The men who weren't even there: Legislative voting with absentees*

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Abstract

Voting power in voting situations is measured by the probability of changing decisions by altering the cast 'yes' or 'no' votes. Recently this analysis has been extended by strategic abstention. Abstention, just as 'yes' or 'no' votes can change decisions.

This theory is often applied to weighted voting situations, where voters can cast multiple votes. Measuring the power of a party in a national assembly seems to fit this model, but in fact its power comprises of votes of individual representatives each having a single vote. These representatives may vote yes or no, or may abstain, but in some cases they are not even there to vote. We look at absentees not due to a conscious decision, but due to illness, for instance. Formally voters will be absent, say, ill, with a certain probability and only present otherwise. As in general not all voters will be present, a thin majority may quickly melt away making a coalition that is winning in theory a losing one in practice. A simple model allows us to differentiate between winning and more winning and losing and less losing coalitions reflected by a voting game that is not any more simple. We use data from Scotland, Hungary and a number of other countries both

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to illustrate the relation of theoretical and effective power and show our results working in the practice.

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1 Introduction

When we think of voting, we think 'yes' and 'no' votes and a simple counting of the 'yes' votes determines the outcome. In practice veto rights or weighted voting add some complexity; the UN Security Council or the EU Council of Ministers are well-known examples. The classical theories of power indices (Shapley and Shubik, 1954; Banzhaf, 1965) describe precisely these situations and provide formulas to calculate how much a particular voter can influence the decisions. These theories assume that voters face a binary choice, while there is also a third option: not voting, and this option has lead to many important decisions in the past (Lindner, 2008).

Ternary voting games (Felsenthal and Machover, 1997; Braham and Steffen, 2002; Freixas and Zwicker, 2003), allowing for abstention are richer, but still fail to capture the entire problem. In the records of the Hungarian National Assembly a voting partitions members of the parliament into typically *five* sets: The groups voting 'yes' and 'no' are clear. Those voting 'abstain' exercise the third option perhaps not realising that their vote is practically the same as a 'no' (it could also be a 'yes' in other situations). Then there is another, slightly stronger version of abstention (from the point of our model these are equivalent): being there but not selecting either of the previous options. Finally there are the voters who are not even there. No-show can be strategic (Côrte-Real and Peraira, 2004; Laruelle and Valenciano, 2011), but here we focus on the non-strategic aspect: It is not that the members of parliament (MPs) do not want to vote, but they cannot — due to illness or previous engagement (MPs have many).

Since (Shapley and Shubik, 1954) adopted the Shapley value to measure a priori voting power, we use simple games to model voting situations. In these games a coalition is either *winning* and has a payoff of 1, or *losing* getting 0.

For instance, in the weighted voting situation with a quota of 51 and 3 voters having voting weights 48, 26, 26 any pair has the majority and is a winning coalition, while singletons cannot make decisions alone and hence have a 0 payoff. Due to its symmetry , the power indices derived from the induced simple game will be symmetric, too suggesting that the three players have equal power. This contradicts both one's intuition and the empirical evidence, which shows that larger players tend to have larger influence irrespective of their possible symmetry in the voting game.

In most voting situations weights arise naturally, corresponding to the number of seats a voting block can secure in a legislative body. Such examples include national parliaments, local governments, or the European Parliament. We take the first as our motivating example, our voters are parties, and the actual votes are cast by MPs. Unlike in genuine weighted voting, here the party leaders can speak for their parties, but then MPs decide whether or not to vote and whether they want to vote along party lines. Rebel voting can further enrich the model, it is a natural extension that we do not elaborate here.

We focus on situations where some MPs refuse or are unable to vote with a certain probability. In our model the weights are random variables, and so is the outcome of a voting situation. We call these games generalised voting games, and consider the Shapley-Shubik index as a tool for measuring the power in these situations. In order to give a solid background for the use of the Shapley-Shubik index we show that Young's (1985) axiomatisation of the Shapley value work on the class of generalised voting games.

While the model is formally presented later, the consequences for our simple examples are immediately clear. The coalition of the smaller parties is very insecure, it loses its majority if only a few voters are absent, while a coalition including the large parties has a wide majority, where a few votes do not matter. The coalitional membership of this large player is valued more leading to a higher power.

The paper is then organized as follows. We introduce the usual notation and standard terminology in Section 2. In the third section we discuss the problem of abstention and introduce our model, the concept of generalised voting games. We devote Section 4 to the axiomatisation of the Shapley value for generalised voting games. Then we take a look at absent voters in practice in the US Senate, the Scottish Parliament, the National Assembly of Hungary and a few other legislative bodies. We close with a short summary.

2 Power indices

First we introduce the usual terminology and notation. Let $N = \{1, \ldots, n\}$ be the set of the players. $v : 2^N \to \mathbb{R}$ is a transferable utility (TU) cooperative game (henceforth *game*) with player set N, where $v(\emptyset) = 0$. For any player i and any coalition S: $v'_i(S) = v(S \cup \{i\}) - v(S)$, that is $v'_i(S)$ is player

i's marginal contribution to coalition S in game v. Let v'_i be for player i's marginal contribution function in game v. Player i is a null-player in game v if $v'_i = 0$. We write that $i \sim^v j$, player i is equivalent with player j in game v, if for any coalition S such that $i, j \notin S$: $v'_i(S) = v'_j(S)$. Finally, $|A|$ is for the cardinality of set A.

A voting situation or voting game is a pair $(N, \mathcal{W})$, where the players are the voters and $\mathcal{W}$ denotes the set of *winning coalitions*. We consider *simple voting games* where

1. $\emptyset \notin \mathcal{W}$ and $N \in \mathcal{W}$,
2. if $C \subseteq D$ and $C \in \mathcal{W}$, then $D \in \mathcal{W}$,
3. if $S \in \mathcal{W}$ and $T \in \mathcal{W}$, then $S \cap T \neq \emptyset$.

Condition 3 requires the game to be *proper*, that is, a motion and its opposite cannot be approved simultaneously.

Let $\bar{\Gamma}_N$ denote the set of *proper simple voting games over the player set* N.

We can also write a simple voting game in the form of a transferable utility game v, where $v(S) = 1$ if $S \in \mathcal{W}$ and 0 otherwise. The term "simple" comes from having coalitions with payoffs 0 or 1 only.

Weighted voting games are simple voting games where the importance of each player is expressed by a weight and a coalition is winning if the total weight of its members exceeds the quota characteristic of the game. A weighted voting game can be expressed by an $n+1$ tuple $(q, w) = (q, w_1, \ldots, w_n)$ consisting of the quota to pass a resolution and a vector w of *voting weights*. Let $w_S = \sum_{i \in S} w_i$ and let $\hat{\Gamma}_N$ denote the set of weighted voting games with player set N. Weighted voting games with $q > w_N/2$ are proper simple voting games. While there are infinitely many weighted voting games, since there are only a finite number of simple voting games, many of these are equivalent.

We study the players' ability to change decisions. If, by joining a losing coalition, a player can turn it winning, we call the player swing. *Voting power* then refers to this ability to change decisions.

Given a game v of $\bar{\Gamma}_N$, an *a priori measure of voting power* or *power measure* $\kappa : \bar{\Gamma}_N \to \mathbb{R}^N_+$ assigns to each player i a non-negative real number $\kappa_i(v)$, its *power* in game v; if for any game v of $\bar{\Gamma}_N$: $\sum\limits_{i \in N} \kappa_i(v) = 1$, then it is also a power *index*.

In the following we explain some of the well-known indices. The *Shapley-Shubik index* ϕ (Shapley and Shubik, 1954) applies the Shapley value (Shapley, 1953) to simple games: Voters arrive in a random order; if and when a coalition turns winning the full credit is given to the last arriving, the *pivotal* player. A player's power is given as the proportion of orderings where it is pivotal, formally for any simple voting game v player i's Shapley-Shubik index in game v is as follows

$$\phi_i(v) = \sum_{S \ni i, S \subseteq N} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S \setminus \{i\})) ,$$

where $s = |S|$.

The *Banzhaf measure* ψ (Penrose, 1946; Banzhaf, 1965) is the vector of probabilities that a party is *critical* for a coalition, that is, the probabilities that it can turn winning coalitions into losing ones. Formally, for any simple voting game v player i's Banzhaf-measure in game v is as follows

$$\psi_i(v) = \frac{\eta_i(\mathcal{W})}{2^{n-1}} ,$$

where $\eta_i(\mathcal{W})$ is the number of coalitions in $\mathcal{W}$ in which i is critical. When normalized to 1, we get the *Banzhaf index* β (Coleman, 1971). Formally, for any simple voting game v player i's Banzhaf index in game v is as follows

$$\beta_i(v) = \frac{\eta_i(\mathcal{W})}{\sum\limits_{j \in N} \eta_j(\mathcal{W})} .$$

3 The men who weren't even there

The mainstream voting power literature takes it for granted that if a player has the right to vote, he or she will use this right and express a clear opinion. Indeed, the general approach is to model voting situations by voting games. This model has been enriched by models with abstention (Braham and Steffen, 2002; Machover and Felsenthal, 1997; Lindner, 2008): Machover and Felsenthal (1997) consider voters facing a ternary decision: vote 'yes', 'no' or 'abstain' and the choice is completely symmetric with respect to these options, thus abstention is one of the voting alternatives. In practice abstention is often regarded as a 'yes' or a 'no' vote, depending on circumstances. The abstention of some of the voters usually reduces the chances of the passing of a motion, but there are examples of the other kind too, the most striking being probably the USSR's boycott of the UN Security Council that has led to sending UN forces to Korea, a motion the Soviet Union strongly opposed and would have been able to veto. Braham and Steffen (2002) argue that abstention is a totally different action and voters first decide whether they want to form an opinion on the issue and if the answer is yes, then they vote.

While abstention may have the same effect as before, here voters do not care about the consequences. Both models, however, assume that voters abstain strategically.

Suppose not all voters are present, and their absence is not a strategic decision, but it is unplanned. One might say that if voting is important other engagements could be postponed and thus being absent is essentially the same as abstaining. In some legislative bodies, however, majority is relative to those present, and hence abstention corresponds to a 'no' vote, while not voting is the true abstention. In voting statistics, however, MPs are usually listed in 4 categories: those in favour, those against, those abstaining and those present, but not voting, not to mention those who *weren't even there*. Here we focus on this latter group and for simplicity ignore abstention, both genuine and strategic.

3.1 Model

We study the power of voting blocks, such as parties in a legislation while allowing individual members to be absent from voting. In the following we formalise the model. Consider a weighted voting game $(q, w_1, w_2, \ldots, w_n)$ or briefly (q, w), where q is the quota, and w_i is the weight of player i (the number of MPs of party i). At a particular vote some are sick (or busy elsewhere) and only $w' = (w'_1, w'_2, \ldots, w'_n)$ are present, where $w' \leq w$. The rules of the legislation determine a possibly different quota q' for these reduced number MPs, leading to the weighted voting game (q', w') that can then be solved using the standard theories applicable to weighted voting games. However, it also can happen that too many MPs are absent, therefore the given legislation is not able to pass any proposal. This situation cannot be modelled by a weighted voting game, because the grand coalition is not a winning coalition. However, this situation can be described by the 0 game, where every coalition has 0 value.

Going a step beyond the usual approaches we assign probabilities $p(q, w)$ to each of the games (q, w), where the parties have respectively $w_1, \ldots, w_n$ voters present and q is the quota. Generally speaking, we consider the case, where the numbers w of MPs that are present and the quota q are random variables with the joint probability distribution $p(q, w)$. Every outcome of (q, w) determines a simple game (more precisely a weighted voting game or the 0 game), and the generalised voting game $\tilde{v}$ is the weighted sum of the considered simple games with weights of $p(q, w)$.

Note that not all simple voting games can be generated by weighted voting games (Taylor and Zwicker, 1992; Elkind, Goldberg, Goldberg, and Wooldridge, 2008).

Definition 1. The generalised voting game $\tilde{v}$ is defined as follows

$$\tilde{v} = \sum_{v \in \hat{\Gamma}_N \cup \{0\}} v \ \hat{p}(v) \ ,$$

where $\hat{p}$ is an arbitrary probability distribution on the set $\hat{\Gamma}_N \cup \{0\}$. Moreover, let $\tilde{\Gamma}_N$ denote the class of generalised voting games with player set N.

Notice that every weighted voting game is a generalised voting game, and 0 is also a generalised voting game.

Next we give two examples for illustration.

Example 1. Assume that in a legislation the parties have $w_1, \ldots, w_n$ MPs, the quota q is independent of the number of MPs present, and the MPs are present with same probability p and independently of one another. Then the probability that $w' \in \mathbb{R}^N$, $w' \leq w$ MPs are present is

$$p(w') = \prod_{i=1}^{n} \binom{w_i}{w'_i} p^{w'_i} (1-p)^{w_i - w'_i} \ , \tag{3.1}$$

Since the quota q does not change, for any coalition S

$$\tilde{v}(S) = \sum_{i=q}^{w_S} \binom{w_S}{i} p^i (1-p)^{w_S - i} \ ,$$

where $w_S = \sum_{i \in S} w_i$.

The Hungarian National Assembly of 2006 had six parties and 386 representatives. The game of simple, absolute majority[1] can be written as $(q, w) = (194; 190, 141, 23, 20, 11, 1)$. Let the probability of missing a session be $1 - p = 0.1$. We look at three coalitions: $w_{\{1\}} = 190$, $w_{\{1,3\}} = 213$ and $w_{\{1,2\}} = 331$. The latter two are winning coalitions, but $\tilde{v}(\{1\}) = 0$, $\tilde{v}(\{1,3\}) = 0.35$, and $\tilde{v}(\{1,2\}) = 1$.

Example 2. Now consider *relative* majority. To remain winning coalition S must outnumber the complementary coalition $N \setminus S$ or generally: it must have a share $\rho = \frac{q}{w_N}$ of the total. Suppose that S has i, the complementary coalition j members present. Then S is winning, if $i \geq \rho(i + j)$, that is, if $j \leq \frac{1-\rho}{\rho} i$. In fact $j \leq \left\lfloor \frac{1-\rho}{\rho} i \right\rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part. The value of coalition S is calculated as

[1] Voting by absolute majority is rare in Hungary, but is the common voting method in, for instance, Luxembourg.

$$
\begin{aligned}
\tilde{v}(S) = \sum_{i=1}^{w_S} \sum_{j=0}^{\left\lfloor \frac{1-\rho}{\rho} i \right\rfloor} \binom{w_S}{i} p^i (1-p)^{w_S - i} & \binom{w_N - w_S}{j} p^j (1-p)^{w_N - w_S - j} \\
&= \sum_{i=1}^{w_S} \sum_{j=0}^{\left\lfloor \frac{1-\rho}{\rho} i \right\rfloor} \binom{w_S}{i} \binom{w_N - w_S}{j} p^{i+j} (1-p)^{w_N - (i+j)} \ .
\end{aligned} \tag{3.2}
$$

We continue the example of the Hungarian National Assembly. While in the previous example all zero coalition in the ordinary weighted voting game remained zero coalition in the generalised voting game, with relative majority there are zero coalitions whose values become positive. Such is the case for $\tilde{v}(\{1\}) = 0.16$ (in an ordinary weighted game it is $v(\{1\}) = 0$).

In the following we drop the tilde, v can mean a voting game or a generalised voting game – this will lead to no confusion.

4 The Shapley-Shubik index

The generalised voting games are not necessarily simple games, the values of coalitions can be strictly between 0 and 1, so the usual power indices cannot be applied. We can, however return to TU games and use *values* to determine the power distribution. Instead of the Shapley-Shubik index, we can use the Shapley value, or instead of the Banzhaf index (or measure) we can use the Banzhaf value, but to stress the parallel features with simple voting games we keep on using the term power index for values.

There are many values for TU games and the choice is not easy. Which value should one use? We can answer this question by *characterising* the values by elementary axioms, reasonable properties, and examining which values meet the given axioms. We conclude that the Shapley-Shubik index (Shapley value) is the *only* solution concept which meets the recommended properties.

Therefore, in this section we characterise the Shapley value on the class of generalised voting games. We show that the Shapley value is the only solution on the class of generalised voting games which meets three axioms: Efficiency, Symmetry and Marginality. Put it differently, we show that (Young, 1985)'s axiomatisations of the Shapley value works on the class of generalised voting games.

Note that an axiomatisation for a class of TU games does not imply a similar axiomatisation on super- or subclasses of TU games. The fact that

Young (1985)'s axiomatisation works on the class of monotone TU games concludes nothing about the validity on the class of generalised voting games, a subset of the class of monotone games. Moreover, even if Young (1985)'s axiomatisation works on the class of weighted voting games and the class of weighted voting games is a subset of the class of generalised voting games, these still do not imply anything for Young (1985)'s axiomatisation on the class of generalised voting games.

We stress that we do not only show that Young (1985)'s axiomatisation works in this case, but we apply, with certain modifications, the concept of his proof as well. Therefore, although there are some alternative proofs for Young (1985)'s axiomatisation (Moulin, 1988; Pintér, 2011), ours is based on the original one.

For any quota q the weighted voting game v is a *monotone* simple game. For the games $v, w \in \hat{\Gamma}_N$ let $v \vee w = \max\{v, w\}$. Notice that if $i \sim^v j$ and $i \sim^w j$, then $i \sim^{v \vee w} j$.

Furthermore, for any simple voting game v let $\mathcal{M}_v \subseteq 2^N$ be the set of the minimal winning coalitions of a game v. Then

$$v = \bigvee_{T \in \mathcal{M}_v} u_T \ ,$$

where u_T is the unanimity game on coalition T. Later, we also need the following result.

Lemma 2. *For any simple voting game v and $T^* \in \mathcal{M}_v$ let*

$$w = \bigvee_{T \in \mathcal{W}_v \setminus \{T^*\}} u_T \ .$$

Then for any $i \notin T^*$: $w_i' = v_i'$.

Proof. Let S be an arbitrary coalition. Two cases can happen: (1) $T^* \nsubseteq S$. Since $i \notin T^*$ $u_{T^*}(S) = u_{T^*}(S \cup \{i\}) = 0$, therefore $v(S) = w(S) \vee u_{T^*}(S) = w(S)$ and $v(S \cup \{i\}) = w(S \cup \{i\}) \vee u_{T^*}(S \cup \{i\}) = w(S \cup \{i\})$. Put it differently, $v_i'(S) = w_i'(S)$.

(2) $T^* \subseteq S$. Then $v(S \cup \{i\}) = v(S) = 1$, i.e $v_i'(S) = 0$. Indirectly assume that $w(S \cup \{i\}) - w(S) > 0$. Then $w(S \cup \{i\}) = 1$ and $w(S) = 0$, that is $S \cup \{i\}$ is a minimal winning coalition in game w. Since $\mathcal{W}_w = \mathcal{W}_v \setminus \{T^*\}$ it means that $S \cup \{i\}$ is a minimal winning coalition in game v, which contradicts that T^* is a minimal winning coalition in game v. □

Furthermore, let $1_T^v = 1$ if T is a winning coalition in simple voting game v and $1_T^v = 0$ otherwise. Then

$$v = \bigvee_{T \subseteq N} 1_T^v u_T \ . \tag{4.1}$$

In the following we introduce the three axioms which we use in our characterisation result.

Definition 3 (Axioms)**.** The solution $\kappa : \tilde{\Gamma}_N \to \mathbb{R}^N$ satisfies

- Efficiency, if for all $\tilde{v} \in \tilde{\Gamma}_N$: $\tilde{v}(N) = \sum_{i=1}^{n} \kappa_i(\tilde{v})$,
- Symmetry, if for all $\tilde{v} \in \tilde{\Gamma}_N$ and for all $i, j \in N$ such that $i \sim^{\tilde{v}} j$: $\kappa_i(\tilde{v}) = \kappa_j(\tilde{v})$,
- Marginality, if for all $\tilde{v}, \tilde{w} \in \tilde{\Gamma}_N$, for all $i \in N$ such that $\tilde{v}'_i = \tilde{w}'_i$: $\kappa_i(\tilde{v}) = \kappa_i(\tilde{w})$.

The axiom of Efficiency goes back to Shapley (1953). Efficiency is a very natural axiom for a solution on the class of voting games or of weighted voting games. It says that the whole power of the given legislation must be distributed among the parties. Since the generalised voting games are natural generalisations of weighted voting games, it is very reasonable to assume this axiom for this class of games either.

Symmetry applies only to equivalent players and is, therefore, weaker than Anonymity (Shapley, 1953). This axiom expresses that if two parties (players) are equivalent from the viewpoint of a voting situation, then their values must be equal, that is, no discrimination is permitted. As Anonymity, Symmetry is also a plausible axiom.

The Marginality axiom is due to Young (1985). Although he formally introduced a stronger axiom, Strong Monotonicity using inequalities, at the end of his paper he mentioned that inequalities can be replaced by equalities in the results he provides. He does not give a name for this weaker axiom; *Marginality* is our term. This axiom states that if in two voting situations a given party (player) performs with the same success, then it must be evaluated in the two situations identically. This axiom expresses that only the performance of the party matters, no other factors are to be taken into account.

Remark 4. It is important to note that since the Shapley value (Shapley-Shubik index) is (completely) determined by the marginal contributions, it is true that for any game v and w, and for any player i: if $v'_i = w'_i$ and $\kappa_i(w) = \phi_i(w)$, then the Marginality axiom implies that $\kappa_i(v) = \phi_i(v)$.

In the following proposition, without proof, we present a well known result.

Proposition 5. *The Shapley-Shubik index (Shapley solution) meets the axioms Efficiency, Symmetry and Marginality.*

The following lemma is about a key observation. It states that the property that every weighted voting game is the maximum of unanimity games on winning coalitions in the given simple voted game, see Equation (4.1), is practically true for generalised voting games as well.

Lemma 6. *For any generalised voting game $\tilde{v}$*

$$\tilde{v} = \sum_{w \in \hat{\Gamma}_N \cup \{0\}} \bigvee_{T \subseteq N} p_{\tilde{v}}(w) 1_T^w u_T = \bigvee_{T \subseteq N} \sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{v}}(w) 1_T^w u_T \ . \tag{4.2}$$

Proof. Let $S \subseteq N$ be an arbitrary coalition. Then

$$\begin{aligned} \sum_{w \in \hat{\Gamma}_N \cup \{0\}} \bigvee_{T \subseteq N} p_{\tilde{v}}(w) 1_T^w u_T(S) &= \sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{v}}(w) \max_{T \subseteq N} \{1_T^w u_T(S)\} \\ &= \sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{v}}(w) 1_S^w u_S(S) \ , \end{aligned}$$

that is, this is the probability that coalition S is a winning coalition. Furthermore,

$$\begin{aligned} \bigvee_{T \subseteq N} \sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{v}}(w) 1_T^w u_T(S) &= \max_{T \subseteq N} \left\{ \sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{v}}(w) 1_T^w u_T(S) \right\} \\ &= \sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{v}}(w) 1_S^w u_S(S) \ , \end{aligned}$$

that is, this is again the probability that coalition S is a winning coalition. □

Notice that $\sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{v}}(w) 1_T^w u_T$ is a generalised voting game for any coalition T.

Next we consider Young (1985)'s axiomatisation of the Shapley value. The following theorem is our axiomatisation result.

Theorem 7. *On the class $\tilde{\Gamma}_N$ solution κ satisfies Efficiency, Symmetry and Marginality if and only if $\kappa = \phi$, that is κ_i is the Shapley-Shubik index for any generalised voting game and for any player i.*

The proof of Theorem 7 – a non-trivial modification of Young (1985)'s proof to our setting – is relegated to Appendix A.

Summing up the above discussion, by assuming that an ideal power index must meet the axioms Efficiency, Symmetry and Marginality we get the Shapley-Shubik index, therefore in the following we consider only the Shapley-Shubik index.

Unfortunately alternative axiomatisations of the Shapley value (Shapley, 1953; Chun, 1991; van den Brink, 2001) could not be translated into our setting. Shapley (1953) uses Additivity as an axiom: Even though we can modify this axiom by replacing $+$ by operator $\vee$, it is very difficult to give any reasonable interpretation to this axiom in our generalised voting game setting. In Chun (1991)'s and van den Brink (2001)'s characterisations, the Shapley-Shubik index (the Shapley value) does not even satisfy the similarly modified axioms. Therefore, in these cases, the reasonable and interpretable modifications of the original axioms do not go with the Shapley-Shubik index (the Shapley value), hence we do not consider these axiomatisations either.

5 Absenteeism in practice

In this section we discuss several examples from various democracies to investigate how missing representatives can influence the distribution of power. While voting statistics should be of general interest, it proved to be rather difficult to obtain such data. In special cases these data are meaningless: the House of Commons of the United Kingdom uses a very ancient voting method, where the quorum refers to the number of votes cast rather than the voters present and the possibility that an MP votes on both sides cannot be excluded.

The data we did manage to obtain show a great variance in the attitude to absent representatives.

5.1 US Senate

Our first example is the *US Senate*. According to the voting statistics for 111th Congress (Senate Legislative Information System, 2009, 2010) there have been a total of 696 votes and a total of 2197 instances of not voting. Therefore on average 3% of the Senators have not voted. In addition senators "present, giving *live pair*" are recorded separately: when a senator is unable to attend a critical vote, a senator with the opposite voting intention does not vote. This way an illness or a commitment of higher importance does not upset the majority decision of the Senate. While in about 1% of the decisions the winning (relative) majority fails to have absolute majority, these figures suggest that opportunism plays little role in the legislation of the Senate.

5.2 Scottish Parliament

The Scottish Parliament does not have the full legislative power of a national parliament, but the agenda of the of the Scottish National Party that obtained majority in May 2011 contains issues as ambitious as independence so the stakes are equally high. According to our data (Scottish Parlament Informations Centre, 2010) on average 8.2% of the MSP's have not been present at the 698 votes during the period May 2007 – May 2010.

Party	Seats	SS-index (%)	GSS-index (%)
Scottish National Party	69	100	99.834
Scottish Labour Party	37	0	0.035
Scottish Conservative Party	15	0	0.035
Scottish Liberal Democrats	5	0	0.035
Scottish Green Party	2	0	0.034
independent	1	0	0.026
Total	129	100	100.000

Table 1: Power distribution in the Scottish Parliament in 2011

In Table 1 we present the power distributions given by the Shapley-Shubik index and the generalised Shapley-Shubik index using the current distribution of the seats assuming that the absence rate has not altered with the elections. It is clear that the Scottish National Party holds the majority of seats, and despite the fact that the majority extends to several seats, the SNS will only be decisive in 99.83% of cases.

5.3 France and Italy

France has a presidential system with a bicameral parliament. We focus on the National Assembly, the lower house of the Parliament. It consists of 577 members each elected in single member constituencies in a two-round ballot. Such a system tends to give a wide majority for the ruling parties.

The statistics of the 13th Legislation (2009-2010) of the Assemblée Nationale (Assemblée Nationale, 2010, p. 61) reveal that on the average vote 16.3% of the representatives have been absent! This would leave much room for the opposition to obstruct the lawmaking had the prime minister not had a safe majority in the past years. Of course the safe majority may also explain the high rate of absence.

While this exists in France, too, Italian laws blocked by the opposition are not uncommon, although we found that these are not so much due to absent, but rebel MPs. Such a rebel MP votes against the party position and thereby does not only decrease the size of its own party, but increases the size of the opposition. The inclusion of rebel MPs is a very natural extension of our model.

5.4 Finland, Australia

Finland provides an interesting example of a minority government, a minority by only 2 seats. Since the parliament is not large, the minority can easily become a majority. Ignoring political realities we simply take the number of seats obtained in the April 2011 elections, assume that the government acts as a single party and assume that $p = 0.9$. We find (Table 2 that with absent MPs, the power of the minority government increases significantly.

Party	Seats	SS-index (%)	GSS-index (%)
Government	99	60	73.26
Social Democratic Party	42	10	6.76
True Finns	39	10	6.76
Left Alliance	14	10	6.76
Christian Democrats	6	10	6.46
Total	200	100	100,00

Table 2: Power distribution in the Finnish Parliament, 2011 with minority government.

The House of Representatives of Australia consists of 150 representatives elected in single member constituencies by the so-called preferential system. Such a system is very sensitive to voter preferences, which usually guarantees a strong majority for one of the (two) major parties. In the 2010 elections, however, none of the large parties obtained majority and currently the country is lead by a minority Labour government. While four of the 6 crossbenchers have given confidence and supply to provide nominal majority, with full support from these representatives a similar rate of absence as in other countries the government would have only 77.7% of power.

5.5 The National Assembly of Hungary

Finally we discuss the case of Hungary in some length as, with thin majorities and a higher rate of abstention she illustrates several of the interesting cases rather well.

Party	Seats	SS-index (%)	GSS-index (%)
Labor, Green, 3 independents	76	100	77,7
Coalition	72	0	14,3
National Party WA	1	0	4,0
independent	1	0	4,0
Total	150	100	100,0

Table 3: Power distribution in the Australian House of Representatives, 2010 with four crossbenchers supporting the minority Labor government.

The Hungarian National Assembly has 386 seats with MPs elected in a very complex system with both single member constituencies, some compensation for the losing votes but also party-lists with proportional representation are used to fill the places.

Of the 4104 ballots in 2009 the highest number of votes cast was only 382 and on average a mere 353 MPs were present and voting. This means that only $p = 91.55\%$ of the MPs are normally present. We use this value in our calculations.

Government:		Single-party		Coalitional	
	Seats	SS	GSS	SS	GSS
Hungarian Socialist Party	209	100	88,4	100	100
Alliance of Free Democrats	70	0	2,3		
Hungarian Democratic Forum	38	0	2,3	0	0
Independent Smallholders' Party	26	0	2,3	0	0
Christian Democratic People's Party	22	0	2,3	0	0
Federation of Young Democrats	20	0	2,3	0	0
Total	385	100	100,0	100	100

Table 4: Hungarian National Assembly in 1994: Thin majority requires coalitional government. (Power indices in percents.)

Our first example is the composition of the parliament after the 1994 elections, when the Socialists obtained 54.3% of the seats. Table 4 shows that despite having an absolute majority, the Hungarian Socialist Party would alone have only about 88.4% of power due to absent members, so they formed a coalitional government with the Alliance of Free Democrats, the only party that did not refuse to cooperate with the former Communists. The resulting majority with 279 seats was not only sufficient to be absence-proof for simple majority, but also for constitutional changes. Interestingly Mr Orbán's current cabinet (Table 5) has a similar ability having 99.45% of power for laws that require qualified majority, falling short of the Horn-cabinet of 1994 that left only about 10^{-15} to the opposition.

Government:		Single-party		Coalitional	
	Seats	SS	GSS	SS	GSS
Federation of Young Democrats	227	75,0	74,941	100	99,451
Christian Democratic Pple's Party	36	8,3	8,274		
Hungarian Socialist Party	59	8,3	8,367	0	0,159
Movement for a Better Hungary	47	8,3	8,367	0	0,159
Politics Can Be Different	16	0,0	0,034	0	0,159
independent	1	0,0	0,017	0	0,073
Total	386	99,99	100,000	100	100,000

Table 5: Power distribution for double-majority laws in the HNA, 2010.

	Seats	SS	GSS
Hungarian Socialist Party	190	83,3	70,72
Alliance of Free Democrats	18	3,3	6,22
Federation of Young Democrats	141	3,3	6,22
Christian Democratic People's Party	22	3,3	6,22
Hungarian Democratic Forum	9	3,3	5,99
independent	6	3,3	4,64
Total	386	100,0	100,0

Table 6: Power distribution in the HNA in 2009.

Finally, in last year of the 2006-2010 legislature the Socialist formed a cabinet with minority support. Table 6 shows that while under the standard theories any party would be sufficient for the HSP to obtain majority, with absent MPs the majority is more difficult to obtain.

6 Discussion

Ours is not the first paper to study voting power in an assembly, in fact a long series of papers used national or regional parliaments as examples. Rusinowska (2002); van Deemen and Rusinowska (2003); Rusinowska and van Deemen (2005) respectively study paradoxes of voting power in Polish, Dutch and German politics; Lane and Mæland (1995) analyze voting power in Scandinavian countries. Bilbao, Jiménez, and López (1998); Alonso-Meijide, Carreras, and Fiestras-Janeiro (2005) use the Spanish and Catalan parliaments as illustration for games on convex geometries and to voting power with a coalition structure. These approaches, however treat parties as unanimously voting blocks, that is, as a single voters with weights corresponding to the number of representatives. Our approach is, to our best knowledge, original.

Most of the aforementioned papers calculate voting power under the assumption that the parties act independently. While Lane and Mæland (1995) find ample evidence of surviving minority governments, outside the Nordic countries a typical government is formed by a majority coalition of some of the parties. These parties vote unanimously in support of the government and thus act as a single party with the majority of the seats. Calculating power indices with the coalition taken into account is not very exciting: the majority takes all all the power. This paints a rather grim picture of (most) parliamentary democracies where majority coalitions act as dictators between elections.

A The proof of Theorem 7

In order to apply the concept of Young (1985)'s proof we need the following result:

Lemma 8. *Let v be an arbitrary weighted voting game, and player i be such that there exists $T^* \in \mathcal{W}_v$ such that $i \notin T^*$. Then player i is either a null-player or there exists a weighted voting game w such that $T^* \notin \mathcal{W}_w \subseteq \mathcal{W}_v$ and $w'_i = v'_i$.*

Proof. Let $v = (q, N_1, \ldots, N_n)$ be a weighted voting game, T^* be a minimal winning coalition in game v and player i be such that $i \notin T^*$.

Two cases can occur: (1) player i is not in any minimal winning coalition in game v. Then player i is a null-player, hence $v'_i = 0 = 0'_i$.

(2) Player i is a member of a minimal winning coalition in game v. Let $k = \sum\limits_{i \in T^*} N_i$, that is k is the number of votes coalition T^* has. Let $e = k - q + 1$ (notice that $k \geq q$), moreover, let $q' = q + e$, $N'_i = N_i + e$ and for all $j \in N \setminus \{i\}$: let $N'_j = N_j$. Every minimal winning coalition in game v which player i is a member of, is a minimal winning coalition in the voting situation $(q', N'_1, \ldots, N'_n)$, so $w = (q', N'_1, \ldots, N'_n)$ is a weighted voting game such that T^* is not a winning coalition and $\mathcal{W}_w \subseteq \mathcal{W}_v$. Furthermore, it is easy to see that for all $T \in \mathcal{W}_v \setminus \mathcal{W}_w$: i is a null-player in game u_T, therefore Lemma 2 implies that $w'_i = v'_i$. □

The proof of Theorem 7.

If: Proposition 5 states that the Shapley-Shubik index satisfies the axioms Efficiency, Symmetry and Marginality.

Only if: The proof goes by induction on the number of minimal winning coalitions.

Notice that 0 is a generalised voting game. Furthermore, axioms Efficiency and Symmetry imply that $\kappa(0) = 0 = \phi(0)$.

Let $\tilde{v}$ be an arbitrary generalised voting game. From Lemma 6

$$\tilde{v} = \bigvee_{T \subseteq N} \sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{v}}(w) 1_T^w u_T \ .$$

For any $T \subseteq N$ let $u_T^{\tilde{v}} = \sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{v}}(w) 1_T^w u_T$. Then for all $i, j \in N$ such that $i, j \in T$ or $i, j \notin T$: $i \sim^{u_T^{\tilde{v}}} j$, and if $i \notin T$ then i is a null-player.

Let $\mathcal{W}_{\tilde{v}} = \left\{ T \subseteq N : \sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{v}}(w) 1_T^w > 0 \right\}$, that is the minimal winning coalitions in $\tilde{v}$ are the coalitions which are minimal winning coalitions with positive probability.

Let n be an arbitrary natural number, and assume that for all generalised voting games $\tilde{v}$ such that $|\mathcal{W}_{\tilde{v}}| \leq n$ it is true that $\kappa(v) = \phi(v)$. Moreover, let $\tilde{w}$ be a generalised voting game such that $|\mathcal{W}_{\tilde{w}}| = n + 1$.

Divide the player set N into two subsets N^1 and N^2. Let N^1 contain the players which are not member in all minimal winning coalitions in game $\tilde{w}$, that is $N^1 = \{i \in N : \exists T^* \in \mathcal{W}_{\tilde{w}},\ i \notin T^*\}$. Let $N^2 = N \setminus N^1$.

Let $i^* \in N^1$ be an arbitrary player and $T^* \in \mathcal{W}_{\tilde{w}}$ be such that $i^* \notin T^*$. We know that

$$\tilde{w} = \sum_{w \in \hat{\Gamma}_N \cup \{0\}} \bigvee_{T \subseteq N} p_{\tilde{w}}(w) 1_T^w u_T \ .$$

Then for any $w \in \hat{\Gamma}_N \cup \{0\}$ two cases can happen: (1) $p_{\tilde{w}}(w) 1_{T^*}^w = 0$, in this case let $z(w) = w$, (2) $p_{\tilde{w}}(w) 1_{T^*}^w > 0$. In the latter case, from Lemma 8 either there exists a weighted voting game $z(w)$ such that $T^* \notin \mathcal{W}_{z(w)}$ and $v'_{i^*} = z(v)'_{i^*}$, or i^* is a null-player in game w. In the letter case (i is a null-player in w) let $z(w) = 0$. Then from Lemma 6

$$\tilde{z} = \bigvee_{T \subseteq N} \sum_{w \in \hat{\Gamma}_N \cup \{0\}} p_{\tilde{w}}(w) 1_T^{z(w)} u_T = \sum_{w \in \hat{\Gamma}_N \cup \{0\}} \bigvee_{T \subseteq N} p_{\tilde{w}}(w) 1_T^{z(w)} u_T$$

is a generalised voting game and $w'_{i^*} = z'_{i^*}$.

For any $w \in \hat{\Gamma}_N \cup \{0\}$ such that $p_{\tilde{w}}(w) > 0$, $T^* \notin \mathcal{W}_{z(w)}$, hence $|\mathcal{W}_{\tilde{z}}| \leq n$. Then from the induction hypothesis: $\kappa(\tilde{z}) = \phi(\tilde{z})$. Therefore, the Marginality axiom implies that $\kappa_{i^*}(\tilde{w}) = \phi_{i^*}(\tilde{z})$.

For any $i, j \in N$ such that $i, j \in T$: $i \sim^{u_T^{\tilde{w}}} j$, and any $i, j \in N^2$ are such that $i, j \in T$ for all $T \in \mathcal{W}_{\tilde{w}}$, hence the Symmetry axiom implies that for any $i, j \in N^2$ $\kappa_i(\tilde{w}) = \kappa_j(\tilde{w})$.

Summing up the above discussion and applying the Efficiency axiom, we get that $\kappa(\tilde{w})$ is well defined (it is uniquely defined). Therefore, Proposition 5 implies that $\kappa(\tilde{w}) = \phi(\tilde{w})$. □

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