Volume Title: Annals of Economic and Social Measurement, Volume 6, number 2

Volume Author/Editor: NBER

Volume Publisher:

Volume URL: http://www.nber.org/books/aesm77-2

Publication Date: April 1977

Chapter Title: Fuzziness in Economic Systems, Its Modeling and Control

Chapter Author: S. S. L. Chang, H. O. Stekler

Chapter URL: http://www.nber.org/chapters/c10514

Chapter pages in book: (p. 165 - 174)

Annals of Economic and Social Measurement, 6/2, 1977

FUZZINESS IN ECONOMIC SYSTEMS, ITS MODELING AND CONTROL

BY S. S. L. CHANG AND H. O. STEKLER

Control engineers have just begun to investigate the properties of "fuzzy" systems. These systems contain stochastic variables whose distributions are not known completely. Only some of the properties of these variables may be known, the bounds for instance. In this case, the theory of fuzzy dynamic programming has shown that there is no unique control solution. Rather there is a set of optimal policies depending on the risk one is willing to take. The least risky policy minimizes the maximum loss.

This paper introduces the concept of fuzzy systems and fuzzy dynamic programming to economic systems. There are a number of possibilities for fuzziness in economic systems. We present a general concept and illustrate it with one case: stochastic coefficients which are bounded but where the entire distribution is not known. As an example, the control laws for a "fuzzy" first-order multiplier-accelerator system are developed, and the solutions are analyzed.

I. INTRODUCTION

In recent years, both economists and control theorists have been working with systems which contain a variety of uncertainties. The simplest approach for introducing uncertainty was to add random variables to the linear equations of the model. The solution to this problem is well known and in economics involved the introduction of the certainty equivalence concept.

The next step was to assume that the coefficients of the model were stochastic. Chow [1] developed the macroeconomic stabilization policy for the case of stochastic coefficients. He showed that the policy involved a linear feedback rule which was dependent upon the joint density of the coefficients in the initial period. Chow presented two methods for computing the means and covariances. The first was Bayesian; the second was an approximation utilizing the asymptotic distribution of the structural parameters from which the reduced form parameters are then derived.

However, Chow noted that in making the control laws dependent upon the joint density of the coefficients in the *initial* period, information about the coefficients obtained during the control period was not being utilized. Methods of efficiently obtaining this information and then using the new information optimally for control come under the heading of adaptive control problems.

However, these techniques do not take into account some of the uncertainties that decision makers face in actuality. First, it is known that there are alternative specifications of economic systems which yield widely different results about policy multipliers. Second, it is a fact that economic data are revised so frequently that the state of the system (even if that could be precisely specified) last period is not known with certainty. Third, when data revisions do occur, they may be substantial. Then the original coefficients of a regression and the regression referring to the same period but using the revised data may differ substantially.[1]

Given the vast uncertainties, it is unlikely that the error should be represented by a probability distribution where the mean and covariances can be computed at a given instant. However, if the probability distribution is now known or cannot be used, none of the aforementioned techniques are applicable.

A new set of techniques have recently been developed to handle stochastic control problems where the probability distribution of the coefficients or errors need not be specified. All that is required is that the error or parameter be bounded. This new technique is known as fuzzy dynamic programming, and the analysis shows that there is not a unique optimal policy, but a set of policies depending on the risk that is acceptable. The most risky policy gives the least cost under favorable conditions, but the largest loss when the uncertain parameters display the worst possible outcome. The least risky policy minimizes the maximum loss (mini-max). However, this policy frequently yields a non-linear complicated control law, but there is an approximate but simple solution to this least risky policy which is called guaranteed cost control. It is also possible to obtain the entire set of optimal controls, with each depending on the risk preference of the decision maker.

In this paper, we shall first develop and explain some of the techniques of fuzzy dynamic programming. We shall then apply this technique to controlling a stochastic multiplier-accelerator system where it is assumed that the distribution of the coefficients is unknown but bounded. Since we are analyzing a simple first-order system, it is feasible to derive the exact minimax solution, and we need not examine the guaranteed cost approximation.

II. Fuzzy Dynamic Programming

A. *An Economic Model*

Assume that our economic system can be represented by the first order system:

(1) $$y_{t+1} = Ay_t + BG_t$$

where both A and B are stochastic coefficients. There are alternative as-

[1] For an example, see the inventory equations in H. O. Stekler's, Economic Forecasting (Praeger, 1970) Appendix A.

sumptions which might be made about these coefficients. First, we could estimate A and B from *past* data using appropriate statistical techniques. However, this would not be an appropriate approach for controlling the economy if the structure of the economy were changing and if the coefficients varied with time. An alternative assumption that has been made is that the coefficients A and B are composed of a known value plus a random disturbance, i.e.

$$A = \bar{A} + e_a \quad \text{and} \quad B = \bar{B} + e_B \tag{2}$$

where e_a and e_B are assumed to be distributed as normal and independent or white. However, given the gross uncertainties inherent in the world we can not realistically assume that we know $\bar{A}$, $\bar{B}$ and the distributions of e_a, e_b precisely.

We, therefore, propose another approach, which we feel more realistically reflects the real world in which economic decisions are made. We shall assume that the decision maker must control an economy about which there is some-but imprecise information. For instance, while there may be a consensus about the possible *range* of the A and B coefficients, we explicitly assume that there is no knowledge about the exact *distribution* of the random variables.

Suppose equation (1) had been derived from the multiplier-accelerator model:

$$y_{t+1} = cy_t + I_t + G_t \tag{3}$$

$$I_t = v(y_{t+1} - y_t)$$

Then the A and B coefficients of (1) would then be

$$A = \frac{c - v}{1 - v}, \quad \text{and} \quad B = \frac{1}{1 - v} \tag{4}$$

Some information about c is obviously available. We know that c cannot be greater than 1, nor is it likely to be as low as .7. However, we are not sure which of the values c is likely to take in the interval $0.7 < c < 1.0$. Similar information would be available about v, but our discussion will concentrate on c.

B. *The Theory of Fuzzy Sets* [2] *and Programming* [3]:

This state of incomplete knowledge can be represented by describing c as a fuzzy set: a membership function μ_c on the real line as shown in Figure 1. For any given x, $m_c(x)$ represents the likelihood of $c = x$ given our imperfect knowledge. This membership function may be considered akin to a subjective probability distribution, where these prior probabilities are based on the decision maker's own experience, knowledge and

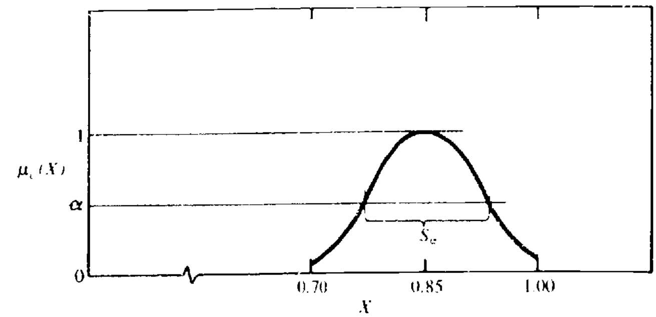

Figure 1 The parameter c as a fuzzy set.

perceptions. If a value between 0.8 and 0.9 is highly likely, then μ_c is close to 1 for $0.8 \leq x \leq 0.9$. Outside this range μ_c falls gradually to zero. The fuzzy set can also be described by its level set $S_\alpha = \{x : \mu_c(x) \geq \alpha\}$. The level set becomes smaller and smaller as α is increased. It is possible to provide an interpretation of α. This parameter determines the cut off between those values of the coefficients which the decision maker would consider and those which are considered irrelevant or extremely unlikely to occur. For each value of α there is a *set* of optimal policies, and, in practice, it would be necessary to undertake computer simulations to determine the sensitivity of the sets of policies to the value of this parameter.[2] In general, the state equation can be written as

$$x_{t+1} = f(x_t, u_t, q_t) \tag{5}$$

where q_t is a r-dimensional vector consisting of r uncertain parameters (corresponding to the number of uncertain coefficients in our model) and is represented as a fuzzy set in E^r space.

While (5) is representative of our imprecise knowledge of the system, we must now explain how a decision maker would proceed. We assume that he selects the values of the coefficients which after careful consideration of all the factors, seem subjectively most appropriate to him. Interpreted in the fuzzy set model, it means the selection of a threshold membership α, and only the possible values of q with a membership α or higher will be considered from this point on:

$$q \in S_\alpha \tag{6}$$

Equations (5), (6) and the usual cost function equation, which may or may not include the parameter q explicitly, now represents the control problem

[2]In Section III below, we show how the policies for our simple model would vary with changes in uncertainty. However, as a caveat, it should be noted that when there are many coefficients in the system, it may be difficult to characterize the boundary of the fuzzy set. As yet, no generally valid solution to this problem has been developed.

to be solved. It is intuitively clear that any control law would give a range of values for the cost, corresponding to the various values of q in S_α. If a control law is such that no other control law can do better for *both* the maximum and minimum costs ($q \in S_\alpha$), it is said to be one of the set of optimum control laws. The one which minimizes the maximum cost is said to be least risky, and its risk parameter $\rho = 0$.

C. *Solution to the Economic Model*

a. Problem restatement

Equation (1) can be written in the form:

(7) $$x_{t+1} = (A_0 + a_t)x_t + (B_0 + b_t)u_t$$

where (A_0, B_0) is the midpoint of S_α and dependent on α, and $|a_t| \le m_a$ and $|b_t| \le m_b$ specify the range of S_α. The variables a_t and b_t are not known at the time of choosing U_t. The cost function is of the form

(8) $$J_t = \sum_{\tau=t}^{\tau=N-1} (x_\tau^2 + Qu_\tau^2) + Px_n^2$$

The problem is to select u_t $t = 0, 1, 2 \ldots N - 1$ to give the lowest J_0. The control law has the form

(9) $$u_t = -k_t x_t$$

with (9), J_t can be expressed as

(10) $$J_t = C_t x_\tau^2$$

where C_t satisfies the recurrent equation

(11) $$C_t = 1 + Qk_t^2 + C_{t+1}(A_t - B_t k_t)^2$$

(12) $$C_n = P$$

b. Risk factors and minmim and Minmax solutions

In the above we have not yet specified the exact values of A_t and B_t. If A_t and B_t are chosen to minimize (maximize) the RHS (right hand side) of (11), the resulting C_t gives the lowest (highest) value of J_t or J_{min}, (J_{max}). Given any sequence k_t, $t = 0, 1, 2 \ldots N - 1$, there is a corresponding J_{min} and J_{max}. There is no unique optimum solution as it depends on how much relative weight we place on the J_{min} and J_{max}. Let J_ρ be defined as

(13) $$J_\rho = (1 - \rho)J_{max} + \rho J_{min}$$

and ρ is called the risk parameter. For any given ρ, $0 \le \rho \le 1$, there is an optimum sequence k_t, $t = 0, 1 \ldots N - 1$, which minimizes J_ρ.

We now present the minimin and minimax solutions:

(1) $\rho = 1$ (minimin solution)

$$A_t = A_0 - m_a = A_-$$

$$B_t = B_0 + m_b = B_+$$

$$k_t^{(1)} = \frac{A_- B_+ C_{t+1}}{Q + C_{t+1} B_+^2}$$

(2) $\rho = 0$ (minimax solution)

$$A_t = A_0 + m_a = A_+$$

$$B_t = B_0 - m_b = B_-$$

(14) $$k_t^{(2)} = \min \left\{ k_t^{(0)}, \frac{A_0}{B_0} \right\}$$

where

(15) $$k_t^{(0)} = \frac{A_+ B_- C_{t+1}}{Q + C_{t+1} B_-^2}$$

We note that if $k_t > A_0/B_0$, then $a = -m_a b = m_b$ would result in a higher cost. Therefore the minimax $k_t^{(2)}$ is A_0/B_0. This is the policy which would be used in a deterministic case if the cost of using policy is ignored.

The coefficients C_t are calculated from (11) and (12) using A_-, B_+, and $k_t^{(1)}$ for the minimum solution and using A_+, B_-, and $k_t^{(2)}$ for the minimax solution.

c. Steady-state solution

If N approaches infinity

$$C_t = C_{t+1} = C$$

We note (8) and (10) are of the same form

(16) $$k = \frac{ABC}{Q + CB^2}$$

Substituting (16) into (11) gives

(17) $$C = 1 + A^2C - A^2B^2C^2(Q + CB^2)^{-1}$$

Equation (16) can be solved for C

(18) $$C = \frac{Qk}{B(A - Bk)}$$

Eliminating C between (17) and (18) gives an equation for k:

$$k^2 + \left(\frac{1 - A^2}{AB} + \frac{B}{QA}\right)k - \frac{1}{Q} = 0$$

(19)
$$k = \sqrt{\frac{1}{Q} + \delta^2} \quad \delta$$

where

(20)
$$\delta = \frac{1}{2}\left(\frac{1 - A^2}{AB} + \frac{B}{QA}\right)$$

Equation (18) can be used only if k satisfies (19). For other values of k, C is obtained directly from (11)

(21)
$$C = \frac{1 + Qk^2}{1 - (A - Bk)^2}$$

The minimum and maximum values of C correspond to the minimum and maximum values of $|A - BK|$ in the allowed ranges of A and B, if $|A - BK| < 1$. Otherwise the maximum cost is for $|A - BK| = 1$ and $C = \infty$.

III. Conclusion and Interpretations:

In the previous sections we developed the rationale and some of the methodology of fuzzy dynamic programming and illustrated it with a simple first-order example. Starting with a membership function, it was possible to define the range of the stochastic coefficients. After a particular value of that function is selected, a *set* of possible policies can be determined, with each policy dependent upon the risk the decision maker is willing to assume. The procedure was illustrated for the riskiest (minimin) and the least risky policies (minimax).

It should be noted that fuzzy dynamic solutions are unlike the conventional stochastic control problems. In the latter case, with an exact known distribution of the random variable, there is only *one* solution which minimizes the expected value of the loss functions. With fuzzy dynamic programming when the range of the coefficients is known, there is a *set* of solutions.

Given the set of solutions, it is desirable to focus on the properties of the extreme cases. We therefore assume reasonable values for A and B, and calculated how the vigor of policy (the gain) was related to the cost associated with changing policy, Q. It can be seen from Figure 2, that policy was used more vigorously in the minimax solution than in the minimin case.

Equation (19) shows that the minimax policy may be even more vigorous than the solution obtained from a deterministic model in which the

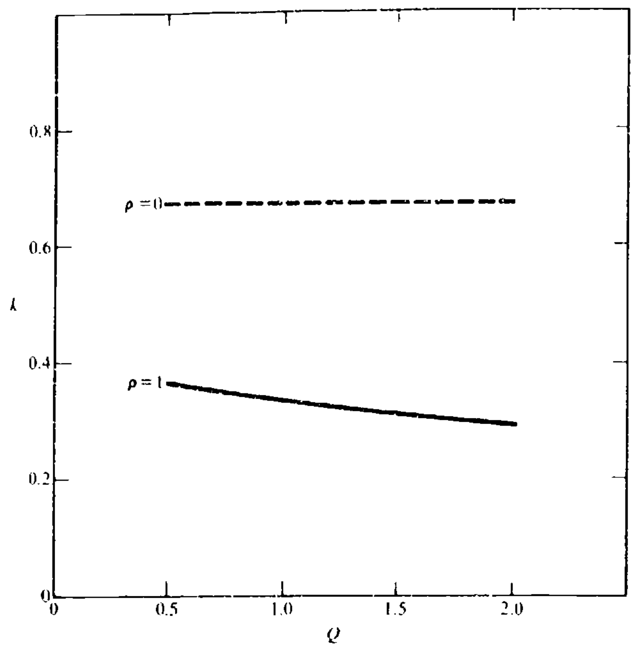

Figure 2 Minimin ($\rho = 1$) and minimax ($\rho = 0$) gain versus policy cost Q.

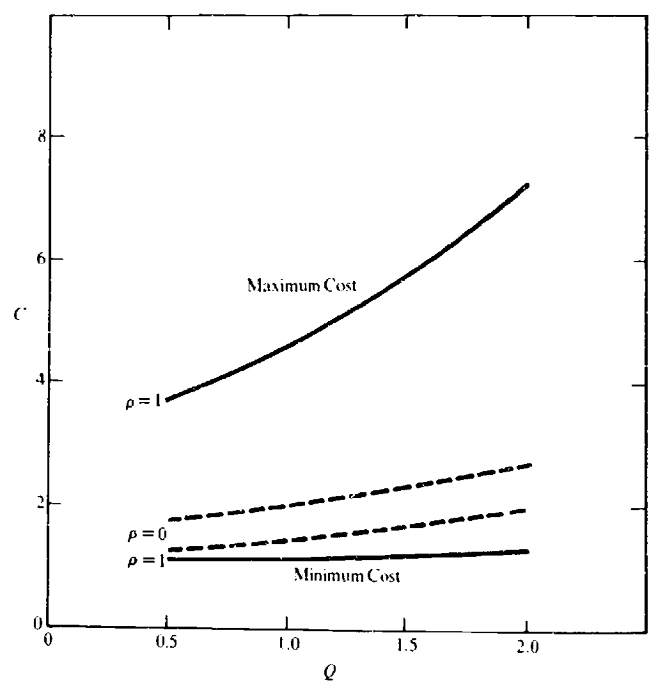

Figure 3 Minimum and maximum cost versus Q.

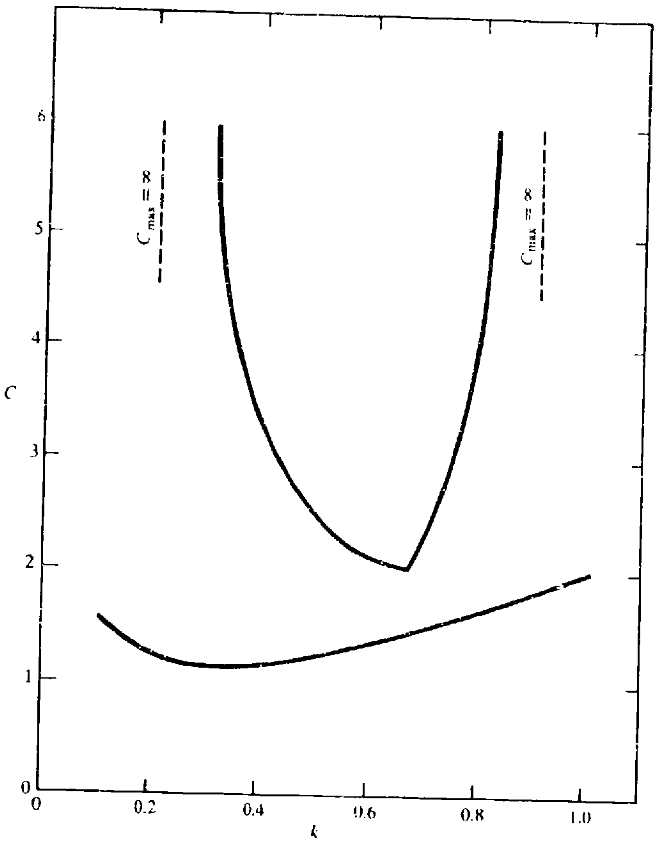

Figure 4 Minimum and maximum cost versus gain.

cost of using policy, Q, was considered. However, the policy never exceeds the full "gap", which would be the deterministic solution when the cost of using policy was ignored.

Figure 3 demonstrates that a minimin policy yields the lowest costs when the most favorable events occur, but an extremely high cost under the most unfavorable circumstances. The two outcomes for the minimax policies lie within these bounds.

In Figure 4, we consider the cost and gain for the entire set of solutions. In our example, the optimum range of the gain[3] lies between .328 and .667. If the gain were larger than .667 BOTH the minimum and the

[3]The result presented here holds only for the first order case. In general there is a set of optimal solutions which have the characteristic that any other solution which gives a lower minimum cost would also give a higher maximum cost, and vise versa. Moreover, in general the derivations of solutions for higher order systems are more complicated than are presented here and the optimum set cannot be represented by the gain.

maximum cost are larger than the corresponding value at .667. Similar results hold for gain less than .327. If we choose a gain within this range, *any other solution which gives a lower minimum cost would then give a higher maximum cost, and vise versa.*

Since all these results have been determined for a given value of α, it would be appropriate to determine how the policies would vary if the uncertainty increased. If α were decreased, the range of possible values of the coefficients would be increased, i.e. the uncertainty is increased. Using Equation (19), it can be shown that regardless of which coefficient displays an increase in uncertainty the vigor of the minimin policy is reduced.[4]

For the minimax policy, equation (19) shows that an increase in uncertainty in the A coefficient, would increase the vigor of policy. An increase in the uncertainty about the policy multipler, the B coefficient, may either increase or decrease the vigor of policy. However with $Q \leq 1$, (i.e the costs of using policy are no greater than those associated with system deviations), an increase in uncertainty would increase the gain, provided that it was not previously at the maximum, A_0/B_0.[5] Again, the result differs from that of stochastic programming, for then an increase in policy uncertainty in the first-order model reduces the vigor of policy.

These results even for a system as simple as a multiplier accelerator model should serve to illustrate both the richness of fuzzy programming and its differences with the conventional stochastic programming solution.

SUNY, Stony Brook

Revised December 1976

[4]This leads to the result that with $\alpha = 0$, no policy at all would be undertaken.
[5]This is the deterministic solution when the costs of using policy are ignored.

REFERENCES

[1] Chow, Gregory C. "Effects of Uncertainty on Optimal Control Problems." *International Economics Review*, Vol. 14, No. 3 Oct. 1973.

[2] L. A. Zadeh, "Fuzzy Sets." *Information and Control* Vol. 8, pp. 338 353, 1965.

[3] There are the following references on fuzzy sets and programming: S.S.L. Chang, "Fuzzy Dynamic Programming and the Decision Making Process". *Third Annual Princeton Conference on Information Sciences and Systems*, 1969; S. S. L. Chang and T. K. C. Peng, "Adaptive Guaranteed Cost Control of Systems with Uncertain Parameters", *IEEE Transon Automatic Control*, Vol AC-17 No. 4, 1972, pp. 474 483, S. S. L. Chang and L. A. Zadeh, "Fuzzy mapping and control." *IEEE Trans. Syst., Man. Cybern.*, Vol SMC-2, pp. 30 34, Jan. 1972. S. S. L. Chang and P. E. Barry, "Optimal Control of Systems with uncertain Parameters". *Fifth IFAC Congress*, Paris, 1972. S. S. L. Chang "On Risk and Decision Making in a Fuzzy Environment". *Fuzzy Sets and Their Applications to Cognitive and Decision Processes Academic Press*, 1975 pp. 219 226. C. V. Negoita and D. Ralescu, "Applications of Fuzzy Sets to System Analysis." Wiley, New York, 1975.