-b)}$$

α is the quotient of (4-3) divided by (4-4). Therefore, α is

$$\alpha = \frac{\frac{d^2Y}{dt^2}}{\frac{dr}{dt}} = \frac{P_0 e^{\lambda t}}{P_0 e^{\lambda t}} \frac{Be^{\mu k}}{Be^{\mu k}} \frac{(\lambda^2 + 2\lambda\mu \frac{dk}{dt} + \mu^2 (\frac{dk}{dt})^2)}{(\lambda\mu + \mu^2 \frac{dk}{dt})}$$

$$\alpha = \frac{\frac{d^2Y}{dt^2}}{\frac{dr}{dt}} = \frac{(\lambda + \mu (\frac{dk}{dt}))^2}{\mu(\lambda + \mu \frac{dk}{dt})} = \frac{(\lambda + \mu \frac{dk}{dt})}{\mu} \ldots \qquad (5\text{-}3)$$

If

$$\mu = \frac{-\lambda}{b} \ldots \quad (5\text{-}4),$$

then (5-3) is represented below.

$$\alpha = \frac{\frac{d^2Y}{dt^2}}{\frac{dr}{dt}} = \frac{(\lambda + \mu \frac{dk}{dt})}{\mu} = \frac{(\lambda + \frac{-\lambda}{b} b)}{\frac{-\lambda}{b}} = \frac{(\lambda - \lambda)}{\frac{-\lambda}{b}} = 0 \ \cdot \ \cdot \ \cdot \qquad (5\text{-}4)$$

In Section 6, the statistical analysis shows that Japan has an equation of income such as in (4-17).

$$Y(t,k) = C_1 \exp(\lambda t + \frac{-\lambda}{b} k) \ldots \quad (4\text{-}17)$$

Eq. (5-4) indicates that $\alpha = 0$. This implies that if α does not exist, the value of its mean is 0, and that of its variance is huge. I calculate the actual value in Section 6. Indeed, the variance is large, but its mean does not equal 0. Thus, I conclude that α exists. This means that when the correct investment is implemented, the acceleration of income and the fluctuation of the interest rate will cease, and the economy will have reached a new equilibrium.

Therefore, in the case where D = $\alpha\lambda = 0$, Eq. (4-19) shows equilibrium income at that time on the condition that $\alpha = 0$.

$$Y^*(t,k) = (C_1 + C_2 k)\exp(\lambda t + \frac{-\lambda}{b} k) \ldots \quad (4\text{-}19)$$

6. Statistical Proof Using Data

In this section, we verify that the equation solved in Section 4 can be reliably applied to actual cases. First, we consider (4-17).

$$Y(t,k) = C_1 \exp(\lambda t + \frac{-\lambda}{b} k) \ldots \quad (4\text{-}17)$$

Putting both sides into a natural logarithm,

$$inY(t,k) = B_0 + \lambda t + \frac{-\lambda}{b} k \ldots (6\text{-}1) \; (B_0 = inC_1).$$

Equation (6-1) shows that the logarithm value of the nominal GDP can be expressed as a linear regression model of time (*t*) and capital (*k*). Therefore, we fix 1980 as a standard and conduct a regression analysis on the nominal GDP shown in Reference List 1 by *t* (1980 as 1) and k. The result is as follows.

Coefficient of determination: 0.89216685

Adjusted coefficient of determination by degree of freedom: 0.883872

Table I　Results of Analysis①

	Coefficient	Standard Error	t	P-value
B_0	4.90583638	0.1397002	35.1169	1.9E-23
B_1	−0.0499785	0.0133574	−3.74164	0.00091
B_2	0.00234751	0.0004156	5.648127	6.1E-06

Thus, a logarithm value of nominal income can be expressed as Eq. (6-2).

$$inY(t,k) = 4.90583638 - 0.0499785t + 0.00234751k \ldots \qquad (6\text{-}2)$$

The coefficient of determination is 0.89, and the adjusted coefficient is 0.88, which are sufficiently high. The negative value of λ, however, is unexpected. Therefore, I must question whether the following formula holds true:

$$B_2 = \frac{-\lambda}{b} \ldots \qquad (6\text{-}3)$$

The results of calculating $\frac{-\lambda}{b}$ are found in Reference List 2.

The value of the mean is 0.002172, which is very close to the value of B_2. The value of the variance is 1.91532×10^{-6}. Before we examine whether both means are equal, we must examine whether both variances are equal. We calculate the standard deviation from the standard error. Standard deviation = Standard error × square root of the number of data. So, σ = 0.002238, and the variance $\sigma^2 = 5.00961 \times 10^{-6}$. Table II below shows the variance with $\frac{-\lambda}{b}$.

Table II Variance with $\frac{-\lambda}{b}$

	Mean	Variance	Data
B_2	0.002347511	5.00961E-06	29
$-\lambda/b$	0.002172425	1.91352E-06	29

We must seek the z value to examine whether both variances are equal. The z value is giveb below.

$$z = \frac{(n-1)s^2}{\sigma^2}$$

in which n is a number of data, s^2 is variance of $\frac{-\lambda}{b}$, and σ^2 is the variance of B_2.

That Z value has a chi-square distribution with n-1 degrees of freedom.

The answer is as follows:

z = 10.6951.

Rejection region of 0.005 left side: $z \le 12.46134$.

Rejection region of 0.005 right side: $z \ge 50.99338$.

Thus, we can reject the hypothesis that the values of variances are equivalent with a 1% hazard ratio.

To test the mean value between sets in which each has different variances, we use the fact that t has a Student's t-distribution with ν degrees of freedom.

The value of t is given below.

$$t = \frac{\frac{-\bar{\lambda}}{b} - B_2}{\sqrt{(\frac{\vartheta_1^2}{n_1} + \frac{\vartheta_2^2}{n_2}}}$$

Here, $\frac{-\bar{\lambda}}{b}$ is the mean of $\frac{-\lambda}{b}$, ϑ_1^2 and ϑ_2^2 are variances, and n_1 and n_2 are numbers of the data of $\frac{-\lambda}{b}$ and B_2, respectively.

The value of ν is given below.

$$\nu = \frac{(\frac{\vartheta_1^2}{n_1} + \frac{\vartheta_2^2}{n_2})^2}{\frac{(\frac{\vartheta_1^2}{n_1})^2}{n_1 - 1} + \frac{(\frac{\vartheta_2^2}{n_2})^2}{n_2 - 1}}$$

The answers are below.

T = 0.358342321

ν = 47

Rejection region of 1% on both sides: $|t| \leq 2.945630052$.

Thus, we cannot reject the hypothesis that both means are equivalent.

Next, we calculate the actual value of α. Reference List 3 shows the process of the calculation. The results follow below.

Table III

Calculation from Reference List 3

Year	α
1980	
1981	
1982	−253.979
1983	−306.041
1984	−829.751
1985	−1731.12
1986	−1273.8
1987	247.5311
1988	−3185.95
1989	−214.671
1990	284.1922
1991	1069.932
1992	−8213.74
1993	1024.186
1994	1033.595
1995	−1493.78
1996	−497.108
1997	−34.6569

1998	−1863.9
1999	441.8159
2000	4339.517
2001	2424.553
2002	−325.836
2003	−560.585
2004	−5948.15
2005	−2381.07
2006	2456.599
2007	723.4678
2008	1270.679

Basic statistical values	
	α
MEAN	−511.039
ST ERROR	477.3842
MEDIAN	−253.979
MODE	#N/A
ST DEVIATION	2480.561
VARIANCE	6153182
KURTOSIS	3.277517
SKEWNESS	−1.25955

RANGE	12553.25
MIN	−8213.74
MAX	4339.517
TOTAL	−13798.1
DATA	27
C-INTER (95.0%)	981.2772

The mean of α is −511 (trillion yen)^2 , which is very close to the actual GDP. Hence, α must be negative to maintain stable growth because λ is negative.
The standard deviation (ST DEVIATION) is 2480.561 (trillion yen),which is quite large.
Its value divided by 10 trillion is very close to $\frac{-\lambda}{b}$.

The skewness is negative. The mass of the distribution is concentrated on the right side of the figure. We find a consecutive series of negative figures from 1982 to 1989 and from 2002 to 2005. We conduct a regression analysis by selecting only the years having negative α. Then we examine whether the capability of this model becomes stronger. The following are the results:

The coefficient of determination increases to 0.93 from 0.89.

The one adjusted by degree of freedom grows to 0.92 from 0.88. We can confirm that this model is best applied to a condition with negative α.

Coefficient of determination: 0.934773.

Coefficient of determination adjusted by degree of freedom: 0.924738.

Table IV　Results of analysis ②

	Coefficient	Standard deviation	t	P-Value
B_0	4.909718	0.1489163	32.96964	6.46E-14
B_1	−0.05517	0.0158882	−3.47251	0.004126
B_2	0.002447	0.0004736	5.166249	0.000181

Next, we verify that (4-19) can be reliably applied to an actual case. Eq. (4-19) calculates real income based on the assumption that real income is the equilibrium income on the condition that α = 0. I confirmed that this income is the real GDP, $Y^*(t,k)$.

$$Y^*(t,k) = (C_1 + C_2 k)\exp(\lambda t + \frac{-\lambda}{b}k)\ldots \qquad (4\text{-}19)$$

Then $C_1 + C_2 k = (\frac{C_1 + C_2 k}{C_2 k}) C_2 k$.

If we define $C_3 = \frac{C_1 + C_2 k}{C_2 k}$,

$C_1 + C_2 k = C_3 C_2 k$.

So we rewrite λ as λ_1 in (4-19)

Thus, (4-19) becomes (6-4).

$$Y^*(t,k) = C_3 (C_2 k)^2 \exp(\lambda_1 t + \frac{-\lambda_1}{b}k)\ldots \qquad (6\text{-}4)$$

We assume real income is expressed as a well-known function, as in (6-5) below, whose capital elasticity of income is *a*.

$$Y^*(k) = Ak^a \ldots \qquad (6\text{-}5)\ (A\text{: constant})$$

Hence, we can expect that C_2^2 is the following function:

$$C_2{}^2 = C_4(\exp\eta)k^{a-2-\frac{\eta}{ink}}\ldots \qquad (6\text{-}6)$$

$$\eta = \frac{-\lambda_1}{b}k$$

We insert that function into (6-4).

$$Y^*(t,k) = C_3 C_4 k^{a-\frac{\eta}{ink}}(\exp\eta)(\exp\lambda_1 t + \frac{-\lambda_1}{b}k)$$

$$Y^*(t,k) = C_3 C_4 k^{a-\frac{\eta}{ink}}\exp(\lambda_1 t + \frac{-2\lambda_1}{b}k)\ldots \qquad (6\text{-}5)$$

Putting both sides into the logarithm, we get:

$$inY^*(t,k) = inC_3 + inC_4 ink^{a-\frac{\eta}{ink}} + \lambda_1 t + \frac{-2\lambda_1}{b}k\ \ldots \qquad (6\text{-}6)$$

We differentiate both sides with respect to *t*.

$$\frac{dY^*/dt}{Y^*} = a\frac{dk/dt}{k} - \eta' + \lambda_1 + \frac{-2\lambda_1}{b}b$$

$$\frac{dY^*/dt}{Y^*} = a\frac{dk/dt}{k} - \frac{-\lambda_1}{b}b + \lambda_1 + \frac{-2\lambda_1}{b}b$$

$$\frac{dY^*/dt}{Y^*} = a\frac{dk/dt}{k}$$

So we confirm that *a* is the capital elasticity of income.

Additionally, (6-6) shows that inY^* can be brought into the regression analysis with respect to ink , *t*, and *k*. Therefore, we use the data of real GDP from Reference List 4, with the results that

Coefficient of determination: 0.989992444.

Adjusted Coefficient of determination by degrees of freedom: 0.9887415.

Table V Results of Analysis ③

	Coefficient	St Error	t	P-value
B_0	−0.515 14704	0.480274	−1.0726	0.2941
B_1	1.119219197	0.090926	12.3091	7E-12
B_2	0.014294401	0.004325	3.30509	0.003
B_3	−0.00128167	0.000226	−5.6828	7E-06

That is, the logarithm of real income can be expressed as the following function:

$$inY^{*}(t,k) = -0.515154704 + 1.119219197ink + 0.014294401t - 0.00128167k \ldots \text{(6-7)}$$

The coefficient of determination is 0.98, as is the adjusted coefficient. Thus, we can confirm the model works remarkably well.

Reference List 5 shows that $\frac{-2\lambda_1}{b}$ s. The value of the mean is

$$2\frac{-\overline{\lambda_1}}{b} = -0.00117895$$

This is very close to B_3. We test these values as we did earlier.

Table VI Variance with $\frac{-2\lambda_1}{b}$

	MEAN	VARIANCE	DATA
B3	−0.001282	1.4243E-06	28
$-2\lambda_1/b$	-0.001179	5.0555E-07	27

First, we examine whether the variances are equal.

We calculate the Z value as before: $Z = 9.2287136$.

Rejection region of 0.005 left side: $Z \leq 11.167237$.

Therefore, we can reject the hypothesis that the values of variances are equivalent.

Then we calculate the *t* value and ν degrees of freedom as before:

$t = 0.3893929$

$\nu = 44$

Rejection region of 1% both sides: $|t| \leq 2.995534$

Therefore, we cannot reject the hypothesis that both means are equivalent.

7. Implications for Financial Policy

For policymakers, the equation implies that the economy would not ascend on a stable course if the product of α and λ is not positive. To maintain economic stability when λ is positive, α should be positive; however, when λ is negative, α should be negative. To sustain a positive value for α, policymakers must increase the interest rate whenever income accelerates and reduce it whenever income decelerates. Yet, to sustain a negative value for α, they must decrease the interest rate when income accelerates and increase the interest rate when income decelerates. In Japan's case, λ is negative, so α should be negative.

Policy may be able to reduce the interest rate whenever there is acceleration, but it may seem impossible to increase the interest rate during deceleration. However, we may have an available strategy for such a policy. First, we assume that the current real growth rate is the equilibrium growth rate. Then, we decrease the interest rate and increase the real growth rate to be more than the equilibrium growth rate, followed by gradually raising the interest rate. We must continue to increase the interest rate until the real growth rate is equal to a new equilibrium growth rate. By this logic, we must quickly attempt to increase the interest rate when the real growth rate exceeds the current equilibrium growth rate.

If the economy shifts to a higher equilibrium as it continues to grow, the economy accelerates further. If so, when the economy reaches the new equilibrium, we will see the higher interest rate. Then we can decrease that interest rate to re-stimulate the economy. Thus, the interest rate will increase as the equilibrium growth rate rises.

We may increase much supply compared to demand on the condition that λ is negative.

If we stimulate the economy by reducing the interest indiscreetly while maintaining a stable equilibrium growth rate, we will see smaller rates of both interest and growth. Currently, Japan's real growth rate is approximately 1% and its interest rate is approximately 1.5%. Both may have reached their lower limits. Japan must soon seek an economic strategy for raising the equilibrium growth rate.

However, is there a way to increase income when interest rates rise? Generally speaking, we must reduce interest rates when we desire to increase the growth rate. However, Japan has pursued a low interest rate policy since the burst of the Bubble Economy, and the growth rate has not recovered. Why does the policy work not work? By way of an answer, I introduce the idea of production period[iii] from *Value and Capital* by John Hicks, who wrote:

> It follows at once from all this that if the average period of the stream of receipts is greater than the average period of the standard stream with which we are comparing it, a fall in the rate of interest will raise the capital value of the receipts stream more than that of the standard stream, and will therefore increase income. But if the average period of the stream of receipts is less than that of the standard stream, it is a rise in the rate of interest which will increase income.

To explain his statement briefly, let us assume that there exists the stream of inputs $R_t\,(R_0, R_1, R_2, R_3, R_4 \ldots)$ and the corresponding stream of earnings $S_t\,(S_0, S_1, S_2, S_3, S_4 \ldots)$.Then the present values of both discounted by interest rate are equal. So we find

$$R_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \ldots = S_0 + \frac{S_1}{(1+r)} + \frac{S_2}{(1+r)^2} + \ldots$$

We calculate the capital elasticity of the interest rate as

$$-A_s = \frac{\dfrac{S_1}{(1+r)} + 2\dfrac{S_2}{(1+r)^2} + 3\dfrac{S_3}{(1+r)^3} + \cdots}{S_0 + \dfrac{S_1}{(1+r)} + \dfrac{S_2}{(1+r)^2} + \dfrac{S_3}{(1+r)^3} + \cdots}$$

This value is the average period of earnings weighted by the terms.
Then the average period of input is

$$-A_R = \frac{\dfrac{R_1}{(1+r)} + 2\dfrac{R_2}{(1+r)^2} + 3\dfrac{R_3}{(1+r)^3} + \cdots}{R_0 + \dfrac{R_1}{(1+r)} + \dfrac{R_2}{(1+r)^2} + \dfrac{R_3}{(1+r)^3} + \cdots}$$

Then, if $A_s > A_R$, a decline in interest rates causes income to increase, and if $A_s < A_R$, an increase in the interest rate leads to an increase in income.

Now we calculate the average period when inputs or earnings are constant.

If

$R_0 = R_1 = R_3 = R_4 = \ldots$

or

$S_0 = S_1 = S_2 = S_3 = S_4 = \ldots$

So the average period of inputs or earnings resembles the following equation:

$$A = \frac{\frac{1}{(1+r)} + 2\frac{1}{(1+r)^2} + 3\frac{1}{(1+r)^3} + \ldots}{1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots} = \frac{\frac{1}{(1+r)}}{(1-\frac{1}{(1+r)})^2} \Big/ \frac{1}{(1-\frac{1}{(1+r)})} = \frac{\frac{1}{(1+r)}}{1-\frac{1}{(1+r)}} = \frac{1}{r}$$

That is equal to the reciprocal of the interest rate. If the stream has a crescendo, the average period of the stream is longer than the reciprocal of interest rate. If the stream has a decrescendo, its average period is shorter than the reciprocal of the interest rate. I think that the average period of inputs generally extends as the economy progresses, but that of earnings becomes longer; therefore, a fall in the interest rate causes an increase in income.

However, if we assume the stream of inputs is constant, then an average period of inputs is equal to the reciprocal of interest rate, and it is shorter than the average period of earnings. Then we find the equation

$A_s < A_R$

In that case, in general, the interest rate should rise to increase income. We must pay attention not to the absolute level of interest rate, but rather, to the magnitude of the relation between average periods of inputs and earnings.

If the interest rate is 1.5%, the reciprocal is approximately 67 years. So to reduce interest rates further is unproductive if the stream of inputs is constant and the average period of earnings is less than 67 years. As Hicks demonstrated, one cannot expect income to increase by decreasing interest rates indiscreetly.

8. Conclusion

Nominal GDP can be expressed as the equation

If $\alpha\lambda > 0$

$$Y(t,k) = C_1 \exp(\lambda t + \frac{\lambda}{\alpha - b}k) \ldots \qquad (4\text{-}16)$$

or

$$Y(t,k) = C_1 \exp(\lambda t + \frac{-\lambda}{b}k) \ldots \qquad (4\text{-}17)$$

If $\alpha\lambda < 0$

$$Y(t,k) = P_1 \exp(\lambda t + \frac{\lambda(2b-\alpha)}{2b(\alpha - b)}k)(C_2 \cos(\frac{\alpha\lambda}{2b(\alpha - b)}k) + C_3 \sin(\frac{\alpha\lambda}{2b(\alpha - b)}k) \ldots \qquad (4\text{-}20)$$

If (4-16) can be applied, real income can be expressed as (6-8)

$$Y^*(t,k) = C_3 C_4 k^{a - \frac{\eta}{ink}} \exp(\lambda t + \frac{-2\lambda_1}{b}k) \ldots \qquad (6\text{-}8)$$

$$\eta = \frac{-2\lambda_1}{b}k \quad \text{(a: income elasticity of capital)}$$

Statistical research shows Japan's GDP can be expressed by the following equations:

Nominal income Y

$$inY(t,k) = 4.90583638 - 0.0499785t + 0.00234751k \ldots \qquad (6\text{-}2)$$

Real income Y^*

$$inY(t,k) = -0.515154704 + 1.119219197ink + 0.014294401t - 0.00128167k \ldots \qquad (6\text{-}10)$$

From Section 4, we can show that a sign of $\alpha\lambda$ determines which type of equation can be applied:

If $\alpha\lambda > 0$: (4-16) or (4-17)

If $\alpha\lambda < 0$: (4-20)

The solution to this equation may provide a firm underpinning to economic policy by the following logic. Assuming that the trend of time is positive ($\lambda > 0$), policymakers must increase the interest rate whenever economic growth accelerates, and decrease it when growth decelerates. This action can lead the economy to a stable course because it preserves the condition that $\alpha > 0$, as shown by (4-16) or (4-17). In other words, we must increase the interest rate rather than decrease it to keep acceleration positive whenever the real growth rate is lesser than the equilibrium growth rate.

However, if $\lambda < 0$, the recommendation is reversed. We must decrease the interest rate whenever the economy accelerates and increase it in the event of deceleration. Thus, since Japan has negative λ, we may be able to decrease the interest rate whenever there is acceleration, but it may seem impossible to increase the interest rate in the event of deceleration.

Nonetheless, we may have a strategy available for such a policy. First, we assume the current real growth rate is the equilibrium growth rate. Then, we decrease the interest rate and increase the real growth rate making it more than the equilibrium growth rate, followed by gradually increasing the interest rate. We must continue to increase the interest rate until the real growth rate is equal to a new equilibrium growth rate. By this logic, we must quickly attempt to increase the interest rate when the real growth rate becomes higher than the current equilibrium growth rate.

If the economy shifts to a higher equilibrium as it continues to grow, the economy will accelerate further. If so, when the economy reaches the new equilibrium, we will see higher interest rates. Then we can reduce that interest rate to re-stimulate the economy. Thus, the interest rate will increase as the equilibrium growth rate rises.

We may increase much supply compared to demand on the condition that λ is negative.

However, our condition that both λ and α are negative to maintain economic stability can be considered a shrinking economy. If we stimulate the economy by reducing the interest rate indiscreetly while maintaining a stable equilibrium growth rate, we will achieve smaller rates of both interest and growth. Currently, Japan's real growth rate is approximately 1% and its interest rate is approximately 1.2%. Both may have reached their lower limits. Japan's policymakers must soon seek an economic strategy for raising the equilibrium growth rate.

It is difficult to change basic beliefs about financial policy. But if policymakers hope to effect real-world change, they first must change their economic thought. John Maynard Keynes long ago noted the stubbornness of economic thought:

> But besides this contemporary mood, the ideas of economists and political philosophers, both when they are right and when they are wrong, are more powerful than is commonly understood. Indeed the world is ruled by little else. Practical men, who believe themselves to be quite exempt from any intellectual influences, are usually the slaves of some defunct economist.

Japan has imposed a low interest rate policy for many years, and the result has been persistent deflation. If we maintain a low interest rate policy and do not fully consider the alternative presented in this research, we are acting unreasonably. Now is the time to try new theories and new economic thoughts.

9. References

List 1

Year	GDP （Nominal）	Capital Stock （Begin）
1980	242.8387	358.4012
1981	261.0682	382.292
1982	274.0866	405.8706
1983	285.0583	427.7703
1984	302.9749	447.2519
1985	325.4019	471.3132
1986	340.5595	524.3229
1987	354.1702	554.6213
1988	380.7429	595.5875
1989	410.1222	632.2497
1990	442.781	677.281
1991	469.4218	726.7462
1992	480.7828	786.1105
1993	483.7118	826.5722
1994	488.4503	857.0921
1995	495.1655	888.7079
1996	505.0118	918.2468
1997	515.6441	948.9018
1998	504.9054	980.756
1999	497.6286	1005.381

2000	502.9899	1026.533
2001	497.7197	1051.391
2002	491.3122	1068.489
2003	490.294	1082.417
2004	498.3284	1089.336
2005	501.7344	1120.987
2006	507.3648	1135.384
2007	515.5204	1162.551
2008	505.1119	1193.615

¥ Trillion

List 2

YEAR	$-\lambda / b$
1980	0.002092
1981	0.00212
1982	0.002282
1983	0.002565
1984	0.002077
1985	0.000943
1986	0.00165
1987	0.00122
1988	0.001363
1989	0.00111
1990	0.00101
1991	0.000842
1992	0.001235
1993	0.001638
1994	0.001581
1995	0.001692
1996	0.00163
1997	0.001569
1998	0.00203
1999	0.002363

2000	0.00201
2001	0.002923
2002	0.003588
2003	0.007224
2004	0.001579
2005	0.003472
2006	0.00184
2007	0.001609
2008	0.005744

List 3

Year	nGDP	ΔY	Δ^2Y	Prime rate	CPI	REAL	D	α
1980	242.8387			9.163561644	0.077	0.014636		
1981	261.0682	18.2295		8.636164384	0.049	0.037362	0.022726	
1982	274.0866	13.0184	−5.2111	8.587945205	0.028	0.057879	0.020518	−253.979
1983	285.0583	10.9717	−2.0467	8.356712329	0.019	0.064567	0.006688	−306.041
1984	302.9749	17.9166	6.9449	7.919726027	0.023	0.056197	−0.00837	−829.751
1985	325.4019	22.427	4.5104	7.359178082	0.02	0.053592	−0.00261	−1731.12
1986	340.5595	15.1576	−7.2694	6.529863014	0.006	0.059299	0.005707	−1273.8
1987	354.1702	13.6107	−1.5469	5.404931507	0.001	0.053049	−0.00625	247.5311
1988	380.7429	26.5727	12.962	5.598082192	0.007	0.048981	−0.00407	−3185.95
1989	410.1222	29.3793	2.8066	5.890684932	0.023	0.035907	−0.01307	−214.671
1990	442.781	32.6588	3.2795	7.844657534	0.031	0.047447	0.01154	284.1922
1991	469.4218	26.6408	−6.018	7.482191781	0.033	0.041822	−0.00562	1069.932
1992	480.7828	11.361	−15.2798	5.968219178	0.016	0.043682	0.00186	−8213.74
1993	483.7118	2.929	−8.432	4.844931507	0.013	0.035449	−0.00823	1024.186
1994	488.4503	4.7385	1.8095	4.42	0.007	0.0372	0.001751	1033.595
1995	495.1655	6.7152	1.9767	3.487671233	−0	0.035877	−0.00132	−1493.78
1996	505.0118	9.8463	3.1311	3.057808219	0.001	0.029578	−0.0063	−497.108
1997	515.6441	10.6323	0.786	2.489863014	0.018	0.006899	−0.02268	−34.6569
1998	504.9054	−10.7387	−21.371	2.436438356	0.006	0.018364	0.011466	−1863.9
1999	497.6286	−7.2768	3.4619	2.32	−0	0.0262	0.007836	441.8159

2000	502.9899	5.3613	12.6381	2.211232877	−0.01	0.029112	0.002912	4339.517
2001	497.7197	−5.2702	−10.6315	1.772739726	−0.01	0.024727	−0.00438	2424.553
2002	491.3122	−6.4075	−1.1373	1.921780822	−0.01	0.028218	0.00349	−325.836
2003	490.294	−1.0182	5.3893	1.560410959	−0	0.018604	−0.00961	−560.585
2004	498.3284	8.0344	9.0526	1.708219178	0	0.017082	−0.00152	−5948.15
2005	501.7344	3.406	−4.6284	1.60260274	−0	0.019026	0.001944	−2381.07
2006	507.3648	5.6304	2.2244	2.293150685	0.003	0.019932	0.000905	2456.599
2007	515.5204	8.1556	2.5252	2.342191781	0	0.023422	0.00349	723.4678
2008	505.1119	−10.4085	−18.5641	2.281232877	0.014	0.008812	−0.01461	1270.679

＊1 Prime rates weighted averages by term

＊2 CPI stands for consumer price index

＊3 Real is the real interest rate, that is, the difference between CPI and the prime rate

＊4 D is the difference in real interest rates

List 4

Real GDP

YEAR	GDP ∗
1980	284.375
1981	296.2529
1982	306.2562
1983	315.6299
1984	329.7193
1985	350.6016
1986	360.5274
1987	375.3358
1988	402.1599
1989	423.7565
1990	447.3699
1991	462.242
1992	466.0279
1993	466.8251
1994	470.8565
1995	479.7164
1996	492.3679
1997	500.0644
1998	489.8207
1999	489.13

2000	503.1198
2001	504.0475
2002	505.3694
2003	512.513
2004	526.5777
2005	536.7622
2006	547.7093
2007	560.8164

¥ Trillion

List 5

YEAR	$-2\lambda_1/b$
1980	
1981	−0.0012
1982	−0.00121
1983	−0.00131
1984	−0.00147
1985	−0.00119
1986	−0.00054
1987	−0.00094
1988	−0.0007
1989	−0.00078
1990	−0.00063
1991	−0.00058
1992	−0.00048
1993	−0.00071
1994	−0.00094
1995	−0.0009
1996	−0.00097
1997	−0.00093
1998	−0.0009
1999	−0.00116

2000	−0.00135
2001	−0.00115
2002	−0.00167
2003	−0.00205
2004	−0.00413
2005	−0.0009
2006	−0.00199
2007	−0.00105

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〔3〕John Hicks, Value and Capital, 171-188, Oxford At The Clarendon Press. 1939.