POLICY RESEARCH WORKING PAPER 1985

The Optimality of Being Efficient

Designing Auctions

Lawrence M. Ausubel
Peter Cramton

This paper provides a new defense for emphasizing efficient auction design rather than *optimal* auction design. Because in auction markets followed by perfect resale, it is "optimal" to be "efficient."

The World Bank
Private Sector Development Department
Private Participation in Infrastructure Division
September 1998

Summary findings

In an optimal auction, a revenue-optimizing seller often awards goods inefficiently, either by placing them in the wrong hands or by withholding them from the market. This conclusion rests on two assumptions: (1) the seller can prevent resale among bidders after the auction, and (2) the seller can commit to not selling the withheld goods after the auction.

Ausubel and Cramton examine how the optimal auction problem changes when those assumptions are relaxed. In sharp contrast to the no-resale assumption, they assume perfect resale: all gains from trade are exhausted in resale. In a multiple-object model with independent signals, they characterize optimal auctions with resale. They prove generally that with perfect resale, the seller can do no better than assign goods efficiently. Moreover, any misassignment of goods strictly lowers the seller's revenue from the optimum. In auction markets followed by perfect resale, it is optimal to be efficient.

The authors' results provide a new defense for emphasizing efficient auction design rather than optimal auction design. The presence of a perfect resale market forces even the most selfish seller, whose sole objective is maximizing revenues, to focus — out of necessity — on efficiency. Given the vast and active resale market in Treasury securities, it seems safe to assert that the model with perfect resale is a better description of the U.S. Treasury market than the model without any resale — so its predictions ought to be taken more seriously.

This paper — a product of the Private Participation in Infrastructure Division, Private Sector Development Department — is part of a larger effort in the department to study issues arising from private participation in infrastructure. The study was funded by the Bank's Research Support Budget under the research project "Auctions in Infrastructure" (RPO 682-58). Copies of this paper are available free from the World Bank, 1818 H Street NW, Washington, DC 20433. Please contact Sandra Vivas, room Q7-005, telephone 202-458-2809, fax 202-522-2029, Internet address svivas@worldbank.org. The authors may be contacted at ausubel@econ.umd.edu or peter@cramton.umd.edu. September 1998. (23 pages)

The Optimality of Being Efficient

Lawrence M. Ausubel and Peter Cramton*

*The authors gratefully acknowledge the support of the National Science Foundation and the World Bank.

TABLE OF CONTENTS

Page No.

THE OPTIMALITY OF BEING EFFICIENT

1 INTRODUCTION

A cornerstone of the auction literature is the theory of "optimal auctions."[1] This theory uses mechanism design techniques to characterize, in general settings, the auction that maximizes the seller's expected revenues. One feature of the solution is that typically there is a conflict between the goals of revenue maximization and efficiency. The revenue-optimizing seller often either places goods in hands other than those who value them the most or withholds goods entirely from the market. However, the conclusion that the seller gains by assigning goods inefficiently depends critically on two strong assumptions: (1) the seller can prevent resale among bidders from occurring after the auction; and (2) the seller can commit to not sell the withheld goods after the auction. In this paper, we examine how the optimal auction problem changes when one or both of these assumptions are relaxed.

When the seller cannot ban resale, the bidders may undo the seller's inefficient assignment from the auction. If agents understand and anticipate this, the incentives that the seller attempted to create in the solution to the mechanism design problem are undermined, and so the "optimal" auction may cease to be optimal. Coase (1960) has criticized standard economic analyses of the law that assume away the possibility that economic agents may recognize *any* gains from trade, by instead making the opposite extreme assumption that *all* gains from trade are realized. For most of this paper, we will adopt the Coase Theorem by assuming perfect resale. Resale causes any misassignment of the goods to be corrected. This is an extreme assumption. Certainly, there are settings where perfect resale is not possible, because of private information that the auction winners have after the auction (Myerson and Satterthwaite 1983; Cramton, Gibbons and Klemperer 1987). However, there are other settings where perfect resale is possible. Perfect resale has the significant advantage that it is a simple and general assumption on the resale market. Moreover, since resale is voluntary, resale inevitable shifts outcomes toward the efficient assignment. We view perfect resale as a good first-approximation of many resale markets.

When the seller cannot commit to refrain from selling the withheld objects after the auction, the seller may himself undo the inefficient allocation of the auction. Again, if agents understand and anticipate this, the "optimal" auction may cease to be optimal. Coase (1972) has criticized standard economic analyses of durable goods monopoly that assume the seller has *full* commitment powers, by instead making the opposite extreme assumption that the seller has *no* commitment powers. In parts of this paper, we will take the Coase Conjecture seriously and explore the implications of no commitment powers by the seller. Without commitment, all inefficient withholding of the goods is corrected. Again, this is an extreme assumption. Certainly, there are reasons why the seller may credibly withhold goods from the market for long periods of time (Ausubel and Deneckere 1989). That said, the Coase Conjecture remains a simple assumption on post-auction behavior whose implications are certainly worth exploring,

[1] This research began with Myerson (1981) and has since been extended by many others, for example, Engelbrecht-Wiggans (1988), Cremer and McLean (1985, 1988), Maskin and Riley (1989), McAfee and Reny (1992), and Bulow and Klemperer (1996).

and may at least be an acceptable assumption for modeling real-world situations where the seller chooses not to utilize any reserve price.

Armed with the Coasean assumptions, we state and solve three optimal auction programs:

1. *Unconstrained optimal auction.* The seller can forbid resale and commit to not selling additional goods after the auction. Hence, the seller maximizes revenues, under the hypothesis that resale among buyers is impossible.

2. *Resale-constrained optimal auction.* The seller can withhold supply, but cannot prevent resale. Thus, the seller maximizes revenues, subject to the constraint that there will be perfect resale among bidders after the auction.

3. *Efficiency-constrained optimal auction.* The seller can neither withhold supply, nor prevent resale. Hence, the seller maximizes revenues, subject to the constraint that there will be perfect resale among the seller and bidders after the auction.

We analyze an "independent signals" model with multiple identical objects. Each risk-neutral bidder has a private signal about its demand for the good. A bidder's demand depends on everyone's signal, and the signals are independent. This model includes both private value and common value models as special cases. It allows ex ante asymmetries among bidders.

Each of the optimal auctions is solved via a general version of the Revenue Equivalence Theorem: Any auction that results in the same assignment of the goods yields the same seller revenues, provided that the lowest bidder types get the same payoff. Moreover, when the lowest bidder types are given zero surplus (as they are in any optimal auction), this revenue equals the marginal revenues integrated over the quantity won and summed over bidders. Marginal revenue is what the seller gets from awarding additional quantity to a bidder. It is equal to the bidder's marginal value less the informational rent that the bidder is able to capture from its private information.

In the unconstrained optimal auction, the seller simply assigns the good in decreasing order of marginal revenue, until the good is exhausted or marginal revenue turns negative. Goods are assigned by moving down the aggregate marginal revenue curve.

In the efficiency-constrained optimal auction, the seller is forced to assign the goods efficiently. The seller's only discretion occurs when there is a tie in marginal values (the aggregate demand curve is flat). Then the seller assigns first to those with the higher marginal revenue. Goods are assigned by moving down the aggregate demand curve, until the quantity available is exhausted.

In the resale-constrained optimal auction, the seller has more discretion. Because of perfect resale, the seller is forced to award in decreasing order of marginal value (i.e., by moving down the aggregate demand curve), but the seller can withhold quantity. The choice of the optimal quantity to award is a simple one-variable calculation. The seller's choice of aggregate quantity depends on the bidders' reports of private information, and is equivalent to setting an ex post reserve price.

We show that, when a seller cannot prevent resale, the seller no longer has any incentive to misassign goods. The seller can do no better than assigning goods to exhaust all gains from trade among the bidders. This is an extremely general result, which does not rely on any of the standard assumptions of the optimal auction literature. We, therefore, prove it in a general model. The key to the argument is that any equilibrium of the auction-plus-resale game must satisfy all the constraints of the resale-constrained optimal auction program. Thus, an equilibrium of the two-stage game cannot result in greater revenues.

We next consider whether misassignment actually hurts the seller. Does the seller necessarily get strictly lower revenues by any misassignment of the good? We show that the answer is yes in the identical objects model. The seller does strictly better by assigning the goods to those with the highest values. Intuitively, misassigning the goods results in the seller foregoing a share of the gains from trade that are ultimately captured by the bidders in resale.

Finally, we explore how a seller can implement optimal auctions with resale. We prove in a private-value setting that the Vickrey auction is not distorted by resale. When the seller uses a Vickrey auction, sincere bidding — followed by no resale — is an equilibrium in the auction-plus-resale game. Thus, a Vickrey auction implements the efficiency-constrained optimal auction, if the lowest bidder types get a payoff of 0.[2]

Our results thus provide a new defense for emphasizing *efficient* auction design rather than *optimal* auction design.[3] The presence of a perfect resale market forces even the most selfish seller, whose sole objective is maximizing revenues, to focus — out of necessity — on efficiency. While the Coasean assumption of *perfect resale* is extreme, it is no more extreme than the standard assumption of *no resale* which the auction literature routinely makes.

In this paper, we focus solely on the Coasean critiques of the unconstrained optimal auction. Other available critiques strengthen our conclusion. For example, when one recognizes that bidder participation is affected by the auction design, then the case for an efficient auction improves. McAfee and McMillan (1987), Harstad (1990, 1993), and Levin and Smith (1994, 1995, 1996) provide justification why a revenue-maximizing seller should care about efficiency. With endogenous bidder participation and symmetric bidders, efficiency and revenue-maximization are equivalent. Bulow and Klemperer (1996) demonstrate that if a reserve price discourages even a single potential bidder from participating, the reserve makes the seller worse off.

Another critique of the optimal auction approach is the severe informational requirement placed on the mechanism designer. The approach assumes that the distributions of private information are common knowledge, and the optimal auction makes explicit use of this information. If the seller does not know the distributions or is constrained to adopt auction rules that are independent of the distributions, then implementing the optimal auction may be impossible (but see Caillaud and Robert, 1998). In contrast, the informational requirements of the efficient auction are often less severe. In interesting cases, the efficient auction rules may be independent of the distributions of private information (see, for example, Ausubel, 1997).

Our paper, by introducing a resale constraint into the optimal auction, is connected to both the resale literature and the optimal auction literature. The study of resale in auction markets is just emerging. Bikhchandani and Huang (1989) and Viswanathan and Wang (1996) develop a model of

[2] Krishna and Perry (1997) show that in a market without resale the Vickrey auction implements the efficiency-constrained optimal auction, when the lowest types get 0.

[3] Recent papers emphasizing efficiency rather than revenue maximization as the seller's objective include Ausubel (1997), Ausubel and Cramton (1998), Dasgupta and Maskin (1997), and Krishna and Perry (1997). The real-world discussion of how best to structure the FCC spectrum auctions tended also to emphasize efficiency over revenue maximization.

Treasury auctions, where resale is especially important. Bidding behavior is significantly affected by resale. Agastya and Daripa (1998) also focus on Treasury auctions, emphasizing the interaction of the futures market, the auction, and resale. Haile (1998), using a reduced-form representation of the resale market, characterizes equilibrium bidding behavior in standard single-good auctions. Haile (1997) examines resale in a setting where bidders acquire additional information after the auction. Haile (1996) empirically tests the model using U.S. Forest Service timber data. Horstmann and LaCasse (1997) consider resale by the seller to a potentially different set of bidders. Our paper is most closely related to several recent studies of optimal auctions with multiple goods. For example, Krishna and Perry (1997) find conditions under which the Vickrey auction is optimal among efficient mechanisms. Armstrong (1997) is also interested in when an optimal auction is efficient. He shows that with two goods and two types, the optimal auction is always efficient (this result, however, does not generalize to three types). Avery and Hendershott (1998) analyze optimal bundling in a multiple objects setting.

Our paper is organized as follows. In section 2, we establish the seller's general incentive to misassign goods and we identify settings where the optimal auction is efficient. In section 3, we solve two variations on the optimal auction, which recognize the possibility of resale. Section 4 proves that perfect resale destroys the seller's incentive to misassign goods. Section 5 establishes that, with perfect resale, any misassignment of goods results in strictly lower seller revenues than the best efficient assignment. In section 6, we show that the Vickrey auction is not distorted by the possibility of resale.

2 THE INCENTIVE TO MISASSIGN THE GOOD

There are two ways an optimal auction can fail to be efficient: (1) the seller can withhold some quantity; and (2) the seller can award quantity to a bidder with a lower marginal value instead of a bidder with a higher marginal value. Myerson (1981) demonstrates both inefficiencies in deriving the optimal auction in an independent private value auction for a single good. We begin by examining the incentive to misassign goods in a multiple object setting.

2.1 *Identical objects model*

For most of the paper, we consider a model with multiple identical objects or close substitutes. The seller has a quantity 1 of a divisible good to sell to n bidders. The seller's valuation for the good equals zero. Each bidder i can consume any quantity $q_i \in [0,\lambda_i]$, where $\lambda_i \in (0,1]$. We can interpret q_i as bidder i's share of the total quantity being auctioned, and λ_i as i's capacity or quantity restriction. Let $q = (q_1,\ldots,q_n)$, $Q = \{q \mid q_i \in [0,\lambda_i] \text{ and } \sum_i q_i \leq 1\}$, and $\overline{Q} = \{q \mid q_i \in [0,\lambda_i] \text{ and } \sum_i q_i = 1\}$. Then Q is the set of all feasible assignments and $\overline{Q}$ is the set of all feasible assignments in which the seller sells the entire quantity available. Bidder i has a diminishing marginal value, which may depend on all the bidders' private information. Let $t_i \in T_i$ be bidder i's type, $t = (t_1,\ldots,t_n)$, $t_{-i} = t\backslash t_i$, and $\tau_i = \min \{t_i \mid t_i \in T_i\}$. The bidders' types are drawn independently from the distribution functions F_i with full support on T_i. A bidder's type is private information; whereas, the value functions, capacities, and distributions of types are common knowledge. The bidders are risk neutral. A bidder i with marginal value $v_i(t,q_i)$ who receives quantity $q_i \in [0,\lambda_i]$ and pays x for it has a payoff $\int_0^{q_i} v_i(t,y)\,dy - x$.

We require marginal value to satisfy

Value monotonicity. For all i, j, t, q_i, $v_i(t,q_i) \geq 0$, $\partial v_i(t,q_i)/\partial t_i > 0$, $\partial v_i(t,q_i)/\partial t_j \geq 0$, $\partial v_i(t,q_i)/\partial q_i \leq 0$.

Value regularity. For all i, j, q_i, q_j, t_{-i}, and $t_i' > t_i$, $v_i(t_i,t_{-i},q_i) > v_j(t_i,t_{-i},q_j) \Rightarrow v_i(t_i',t_{-i},q_i) > v_j(t_i',t_{-i},q_j)$.

These conditions guarantee that if goods are assigned in order of marginal values, then $q_i(t)$ will be weakly increasing in t_i. This model includes both private value and common value models as special cases. In the private value model, v_i only depends on t_i. In the common value model, $v_i(t) = v_j(t)$. The model allows ex ante asymmetries among the bidders, both in the bidder's capacity, λ_i, and more importantly, in the value functions and the distributions of types.

2.2 The optimal auction with identical objects

We begin by determining the optimal auction. This extends Maskin and Riley (1989), which assumes symmetry and private values, and Bulow and Klemperer (1996), which assumes symmetry and a single good. (See Engelbrecht-Wiggans (1988) and Krishna and Perry (1997) for more general treatments of revenue equivalence.)

Define bidder i's marginal revenue as

$$MR_i(t,q_i) = v_i(t,q_i) - \frac{1-F_i(t_i)}{f_i(t_i)}\frac{\partial v_i(t,q_i)}{\partial t_i}.$$

We interpret $MR_i(t,q_i)$ as the marginal revenue the seller gets from awarding quantity to bidder i after deducting the informational rent that i is able to capture from its private information. This interpretation is justified by the following revenue equivalence theorem. Any auction that results in the same assignment yields the same seller revenue, provided that the lowest bidder types get the same payoff. Moreover, this revenue is simply the marginal revenues integrated over the quantity won and summed over bidders, when the lowest bidder types are given no surplus.

THEOREM 1 ("Revenue Equivalence"). *In any equilibrium of any auction game in which the lowest-type bidders receive an expected payoff of zero, the seller's expected revenue equals*

$$\text{(R)} \qquad E_t\left[\sum_{i=1}^{n}\int_0^{q_i(t)} MR_i(t,y)\,dy\right].$$

PROOF. [Note: The current draft assumes "flat" demands.] Incentive compatibility requires that t_i' does not want to report t_i: $U_i(t_i') \geq U_i(t_i) + E_{t_{-i}}[(v_i(t_i',t_{-i}) - v_i(t))q_i(t)]$, so $U_i(t_i)$ has derivative $\frac{dU_i(t_i)}{dt_i} = E_{t_{-i}}[\frac{\partial v_i(t)}{\partial t_i}\cdot q_i(t)] \equiv w_i(t_i)$, a.e., and $U_i(t_i) = U_i(\tau_i) + \int_0^{t_i} w_i(\tau)\,d\tau$. Thus,

$$\begin{aligned} E_{t_i}[U_i(t_i)] &= U_i(\tau_i) + \int_0^1 \int_0^{t_i} w_i(\tau)\,d\tau\, f_i(t_i)\,dt_i \\ &= U_i(\tau_i) + \int_0^1 (1 - F_i(t_i)) w_i(t_i)\,dt_i \quad \text{(by parts)} \\ &= U_i(\tau_i) + E_{t_i}\left(\frac{1 - F_i(t_i)}{f_i(t_i)} \cdot w_i(t_i)\right). \end{aligned}$$

Expected revenue is the expected value of the good to the winning bidders, $E_t(\sum_{i=1}^n q_i(t)v_i(t))$, less the expected payoff to the n bidders, $\sum_{i=1}^n E_{t_i}(U_i(t_i))$. Hence, expected revenue is

$$E_t\left(\sum_{i=1}^n \left(q_i(t)\left(v_i(t) - \frac{1 - F_i(t_i)}{f_i(t_i)} \cdot \frac{\partial v_i(t)}{\partial t_i}\right) - U_i(\tau_i)\right)\right) = E_t\left(\sum_{i=1}^n [q_i(t) MR_i(t) - U_i(\tau_i)]\right).$$

From Theorem 1, a revenue-maximizing seller will assign quantity in descending order of marginal revenue, and stop assigning when the good is exhausted or marginal revenue turns negative. Such an assignment can be made incentive compatible if bidder i's quantity $q_i(t)$ is weakly increasing in t_i. To guarantee this we require marginal revenue to satisfy

MR monotonicity. For all i, j, t, q_i, $\partial MR_i(t,q_i)/\partial t_i > 0$, $\partial MR_i(t,q_i)/\partial t_j \geq 0$, $\partial MR_i(t,q_i)/\partial q_i \leq 0$.

MR regularity. For all i, j, q_i, q_j, t_{-i}, and $t_i' > t_i$, $MR_i(t_i,t_{-i},q_i) > MR_j(t_i,t_{-i},q_j) \Rightarrow MR_i(t_i',t_{-i},q_i) > MR_j(t_i',t_{-i},q_j)$.

THEOREM 2. *Suppose that MR monotonicity and MR regularity are satisfied. The seller's expected revenue is maximized by awarding the good to those with the highest marginal revenues, until the good is exhausted or marginal revenue becomes negative.*

PROOF. Individual rationality requires $U_i(\tau_i) \geq 0$, so the best the seller can do is set $U_i(\tau_i) = 0$. Thus, the seller's optimization problem is to select $q(t) = (q_1(t),\ldots,q_n(t))$ to maximize (R), where for all t, $q(t) \in Q$. This problem is solved by pointwise optimization. Fix t. The seller should allocate the good to those with the highest marginal revenues, until quantity is exhausted or marginal revenue becomes negative. Since quantity is awarded in descending order of MR, MR regularity implies that $q_i(t)$ is weakly increasing in t_i, which is sufficient for $q(t)$ to be consistent with incentive compatibility.

Theorem 2 illustrates both inefficiencies of the optimal auction. First, since $v_i(t,q_i) > MR_i(t,q_i)$, it is possible for $v_i(t,q_i) > 0 > MR_i(t,q_i)$, in which case the seller inefficiently holds back quantity. Second, since the distribution of types differs across bidders, it is possible that $v_i(t,q_i) > v_j(t,q_j)$ and yet $MR_i(t,q_i) < MR_j(t,q_j)$. In this case, the seller may misassign quantity to j when i has a higher value. For example, if one of the bidders has a higher value ex ante, then the seller may improve revenues by requiring the ex ante strong bidder's bid to beat the others by a particular margin.

2.3 Settings without an incentive to misassign the good

The incentive to misassign the good is quite general. However, misassignment does vanish in some important special cases.

First, suppose bidders have flat demands and are ex ante symmetric:

Flat demands (constant marginal values). $\partial v_i(t,q_i)/\partial q_i = 0$, for $q_i \in [0,\lambda_i]$, so marginal value is $v_i(t)$.

Symmetry. For all i, j, $v_i(\ldots,t_i,\ldots,t_j,\ldots) = v_j(\ldots,t_j,\ldots,t_i,\ldots)$ and $F_i = F_j = F$.

In this case, we can restate the regularity conditions as:

Value regularity. A higher type has a weakly higher value: $t_i > t_j \Rightarrow v_i(t) \geq v_j(t)$.

MR regularity. A higher type has a higher marginal revenue: $t_i > t_j \Rightarrow MR_i(t) > MR_j(t)$.

PROPOSITION 1. *In a symmetric, flat demands model satisfying both value and MR regularity, then the seller can maximize revenues by awarding the good to those with the highest values.*

PROOF. From Theorem 1, the seller wants to assign the good to those with the highest marginal revenue, but by MR regularity, the highest types have the highest marginal revenues, and by value regularity the highest types have the highest values. Hence, assigning the good in order of marginal revenue also assigns the good in order of value. Moreover, MR regularity implies that $q_i(t)$ is weakly increasing in t_i, which is sufficient for $q(t)$ to be consistent with incentive compatibility.

From Myerson's (1981) single-good analysis, it is clear that the symmetry assumption is essential to Proposition 1. Can we relax the flat demands assumption to downward-sloping demands? The optimal selling procedure assigns the good based on the aggregate marginal revenue curve; whereas, an efficient auction assigns the goods based on the aggregate demand curve. With flat demands, the assignments based on aggregate demand and marginal revenue are identical, assuming ex ante symmetry. However, with downward-sloping demands, this typically is not be the case.

How the revenue-maximizing assignment distorts the efficient assignment depends on the distribution of private information. Suppose the bidders have separable inverse demands, $p_i(t,q_i) = v_i(t) - g_i(q_i)$, where $dg_i/dq_i > 0$ for all q_i. Further suppose that the intercept $v_i(t)$ satisfies the symmetry and value regularity assumptions of Proposition 1, and that the bidders' types are drawn independently from the distribution F. In this setting, any distortion depends on the hazard rate on types, $f(t_i)/(1-F(t_i))$.

PROPOSITION 2. *In the symmetric model with downward-sloping demands, assigning the good to those with the highest values maximizes revenue if the hazard rate on types is constant. However, if the hazard rate is increasing (decreasing), the optimal auction distorts the efficient assignment by shifting quantity away from (toward) low types.*

PROOF. From Theorem 1, the seller's expected revenue from an allocation $q(t) = (q_1(t),\ldots,q_n(t))$, where the lowest type bidders get 0, is

$$E_t\left[\sum_{i=1}^{n}\left(\int_0^{q_i(t)}\left[v_i(t) - g_i(x) - \frac{1-F(t_i)}{f(t_i)}\right]dx\right)\right].$$

If F has a constant hazard rate, then F is the exponential distribution, $F(t_i) = 1 - \exp(-t_i/\alpha)$ and $[1 - F(t_i)]/f(t_i) = \alpha$. Hence, the seller's optimization problem is to select an assignment $q(t) \in Q$ to maximize

$$E_t\left[\sum_{i=1}^{n}\left(\int_0^{q_i(t)}\left[v_i(t) - g_i(x) - \alpha\right]dx\right)\right].$$

By pointwise optimization, the solution is to assign the good to those with the highest marginal values subject to the reserve price $r = \alpha$, or if constrained to sell all units, the seller simply assigns the good to those with the highest marginal values. There is no incentive to misassign since marginal revenue for all bidders is just the true demand shifted down by a constant. If F has an increasing hazard rate, then $[1 - F(t_i)]/f(t_i)$ is decreasing in t_i. Thus, a high type's marginal revenue curve is shifted down less than a low type's marginal revenue curve. Hence, the seller, by assigning on the basis of marginal revenue rather than marginal value, misassigns in favor of the high types. Quantity is shifted away from low types.

Proposition 2 demonstrates that when bidders have downward-sloping demands the seller typically does have an incentive to misassign the good, except for a very special case. Proposition 2 also provides some intuition for how revenues from an efficient auction may compare with revenues from a uniform price auction. For example, if a bidder's type has an increasing hazard rate (e.g., is uniformly distributed), the revenue-maximizing assignment differs from an efficient assignment by shifting quantity away from the low-demand bidders (low types). However, a uniform-price auction tends to shift quantity *toward* small bidders, because of greater demand reduction by large bidders (Ausubel and Cramton 1998). Hence, this suggests that, with ex ante symmetric bidders, an efficient auction will revenue-dominate the uniform-price auction in the more typical case where the hazard rate is increasing.[4] At the very least, there should not be a presumption that efficient auctions perform poorly relative to other standard auctions, such as the uniform-price auction or the pay-your-bid auction. There is little evidence that these other standard auctions distort outcomes in ways that enhance revenues.

A final setting in which there is no conflict between efficiency and revenue maximization is where bidders receive no informational rents. Then marginal values and marginal revenues coincide. Cremer and McLean (1985, 1988) and McAfee and Reny (1992) show how the seller can extract the full surplus when bidders are risk neutral, when there is unlimited liability, and when private information is correlated. If the seller can extract all the surplus, then the seller can do no better than an efficient auction, since this maximizes the gains from trade, all of which are received by the seller.

3 OPTIMAL AUCTIONS RECOGNIZING RESALE

The optimal auction described above may be difficult for a seller to implement on two grounds: (1) it assumes that the bidders cannot engage in resale following the auction, and (2) it assumes that the

[4] When there are ex ante asymmetries among the bidders, then it would seem possible for the uniform-price auction to yield more revenue than an efficient auction. For example, if there are a number of ex ante weak bidders (low demands), then competition may be stimulated in an auction that gives these weak bidders more favorable treatment. The uniform-price auction effectively does just that. Participation by small bidders is encouraged, since they win larger quantities due to demand reduction by the stronger bidders. A uniform-price auction also has the advantage that it yields a greater diversity of winners, which reduces market power in the aftermarket.

seller can commit to not selling additional quantity after the initial auction. In this section, we will relax both of these assumptions. This will result in a total of three optimal auction programs:[5]

1. *Unconstrained optimal auction.* The seller can prevent resale and credibly hold back quantity.

2. *Resale-constrained optimal auction.* The seller can credibility hold back quantity but cannot prevent resale.

3. *Efficiency-constrained optimal auction.* The seller can neither prevent resale nor withhold quantity.

How resale effects the auction depends on what we assume about the resale market. We take the Coase (1960) theorem seriously and assume perfect resale. Resale causes any misassignment of the goods to be corrected. This is an extreme assumption. Certainly, there are settings where perfect resale is not possible, because of private information that the auction winners have after the auction (Myerson and Satterthwaite 1983; Cramton, Gibbons, Klemperer 1987). However, there are other settings where perfect resale is possible. Perfect resale has the significant advantage that it is a simple and general assumption on the resale market. Moreover, resale inevitable shifts outcomes toward the efficient assignment. Since resale is voluntary, resale can only occur if it creates gains from trade by shifting goods to higher value uses. To the extent that resale is successful, we view perfect resale as a first approximation of the outcome of the resale market.

We can apply Theorem 1 (Revenue Equivalence) to solve for each of the optimal auctions. In particular, we can focus solely on the assignment rule $q(t)$, since the payment rule $x(t)$ will be determined from incentive compatibility and the requirement that the lowest buyer types get a net payoff of 0. Consider an assignment rule $q(t) \in Q$. A reassignment q' of $q(t)$ is feasible if goods are not created or destroyed: $\sum_i(q_i(t) - q_i') = 0$. An assignment $q(t) \in Q$ is *resale-efficient* if for every $t \in T$, there does not exist a feasible reassignment q' of $q(t)$ such that, for all i, $v_i(t,q_i') \geq v_i(t,q_i(t))$ with at least one strict inequality. This definition requires all gains from trade *among bidders* to be realized. It permits the seller to inefficiently withhold quantity. An assignment rule $q(t)$ is *ex post efficient* if it is resale-efficient and for every $t \in T$, $q(t) \in \overline{Q}$. Ex post efficiency requires all gains from trade among the seller and bidders to be realized. Let Q^R be the set of all resale-efficient assignment rules and let $\overline{Q}^R$ be the set of all ex post efficient assignment rules.

Let $q^*(t)$, $q^R(t)$, and $\overline{q}^R(t)$ denote the assignment rule in the unconstrained, the resale-constrained, and the efficiency-constrained optimal auctions. Then provided an appropriate regularity condition is satisfied (which we discuss below), the optimal auctions can be stated as follows:

UNCONSTRAINED OPTIMAL AUCTION. Maximize the seller's expected revenues, under the hypothesis that the resale of objects among buyers is impossible:

[5] We do not treat the fourth case where the seller can forbid resale but cannot commit to restricting quantity, since we view forbidding resale as a more difficult task. The seller can unilaterally restrict quantity, but forbidding resale is a restriction on others. It may require enforcement mechanisms not available to the seller. Some procurement auctions are exceptions.

$$q^*(t) \in \arg\max_{q(t)\in Q} E_t\left[\sum_{i=1}^{n}\int_0^{q_i(t)} MR_i(t,y)\,dy\right].$$

RESALE-CONSTRAINED OPTIMAL AUCTION. Maximize the seller's expected revenues, subject to the constraint that there will be perfect resale among bidders after the auction:

$$q^R(t) \in \arg\max_{q(t)\in Q^R} E_t\left[\sum_{i=1}^{n}\int_0^{q_i(t)} MR_i(t,y)\,dy\right].$$

EFFICIENCY-CONSTRAINED OPTIMAL AUCTION. Maximize the seller's expected revenues, subject to the constraint that there will be perfect resale among *the seller and* bidders after the auction:

$$\bar{q}^R(t) \in \arg\max_{q(t)\in \bar{Q}^R} E_t\left[\sum_{i=1}^{n}\int_0^{q_i(t)} MR_i(t,y)\,dy\right].$$

In each case, the optimal assignment rule is found by pointwise optimization. Fix t. For ease of notation, drop the dependence on t and assume that marginal values and marginal values are strictly decreasing in quantity. Let $d_i(p)$ be i's demand curve (the inverse of $v_i(q_i)$); similarly, let $r_i(p)$ be the inverse of $MR_i(q_i)$. Then aggregate demand is $D(p) = \Sigma_i\, d_i(p)$ and $R(p) = \Sigma_i\, r_i(p)$. Inverting these curves, results in the aggregate inverse demand $p(\hat{q})$ and the aggregate marginal revenue $MR(\hat{q})$, where $\hat{q} = \Sigma_i\, q_i$. Both of these functions are continuous and strictly decreasing in $\hat{q}$. Let $\hat{q}^* = \min\{1,\ \hat{q} \text{ s.t. } MR(\hat{q}) = 0\}$.

The unconstrained problem is solved by assigning quantity in order of marginal revenue, until the good is exhausted or marginal revenue turns negative (Theorem 2): $q_i^* = r_i(MR(\hat{q}^*))$.

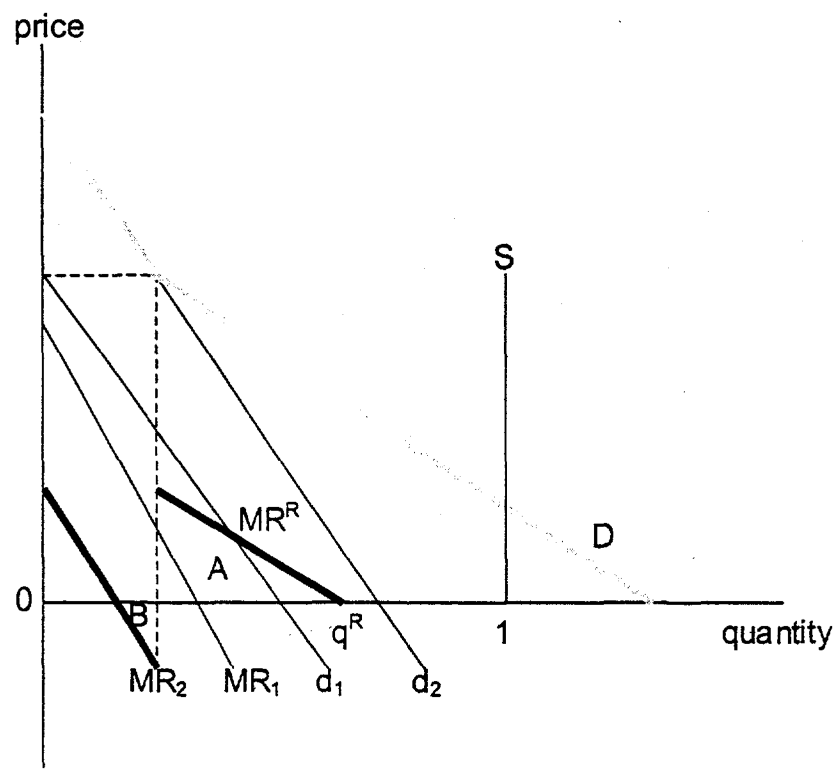
The efficiency-constrained problem is solved by assigning quantity in order of marginal value, until the good is exhausted: $\bar{q}_i^R = d_i(p(1))$.

The resale-constrained problem is solved by assigning the optimal quantity $\hat{q}^R$ in order of marginal value: $q_i^R = d_i(p(\hat{q}^R))$ .We determine the optimal quantity $\hat{q}^R$ as follows. As additional quantity is awarded, the fraction that is assigned to bidder i depends on the ratio of the slopes of bidder i's demand curve and the aggregate demand curve. Hence, the resale-constrained marginal revenue curve is simply the following weighted average of the marginal revenue curves:

$$MR^R(\hat{q}) = \sum_{i=1}^{n}\left(\frac{d_i'(p(\hat{q}))}{D'(p(\hat{q}))} MR_i(d_i(p(\hat{q})))\right).$$

Then $\hat{q}^R \in \arg\max_{\hat{q}\le 1} \int_0^{\hat{q}} MR^R(y)\,dy$. The optimum occurs either at 1 or at a point where MR^R is 0. Figure 1 gives an example with two bidders. The resale-constrained marginal revenue curve is neither continuous nor decreasing. It has a jump at every kink in the demand function, where a bidder is added to or dropped from the set of bidders that is receiving additional quantity at $\hat{q}$. In the figure, quantity is first awarded to bidder 2 and then to bidder 1, since bidder 2 has the higher demand curve. At the kink in the demand curve, bidder 1 begins receiving quantity, which causes a large jump up in MR^R, since bidder 1 has a high marginal revenue. Since the area in triangle A is bigger than the area in triangle B, the seller continues to award quantity until q^R is reached.

Figure 1



THEOREM 3. *Consider the mechanism* $\langle q,x\rangle$ *with* $q(t)$ *as specified below and* $x(t)$ *chosen to satisfy incentive compatibility such that the lowest type of each bidder gets a payoff of* 0. *Then:*

(i) $q^*(t)$ *solves the unconstrained optimal auction if MR is monotone and regular.*

(ii) $q^R(t)$ *solves the resale-constrained optimal auction if value is regular and MR is monotone.*

(iii) $\bar{q}^R(t)$ *solves the efficiency-constrained optimal auction if value is monotone and regular.*

PROOF. (i), (ii), and (iii) follow from Theorem 1, provided $q_i(t_i,t_{-i})$ is weakly increasing in t_i in each case. In case (i), from MR regularity, as i's type increases its MR ranking improves. Since quantity is assigned in order of marginal revenue, $q_i(t_i,t_{-i})$ is weakly increasing in t_i. In cases (ii) and (iii), from value regularity, as i's type increases its value ranking improves. Since quantity is assigned in order of marginal value, $q_i(t_i,t_{-i})$ is weakly increasing in t_i.

4 AN EFFICIENT AUCTION IS OPTIMAL WITH PERFECT RESALE

In general, optimal auctions take advantage of any ex ante asymmetries among bidders, the precise shape of demand curves, and the form and distribution of private information. Sellers generally have an incentive to misassign the good. However, misassignment means that there are gains from trade in the resale market. In the remainder of the paper, we assume that the seller cannot prevent resale. We show that the possibility of resale undermines the seller's ability to gain by misassigning the good. The best that the seller can do is conduct an efficient auction.

Our main result is that when a seller cannot prevent resale, the seller no longer has an incentive to misassign the good. The seller can do no better than to assign the good to those with the highest values. This is an extremely general result that does not rely on any of the standard assumptions of the optimal auction literature. We, therefore, introduce a general auction model.

There are n bidders ($i = 1,\ldots,n$) and one seller ($i = 0$). An assignment of goods is $q \in Q \equiv Q_1\times\cdots\times Q_n$ for the bidders and $q_0 \in Q_0$ for the seller. An assignment of money is $r \in \Re^n$ for the bidders and $r_0 \in \Re$ for the seller. An allocation of goods and money is $a \in A \equiv Q\times\Re^n$ for the bidders and $a_0 \in A_0 \equiv Q_0 \times\Re$ for the seller. Trade neither creates nor destroys money or goods. Given an initial allocation $(\hat{a}_0,\hat{a})$, the final allocation (a_0,a) is feasible if budget balance is satisfied in both money and goods:

(BB) $$\sum_{i=0}^{n}(a_i - \hat{a}_i) = 0 .$$

Given an allocation a, a reallocation a' among the bidders is feasible if budget balance is satisfied in both money and goods:

(BB′) $$\sum_{i=1}^{n}(a_i' - a_i) = 0 .$$

Bidder i's private information is its type $t_i \in T_i$ (possibly multidimensional). A realization of types is $t \in T \equiv T_1\times\cdots\times T_n$, and $t_{-i} = t\backslash t_i$. Types are drawn from the probability measure F on T and $dF_i(t_{-i}|t_i)$ is the conditional probability of t_{-i} given t_i. Utility is $u_i(t,a)$ for bidder i and $u_0(a_0)$ for the seller. This specification allows for asymmetries, externalities, complementarities, and risk aversion.

A direct mechanism, $\phi(a|t)$, is a probability measure on A for each $t \in T$. Bidder i's interim utility from reporting t_i' when its true type is t_i in the direct mechanism is

$$U_i(t_i'|t_i) = \int_{t_{-i}\in T_{-i}} \int_{a\in A} u_i(t_i,t_{-i},a)\,d\phi(a|t_i',t_{-i})\,dF_i(t_{-i}|t_i),$$

and $U_i(t_i) \equiv U_i(t_i|t_i)$ is t_i's equilibrium payoff. Utility is normalized so that the status quo yields an interim payoff of 0 for each bidder. The direct mechanism ϕ is incentive compatible if

(IC) $U_i(t_i) \geq U_i(t_i'|t_i)$ for all $t_i,t_i' \in T_i$,

and individually rational if

(IR) $U_i(t_i) \geq 0$ for all $t_i \in T_i$.

An allocation rule $a(t)$ is *resale-efficient* if for every $t \in T$, there does not exist a feasible reallocation a' of $a(t)$ such that for all $i=1,\ldots,n$, $u_i(t,a') \geq u_i(t,a(t))$ with at least one strict inequality. An allocation rule $(a_0,a(t))$ is *ex post efficient* if for every $t \in T$, there does not exist a feasible reallocation (a_0',a') of $a(t)$ such that for all $i=1,\ldots,n$,

(EE) $u_i(t,a') \geq u_i(t,a(t))$ and $u_0(a_0') \geq u_0(a_0)$ with at least one strict inequality.

A reallocation a' of $a(t)$ is an individually rational reallocation if $u_i(t,a') \geq u_i(t,a)$. Define:

$Eff(t,a) = \{a' \in A$ s.t. a' is a feasible and IR reallocation from $a(t)$ and a' is resale-efficient$\}$

$Eff(t) = \{a' \in A$ s.t. a' is a feasible allocation and a' is resale-efficient$\}$

Given an initial allocation $(\hat{a}_0,\hat{a})$, the seller's optimal auction program is

$$\max_{\phi} \int_{t\in T} \int_{a\in A} u_0(a_0)\,d\phi(a,t)\,dF(t)$$

subject to (IC), (IR), and (BB).

The seller's resale-constrained optimal auction program is the same with the added constraint:

(RE) $$\phi(A'|t) = 0 \text{ if } A' \cap Eff(t) = \varnothing.$$

We need to define an auction followed by perfect resale. Let $a \in A$ be the allocation entering the resale round. Let $t \in T$ be the type vector expressed by the bidders in the resale round, and let $\phi(\cdot|t)$ be the probability measure on bidder allocations at the end of resale. We call this perfect resale if, for every $A' \subset A$: $A' \cap Eff(t,a) = \varnothing \Rightarrow \phi(A'|t) = 0$.

THEOREM 4. *The seller's expected utility from any auction followed by perfect resale can be no greater than the solution to the seller's resale-constrained optimal auction program. Hence, a revenue-maximizing seller assigns goods to bidders so as to exhaust all gains from trade among the bidders.*

PROOF. Consider any equilibrium σ of the auction plus perfect resale and let $\phi(a|t)$ denote the probability measure on outcomes of σ as a function of the type realization. $\phi(a|t)$ is resale-efficient, since for any $A' \subset A$, if $A' \cap Eff(t) = \varnothing$, then $A' \cap Eff(t,a) = \varnothing$, for all $a \in A$, and so by the definition of perfect resale, $\phi(A'|t) = 0$. Since participation in the auction plus resale is voluntary, $\phi(a|t)$ must satisfy (IR). Finally, one deviation available to type t_i of bidder i, but by no means the only available deviation, is to pose as type t_i' in both the auction and the resale round. In order for all such deviations to be unprofitable, $\phi(a|t)$ must satisfy (IC). Since the outcome of σ satisfies all the constraints of the seller's resale-constrained optimal auction program, the seller's expected utility from σ can be no greater than the solution to this program.

5 THE SUBOPTIMALITY OF BEING INEFFICIENT

In the prior section, we proved generally that a seller, faced with a perfect resale market, does best by holding an efficient auction. In this section, we demonstrate the stronger result that an inefficient auction, when followed by perfect resale, yields strictly lower expected revenues than an efficient auction. This suggests a general prescription for auction design that, when perfect resale is a good approximation, a revenue-maximizing seller may do best by selecting an auction which makes resale unnecessary.

To keep the analysis manageable, we will consider the identical-object model of Section 3, but with discrete types. We begin by modifying the usual optimal-auctions apparatus to accommodate discrete types.

5.1 The optimal auction with discrete types

There are n bidders and a divisible good. Bidder i's private information is its type $t_i \in T_i$, where we will now assume that $T_i \equiv \{t_i^1,\ldots,t_i^{K_i}\}$ is a finite set, and $t_i^1 < \cdots < t_i^{K_i}$. A realization of types is denoted $t \in T \equiv T_1 \times \cdots \times T_n$, and $t_{-i} = t \backslash t_i$. Types are drawn independently according to the probability distribution $F_i(\cdot)$ on T_i, where $F_i(t_i^k) \equiv \Pr(t_i \le t_i^k)$, $f_i(t_i^k) \equiv \Pr(t_i = t_i^k)$, and $f_{-i}(t_{-i}) \equiv \Pr(t_{-i}) = \prod_{j \ne i} f_j(t_j^k)$. We assume that $F_i(\cdot)$ has full support on T_i, i.e., $f_i(t_i^k) > 0$ for all i and k. As one useful additional piece of notation, if $t_i = t_i^k$, then we will write t_i^+ to mean t_i^{k+1}. When there is no ambiguity, we may also write t^+ to mean (t_i^{k+1}, t_{-i}). Define bidder i's interim value if it is type t_i^k and reports t_i^l to be

$$V_i(t_i^l | t_i^k) \equiv E_{t_{-i}}\left(\int_0^{q_i(t_i^l, t_{-i})} v_i(t_i^k, t_{-i}, y)\, dy \right),$$

and let $X_i(t_i^k) = E_{t_{-i}}[x_i(t_i^k, t_{-i})]$ be bidder i's interim payment from reporting t_i^k.

Analogous to the standard treatment of continuous types, it is possible to define marginal revenue functions as well as regularity conditions so that, when the seller solves any of the unconstrained, resale-constrained, or efficiency-constrained optimal auction programs, the resulting mechanism $\langle q,x \rangle$ has the property that $q_i(t_i,t_{-i})$ is weakly increasing in t_i. We briefly develop these features as follows. The following notation will facilitate the exposition. For any i, k, l, let $\text{IC}_i(k,l)$ denote bidder i's incentive-compatibility constraint that type t_i^k finds mimicking type t_i^l unprofitable:

$$V_i(t_i^k | t_i^k) - X_i(t_i^k) \ge V_i(t_i^l | t_i^k) - X_i(t_i^l).$$

Let $\text{IR}_i(k)$ denote bidder i's individual-rationality constraint that type t_i^k earns a nonnegative payoff from participating:

$$V_i(t_i^k | t_i^k) - X_i(t_i^k) \ge 0.$$

We have

LEMMA 1. *Suppose that $q_i(t_i,t_{-i})$ is a weakly increasing function of t_i, for every $i = 1,\ldots,n$, and $t_{-i} \in T_{-i}$. Also suppose that the transfer function x maximizes the seller's expected profits over all mechanisms $\langle q,x \rangle$ that satisfy IC and IR. Then the incentive constraints for nonconsecutive types are redundant.*

PROOF. For any $k > l > m$, we will demonstrate that $IC_i(k,l)$ and $IC_i(l,m)$ imply $IC_i(k,m)$, establishing that the latter constraint is redundant. Adding $IC_i(k,l)$ and $IC_i(l,m)$ yields

$$V_i(t_i^k|t_i^k) - X_i(t_i^k) \geq V_i(t_i^m|t_i^k) - X_i(t_i^m) + \{[V_i(t_i^l|t_i^k) - V_i(t_i^l|t_i^l)] - [V_i(t_i^m|t_i^k) - V_i(t_i^m|t_i^l)]\}.$$

The expression in braces may be expanded to

$$\sum_{t_{-i}\in T_{-i}} \int_{q_i(t_i^m,t_{-i})}^{q_i(t_i^l,t_{-i})} [v_i(t_i^k,t_{-i},y) - v_i(t_i^l,t_{-i},y)]dy\, f_{-i}(t_{-i}).$$

Since $q_i(t_i,t_{-i})$ is a weakly increasing function of t_i, $q_i(t_i^l,t_{-i}) \geq q_i(t_i^m,t_{-i})$. By value monotonicity, $v_i(t_i^k,t_{-i},y) \geq v_i(t_i^l,t_{-i},y)$, for every t_{-i} and y. Thus, the expression in braces is nonnegative, allowing us to conclude that $IC_i(k,m)$ is automatically satisfied.

Iterative application of this result immediately shows that the incentive constraints for consecutive types imply all the other incentive constraints. This establishes that the incentive constraints for nonconsecutive types are redundant.

LEMMA 2. *Suppose that $q_i(t_i,t_{-i})$ is a weakly increasing function of t_i, for every $i = 1,\ldots,n$. Also suppose that the transfer function x maximizes the seller's expected profits over all mechanisms $\langle q,x\rangle$ that satisfy IC and IR. Then, for every $k = 2,\ldots,K_i$, constraint* $IC_i(k,k-1)$ *is binding.*

PROOF. Suppose not. Then there exists $k\geq 2$ such that

$$\varepsilon \equiv [V_i(t_i^k|t_i^k) - X_i(t_i^k)] - [V_i(t_i^{k-1}|t_i^k) - x_i(t_i^{k-1})] > 0.$$

Consider any alternative payment rule, x', which is selected so that $X_i'(t_i)$ satisfies

$$X_i'(t_i^l) = \begin{cases} X_i(t_i^l) & \text{if } l < k \\ X_i(t_i^l) + \varepsilon & \text{if } l \geq k. \end{cases}$$

Observe that the incentive constraints $IC_i(l,l-1)$ and $IC_i(l-1,l)$, for $l < k$ and $l > k$, continue to be satisfied by $\langle q,x'\rangle$. Meanwhile, $IC_i(k-1,k)$ has been loosened, and $IC_i(k,k-1)$ continues to be satisfied by construction. Finally, $IR_i(1)$ continues to be satisfied by $\langle q,x'\rangle$, while $IC_i(k+1,k)$ and $IR_i(k)$ inductively imply $Ir_i(k+1)$. Since $\langle q,x'\rangle$ yields strictly greater expected revenue than $\langle q,x\rangle$ while still satisfying all the requisite constraints, we conclude that the hypothesis that the transfer function x maximizes the seller's expected profits over all direct mechanisms $\langle q,x\rangle$ is violated. This contradiction proves the lemma.

In light of Lemmas 1 and 2, it is sensible to examine direct mechanisms, $\langle q,x\rangle$, with the properties that constraint $IC_i(k,k-1)$ is binding and $q_i(t_i,t_{-i})$ is a weakly increasing function of t_i. (The latter property will soon be guaranteed by a regularity condition.) Let $U_i(t_i^k)$ denote the equilibrium utility attained by type t_i^k, and define

$$\Delta_i^k \equiv E_{t_{-i}}\left(\int_0^{q_i(t_i^{k-1},t_{-i})}\left[v_i(t_i^k,t_{-i},y) - v_i(t_i^{k-1},t_{-i},y)\right]dy\right).$$

The fact that constraint $IC_i(k,k-1)$ is binding implies $U_i(t_i^k) = U_i(t_i^{k-1}) + \Delta_i^k$, and so

$$U_i(t_i^k) = U_i(t_i^1) + \sum_{j=2}^{k} \Delta_i^j .$$

Steps analogous to the standard derivation lead us to define the discrete version of the marginal revenue function:

$$MR_i(t,y) = v_i(t,y) - \frac{1-F_i(t_i)}{f_i(t_i)}[v_i(t^+,y) - v_i(t,y)] .$$

The seller's problem is then to select $\{q_1(t),\ldots,q_n(t)\}$ which maximizes

$$E_t\left[\sum_{i=1}^{n}\left(-U_i(t_i^1) + \int_0^{q_i(t)} MR_i(t,y)\,dy\right)\right]$$

pointwise, for all $t \in T$.

5.2 An inefficient auction does strictly worse than an efficient auction

We now demonstrate that, in an auction followed by perfect resale, the seller does strictly worse than optimal if the goods are assigned at auction in such a way that resale is required.

Let $v_{-i}(t,q_{-i})$ denote the opportunity cost of bidder i winning additional quantity, i.e., the marginal value of additional quantity allocated efficiently among bidders other than bidder i, when the state is t and a quantity of q_{-i} is already allocated efficiently among bidders other than bidder i. We require

High Type Condition. If λ_i is the maximum quantity that bidder i can win at auction, then

$$v_i(t_i^{K_i}, t_{-i}, \lambda_i) \geq v_{-i}(t_i^{K_i}, t_{-i}, 1-\lambda_i), \text{ for all } t_{-i} \in T_{-i} .$$

DEFINITION. *In a monotonic auction, the quantity assigned to bidder i in state* (t_i,t_{-i}) *is weakly increasing in* t_i.

THEOREM 5. *Consider a monotonic auction followed by strictly-individually-rational, perfect resale, in a setting satisfying the regularity condition and the high type condition. If, in any equilibrium* σ, *the ex ante probability of resale is strictly positive, then the seller's expected revenues are strictly less than the optimum (while an efficient auction attains the optimum).*

PROOF. Let $q_i(t)$ denote the quantity owned by bidder i after resale, and let $x_i(t)$ denote its combined net payment in the auction plus resale, when the bidders' types are t and the equilibrium σ is played in the auction plus resale. Then $\langle q,x \rangle$ may be viewed as a direct mechanism.

Suppose, contrary to the conclusion of the Theorem, that the ex ante probability of resale is strictly positive under σ, but that seller revenues are optimized. We will establish a contradiction. Since resale is assumed perfect, $\langle q,x\rangle$ must solve the seller's resale-constrained optimal auction problem. By Theorem 1, $q_i(t_i,t_{-i})$ is a weakly increasing function of t_i, and by Lemmas 1 and 2, the downward incentive constraints between consecutive types are binding. Let bidder i be one of the bidders whose ex ante probability of reselling is positive, and define

$$\bar{k} = \max \{k \mid \text{type } t_i^k \text{ of bidder } i \text{ resells in equilibrium } \sigma \text{ with positive probability}\}.$$

By the high type condition, observe that $\bar{k} < K_i$. Furthermore, observe from the definition of $\bar{k}$ that

$$(*) \qquad v_i(t_i^{\bar{k}+1},t_{-i},q_i(t_i^{\bar{k}+1},t_{-i})) \geq v_{-i}(t_i^{\bar{k}+1},t_{-i},1-q_i(t_i^{\bar{k}+1},t_{-i})), \text{ for all } t_{-i} \in T_{-i},$$

since otherwise, type $t_i^{\bar{k}+1}$ of bidder i would also have a positive probability of trade in a perfect resale round.

Now suppose that $t_i^{\bar{k}+1}$ mimics $t_i^{\bar{k}}$ in the auction. By inequality (*), even if bidder i owned $q_i(t_i^{\bar{k}+1},t_{-i})$ units at the end of the auction, its marginal unit would be worth no more to bidders $-i$ than to type $t_i^{\bar{k}+1}$ of bidder i. By the hypothesis that the auction is monotonic, $q_i(t_i^{\bar{k}},t_{-i}) \leq q_i(t_i^{\bar{k}+1},t_{-i})$. By the hypothesis of weakly diminishing marginal values, given that bidder i only owns $q_i(t_i^{\bar{k}},t_{-i})$ units at the end of the auction, its marginal unit is certainly worth no more to bidders $-i$ than to type $t_i^{\bar{k}+1}$ of bidder i. Thus, by the hypothesis that resale is strictly individually-rational, type $t_i^{\bar{k}+1}$ of bidder i would find it strictly unprofitable to resell to other bidders.

Let $\tilde{U}_i(t_i^k|t_i^l)$ denote the optimal payoff to t_i^l from mimicking t_i^k in the auction but then continuing optimally (given its true type) in the resale round. By contrast, let $U_i(t_i^k|t_i^l)$ denote the payoff to t_i^l from mimicking t_i^k in the auction and then being forced to continue to mimic t_i^k in the resale round. The previous paragraph has established that $\tilde{U}_i(t_i^k|t_i^{k+1}) > U_i(t_i^k|t_i^{k+1})$. Meanwhile, observe that $\tilde{U}_i(t_i^{k+1}|t_i^{k+1}) = U_i(t_i^{k+1}|t_i^{k+1})$. Consequently, the fact (from Lemma 2) that $U_i(t_i^k|t_i^{k+1}) = U_i(t_i^{k+1}|t_i^{k+1})$ implies that $\tilde{U}_i(t_i^k|t_i^{k+1}) > \tilde{U}_i(t_i^{k+1}|t_i^{k+1})$, yielding a profitable deviation for type t_i^{k+1} in the auction followed by resale, and hence contradicting the hypothesis that σ is an equilibrium. We conclude that the seller's revenues are strictly less than optimal.

6 IMPLEMENTING AUCTIONS WITH RESALE

Our final question is: Can the seller implement the efficiency-constrained optimal auction with an auction followed by resale? Since this two-stage game (auction plus resale) has additional constraints not present in the direct mechanism, adding the possibility of resale may prevent the seller from implementing an efficient auction. We first demonstrate that the Vickrey auction is not distorted by resale. Then we determine the circumstances under which a Vickrey auction implements the efficiency-constrained optimal auction. Unlike some of our earlier results, this result does not depend on perfect resale – any individually-rational resale will do. The result also allows correlated types, asymmetries, and dissimilar objects. However, we assume private values: a bidder's value only depends on its own type and not the types of the others. This represents the most general conditions under which there is a dominant strategy efficient mechanism (Vickrey 1961; Clarke 1971; Groves 1973).

6.1 The Vickrey auction is not distorted by the possibility of resale

Consider the general model of section 3 with the following restrictions. Let Q be the set of all possible assignments of the goods. A bidder's value $v_i(t_i,q)$ for the assignment $q \in Q$ only depends on its own type t_i. Also, a bidder's utility is its value less the amount it pays: $v_i(t_i,q) - x_i$.

In the Vickrey auction, the bidders report their types and then the seller selects the efficient assignment; that is, $q^*(t)$ that maximizes $v_1(t_1,q^*)+\cdots+v_n(t_n,q^*)$. In the Vickrey auction, bidder i pays the opportunity cost of its influence on the assignment: the best that the others can do without bidder i less what the others get in the best assignment with i. Let $v_{-i}(t) = \sum_{j\neq i} v_j(t_j, q^*(t))$ denote what the others get in the best assignment with i. Then bidder i's Vickrey payment is

$$x_i(t) = \max_{q\in Q}\left\{\sum_{j\neq i} v_j(t_j,q)\right\} - v_{-i}(t).$$

With this payment, bidder i gets a payoff $v_i(t_i,q^*) - x_i(t)$ equal to the incremental value that i brings to the auction.

THEOREM 6. *Suppose that each bidder's value depends exclusively on its own type and on the overall assignment of the goods. Consider the Vickrey auction followed by any arbitrary, individually-rational resale. Then sincere bidding – followed by no resale – is an equilibrium of the two-stage game.*

PROOF. From the definition of the Vickrey pricing rule, for all i, t_i, t_i', and t_{-i}:

$$x_i(t_i',t_{-i}) - x_i(t_i,t_{-i}) = v_{-i}(t_i,t_{-i}) - v_{-i}(t_i',t_{-i}).$$

Integrating with respect to $dF_i(t_{-i}|t_i)$:

$$(*) \qquad X_i(t_i'|t_i) - X_i(t_i|t_i) = V_{-i}(t_i|t_i) - V_{-i}(t_i'|t_i).$$

Now define $\Delta_i(t_i'|t_i)$ to be the expected available gains from trade if bidder i misreports t_i' when its true type is t_i: $\Delta_i(t_i'|t_i) = V_i(t_i|t_i) + V_{-i}(t_i|t_i) - V_i(t_i'|t_i) - V_{-i}(t_i'|t_i)$. Given that resale is individually rational, bidder i who misreports t_i' when its true type is t_i cannot expect to earn more than $\Delta_i(t_i'|t_i)$ in the resale round. Let $\pi_i(t_i'|t_i)$ be the gain to bidder i from misreporting t_i' when its true type is t_i. Then

$$\pi_i(t_i'|t_i) \leq V_i(t_i'|t_i) - X_i(t_i'|t_i) + \Delta_i(t_i'|t_i) - [V_i(t_i|t_i) - X_i(t_i|t_i)].$$

Substituting the definition of $\Delta_i(t_i'|t_i)$ gives

$$\pi_i(t_i'|t_i) \leq V_{-i}(t_i|t_i) - V_{-i}(t_i'|t_i) - X_i(t_i'|t_i) + X_i(t_i|t_i),$$

and substituting (*) yields $\pi_i(t_i'|t_i) \le 0$. Thus, we conclude that any misreporting of type remains unprofitable when resale is possible.

6.2 The Vickrey auction implements the efficiency-constrained optimal auction with resale

Now return to the identical objects model of Section 2 (with continuous types), for which the optimal auctions were defined, but restrict the setting to private values: $v_i(t_i,q_i)$. One might guess that Theorem 6 immediately implies that a Vickrey auction implements the efficiency-constrained optimal auction. However, this is not the case unless the lowest type bidders get an expected payoff of 0: $U_i(\tau_i) = 0$. Of course, we could add $U_i(\tau_i)$ to bidder i's payment for all t to assure that the worst-off type of bidder i gets an expected payoff of 0. This modified Vickrey auction would still satisfy incentive compatibility and individual rationality as a direct mechanism. However, a bidder may do better by not participating in the auction, and then participating in resale.

As an example, consider a single-good auction with two bidders, a weak bidder with value v_W on [0,1] and a strong bidder with value v_S on [11,12]. In the Vickrey auction, the strong bidder always gets the good and pays the weak bidder's value, for a net gain of $v_S - v_W$. In the modified Vickrey auction, the strong bidder always gets the good and pays 11, for a net gain of $v_S - 11$. Now suppose the strong bidder decides to bypass the Vickrey auction. Then if we assume the two bidders split-the-difference at resale, they would settle at a price of about 6. But then the strong bidder can do better by avoiding the modified Vickrey auction. In general, this incentive to bypass the auction goes away only if we make the implausible assumption that winners in the auction get all of the gains from trade in resale.

To avoid this possibility, we need a condition that guarantees that $U_i(\tau_i) = 0$ in the Vickrey auction. This condition is essentially that the lowest type of a bidder never adds value to the group of bidders.

Low type condition. For all i and t_{-i},

$$\max_{q\in Q}\left\{v_i(\tau_i,q_i)+\sum_{j\neq i}v_j(t_j,q_j)\right\}=\max_{q\in Q}\sum_{j\neq i}v_j(t_j,q_j).$$

For example, if $v_i(\tau_i,q_i) = 0$ for all i and q_i, then the low type condition is satisfied. Krishna and Perry (1997) assume the low type condition and prove that the Vickrey auction implements the efficiency-constrained optimal auction in a market without resale. Theorem 6 then implies that the Vickrey auction can be successfully embedded in a market with resale. We have

COROLLARY 1. *Suppose the low type condition is satisfied. Then a Vickrey auction implements the efficiency-constrained optimum in an auction followed by resale.*

PROOF. From Theorem 6, sincere bidding, followed by no resale, is an equilibrium in the two-stage game. Hence, the resulting assignment q^* is efficient. The lowest type of bidder i has an ex post payoff of

$$v_i(\tau_i,q_i^*)-x_i(\tau_i,t_{-i})=v_i(\tau_i,q_i^*)+\sum_{j\neq i}v_j(t_j,q_j^*)-\max_{q\in Q}\sum_{j\neq i}v_j(t_j,q_j)=0,$$

where the first equality follows from substituting the Vickrey pricing rule and the second follows from the low type condition. Taking the expectation over t_{-i}, we have $U_i(\tau_i) = 0$, as required.

The low type condition fails in settings where it is commonly known that one bidder has a higher value than the others. The Vickrey auction may perform poorly in this case. This can be overcome by the use of an ex ante reserve price.

7 CONCLUSION

This paper has shown that, in auction markets followed by perfect resale, it is "optimal" to be "efficient." Theorem 4 established that the seller's payoff from using any auction format is never greater than from using the payoff-maximizing efficient auction (followed by no resale). Theorem 5 established that, with somewhat more structure placed on the problem, the seller's payoff from using an inefficient auction format is *strictly less* than from using an efficient auction. The intuition for these results is that the end outcome of the auction-plus-resale process may itself be viewed as a static direct mechanism, and therefore it must satisfy the usual conditions of incentive compatibility and individual rationality. Meanwhile, the two-period trading process introduces the possibility that a bidder may pose as one type in the auction round but as a second type in the resale round, adding extra incentive constraints to the problem.

The analysis of an auction followed by perfect resale motivates a resale-constrained static optimal auction program, while the analysis of an auction followed by *ex post* efficient trade motivates an efficiency-constrained static optimal auction program. Each of these new optimal auction programs is of the same level of difficulty as the standard (unconstrained) optimal auction program in the literature, and possesses an analogous solution (Theorem 3). Naturally, each of the constrained optimal auction programs requires its own "regularity" condition in order to yield a well-behaved solution, but the new regularity conditions are actually less onerous than the regularity condition required for the unconstrained optimal auction program.

Our results thus provide a new defense for emphasizing *efficient* auction design rather than *optimal* auction design. The presence of a perfect resale market forces even the most selfish seller, whose sole objective is maximizing revenues, to focus — out of necessity — on efficiency.

While the Coasean assumption of perfect resale is extreme, it is no more extreme than the standard assumption of no resale that the auction literature routinely makes. Thus, we would argue that the auction model with perfect resale should be used — as a companion to the usual auction model without resale — as an easily-tractable baseline for analyzing auction questions. There seems to be no broadly-convincing reason why one model or the other should be thought to be the more realistic depiction of general environments, yet the policy conclusions from considering the disparate models may often be quite different.

One important example of the differing policy conclusions that the two models yield is in the analysis of the revenue properties of alternative formats for the Treasury auction. In the model without resale, the revenue ranking of the pay-your-bid auction, uniform-price auction and Vickrey auction is inherently ambiguous (Ausubel and Cramton 1998). However, in a private-values model with perfect resale, this paper has shown that the Vickrey auction *unambiguously* outperforms the pay-your-bid and uniform-price auctions, in terms of expected revenues. Given the vast and active resale market in Treasury securities, it seems safe to assert that the model with perfect resale is a better description of the U.S. Treasury market than the model without any resale, and so its predictions ought to be taken more seriously.

REFERENCES

Agastya, Murali and Arupratan Daripa (1998), "Auctions with Futures and Resale: Winner's Curse vs. Losers' Nightmare," Working Paper, University College London.

Armstrong, Mark (1997), "Optimal Multi-Object Auctions," Oxford University, Working Paper.

Ausubel, Lawrence M. (1997), "An Efficient Ascending-Bid Auction for Multiple Objects," Working Paper 97-06, University of Maryland.

Ausubel, Lawrence M. and Peter Cramton (1998), "Demand Reduction and Inefficiency in Multi-Unit Auctions," Working Paper, University of Maryland.

Ausubel, Lawrence M. and Raymond J. Deneckere (1989), "Reputation in Bargaining and Durable Goods Monopoly," *Econometrica*, 57, 511-532.

Avery, Christopher and Terrance Hendershott (1998), "Bundling and Optimal Auctions of Multiple Goods," Working Paper, Harvard University.

Bikhchandani, Sushil and Chi-fu Huang (1989), "Auctions with Resale Markets: An Exploratory Model of Treasury Bill Markets," *Review of Financial Studies*, 2, 311-339.

Bulow, Jeremy and Paul Klemperer (1996), "Auctions vs. Negotiations," *American Economic Review*, 86, 180-194.

Caillaud, Bernard and Jacques Robert (1998), "Implementing the Optimal Auction," University of Montreal, Working Paper.

Clarke, Edward H. (1971), "Multipart Pricing of Public Goods," *Public Choice*, 11, 17-33.

Coase, Ronald H. (1960), "The Problem of Social Cost," *Journal of Law and Economics*, 3, 1-44.

Coase, Ronald H. (1972), "Durability and Monopoly," *Journal of Law and Economics*, 15, 143-149.

Cramton, Peter, Robert Gibbons, and Paul Klemperer (1987), "Dissolving a Partnership Efficiently," *Econometrica*, 55, 615-632.

Cremer, Jacques and Richard McLean (1988), "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56, 1247-1258.

Cremer, Jacques and Michael H. Riordan (1985), "A Sequential Solution to the Public Goods Problem," *Econometrica*, 53, 77-84.

Dasgupta, Partha and Eric Maskin (1997), "Notes on Efficient Auctions," Working Paper, Harvard University.

Engelbrecht-Wiggans, Richard (1988), "Revenue Equivalence in Multi-object Auctions," *Economic Letters*, 26, 15-19.

Groves, Theodore (1973), "Incentives in Teams," *Econometrica*, 41, 617-631.

Haile, Philip A. (1996), "Auctions with Resale Markets: An Application to U.S. Forest Service Timber Sales," Working Paper, University of Wisconsin.

Haile, Philip A. (1997), "Auctions with Private Uncertainty and Resale Opportunities," Working Paper, University of Wisconsin.

Haile, Philip A. (1998), "Auctions with Resale," Working Paper, University of Wisconsin.

Harstad, Ronald M. (1990), "Alternative Common-Value Auction Procedures: Revenue Comparisons with Free Entry," *Journal of Political Economy*, 98, 421-429.

Harstad, Ronald M. (1993), "Auctions with Endogenous Bidder Participation," Working Paper, Rutgers University.

Horstmann, J. Ignatius and Chantale LaCasse (1997), "Secret Reserve Prices in a Bidding Model with a Resale Option," *American Economic Review*, 87, 663-684.

Krishna, Vijay and Motty Perry (1997), "Efficient Mechanism Design," Working Paper, Penn State University.

Levin, Dan and James L. Smith (1994), "Equilibrium in Auctions with Entry," *American Economic Review*, 84, 585-599.

Levin, Dan and James L. Smith (1995), "Auctions with Entry: An Experimental Investigation," Working Paper, Ohio State University.

Levin, Dan and James L. Smith (1996), "Optimal Reservation Prices in Auctions," *Economic Journal*, 106, forthcoming.

Maskin, Eric and John Riley (1989), "Optimal Multi-unit Auctions," in Frank Hahn (ed) *The Economics of Missing Markets, Information, and Games*, Oxford: Oxford University Press, 312-35.

McAfee, R. Preston and John McMillan (1987), "Auctions with Entry," *Economic Letters*, 23, 343-347.

McAfee, Preston R. and Philip Reny (1992), "Correlated Information and Mechanism Design," *Econometrica*, 60, 395-422.

Myerson, Roger B. (1981), "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58-73.

Myerson, Roger B. and Mark A. Satterthwaite (1983), "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 28, 265-281.

Vickrey, William (1961), "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, 16, 8-37.

Viswanathan, S. and James J.D. Wang (1996), "Auctions with When-Issued Trading: A Model of the U.S. Treasury Markets," Working Paper, Duke University.

Policy Research Working Paper Series

	Title	Author	Date	Contact for paper
WPS1965	Manufacturing Firms in Developing Countries: How Well Do They Do, and Why	James Tybout	August 1998	L. Tabada 36869
WPS1966	Sulfur Dioxide Control by Electric Utilities: What Are the Gains from Trade?	Curtis Carlson Dallas Burtraw Maureen Cropper Karen L. Palmer	August 1998	T. Tourougui 87431
WPS1967	Agriculture and the Macroeconomy	Maurice Schiff Alberto Valdés	August 1998	A. Valdés 35491
WPS1968	The Economics and Law of Rent Control	Kaushik Basu Patrick Emerson	August 1998	M. Mason 30809
WPS1969	Protecting the Poor in Vietnam's Emerging Market Economy	Dominique van de Walle	September 1998	C. Bernardo 31148
WPS1970	Trade Liberalization and Endogenous Growth in a Small Open Economy: A Quantitative Assessment	Thomas F. Rutherford David G. Tarr	September 1998	L. Tabada 36896
WPS1971	Promoting Better Logging Practices in Tropical Forests	Marco Boscolo Jeffrey R. Vincent	September 1998	T. Tourougui 87431
WPS1972	Why Privatize? The Case of Argentina's Public Provincial Banks	Geroge R. G. Clarke Robert Cull	September 1998	P. Sintim-Aboagye 38526
WPS1973	The Economic Analysis of Sector Investment Programs	Sethaput Suthiwart-Narueput	September 1998	C. Bernardo 31148
WPS1974	Volatility and the Welfare Costs of Financial Market Integration	Pierre-Richard Agénor Joshua Aizenman	September 1998	S. King-Watson 33730
WPS1975	Acting Globally While Thinking Locally: Is the Global Environment Protected by Transport Emission Control Programs	Gunnar S. Eskeland Jian Xie	September 1998	C. Bernardo 31148
WPS1976	Capital Flows to Central and Eastern Europe and the Former Soviet Union	Stijn Claessens Daniel Oks Rossana Polastri	September 1998	R. Vo 33722
WPS1977	Economic Reforms in Egypt: Emerging Patterns and Their Possible Implications	Rania A. Al-Mashat David A. Grigorian	September 1998	S. Dy 32544
WPS1978	Behavioral Responses to Risk in Rural China	Jyotsna Jalan Martin Ravallion	September 1998	P. Sader 33902

Policy Research Working Paper Series

	Title	Author	Date	Contact for paper
WPS1979	Banking on Crises: Expensive Lessons from Recent Financial Crises	Gerard Caprio, Jr.	September 1998	P. Sintim-Aboagye 38526
WPS1980	The Effect of Household Wealth on Educational Attainment: Demographic and Health Survey Evidence	Deon Filmer Lant Pritchett	September 1998	S. Fallon 38009
WPS1981	Evaluating Public Expenditures When Governments Must Rely on Distortionary Taxation	James E. Anderson Will Martin	September 1998	L. Tabada 36896
WPS1982	Analyzing Financial Sectors in Transition: With Special Reference to the Former Soviet Union	Alan Roe Paul Siegelbaum Tim King	September 1998	D. Cortijo 84005
WPS1983	Pension Reform in Small Developing Countries	Thomas Charles Glaessner Salvador Valdés-Prieto	September 1998	M. Navarro 84722