Linearized Credibility Formula for Exponentially Weighted Squared Error Loss Function

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Is an original paper, which describes techniques for estimating premiums for risks, containing a fraction (a part) of the variance of the risk as a loading on the net risk premium.
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Introduction

One approach is to consider the best risk premium (in the sense of the minimal weighted mean squared error). In the first section it is shown that it can be used as an approximation to the variance loaded premium, by truncating a series expansion. In the second section this premium is derived as an optimal estimator minimizing a suitable loss function. In the credibility theory so far the credibility results described are intended to estimate pure net risk premiums. An important question arises if one is interested in estimating the variance loaded premiums. In a top-down approach, the collective premium can be distributed proportionally to this loaded risk premium. Therefore we also consider credibility estimates for loaded risk premiums.

Theory

Replacing the loss function $(y-x)^2$ when y is estimated instead of x by a slightly more general weighted loss function $(y-x)^2 e^{hy}$, one gets the following best function of X to estimate a random variable Y.

Minimizing weighted mean squared error

When X and Y are two random variables, and Y must be estimated using a function $g(X)$ of X, the choice yielding the minimal weighted mean squared error $E\left[(Y-g(X))^2 e^{hY}\right]$ is the best risk premium (in the sense of the minimal weighted mean squared error) for Y, given X:

$$H[Y \mid X] = E\left[Ye^{hY} \mid X\right] / E\left[e^{hY} \mid X\right] \qquad (1.1)$$

Proof. Write:

$$E\left[(Y-g(X))^2 e^{hY}\right] = E\left\{E\left[(Y-g(X))^2 e^{hY} \mid X\right]\right\} = \int E\left[(Y-g(x))^2 e^{hY} \mid X=x\right] \cdot dF_X(x)$$

For fixed x, the integrand can be written as: $E\left[(Z-p)^2 e^{hZ}\right]$, with $p=g(x)$ and Z distributed as Y, given $X=x$ ($Z \overset{(P)}{\equiv} [Y \mid (X=x)]$). This quadratic form in p is minimized taking $p=E\left[Ze^{hZ}\right]/E\left[e^{hZ}\right]$ or what is the same $g(x)=E\left[Ye^{hY} \mid X=x\right]/E\left[e^{hY} \mid X=x\right]$.

$$\text{Indeed: } \varphi(p) \overset{not.}{=} E\left[(Z-p)^2 e^{hZ}\right] = E\left(Z^2 e^{hZ}\right) + p^2 E\left(e^{hZ}\right) - 2pE\left(Ze^{hZ}\right).$$

We have to solve the following minimization problem:

$$\underset{p}{Min}\, \varphi(p) \qquad (1.2)$$

Since (1.2) is the minimum of a positive definite quadratic form, it is suffices to find a solution with the first derivative equal to zero. Taking the first derivative with respect to p, we get the equation: $2pE\left(e^{hZ}\right) - 2E\left(Ze^{hZ}\right) = 0$.

So: $p = E\left(Ze^{hZ}\right)/E\left(e^{hZ}\right)$, because:

$$\varphi''(p) = 2E\left(e^{hZ}\right) > 0.$$

If the integrand is chosen minimal for each x, the integral over all x is minimized, too.

Remark 1.1 (the best risk premium and variance premium)

For small h, the best premium approaches the variance principle. This can be seen by approximating numerator and denominator of $H[Y|X]$ up to the first order in h:

$$H[Y \mid X] = \frac{E[Y \mid X] + hE[Y^2 \mid X] + O(h^2)}{1 + hE[Y \mid X] + O(h^2)} = \left(E[Y \mid X] + hE[Y^2 \mid X] + O(h^2)\right)\cdot$$
$$\cdot\left(1 - hE[Y \mid X] + O(h^2)\right) = E[Y \mid X] + hVar[Y \mid X] + O(h^2).$$

Indeed, we have:

$$e^{hY} = 1 + h\frac{Y}{1!} + h^2\frac{Y^2}{2!} + \ldots + h^n\frac{Y^n}{n!} + \ldots \approx 1 + hY + O(h^2),$$

and so: $Ye^{hY} \cong Y + hY^2 + O(h^2)$. Therefore, from (1.1) we get:

$$H[Y \mid X] = \frac{E[Ye^{hY} \mid X]}{E[e^{hY} \mid X]} = \frac{E[Y + hY^2 + O(h^2) \mid X]}{E[1 + hY + O(h^2) \mid X]} = \frac{E[Y \mid X] + hE[Y^2 \mid X] + O(h^2)}{1 + hE[Y \mid X] + O(h^2)}.$$

Also: $(1+Z)^{-1} = 1 + \frac{(-1)}{1!}Z + \frac{(-1)(-1-1)}{2!}Z^2 + \ldots + \frac{(-1)(-1-1)\ldots(-1-n+1)}{n!}Z^n + \ldots,$

if: $|Z| < 1$. On taking $Z = hE[Y \mid X] + O(h^2)$ one obtains:

$$\left(1 + hE[Y \mid X] + O(h^2)\right)^{-1} \cong 1 - h\cdot$$
$$E[Y \mid X] + O(h^2),$$ and thus:

$$H[Y \mid X] = E[Y \mid X] + hVar[Y \mid X] + O(h^2) \quad (1.3)$$

In fact any loss function $(y-x)^2 w(y)$ with small $h = w'(0)/w(0)$ leads to the expression (1.3). From this remark we may conclude that to derive credibility estimates for premiums loaded with a fraction of the variance, as well as for the best risk premium, one may consider a weighted loss function. To be able to compute the loaded credibility estimates, we will need (co-) variances of squares of the observations.

1. Credibility for the best risk premium

Consider the original Bühlmann model. Applying (1.1) - of the previous section - to $Y = X_{t+1}$ and $X = \underline{X} = (X_1, \ldots, X_t)'$, we see that the best risk premium - in the sense of weighted mean squared error- to charge for period $(t+1)$ is:

$$H[X_{t+1} \mid \underline{X}] = E\left[X_{t+1}e^{hX_{t+1}} \mid \underline{X}\right] / E\left[e^{hX_{t+1}} \mid \underline{X}\right] \quad (2.1)$$

For small h, the expansion (1.3) for (2.1) can be rewritten as follows:

$$H[X_{t+1} \mid \underline{X}] = E[X_{t+1} \mid \underline{X}] + hVar[X_{t+1} \mid \underline{X}] + O(h^2) \cong E[X_{t+1} \mid \underline{X}] + hVar[X_{t+1} \mid \underline{X}] = E$$
$$[\mu(\theta) \mid \underline{X}] + h\left\{E[\sigma^2(\theta) \mid \underline{X}] + Var[\mu(\theta) \mid \underline{X}]\right\} \quad (2.2)$$

Indeed, we have:

$$E[\mu(\theta) \mid \underline{X}] = E[E(X_{t+1} \mid \theta) \mid \underline{X}] = E[E(X_{t+1} \mid \theta, \underline{X}) \mid \underline{X}] = E(X_{t+1} \mid \underline{X}) \quad (2.3),$$

and also:

$$\mathrm{Var}(X_{t+1} \mid \underline{X}) = E\left(X_{t+1}^2 \mid \underline{X}\right) - E^2[X_{t+1} \mid \underline{X}] \quad (2.4)$$

But: $E[\sigma^2(\theta) \mid \underline{X}] = E[Var(X_{t+1} \mid \theta) \mid \underline{X}] = E\left\{\left[E(X_{t+1}^2 \mid \theta) - E^2(X_{t+1} \mid \theta)\right] \mid \underline{X}\right\} =$

$$= E\left[E\left(X_{t+1}^2 \mid \theta\right) \mid \underline{X}\right] - E\left[E^2(X_{t+1} \mid \theta) \mid \underline{X}\right] = E\left[E\left(X_{t+1}^2 \mid \theta, \underline{X}\right) \mid \underline{X}\right] -$$

$$- E\left[E^2(X_{t+1} \mid \theta) \mid \underline{X}\right] = E\left(X_{t+1}^2 \mid \underline{X}\right) - E\left[E^2(X_{t+1} \mid \theta) \mid \underline{X}\right] \quad (2.5),$$

and:

$$Var[\mu(\theta)|\underline{X}] = E[\mu^2(\theta)|\underline{X}] - E^2[\mu(\theta)|\underline{X}] = E[E^2(X_{t+1}|\theta)|\underline{X}] - $$
$$- E^2(X_{t+1}|\underline{X}) \quad (2.6),$$

because from (2.3) we have: $E[\mu(\theta)|\underline{X}] = E(X_{t+1}|\underline{X})$ and so: $E^2[\mu(\theta)|\underline{X}] = E^2(X_{t+1}|\underline{X})$.
Furthermore:

$$E[\sigma^2(\theta)|\underline{X}] + Var[\mu(\theta)|\underline{X}] = E(X_{t+1}^2|\underline{X}) - E[E^2(X_{t+1}|\theta)|\underline{X}] +$$
$$+ E[E^2(X_{t+1}|\theta)|\underline{X}] - E^2(X_{t+1}|\underline{X}) = E(X_{t+1}^2|\underline{X}) - E^2(X_{t+1}|\underline{X}) =$$
$$= Var(X_{t+1}|\underline{X}) \text{ (see (2.4))}.$$

Therefore,

$$H[X_{t+1}|\underline{X}] = E[\mu(\theta)|\underline{X}] + h\{E[\sigma^2(\theta)|\underline{X}] + Var[\mu(\theta)|\underline{X}]\}$$

. Thus we have (2.2).

Remark 2.1 The first term in (2.2) denotes the expected value part, the second term gives the variance part, the last term the fluctuation part.

Remark 2.2 Apart from the optimal credibility result (2.1) for this situation, we are interested in obtaining a linearized credibility formula for estimating X_{t+1}. Therefore, we prove the following application of the formula (1.1).

The main results of this paper

In the following we present **the main results** leaving the detailed computations to the reader.

An application of the formula (1.1) **- Linearized credibility formula for exponentially weighted squared error loss function -**

The solution to the following problem:

$$\underset{c_0,c_1,\ldots,c_t}{Min} E\left[\left(X_{t+1} - c_0 - \sum_{r=1}^{t} c_r X_r\right)^2 e^{hX_{t+1}}\right] \quad (2.7),$$

gives: $M^{a^*} = z^*\overline{X} + \left\{\dfrac{E_\theta^*[H(X|\theta)]}{E_\theta^*[\mu(\theta)]} - z^*\right\}E_\theta^*[\mu(\theta)] \quad (2.8)$

Here $H[X|\theta]$ is the best risk premium for the conditional distribution of X given θ, and:

$$z^* = tCov^*[H[X|\theta],\mu(\theta)]/\{tVar^*[\mu(\theta)] + E_\theta^*[Var[X|\theta]]\} \quad (2.9)$$

The asterisks denote expectations taken over a weighted structure function U^* satisfying: $dU^*(\theta) = m_h(\theta)dU(\theta)/m_h$ (2.10), with: $m_h(\theta) = E[e^{hX}|\theta]$ (2.11), and

$$m_h = E[m_h(\theta)] = \int m_h(\theta)dU(\theta) \quad (2.12)$$

Proof. To have a minimum in (2.7), the derivative with respect to c_0 must equal to zero, so:

$E[X_{t+1}e^{hX_{t+1}}] = c_0E[e^{hX_{t+1}}] + \sum_{r=1}^{t} c_r E[X_r e^{hX_{t+1}}]$, or: $c_0 = \dfrac{E[X_{t+1}e^{hX_{t+1}}]}{E[e^{hX_{t+1}}]} - \dfrac{\sum_{r=1}^{t} c_r E[X_r e^{hX_{t+1}}]}{E[e^{hX_{t+1}}]}$, that is:

$$c_0 = E_\theta[H[X|\theta]m_h(\theta)/m_h] - \sum_{r=1}^{t} c_r E_\theta[\mu(\theta)m_h(\theta)/m_h] \quad (2.13),$$

where $m_h(\theta)$ and m_h are as in (2.11) and (2.12), because:

$$E[X_{t+1}e^{hX_{t+1}}] = E\{E[X_{t+1}e^{hX_{t+1}}|\theta]\} = E\left[\frac{E(X_{t+1}e^{hX_{t=1}}|\theta)}{E(e^{hX_{t+1}}|\theta)}E(e^{hX_{t+1}}|\theta)\right] =$$
$$= E[H(X|\theta)m_h(\theta)] = E_\theta[H(X|\theta)m_h(\theta)] \text{ (see (1.1))}.$$
$$E[e^{hX_{t+1}}] = E(e^{hX}) = E[E(e^{hX}|\theta)] = E[m_h(\theta)] = m_h$$
$$E[X_r e^{hX_{t+1}}] = E[E(X_r e^{hX_{t+1}}|\theta)] = E[E(X_r|\theta)E(e^{hX_{t+1}}|\theta)] =$$

$$= E\left[E(X \mid \theta)E\left(e^{hX} \mid \theta\right)\right] = E[\mu(\theta)m_h(\theta)] = E_\theta[\mu(\theta)m_h(\theta)].$$

Therefore, the verification of formula (2.13) is readily performed. Using the notation:

$$E_\theta^*[f(\theta)] = \int f(\theta)dU^*(\theta) = \int f(\theta)\frac{m_h(\theta)}{m_h}dU(\theta) = E_\theta[f(\theta)m_h(\theta)/m_h]$$

(2.13) can be written as: $c_0 = E_\theta^*[H(X \mid \theta)] - \sum_{r=1}^{t} c_r E_\theta^*[\mu(\theta)]$ (2.14)

Inserting (2.14) into (2.7) the problem is reduced to:

$$\underset{\underline{c}}{Min}\, E\left[\left\{X_{t+1} - E_\theta^*[H(X \mid \theta)] - \sum_{r=1}^{t} c_r\left(X_r - E_\theta^*[\mu(\theta)]\right)\right\}^2 e^{hX_{t+1}}\right] \quad (2.15),$$

where: $\underline{c} = (c_1, ..., c_t)'$. On taking the derivative of (2.15) with respect to $c_q, q = \overline{1,t}$ and putting the result equal to zero, one obtains:

$$E\left\{\left(X_{t+1} - E_\theta^*[H(X \mid \theta)]\right)\left(X_q - E_\theta^*[\mu(\theta)]\right)e^{hX_{t+1}}\right\} =$$

$$= \sum_{r=1}^{t} c_r \left\{\left(X_r - E_\theta^*[\mu(\theta)]\right)\left(X_q - E_\theta^*[\mu(\theta)]\right)e^{hX_{t+1}}\right\} \quad (2.16)$$

Using conditional expectations over θ an dividing by m_h, this equation can be written as:

$$E_\theta\left\{\left(H(X \mid \theta) - E_\theta^*[H(X \mid \theta)]\right)\left(\mu(\theta) - E_\theta^*[\mu(\theta)]\right)m_h(\theta)/m_h\right\} =$$

$$= \sum_{\substack{r=1 \\ r \neq q}}^{t} c_r E_\theta\left\{\left(\mu(\theta) - E_\theta^*[\mu(\theta)]\right)\left(\mu(\theta) - E_\theta^*[\mu(\theta)]\right)m_h(\theta)/m_h\right\} +$$

$$+ c_q E_\theta\left(Var[X \mid \theta]m_h(\theta)/m_h\right) \quad (2.17),$$

since: $E\left[X_{t+1}X_q e^{hX_{t+1}}\right] = E\left\{E\left[X_{t+1}X_q e^{hX_{t+1}} \mid \theta\right]\right\} = E\left[E\left(X_{t+1}e^{hX_{t+1}} \mid \theta\right)E\left(X_q \mid \theta\right)\right] =$
$= E[H(X \mid \theta)m_h(\theta)\mu(\theta)] = E_\theta[H(X \mid \theta)m_h(\theta)\mu(\theta)]$, from which:
$E\left[X_{t+1}X_q e^{hX_{t+1}}/m_h\right] = E_\theta[H(X \mid \theta)m_h(\theta)\mu(\theta)/m_h]$;
$E\left\{X_{t+1}E_\theta^*[\mu(\theta)]e^{hX_{t+1}}\right\} = E\left(E\left\{X_{t+1}E_\theta^*[\mu(\theta)]e^{hX_{t+1}} \mid \theta\right\}\right) = E\left\{E_\theta^*[\mu(\theta)]E\left(X_{t+1}e^{hX_{t+1}} \mid \theta\right)\right\} =$
$E_\theta^*[\mu(\theta)]E\left[E\left(X_{t+1}e^{hX_{t+1}} \mid \theta\right)\right] = E_\theta^*[\mu(\theta)]E_\theta[H(X \mid \theta)m_h(\theta)] = E_\theta\left\{E_\theta^*[\mu(\theta)]\cdot\right.$
$\left.\cdot H(X \mid \theta)m_h(\theta)\right\}$, from which: $E\left\{X_{t+1}E_\theta^*[\mu(\theta)]e^{hX_{t+1}}/m_h\right\} = E_\theta\left\{E_\theta^*[\mu(\theta)]H(X \mid \theta)\cdot\right.$
$\left.\cdot m_h(\theta)/m_h\right\}$;
$E\left\{E_\theta^*[H(X \mid \theta)]X_q e^{hX_{t+1}}\right\} = E_\theta^*[H(X \mid \theta)]E\left(X_q e^{hX_{t+1}}\right) = ... = E_\theta^*[H(X \mid \theta)]E_\theta[\mu(\theta)\cdot$
$\cdot m_h(\theta)] = E_\theta\left\{E_\theta^*[H(X \mid \theta)]\mu(\theta)m_h(\theta)\right\}$, from which: $E\left\{E_\theta^*[H(X \mid \theta)]X_q e^{hX_{t+1}}/\right.$
$\left./m_h\right\} = E_\theta\left\{E_\theta^*[H(X \mid \theta)]\mu(\theta)m_h(\theta)/m_h\right\}$;
$E\left\{E_\theta^*[H(X \mid \theta)]E_\theta^*[\mu(\theta)]e^{hX_{t+1}}\right\} = E_\theta^*[H(X \mid \theta)]E_\theta^*[\mu(\theta)]E\left(e^{hX_{t+1}}\right) = ... = E_\theta^*[H(X \mid \theta)]\cdot$
$\cdot E_\theta^*[\mu(\theta)]E[m_h(\theta)] = E_\theta^*[H(X \mid \theta)]E_\theta^*[\mu(\theta)]E_\theta[m_h(\theta)] = E_\theta\left\{E_\theta^*[H(X \mid \theta)]\, E_\theta^*[\mu(\theta)]\cdot\right.$
$\left.\cdot m_h(\theta)\right\}$, from which: $E\left\{E_\theta^*[H(X \mid \theta)]E_\theta^*[\mu(\theta)]e^{hX_{t+1}}/m_h\right\} = E_\theta\left\{E_\theta^*[H(X \mid \theta)]\cdot\right.$
$\left.\cdot E_\theta^*[\mu(\theta)]m_h(\theta)/m_h\right\}$; if $r \neq q$ then we obtain:
$E\left(X_r X_q e^{hX_{t+1}}\right) = E\left[E\left(X_r X_q e^{hX_{t+1}} \mid \theta\right)\right] = E\left[E(X_r \mid \theta)E(X_q \mid \theta)E\left(e^{hX_{t+1}} \mid \theta\right)\right] =$
$= E[\mu(\theta)\mu(\theta)m_h(\theta)] = E_\theta[\mu(\theta)\mu(\theta)m_h(\theta)]$, from which: $E\left(X_r X_q e^{hX_{t+1}}/m_h\right) =$
$= E_\theta[\mu(\theta)\mu(\theta)m_h(\theta)/m_h]$; if $r = q$ then we obtain:
$E\left(X_q^2 e^{hX_{t+1}}\right) = E\left[E\left(X_q^2 e^{hX_{t+1}} \mid \theta\right)\right] = E\left[E\left(X_q^2 \mid \theta\right)E\left(e^{hX_{t+1}} \mid \theta\right)\right] = E\left[E\left(X^2 \mid \theta\right)m_h(\theta)\right] =$

$= E_\theta\left[E\left(X^2 \mid \theta\right)m_h(\theta)\right]$, from which: $E\left(X_q^2 e^{hX_{t+1}} / m_h\right) = E_\theta\left[E\left(X^2 \mid \theta\right)m_h(\theta)/m_h\right]$; also, we have $E\left\{X_r E_\theta^*[\mu(\theta)]e^{hX_{t+1}}\right\} = E\left(E\left\{X_r E_\theta^*[\mu(\theta)]e^{hX_{t+1}} \mid \theta\right\}\right) = E\left\{E_\theta^*[\mu(\theta)]E(X_r \mid \theta)E\left(e^{hX_{t+1}} \mid \right.\right.$ $\left.\left. \mid \theta\right)\right\} = E\left\{E_\theta^*[\mu(\theta)]\mu(\theta)m_h(\theta)\right\} = E_\theta\left\{E_\theta^*[\mu(\theta)]\mu(\theta)m_h(\theta)\right\}$, from which: $E\ \{X_r \cdot$ $\cdot E_\theta^*[\mu(\theta)]e^{hX_{t+1}} / m_h\} = E_\theta\left\{E_\theta^*[\mu(\theta)]\mu(\theta)m_h(\theta)/m_h\right\}$, where $r = \overline{1,t}$; similarly, we have: $E\left\{E_\theta^*[\mu(\theta)]X_q e^{hX_{t+1}} / m_h\right\} = E_\theta\left\{E_\theta^*[\mu(\theta)]\mu(\theta)m_h(\theta)/m_h\right\}$, where $q = \overline{1,t}$; we write: $E\left\{E_\theta^{*2}[\mu(\theta)]e^{hX_{t+1}}\right\} = E_\theta^{*2}[\mu(\theta)]E\left(e^{hX_{t+1}}\right) = E_\theta^{*2}[\mu(\theta)]E[m_h(\theta)] = E\left\{E_\theta^{*2}[\mu(\theta)] \cdot\right.$ $\left.m_h(\theta)\right\} = E_\theta\left\{E_\theta^{*2}[\mu(\theta)]m_h(\theta)\right\}$, from which: $E\left\{E_\theta^{*2}[\mu(\theta)]e^{hX_{t+1}} / m_h\right\} = E_\theta\left\{E_\theta^{*2}[\mu(\theta)] \cdot\right.$ $\left.\cdot e^{hX_{t+1}} / m_h\right\}$; finally, we observe that:

$$E\left\{\left(X_q - E_\theta^*[\mu(\theta)]\right)^2 e^{hX_{t+1}}\right\} = E_\theta\left\{\left[E\left(X^2 \mid \theta\right) + E_\theta^{*2}[\mu(\theta)] - 2E_\theta^*[\mu(\theta)]E(X \mid \theta)\right] \cdot\right.$$

$$\left.\cdot m_h(\theta)\right\} = E_\theta\left\{\left[E\left(X^2 \mid \theta\right) + E^2(X \mid \theta) - 2E^2(X \mid \theta)\right]m_h(\theta)\right\} = E_\theta\left\{\left[E\left(X^2 \mid \theta\right) -\right.\right.$$

$$\left.\left.- E^2(X \mid \theta)\right]m_h(\theta)\right\} = E_\theta\left[Var(X \mid \theta)m_h(\theta)\right]$$, from which: $E\left\{\left(X_q - E_\theta^*[\mu(\theta)]\right)^2 \cdot\right.$ $\left.\cdot e^{hX_{t+1}} / m_h\right\} = E_\theta\left[Var(X \mid \theta)m_h(\theta)/m_h\right]$, where: $E\left(X_q^2 e^{hX_{t+1}}\right) = E\left[E\left(X_q^2 e^{hX_{t+1}} \mid\right.\right.$ $\left.\left. \mid \theta\right)\right] = E\left[E\left(X_q^2 \mid \theta\right)E\left(e^{hX_{t+1}} \mid \theta\right)\right] = E\left[E\left(X^2 \mid \theta\right)m_h(\theta)\right] = E_\theta\left[E\left(X^2 \mid \theta\right) m_h(\theta)\right]$, $E\left\{E_\theta^{*2}[\mu(\theta)]e^{hX_{t+1}}\right\} = E_\theta^{*2}[\mu(\theta)]E\left(e^{hX_{t+1}}\right) = \ldots = E_\theta^{*2}[\mu(\theta)]E[m_h(\theta)] = E\left\{E_\theta^{*2}[\mu(\theta)] \cdot m_h(\theta)\right\} = E_\theta$ $\left\{E_\theta^{*2}[\mu(\theta)]m_h(\theta)\right\}$, $E\left\{X_q E_\theta^*[\mu(\theta)]e^{hX_{t+1}}\right\} = E_\theta^*[\mu(\theta)]E\left(X_q e^{hX_{t+1}}\right) =$ $= \ldots = E_\theta^*[\mu(\theta)]E_\theta[\mu(\theta)m_h(\theta)] = E_\theta\left\{E_\theta^*[\mu(\theta)]\mu(\theta)m_h(\theta)\right\} = E_\theta\left\{E_\theta^*[\mu(\theta)]E(X \mid \theta) \cdot\right.$ $\left.\cdot m_h(\theta)\right\}$, with $q = \overline{1,t}$.

Defining Cov^* and Var^* by using E_θ^* instead of E_θ (2.17) becomes:

$$Cov^*[H(X \mid \theta), \mu(\theta)](= E_\theta^*[(H(X \mid \theta) - E_\theta^*[H(X \mid \theta)])(\mu(\theta) - E_\theta^*[\mu(\theta)])]) =$$

$$= \sum_{\substack{r=1 \\ r \neq q}}^{t} c_r Cov^*[\mu(\theta), \mu(\theta)] + c_q E_\theta^*[Var(X \mid \theta)] = \sum_{\substack{r=1 \\ r \neq q}}^{t} c_r Var^*[\mu(\theta)] + c_q E_\theta^*[Var(X \mid \theta)],$$

where $q = \overline{1,t}$.

Because of the symmetry of this system of equations in the variables: $c_1, c_2, ..., c_t$ one obtains $c_1 = c_2 = ... = c_t$ and therefore:

$$c = Cov^*[H(X \mid \theta), \mu(\theta)] / \left\{tVar^*[\mu(\theta)] + E_\theta^*[Var(X \mid \theta)]\right\} \quad (2.18)$$

Inserting (2.18) into (2.14) and taking $z^* = ct$ as in (2.9) one obtains:

$$c_0 = E_\theta^*[H[X \mid \theta]] - c\sum_{r=1}^{t} E_\theta^*[\mu(\theta)] = E_\theta^*[H[X \mid \theta]] - \frac{z^*}{t} tE_\theta^*[\mu(\theta)] = E_\theta^*[H[X \mid \theta]] -$$

$- z^* E_\theta^*[\mu(\theta)]$. Consequently:

$$M^{a^*} = c_0 + \sum_{r=1}^{t} c_r X_r = E_\theta^*[H[X \mid \theta]] - z^* E_\theta^*[\mu(\theta)] + \frac{z^*}{t}\sum_{r=1}^{t} X_r = z^* \overline{X} +$$

$+ \left(\frac{E_\theta^*[H[X \mid \theta]]}{E_\theta^*[\mu(\theta)]} - z^*\right)E_\theta^*[\mu(\theta)]$, as was to be proven. Taking the limit for: $h \to 0$, the original Bühlmann credibility formula results (see (1.1)).

Conclusions

In this paper we have obtained the best risk premium - in the sense of weighted mean squared error - to charge for period $(t+1)$, by truncating a series expansion. To be able to compute the loaded credibility estimates, we demonstrated the relevant (co-) variances of squares of the observations. The fact that it is based on complicated mathematics, involving conditional expectations, needs not bother the user more than it does when he applies statistical tools like SAS, GLIM, discriminant analysis, and scoring models. Apart from the optimal credibility result (2.1) for this situation, we have presented a linearized credibility formula for exponentially weighted squared error loss function (a linearized credibility formula for estimating X_{t+1}), using the greatest accuracy theory.

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