Sequential Two-Prize Contests

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May 2008

Abstract

We study two-stage all-pay auctions with two identical prizes. In each stage, players compete for one prize. Each player may win either one or two prizes. We analyze the equilibrium strategies where players' marginal values for the prizes are either declining or inclining.

JEL CLASSIFICATION: D44, D82, J31, J41.

KEYWORDS: Multi-prize contests, All-pay auctions.

1 Introduction

In winner-take-all contests, all contestants including those who do not win the prize, incur costs as a result of their efforts. However, only the winner receives the prize. Winner-take-all contests have been applied to many settings including, rent-seeking and lobbying in organizations, R&D races, political contests, promotions in labor markets, trade wars, military conflicts and biological wars of attrition. The most common winner take-all contest is the all-pay auction. In the all-pay auction each player submits a bid (effort) for the prize and the player who submits the highest bid receives the prize, but, independently of success, all players bear the cost of their bids. In the economic literature, all-pay auctions are usually studied under complete information where each player's value for the prize is common knowledge,[1] or under incomplete information

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[1] See, for example, Hillman and Samet (1987), Hillman and Riley (1989), Baye, Kovenock and de Vries (1993), Che and Gale (1998)),Kaplan, Luski and Wettstein (2003) and Konrad (2006).

where each player's value for the prize is private information to that player and only the distribution of the players' values is common knowledge.[2] Most of this literature has focused on all-pay auctions with a single prize which is awarded to the player with the highest bid.[3] In the real world, however, we can find numerous contests with several prizes where the prizes can be awarded either simultaneously or sequentially. Some examples would be where employees spend effort in order to be promoted in organizational hierarchies which often consist of several types of well-defined positions, and competition settings such as where students compete for grades in exams.

In this paper, we analyze the equilibrium behavior in sequential two-prize all-pay auctions under complete information where the prizes are identical. Each player may win more than one prize. Players' marginal values for the first and the second prize are either declining or inclining. Because it is possible to win more than one prize our model is more complicated than the standard one-prize all-pay auction. In one-prize contests, the players' strategies are quite simple since the players have only two options, either to win the prize or to lose it, while in sequential multi-prize contests more complex strategies are involved since each player may win more than one prize and therefore players may face many options that depend on the identity of the winner in each stage, and each of these options may have a different effect on the chance of each player to win the other prizes in the later stages. In particular, in sequential multi-prize contests, each player has to decide in which stages he will compete to win and in which stages he will quit and keep his effort (resources) for the other rounds. In our sequential two-prize all pay auction with two players, we find a sub-game perfect equilibrium where no player quits in the first stage such that both players compete to win both of the prizes. The players use mixed strategies in both stages and therefore each has the chance to win both the prizes as well as none of them. The players' behavior in the second stage is similar to that of the one-prize all-pay auction, but their behavior in the first stage is completely unique in our sequential contest. For example, in contrast to the one-prize all-pay auction, in our sequential model even if a player is weaker than the other players (he has lower marginal values than those of his opponents) he may have a positive expected payoff

[2] See, for example, Hillman and Riley (1989), Amman and Leininger (1996), Krishna and Morgan (1998), Gavious, Moldovanu and Sela (2003) and Moldovanu and Sela (2006).

[3] Baye, Kovenock and de Vries (1996) provided a complete characterization of equilibrium behavior in the complete information all-pay auction with one prize.

in the first stage of the contest.

In multi-prize contests, the literature has focused on the optimal prize structure.[4] In fixed-prize contests, the designer can determine the number of prizes having a positive value and the distribution of the fixed total prize sum among the different prizes. In all-pay auctions with complete information, Barut and Kovenock (1998) show that the revenue maximizing prize structure allows any combination of $n-1$ prizes, where n is the number of contestants. In particular, allocating the entire prize sum to a unique first prize is optimal. On the other hand, Cohen and Sela (2008) show that in all-pay auctions under complete information where players have asymmetric values for the prizes, allocation of several prizes might be profitable for the contest designer who maximizes the total effort. In an all-pay auction with incomplete information, Moldovanu and Sela (2001) show that when cost functions are linear or concave in effort, it is optimal to allocate the entire prize sum to a single "first" prize but when cost functions are convex, several positive prizes may be optimal. Che and Gale (2003) study a contest where the contestants choose a prize from a menu of fixed prizes and the winner is determined according to the best combination of effort and prize. Cohen, Kaplan and Sela (2007) study all-pay auctions with effort-dependent rewards under incomplete information. They find that when the designer maximizes total effort and there is a sufficiently large number of contestants, the optimal reward decreases in the contestants' effort. They also find that the designer's payoff depends only upon the expected value of the effort-dependent rewards and not on the number of rewards.

Another important issue dealt with in the literature concerning the allocation of prizes in contests is whether to distribute the prizes simultaneously or sequentially. Clark and Riis (1998) analyze contests with multiple identical prizes and compare simultaneous versus sequential designs from the point of view of a revenue-maximizing designer. They show that if there is a dominant player (one who has a much higher value than his colleagues) a designer would maximize the expected total bid in the contest by distributing prizes simultaneously, whereas if no player were dominant, the designer would prefer a sequential distribution. The point common to the papers on multi-prize contests mentioned above is that in all of them a player can win only one prize, while, as was already mentioned, in our sequential two-prize all-pay auction, every player may win more than one prize.

[4] The tournament literature has shown how prizes based on rank-orders of performance can be effectively used to provide incentives (see Lazear and Rosen, 1981, Green and Stokey, 1983, and Nalebuff and Stiglitz, 1983).

The paper is organized as follows: in Section 2 we introduce our sequential two-prize all pay auction. In Section 3 we analyze the equilibrium behavior in two-player contests where players' marginal values are declining, and in Section 4 we analyze the equilibrium behavior in two-player contests where players' marginal values are inclining. In Section 5 we show the generalization of our results to the case with more than two players. Section 6 concludes.

2 The model

We consider a sequential all-pay auction with n players and two prizes. Player i's value for the prizes is given by the vector $V^i = (v_1, v_2)$, where v_j represents the marginal value of obtaining the j-th prize. That is, if player i wins only one prize his value is v_1 and if he wins two prizes his value is $v_1 + v_2$. We assume that all players' marginal values are either declining or inclining and they are common knowledge.

Each player i submits a bid (effort) $x \in [0, \infty)$ in the first stage. The player with the highest bid wins the first prize and all the players pay their bids. All the players know the identity of the winner in the first stage before the beginning of the second stage. In other words, the players' values in the second stage are common knowledge. The player with the highest bid in the second stage wins the second prize and all the players pay their bids.

3 The second stage

Assume that player i's marginal value in the second stage is v_i, and without loss of generality assume that the players' marginal values satisfy, $v_1 \geq v_2 \geq v_l$ for all $l \neq 1, 2$. According to Baye, Kovenock and de Vries (1993,1996), there is always a mixed-strategy equilibrium in which all the players except players 1 and 2 stay out of the contest. Players 1 and 2 randomize on the interval $[0, v_2]$ according to their effort cumulative distribution functions, which are given by

$$\begin{aligned} v_1 F_2(x) - x &= v_1 - v_2 \\ v_2 F_1(x) - x &= 0 \end{aligned}$$

Thus, player 1's equilibrium effort is uniformly distributed, that is

$$F_1(x) = \frac{x}{v_2}$$

while player 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{v_1 - v_2 + x}{v_1}$$

The respective expected payoffs are $u_1 = v_1 - v_2$ and $u_2 = 0$. Hence, in the second stage, the expected payoffs of all the players except the player with the highest value are zero and the expected payoff of the player with the highest value is equal to the difference between the two highest values. Now, given the equilibrium strategies in the second stage we can analyze the equilibrium strategies in the first stage.

4 The first stage - declining values

In this section, we analyze an equilibrium in which all the players compete in the first stage to win the first prize. An important point of the equilibrium analysis is that if a player competes in the first stage, his expected payoff in this stage should be larger or equal to his expected payoff if he would quit in the first stage and compete only in the second stage.

Assume first that there are only two players, 1 and 2 with values V^1 and V^2, and for each player the marginal values are declining. Suppose that $v_1 \geq v_2 \geq v_3 \geq v_4$. As such, we arrive at the following three possible cases.

4.1 Case A1: $V^1 = (v_1, v_4)$, $V^2 = (v_2, v_3)$.

Proposition 1 *Consider a sequential two-prize all-pay auction with two players where the players' marginal values are* $V^1 = (v_1, v_4)$, $V^2 = (v_2, v_3), v_1 \geq v_2 \geq v_3 \geq v_4$. *Then, type* 1*'s equilibrium effort is distributed according to*

$$F_1(x) = \frac{x}{v_4}$$

while type 2's equilibrium effort is distributed according to

$$F_2(x) = \frac{x + v_3 - v_4}{v_3}$$

The respective expected payoffs are $u_1 = v_1 - v_4$ *and* $u_2 = v_2 - v_4$.

Proof. If player 1 wins in the first stage his payoff is v_1, but if he doesn't win in the first stage, his expected payoff in the second stage is $v_1 - v_3$. Similarly, if player 2 wins in the first stage, his payoff is v_2 but if he doesn't win in the first stage his expected payoff in the next stage is $v_2 - v_4$. Hence, the players randomize on the interval $[0, v_4]$ according to their effort cumulative distribution functions, $F_1(x)$ and $F_2(x)$, which are given by the indifference conditions:

$$\begin{aligned} v_1 F_2(x) + (v_1 - v_3)(1 - F_2(x)) - x &= v_1 - v_4 \\ v_2 F_1(x) + (v_2 - v_4)(1 - F_1(x)) - x &= v_2 - v_4 \end{aligned}$$

■

In this situation, both players have positive expected payoffs in the first stage. Player 2's expected payoff in the first stage, $v_2 - v_4$, is equal to his expected payoff if he would quit in the first stage and compete only in the second stage. Player 1's expected payoff in the first stage, $v_1 - v_4$, is larger than his highest possible expected payoff in the second stage which is $v_1 - v_3$. Thus, both players have incentives to compete in the first stage. Note that player 1's expected payoff in the first stage decreases in his second marginal value, v_4, since the the probability that player 2 will stay out of the contest in the first stage increases in the value of v_4.

4.2 Case A2: $V^1 = (v_1, v_3)$, $V^2 = (v_2, v_4)$.

Proposition 2 *Consider a sequential two-prize all-pay auction with two players where the players' marginal values are* $V^1 = (v_1, v_3)$, $V^2 = (v_2, v_4), v_1 \geq v_2 \geq v_3 \geq v_4$. *Then, type 1's equilibrium effort is distributed according to*

$$F_1(x) = \frac{x + v_3 - v_4}{v_3}$$

while type 2's equilibrium effort is distributed according to

$$F_2(x) = \frac{x}{v_4}.$$

The respective expected payoffs are $u_1 = v_1 - v_4$ and $u_2 = v_2 - v_4$.

Proof. If player 1 wins in the first stage his payoff is v_1 but if he doesn't win in the first stage, his expected payoff in the second stage is $v_1 - v_4$. Similarly, if player 2 wins in the first stage his payoff is v_2, but if he doesn't win in the first stage his expected payoff in the next stage is $v_2 - v_3$. Hence, the players randomize on the interval $[0, v_4]$ according to their effort cumulative distribution functions, $F_1(x)$ and $F_2(x)$, which are given by the indifference conditions:

$$\begin{aligned} v_1 F_2(x) + (v_1 - v_4)(1 - F_2(x)) - x &= v_1 - v_4 \\ v_2 F_1(x) + (v_2 - v_3)(1 - F_1(x)) - x &= v_2 - v_4 \end{aligned}$$

■

The players' expected payoffs are exactly as in case A1, but they have the opposite strategies with respect to case A1, namely, player 1 uses player 2's strategy, and player 2 uses player 1's strategy. It is interesting that player 1, the stronger player in this case (he has the highest marginal value in each stage), chooses an effort of zero with a probability of $\frac{v_3 - v_4}{v_3}$. The reason is that player 1's expected payoff in the first stage is the same as if player 1 would compete only in the second stage. Thus, he is indifferent between whether to stay out or to compete in the first stage.

Similar to case A1, player 2's expected payoff decreases in his second marginal value v_4, since the probability that player 1 will stay out of the contest in the first stage increases in v_4.

4.3 Case A3: $V^1 = (v_1, v_2)$, $V^2 = (v_3, v_4)$.

Proposition 3 *Consider a sequential two-prize all-pay auction with two players where the players' marginal values are $V^1 = (v_1, v_2)$, $V^2 = (v_3, v_4)$, $v_1 \geq v_2 \geq v_3 \geq v_4$.*

1) If $v_2 + v_4 \geq 2v_3$, then type 1's equilibrium effort is distributed according to

$$F_1(x) = \frac{x}{v3}$$

while type 2's equilibrium effort is distributed according to

$$F_2(x) = \frac{x + v_2 + v_4 - 2v_3}{v_2 + v_4 - v_3}.$$

The respective expected payoffs are $u_1 = v_1 + v_2 - 2v_3$ *and* $u_2 = 0$.

2) If $v_2 + v_4 < 2v_3$*, then, type* 1*'s equilibrium effort is distributed according to*

$$F_1(x) = \frac{x + 2v_3 - v_2 - v_4}{v3}$$

while type 2*'s equilibrium effort is distributed according to*

$$F_2(x) = \frac{x}{v_2 + v_4 - v_3}$$

The respective expected payoffs are $u_1 = v_1 - v_4$ *and* $u_2 = 2v_3 - v_2 - v_4$.

Proof. If player 1 wins in the first stage his payoff is v_1 and then he also has an expected payoff in the second stage of $v_2 - v_3$. If player 1 doesn't win in the first stage, his expected payoff in the second stage is $v_1 - v_4$. If player 2 wins in the first stage his payoff is v_3, but if he doesn't win in the first stage his expected payoff in the next stage is zero.

Hence, if $v_2 + v_4 \geq 2v_3$ the players randomize on the interval $[0, v_3]$ according to their effort cumulative distribution functions, $F_1(x)$ and $F_2(x)$, which are given by the indifference conditions:

$$\begin{aligned} (v_1 + v_2 - v_3)F_2(x) + (v_1 - v_4)(1 - F_2(x)) - x &= v_1 + v_2 - 2v_3 \\ v_3 F_1(x) - x &= 0 \end{aligned}$$

Similarly, if $v_2 + v_4 - 2v_3 < 0$, the players randomize on the interval $[0, v_2 + v_4 - v_3]$ according to their effort cumulative distribution functions, $F_1(x)$ and $F_2(x)$, which are given by the indifference conditions:

$$\begin{aligned} (v_1 + v_2 - v_3)F_2(x) + (v_1 - v_4)(1 - F_2(x)) - x &= v_1 - v_4 \\ v_3 F_1(x) - x &= 2v_3 - v_2 - v_4 \end{aligned}$$

■

If $v_2 + v_4 \geq 2v_3$, player 1's marginal values are larger than those of player 2, such that player 2 doesn't have any chance to win in each of the stages and therefore his expected payoff is zero. Note that player 1 has an incentive to compete in the first stage, since if he quits in the first stage and competes only in the second stage he will win $v_1 - v_4$ which is smaller than his expected payoff $v_1 + v_2 - 2v_3$ in the first stage given our condition that $v_2 + v_4 \geq 2v_3$.

If $v_2 + v_4 < 2v_3$, player 1 has an expected payoff of $v_1 - v_4$ in the first stage which is equal to his expected payoff if he competes only in the second stage. Thus, player 1 is indifferent between staying out or competing in the first stage, and he chooses each of these options with a positive probability. Consequently, player 2 has a positive expected payoff in the first stage although he is weaker than player 1 given that both of his marginal values are smaller than both of player 1's marginal values.

Based on cases A1, A2 and A3 we obtain the following result:

Conclusion 1 *In sequential two-prize all-pay auctions with two players where the players' marginal values are declining, the player with the highest marginal value has the higher expected payoff in the first stage which is larger or equal to the difference between the highest and lowest marginal values of the players.*

5 The first stage - inclining values

We assume now that there are only two players, 1 and 2 with values V^1 and V^2 respectively, and for each player the marginal values are inclining. Suppose that $v_1 \geq v_2 \geq v_3 \geq v_4$. As such we arrive at the following three possible cases.

5.1 Case B1: $V^1 = (v_2, v_1)$, $V^2 = (v_4, v_3)$.

Proposition 4 *Consider a sequential two-prize all-pay auction with two players where the players' marginal values are $V^1 = (v_2, v_1)$, $V^2 = (v_4, v_3), v_1 \geq v_2 \geq v_3 \geq v_4$. Then, type 1's equilibrium effort is distributed according to*

$$F_1(x) = \frac{x}{v_4}$$

while type 2's equilibrium effort is distributed according to

$$F_2(x) = \frac{x + v_1 + v_3 - 2v_4}{v_1 + v_3 - v_4}.$$

The respective expected payoffs are $u_1 = v_1 + v_2 - 2v_4$ and $u_2 = 0$.

Proof. If player 1 wins in the first stage his payoff is v_2 and then he also has an expected payoff in the second stage of $v_1 - v_4$. If player 1 doesn't win in the first stage, his expected payoff in the second stage is

$v_2 - v_3$. If player 2 wins in the first stage his payoff is v_4, but if he doesn't win in the first stage his expected payoff in the next stage is zero. Hence, the players randomize on the interval $[0, v_4]$ according to their effort cumulative distribution functions, $F_1(x)$ and $F_2(x)$, which are given by the indifference conditions:

$$\begin{aligned}(v_2 + (v_1 - v_4))F_2(x) + (v_2 - v_3)(1 - F_2(x)) - x &= v_1 + v_2 - 2v_4 \\ v_4 F_1(x) - x &= 0\end{aligned}$$

■

In this case, player 1 competes in the first stage since he knows that he has the highest value in the second stage. This situation leaves player 2, the weaker player (he has lower marginal values), with an expected profit of zero in both stages.

5.2 Case B2: $V^1 = (v_3, v_1)$, $V^2 = (v_4, v_2)$.

Proposition 5 *Consider a sequential two-prize all-pay auction with two players where the players' marginal values are $V^1 = (v_3, v_1)$, $V^2 = (v_4, v_2), v_1 \geq v_2 \geq v_3 \geq v_4$. Then, type 1's equilibrium effort is distributed according to*

$$F_1(x) = \frac{x}{v_4 + v_2 - v_3}$$

while type 2's equilibrium effort is distributed according to

$$F_2(x) = \frac{x + v_1 - v_2 + 2(v_3 - v_4)}{v_4 + v_2 - v_3}.$$

The respective expected payoffs are $u_1 = v_1 - v_2 + 2(v_3 - v_4)$ and $u_2 = 0$.

Proof. If player 1 wins in the first stage his payoff is v_3 and then he also has an expected payoff of $v_1 - v_4$ in the second stage. If player 1 doesn't win in the first stage, his expected payoff in the second stage is zero. If player 2 wins in the first stage his payoff is v_4 and then he also has an expected payoff of $v_2 - v_3$. If he doesn't win in the first stage his expected payoff in the next stage is zero. Hence, the players randomize on the interval $[0, v_2 + v_4 - v_3]$ according to their effort cumulative distribution functions, $F_1(x)$ and $F_2(x)$, which are given by the indifference conditions:

$$\begin{aligned}(v_3 + (v_1 - v_4))F_2(x) - x &= v_1 - v_2 + 2(v_3 - v_4) \\ (v_4 + (v_2 - v_3)F_1(x) - x &= 0\end{aligned}$$

■

Here player 1 knows that he is stronger than his opponent in the first stage and that he will be even stronger in the second stage. Thus, he competes in the first stage and tries to win the first prize. Therefore, player 2 (like in case B1) will not have a positive expected profit in both stages of the contest.

5.3 Case B3: $V^1 = (v_4, v_1)$, $V^2 = (v_3, v_2)$.

Proposition 6 *Consider a sequential two-prize all-pay auction with two players where the players' marginal values are $V^1 = (v_4, v_1)$, $V^2 = (v_3, v_2)$, $v_1 \geq v_2 \geq v_3 \geq v_4$.*

1) If $2v_4 + v_1 \geq 2v_3 + v_2$, then, type 1's equilibrium effort is distributed according to

$$F_1(x) = \frac{x}{v_3 + v_2 - v_4}$$

while type 2's equilibrium effort is distributed according to

$$F_2(x) = \frac{x + v_1 - v_2 + 2(v_4 - v_3)}{v_4 + v_1 - v_3}.$$

The respective expected payoffs are $u_1 = v_1 - v_2 - 2(v_3 - v_4) \geq 0$ and $u_2 = 0$.

2) If $2v_4 + v_1 < 2v_3 + v_2$, then, type 1's equilibrium effort is distributed according to

$$F_1(x) = \frac{x + v_2 - v_1 + 2(v_3 - v_4)}{v_2 + v_3 - v_4}$$

while type 2's equilibrium effort is distributed according to

$$F_2(x) = \frac{x}{v_4 - v_3 + v_1}$$

The respective expected payoffs are $u_1 = 0$ and $u_2 = v_2 - v_1 + 2(v_3 - v_4) > 0$.

Proof. If player 1 wins in the first stage his payoff is v_4 and then he also has an expected payoff of $v_1 - v_3$ in the second stage. If player 1 doesn't win in the first stage, his expected payoff in the second stage is zero. Similarly, if player 2 wins in the first stage his payoff is v_3 and then he also has an expected payoff of $v_2 - v_4$. If he doesn't win in the first stage his expected payoff in the next stage is zero.

Hence, if $v_4+(v_1-v_3)\geq v_3+(v_2-v_4)$, the players randomize on the interval $[0,v_3+v_2-v_4]$ according to their effort cumulative distribution functions, $F_1(x)$ and $F_2(x)$, which are given by the indifference conditions:

$$\begin{aligned}(v_4+(v_1-v_3))F_2(x)-x &= v_1-v_2-2(v_3-v_4)\\ (v_3+(v_2-v_4))F_1(x)-x &= 0\end{aligned}$$

If $v_4+(v_1-v_3)<v_3+(v_2-v_4)$, the players randomize on the interval $[0,v_4+v_1-v_3]$ according to their effort cumulative distribution functions, $F_1(x)$ and $F_2(x)$, which are given by the indifference conditions:

$$\begin{aligned}(v_4+(v_1-v_3))F_2(x)-x &= 0\\ (v_3+(v_2-v_4))F_1(x)-x &= v_2-v_1+2(v_3-v_4)\end{aligned}$$

■

If $v_4+(v_1-v_3)\geq v_3+(v_2-v_4)$, although player 1 has a lower value than player 2 in the first stage, he wins the first stage with a higher probability than player 2. The reason is that player 1 knows that if he wins in the first stage he will have a high probability to win also in the second stage, since in the second stage he will have the highest value. Otherwise, if he loses in the first stage, he will lose with a high probability in the second stage as well.

If $v_4+(v_1-v_3)<v_3+(v_2-v_4)$, player 1 knows that if he wins in the first stage he will have the higher value in the second stage and therefore he will have the higher probability to win in the second stage. Nevertheless, player 1 wins in the first stage with a lower probability than player 2 and he has an expected payoff of zero in the first stage. The reason is that his value in the first stage is relatively low and/or his value in the second stage is not high enough in relation to player 2's values. Thus, he loses with a higher probability than his opponent in the first stage and after that in the second stage as well.

In contrast to the case of declining values, here we obtain,

Conclusion 2 *In sequential two-prize all-pay auctions with two players, if the players' marginal values are inclining, the player with the highest marginal value does not necessarily have the higher expected payoff in the first stage, and he may even have an expected payoff of zero.*

6 The first stage - N players

Up to this point we considered contests with only two players. The generalization to the case with more than two players is straightforward and we present it for where the marginal values are declining. The generalization to the case where the marginal values are inclining is similar and therefore it is omitted. It is important to note that similar to one-prize all-pay auctions with more than two players (see Baye, Kovenock and de Vries, 1996), in our model with more than two players, the equilibrium strategies are not necessarily unique. We assume that there are $N > 2$ players. In the equilibrium that we present, only the three players $1, 2$, and 3 have an effect on the equilibrium and only 1 and 2 are active. For each player the marginal values are declining. Assume that $v_1 \geq v_2 \geq v_3 \geq v_4$ are the players' highest marginal values. Then we have the following four cases.

6.1 Case C1: $V^1 = (v_1, \ )$, $V^2 = (v_2, v_3)$, $V^3 = (\ , \ )$.

This case is equivalent to case A1 with two players where $V^1 = (v_1, v_4)$, $V^2 = (v_2, v_3)$. Thus, like in case A1, players 1 and 2 randomize on the interval $[0, v_4]$ according to their effort cumulative distribution functions,

$$\begin{aligned} F_1(x) &= \frac{x}{v_4} \\ F_2(x) &= \frac{x + v_3 - v_4}{v_3} \end{aligned}$$

The respective expected payoffs are $u_1 = v_1 - v_4$ and $u_2 = v_2 - v_4$.

6.2 Case C2: $V^1 = (v_1, \ v_3)$, $V^2 = (v_2, \ )$, $V^3 = (\ , \ )$.

This case is equivalent to case A2 with two players where $V^1 = (v_1, v_3)$, $V^2 = (v_2, v_4)$. Thus, like in case A2, players 1 and 2 randomize on the interval $[0, v_4]$ according to their effort cumulative distribution functions,

$$\begin{aligned} F_1(x) &= \frac{x + v_3 - v_4}{v_3} \\ F_2(x) &= \frac{x}{v_4}. \end{aligned}$$

The respective expected payoffs are $u_1 = v_1 - v_4$ and $u_2 = v_2 - v_4$.

6.3 Case C3: $V^1 = (v_1,\ ),\ V^2 = (v_2,\ ), V^3 = (v_3,\ )$.

This case is the only one without an equivalent one for the case of two players. In this case, players 1 and 2 randomize on the interval $[0, v_3]$ according to their effort cumulative distribution functions, $F_1(x)$ and $F_2(x)$, which are given by the indifference conditions:

$$\begin{aligned} v_1 F_2(x) + (v_1 - v_3)(1 - F_2(x)) - x &= v_1 - v_3 \\ v_2 F_1(x) + (v_2 - v_3)(1 - F_1(x)) - x &= v_2 - v_3 \end{aligned}$$

Thus, both players' efforts are distributed according to

$$F(x) = \frac{x}{v_3}.$$

The respective expected payoffs are $u_1 = v_1 - v_3$ and $u_2 = v_2 - v_3$.

6.4 Case C4: $V^1 = (v_1,\ v_2),\ V^2 = (v_3,\ ),\ V^3 = (\ ,\ )$.

This case is equivalent to case A3 with two players where $V^1 = (v_1, v_2)$, $V^2 = (v_3, v_4)$. Thus, like in case A3, we obtain that:

1) If $v_2 + v_4 \geq 2v_3$, players 1 and 2 randomize on the interval $[0, v_3]$ according to their effort cumulative distribution functions,

$$\begin{aligned} F_1(x) &= \frac{x}{v_3} \\ F_2(x) &= \frac{v_2 + v_4 - 2v_3 + x}{v_2 - v_3 + v_4} \end{aligned}$$

The respective expected payoffs are $u_1 = v_1 + v_2 - 2v_3$ and $u_2 = 0$.

2) If $v_2 + v_4 < 2v_3$, players 1 and 2 randomize on the interval $[0, v_2 + v_4 - v_3]$ according to their effort cumulative distribution functions,

$$\begin{aligned} F_1(x) &= \frac{x + 2v_3 - v_2 - v_4}{v3} \\ F_2(x) &= \frac{x}{v_2 + v_4 - v_3}. \end{aligned}$$

The respective expected payoffs are $u_1 = v_1 - v_4$ and $u_2 = 2v_3 - v_2 - v_4$.

As we can see, in sequential two-prize all-pay auctions with more than two players there is always an equilibrium in which only two players compete in each stage of the contest.

7 Concluding remarks

We analyzed a sub-game perfect equilibrium in a sequential two-prize all-pay auction with two players. The equilibrium strategies in the second stage are equivalent to those in one-prize all-pay auctions, but the equilibrium strategies in the first stage are more complex. We showed that independent of the players' values, there is always an equilibrium in which no player decides to quit in the first stage and then competes only for one prize in the second stage. Each of the players, therefore, has a positive chance to win both of the prizes as well as none of them.

We also showed that if the players' marginal values are declining, the player with the highest marginal value, independent of the other marginal values, has the highest expected payoff in the first stage. On the other hand, if the marginal values are inclining, the player with the highest marginal value does not necessarily have the highest expected payoff in the first stage.

The characteristics of the equilibria in one-prize and sequential two-prize all-pay auctions are not the same. For example, in one-prize all-pay auctions the player with the lowest value necessarily has an expected payoff of zero for any equilibrium. On the other hand, in sequential two-prize all pay auctions with two players and declining marginal values, even if a player has two marginal values which are both smaller than those of his opponent, he may have a positive expected payoff in the first stage. Furthermore, while the expected payoff of the players in the one-prize all-pay auction is obvious, namely, the weaker player has an expected payoff of zero and the stronger player has an expected payoff equal to the difference of the players' values, in the sequential two-prize all-pay auction the players' expected payoffs as well as their strategies are not clear at all.

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