

# Taking into account extreme events in European option pricing

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*According to traditional option pricing models,<sup>1</sup> financial markets underestimate the impact of tail risk.*

*In this article, we put forward a European option pricing model based on a set of assumptions that ensure, inter alia, that extreme events are better taken into account. Using simulations, we compare the option prices obtained from the standard Black and Scholes model with those resulting from our model. We show that the traditional model leads to an overvaluation of at-the-money options, which are the most traded options, while the less liquid in-the-money and out-of-the-money options are undervalued.*

NB: This article reflects the opinions of the authors and does not necessarily express the views of the Banque de France.

<sup>1</sup> See Black and Scholes (1973).

The literature on the theory of financial asset pricing is mainly developed by Merton (1973a, 1973b, 1974, 1976), Black and Scholes (1973) and Cox, Ingersoll and Ross (1985a, 1985b). As regards option pricing, the reference model is still that of Black and Scholes (BS) (1973). However, the assumptions of this model are inappropriate and largely rejected by the data. For example, in the initial formulation of the model, the volatility of the rate of return on the risky asset underlying the option is assumed to be constant. This is not verified empirically. In addition, the rate of return of the underlying asset is assumed to follow a normal distribution. Yet, the assumption of a normal distribution of the rates of return on financial variables is largely contested, even rejected, in particular because it underestimates the frequency of extreme events.<sup>2</sup>

In this article, we mainly focus on this last criticism of the Black and Scholes model and put forward an option pricing model for European options based on more realistic assumptions.

Our model fits into the more general framework of discrete-time factor models for financial or physical asset pricing<sup>3</sup> according to the no-arbitrage principle.

Under the no-arbitrage assumption, the price of a financial asset is equal to expected future cash flows discounted by a discount factor representing both risk aversion and preference for the present. This principle implies two types of modelling approaches:

- first, define factors that represent the information held by investors and model the dynamics of these factors;
- second, select a model of the discount factor according to these factors.

In particular, these two elements are used to define the "virtual" dynamics of the factors (the so-called "risk-neutral" dynamics) for which the asset price

becomes equal to the expected future cash flows discounted by the risk-free rate.

In this article, we show how these general principles can be applied to calculate the price of a European option. More specifically, one of the key assumptions of the model is linked to the definition of the factor's historical dynamics, which are supposed to be a mixture of Gaussian distributions. This assumption ensures a better modelling of extreme events. To simplify the presentation, we choose a static framework. We show that the model presented here expands on the Black and Scholes model (See Box 1) and enables us to take better account of tail risk. It emerges that at-the-money options are overpriced while in-the-money and out-of-the-money options are underpriced.

## 1 | DEFINITION OF THE INFORMATION SET AND ITS PROBABILITY DISTRIBUTION

In order to price an asset, the investor defines a set of fundamental factors that are likely to have an effect on the price.  $w_t$  denotes the value of the factors at date  $t$  and  $\underline{w}_t$  its historical dynamics.<sup>4</sup> The future cash flows generated by the asset are assumed to depend on the future realisations of these factors.

In the option pricing model presented in this article, the information available to investors at every date is the rate of return on the underlying asset. It is an observable factor for which the historical dynamics can be derived from a sample of observations.

A commonly used –albeit widely contested– assumption is that the rate of return on the risky asset follows a Gaussian distribution.<sup>5</sup> We discuss this assumption and make an alternative proposal.

<sup>2</sup> See Mandelbrot (1962, 1963, 1967); Fama (1965).

<sup>3</sup> See Bertholon, Monfort and Pegoraro (2006) and Pegoraro (2006).

<sup>4</sup> This factor can be considered to be the information available to the investor at date  $t$ . This information may be observable (the yield on an underlying asset, macroeconomic factors), partially or totally unobservable (volatility regimes) by the econometrician.

<sup>5</sup> See Black and Scholes (1973).

**Box 1****The Black and Scholes model****Assumptions**

- The price of the underlying asset  $S_t$  follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where  $\mu$  and  $\sigma$  are constant.

The rate of return on the underlying asset,  $x_{t+1} = \ln(S_{t+1} / S_t)$  therefore follows a Gaussian distribution with a mean of  $\mu - \frac{\sigma^2}{2}$  and a standard deviation of  $\sigma$ ;

- there are no restrictions on short-selling;
- no commissions or taxes are charged;
- all the underlying assets are perfectly divisible;
- the underlying asset pays no dividends;
- there are no arbitrage opportunities;
- the market operates continuously;
- the risk-free interest rate,  $r^f$ , is constant.

**The Black and Scholes formula**

It is used to calculate the theoretical value  $C_t$  of a European option at date  $t$  with the five following variables:

- $S_t$  the price of the underlying asset at date  $t$ ;
- $T$  the expiration date of the option;
- $K$  the option exercise price;
- $r^f$  the risk-free interest rate;
- $\sigma$  the volatility of the price of the underlying asset.

The relative theoretical price at date  $t$   $\left(\frac{C_t}{S_t}\right)$  of a call option, denoted  $c_t$  with maturity at  $t$  and relative exercise price  $\kappa = \frac{K}{S_t}$  is:

$$c_t = N(d_1) - \kappa e^{-r^f(T-t)} N(d_2).$$

Similarly, the relative price of a put option is:

$$p_t = \kappa e^{-r^f(T-t)} N(-d_2) - N(-d_1),$$

where:

- $N$  is the standard normal cumulative distribution function,  $N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$ ,

$$d_1 = \frac{-\ln(\kappa) + (r^f + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

More specifically, in the case of a call option at date  $t$  with a maturity equal to one period ( $T=t+1$ ), we have:

$$c_t = N(\tilde{d}_1) - \kappa e^{-r^f} N(\tilde{d}_2),$$

where:

$$\tilde{d}_1 = \frac{r^f - \ln(\kappa)}{\sigma} + \frac{\sigma}{2},$$

$$\tilde{d}_2 = \tilde{d}_1 - \sigma.$$

Then,  $c_{BS}(\sigma^2, \kappa) = N(\tilde{d}_1) - \kappa e^{-r^f} N(\tilde{d}_2)$ .

## 1|1 Stylised facts and limitations of the Gaussian distribution

To illustrate the limitations of the normal distribution assumption, we take a look at the weekly rate of return on the CAC40 between 3 January 1996 and 30 April 2008.

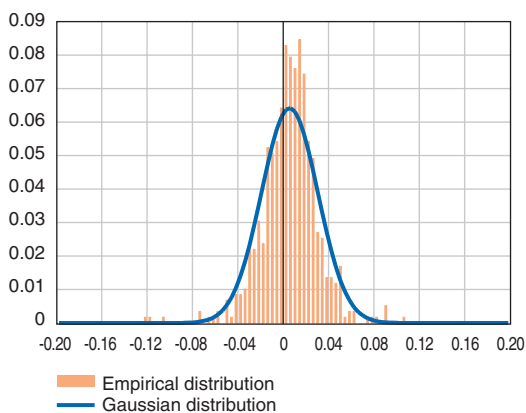
Chart 1 shows the empirical distribution of this rate of return. We first approximate this distribution to a Gaussian distribution  $N(\mu, \sigma^2)$ . The estimated values of  $\mu$  and  $\sigma^2$  are respectively the mean and empirical variance of the rates of return.

The theoretical distribution, thus estimated, provides for an exact reproduction of the observed mean and variance. However, some empirical characteristics of the distribution of returns cannot be reproduced with a Gaussian distribution (see Table 1):

- the empirical distribution tails are thicker than the Gaussian distribution tails: high returns (positive or negative) are more frequent than what the Gaussian distribution would predict. The kurtosis coefficient is thus above 3, which is that of a normal distribution. Furthermore, it has a more acute "peak" around zero and fat tails (see Table 1). Consequently, the theoretical probability of an extreme value occurring

**Chart 1**  
Empirical distribution of weekly rate of return on the CAC40 and Gaussian distribution

(X-axis: rate of return on the CAC40; Y-axis: probability density)



**Table 1**

	Empirical distribution	Gaussian distribution
Mean	0.13%	0.13%
Standard deviation	2.48%	2.48%
Skewness	-0.41	0
Kurtosis	6.61	3

Note: The mean and standard deviation are calculated relative to the weekly rate of return on the CAC 40 (03.01.96 to 30.04.08).

is underestimated when the rate of return is assumed to follow a Gaussian distribution;

- the empirical distribution is not symmetrical, unlike the Gaussian distribution: negative returns are more frequent than what the Gaussian distribution would predict. The skewness coefficient is negative for the empirical distribution, reflecting a longer left-hand tail, and equal to zero for the Gaussian distribution.

## 1|2 Mixtures of Gaussian distributions: definition and interpretation

In the literature, several categories of distributions have been put forward to compensate for the shortcomings of the Gaussian distribution: alpha-stable distributions;<sup>6</sup> finite mixtures of distributions, such as Gaussian mixtures;<sup>7</sup> simple and generalised Student distributions;<sup>8</sup> hyperbolic distributions.<sup>9</sup>

In this article, we focus on the mixture of Gaussian distributions for several reasons:

- it is an adequate proxy for all of the alternative distributions mentioned above;
- its theoretical properties are such as to facilitate manipulations in the framework of a theoretical asset pricing model, such as an option pricing model;
- it is very easy to simulate;
- it enables us to reproduce various characteristics (mean, variance, skewness and kurtosis) observed in the data, including in the simplest case where the mixture only includes two Gaussian distributions.

6 See Mandelbrot (1997); Mittnick and Rachev (1993); Adler et al. (1998).

7 See Kon (1984); Akgiray and Booth (1987); Tucker and Pond (1988).

8 See Bollerslev (1987); Baillie and Bollerslev (1989); Lambert and Laurent (2000, 2001).

9 See Barndorff-Nielsen (1994); Eberlein and Keller (1995); Kuechler et al. (1999).

More formally, making the assumption that the distribution is a mixture of two Gaussian distributions amounts to assuming that the random variable  $x$  (the rate of return for example) can take on values from two different regimes: regime 1 with a probability of occurrence equal to  $p$  and regime 2 with a probability of occurrence of  $1 - p$ . The probability distribution under regime 1 is a Gaussian distribution, with a mean of  $\mu_1$  and a variance of  $\sigma_1^2$ , denoted  $N(\mu_1, \sigma_1^2)$ . Under regime 2, the probability distribution is a Gaussian distribution with a mean of  $\mu_2$  and a variance of  $\sigma_2^2$ , denoted  $N(\mu_2, \sigma_2^2)$ . Overall, the probability distribution of the random variable  $x$  (a mixture of two Gaussian distributions) depends on five parameters,  $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2$  and  $p$ . The probability density of the mixture of two Gaussian distributions can be written as:

$$f(x) = pn(x; \mu_1, \sigma_1^2) + (1-p)n(x; \mu_2, \sigma_2^2) \quad (1)$$

$$\text{with } n(x; \mu_i, \sigma_i^2) = \frac{1}{\sigma_i \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(x-\mu_i)^2}{\sigma_i^2}},$$

the probability density of a Gaussian distribution with a mean of  $\mu_i$  and a variance of  $\sigma_i^2$ . This type of reasoning may be applied to more than two regimes, the distribution being a Gaussian distribution under each regime (see Assumption 1 in Box 2).

Another advantage of a mixture of Gaussian distributions relates to its interpretation, which is not always the case with other distributions, such as hyperbolic distributions or Student distributions. For example, in the case of a mixture of two Gaussian distributions, each regime may represent market states with different levels of volatility. The regime with the highest volatility may be interpreted as a financial crisis regime.

We have shown that a Gaussian distribution cannot reproduce the entire empirical distribution of returns (see Chart 1).

To obtain a better estimation of the empirical distribution of the rate of return on the CAC40, we estimate the parameters of the above-mentioned mixture of two Gaussian distributions.<sup>10</sup> The estimated values of the parameters are shown in Table 2. In this example, the probability of being in regime 1, *i.e.* the high-volatility regime, is 0.12.

<sup>10</sup> To do this we use maximum likelihood method.

**Table 2**

	Gaussian distribution Regime 1	Gaussian distribution Regime 2	Mixture of Gaussian distributions	Empirical distribution
Mean	-0.34%	0.19%	0.08%	0.13%
Standard deviation	4.96%	1.91%	2.96%	2.48%
Skewness	0	0	-0.21	-0.41
Kurtosis	3	3	5.42	6.61

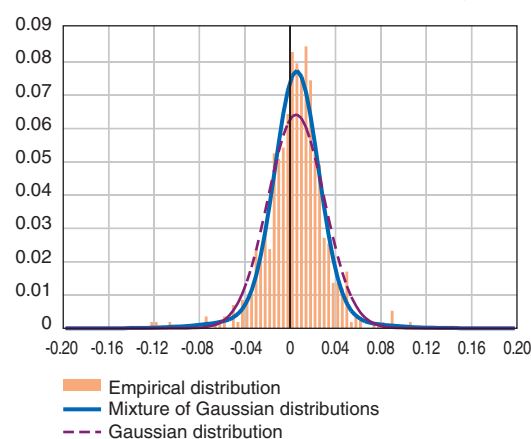
Note:  $p=0,12$  is the probability of occurrence of regime 1, a high-volatility regime. The mean and standard deviation are calculated relative to the weekly rate of return on the CAC 40 (03.01.96 to 30.04.08).

In this regime, the volatility is 4.96% and the mean 0.34%, *i.e.* an annualised volatility and mean of 35.7% and -17.7% respectively. This regime may be considered as a financial crisis regime. In regime 2, the volatility and the mean are 1.91% and 0.19%, *i.e.* 13.7% and 9.88% in annualised terms.

Chart 2 shows the empirical data distribution and its approximations *via* the mixture of Gaussian distributions (full line) and the Gaussian distribution (dotted line). The mixture of Gaussian distributions is better able to reproduce the distribution of data. In particular, it provides a better estimate of the tails and asymmetry of the empirical distribution. For example, a rate of return of three standard deviations (extreme event) is observed on average every 24 weeks. This event is forecasted on average every 160 weeks with the Gaussian distribution and on average every 22 weeks with the Gaussian mixture.

**Chart 2**  
Empirical distribution of the rate of return on the CAC 40 and approximation by a mixture of Gaussian distributions

(X-axis: rate of return on the CAC40; Y-axis: probability density)



## Box 2

## The option pricing model of Bertholon, Monfort, Pegoraro (2006)

## Assumptions

•  $H_1$ : the historical distribution of the rate of return on the underlying asset,  $x_{t+1} = \ln \left( \frac{S_{t+1}}{S_t} \right)$  where  $S_t$  is the price

at date  $t$  of the underlying asset, is a mixture of  $J$  Gaussian distributions. Its probability density is given by.

$$f(x) = \sum_{j=1}^J p_j n(x, \mu_j, \sigma_j^2),$$

where, for  $j = 1, \dots, J$ :

$$n(x, \mu_j, \sigma_j^2) = \frac{1}{\sigma_j \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(x-\mu_j)^2}{\sigma_j^2}} \text{ is the density of a Gaussian distribution with a mean of } \mu_j \text{ and a variance of } \sigma_j^2;$$

$$-0 \leq p_j \leq 1 \text{ et } \sum_{j=1}^J p_j = 1.$$

•  $H_2$ : the stochastic discount factor is exponential-affine  $M_{t,t+1} = \exp(\alpha_t x_{t+1} + \beta_t)$ .

## Results

**Result 1:** under assumptions  $H_1$  and  $H_2$ , the stochastic discount factor allows for a unique solution for  $(\alpha, \beta)$ , denoted  $(\alpha, \beta)$  that meets the two no-arbitrage conditions:

$$\begin{cases} E_t(M_{t,t+1} \exp(r_{t+1}^f)) = 1 \\ E_t(M_{t,t+1} \exp(x_{t+1})) = 1 \end{cases}, \text{ and under } H_2: \begin{cases} \exp(r_{t+1}^f + \beta_t) E_t(\exp(\alpha_t x_{t+1})) = 1, \\ \exp(\beta_t) E_t(\exp(\alpha_t + 1) x_{t+1}) = 1, \end{cases}$$

where  $r_{t+1}^f$  is the risk-free interest rate between  $t$  and  $t+1$  (known at  $t$ ).

The first condition is the no-arbitrage condition applied to the risk-free asset. The second is the no-arbitrage condition applied to the underlying asset.

**Result 2:** under assumptions  $H_1$  et  $H_2$ , the risk-neutral distribution of the factor is unique; it is also a mixture of Gaussian

distributions. Its probability density,  $f^Q(x)$ , is given by:  $f^Q(x) = \sum_{j=1}^J v_j n(x, \mu_j + \alpha \sigma_j^2, \sigma_j^2)$ ,

where, for  $j = 1, \dots, J$ :

$$v_j = \frac{p_j \exp\left(\alpha \mu_j + \frac{\alpha^2}{2} \sigma_j^2\right)}{\sum_{j=1}^J p_j \exp\left(\alpha \mu_j + \frac{\alpha^2}{2} \sigma_j^2\right)}, \quad 0 \leq v_j \leq 1, \quad \sum_{j=1}^J v_j = 1.$$

**Result 3:** under assumptions  $H_1$  et  $H_2$ , the theoretical price of a European call option with a one-period maturity is:

$$c_t = \sum_{j=1}^J v_j \gamma_j c_{BS} \left( \sigma_j^2, \frac{K}{\gamma_j} \right),$$

where  $c_{BS}(\dots)$  is the Black-Scholes formula with a period defined in Box 1 and  $\gamma_j = \exp\left(\mu_j + \alpha \sigma_j^2 - r^f + \frac{\sigma_j^2}{2}\right)$ .



The mixture of Gaussian distributions is clearly more appropriate for modelling the historical dynamics of the rate of return on the underlying asset. This is the assumption that will be used throughout the paper.

## 2| DISCOUNTING FUTURE CASH FLOWS GENERATED BY THE ASSET AND APPLICATION TO OPTION PRICING

The fundamental principle underlying the asset price model is the discounting of future cash flows generated by the asset. This raises the question of the discount factor to be used. It may be looked at from two different angles, depending on the "world" one considers. In the "risk-neutral" world, the discount rate is the risk-free rate. In the "real" or "historical" world, the stochastic discount factor is used. We come back to these two approaches to specify the assumptions made in the framework of our model and the links that can be established between the two worlds.

### 2|1 The historical world: the stochastic discount factor

If the no-arbitrage assumption is verified, there exists a positive random variable that enables us to calculate, at any date  $t$ , the price of an asset generating random future cash flows depending on the factors.<sup>11</sup> This variable is called a stochastic discount factor. More specifically, the asset price at date  $t$  is equal to the expected future cash flows generated by the asset, discounted by the stochastic discount factor.

If  $M_{t,t+1}$  is the stochastic discount factor between  $t$  and  $t+1$ ,  $P_t$  the asset price at date  $t$ ,  $g_{t+1} = g(w_{t+1})$  the cash flow generated by the asset between  $t$  and  $t+1$ , then:

$$P_t = E_t(M_{t,t+1}g_{t+1}). \quad (2)$$

<sup>11</sup> See Hansen and Richard (1987).

<sup>12</sup> See Gouriéroux and Monfort (2007).

The first step of the modelling process has made it possible to define and identify the historical conditional probability distribution of factor  $w_{t+1}$ , and therefore that of  $g_{t+1} = g(w_{t+1})$ . The second step involves doing the same for  $(w_{t+1}, M_{t,t+1})$ . Once the conditional distribution for  $(g_{t+1}, M_{t,t+1})$  has been identified, it is possible to determine either analytically or *via* simulations the conditional expectation of  $M_{t,t+1}g_{t+1}$  and therefore  $P_t$ .

The approach adopted in this article is based on an exponential-affine specification of the stochastic discount factor<sup>12</sup> :

$$M_{t,t+1} = \exp(\alpha_t(\underline{w}_t)'w_{t+1} + \beta_t(\underline{w}_t)). \quad (3)$$

In some circumstances, it is possible to determine the coefficients of the linear form,  $\alpha_t(\underline{w}_t)$  and  $\beta_t(\underline{w}_t)$ , in a unique manner, *via* the no-arbitrage condition. The stochastic discount factor is then uniquely defined in the exponential-affine class.

It is necessary to determine coefficients  $\alpha_t$  and  $\beta_t$  in order to obtain a complete specification of the form of the stochastic discount factor. By applying formula (2) to the rate of return on the underlying asset on the one hand, and to the rate of return on the risk-free asset on the other, it is possible to derive two so-called no-arbitrage conditions (see Box 2). We can then show that this system produces a unique solution  $(\alpha_t, \beta_t)$ , making it possible to obtain a complete specification of the form of the stochastic discount factor according to the historical dynamics of the factors.

### 2|2 The risk-neutral world

The "risk-neutral" world corresponds to a virtual economy in which economic agents would be indifferent to risk. The expected rate of return on all assets would then be equal to the risk-free rate. As a result, the discount rate would be equal to the risk-free rate. The risk-neutral world is easy to construct using the historical dynamics of the factors and the stochastic discount factor (see Appendix 1).

## 2|3 Application to option pricing

In our model, it is possible to show that, when the historical distribution of factors is a Gaussian mixture and the stochastic discount factor is exponential-affine, the risk-neutral distribution is unique and is also a mixture of Gaussian distributions. This result enables us, *inter alia*, to obtain an analytical and unique formula for the price of an equity option. Indeed, when the rate of return on the underlying asset is assumed to follow a mixture of Gaussian distributions at any date  $t$  and the stochastic discount factor is exponential-affine, the option pricing formula is shown to be a linear combination of Black and Scholes-type formulas. The price depends on the means and variances of the Gaussian distributions used in the mixture. In the Gaussian case, *i.e.* if we make the assumption of a single regime, we obtain the traditional Black and Scholes option pricing formulas (see Appendix 2 and Box 2). It is also possible to show that this formula can be applied to options with a maturity of over one period.

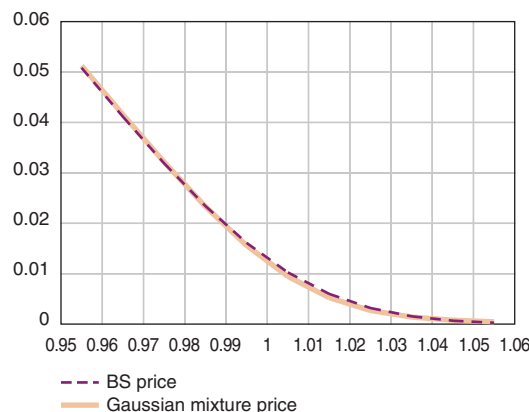
## 3| EXTREME EVENTS AND PRICING: THE CASE OF A EUROPEAN CALL OPTION

In this section, we show, using a numerical example, how incorporating extreme events (such as large variations in the return on risky assets) into a pricing model can have a significant impact on the price of the asset that we are seeking to determine.

We have seen that the mixture of Gaussian distributions is better able to reproduce the high kurtosis and the negative skewness observed empirically with the series of rates of return on the CAC40. This result is also verified in the risk-neutral world where the Gaussian or mixture of Gaussian nature of the factor distribution is preserved thanks to the exponential-affine form of the stochastic discount factor (see Boxes 1 and 2).

**Chart 3**  
Black and Scholes (BS) price and Gaussian mixture, call option

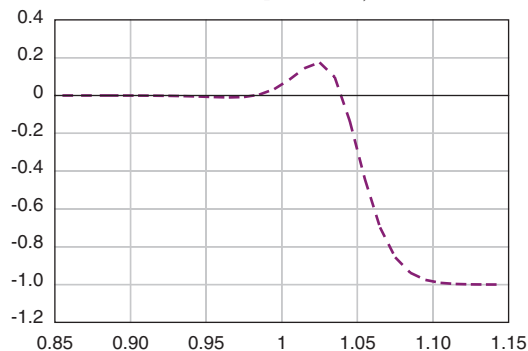
(X-axis: relative strike; Y-axis: relative price call option)



Let us consider a European CAC40 Index call option (observed over the period mentioned in section 1|1), a residual maturity of one week and a constant risk-free rate  $r^f = 0,0007$  (weekly basis). We compare the price obtained using the Black and Scholes model (see Box 1) with that corresponding to a mixture of two Gaussian distributions (see Box 2). The results, presented in Table 3, show that the Black and Scholes model, by underestimating the frequency of extreme events, results in an overvaluation of at-the-money options ( $\kappa \approx 1$ , the most traded options on the market) and an undervaluation of both in-the-money options ( $\kappa < 1$ ) and out-of-the-money options ( $\kappa > 1$ , the least liquid options) (see Charts 3 and 4).

**Chart 4**  
Relative price differential: BS price – Mixture price  
Mixture price

(X-axis: relative strike; Y-axis: relative pricing error)





The relative price differential (see Table 3)<sup>13</sup> between the price obtained with a mixture of Gaussian distributions and the price obtained using the Black and Scholes model is interpreted as the revaluation that occurs when tail risk is better taken into account in European call option pricing (move from a Black and Scholes-type model to a Gaussian mixture model). Price differentials are substantial in this example where the maturity

is short. These price differentials would be less significant for longer maturities.

**Table 3**

	Call option "in the money" $0.85 \leq \kappa < 0.95$	Call option "at the money" $0.95 \leq \kappa < 1.05$	Call option "out of the money" $1.05 \leq \kappa < 1.15$
<i>BS price – Mixture price</i> <i>Mixture price</i>	-0.12%	3.76%	-88.5%

*In this article, we put forward a European option pricing model capable of taking into account tail risk. The numerical examples show that a model based on an underestimation of the frequency of extreme events systematically results in an overvaluation of "at the money" options, the most traded on the market. Conversely, "at the money" and "out of the money" options (the less liquid) are undervalued.*

*For the sake of simplicity, we have used a static model with independent returns and a mixture of two Gaussian distributions. A more realistic model would be a dynamic model<sup>14</sup> in which the conditional time-dependent distribution would always be a Gaussian mixture but with time-dependent parameters. The effect of taking into account extreme events would then depend on the current and past environment in terms of returns and volatility. In this respect, the static model presented here may be considered as reproducing an average effect.*

<sup>13</sup> A lag of 0.01 is used for each relative exercise price range.

<sup>14</sup> See Bertholon, Monfort and Pegoraro (2006).

## APPENDIX 1

### From the historical world to the risk-neutral world

The risk-neutral conditional probability distribution can generally be derived from the historical probability distribution and the specification of the stochastic discount factor. More specifically, if  $f_t(x_{t+1})$  denotes the historical conditional probability density function of the factor, *i.e.* the distribution of the factor observed in the "real" world, and  $f_t^Q(x_{t+1})$  the risk-neutral conditional probability density function of the factor, *i.e.* the distribution of the factor observed in a risk-neutral world, the move from the historical world to the risk-neutral world can be written as:

$$f_t^Q(x_{t+1}) = \frac{M_{t,t+1}}{E_t(M_{t,t+1})} f_t(x_{t+1}), \quad (\text{A1.1})$$

where  $M_{t,t+1}$  is the stochastic discount factor.

If  $P_t^f$  denotes the price of the one period risk-free asset at date  $t$  and by applying the pricing formula (2) to this asset, we obtain:

$$P_t^f = E_t(M_{t,t+1} P_{t+1}^f), \quad (\text{A1.2})$$

where  $P_{t+1}^f = 1$ .

If we define  $r_{t+1}^f = \log\left(\frac{1}{P_t^f}\right)$  the risk-free rate between  $t$  and  $t+1$  (known in  $t$ ), we then have:

$$E_t(M_{t,t+1}) = \exp(-r_{t+1}^f). \quad (\text{A1.3})$$

By re-writing the pricing formula (2) using (A1.1) and (A1.3), we obtain:

$$P_t^f = \exp(-r_{t+1}^f) E_t^Q(g_{t+1}),$$

this is the pricing formula in a risk-neutral world.

## APPENDIX 2

### Price of a European call option with a one-period maturity

Let us consider the case of a European call option with a one-period maturity.  $c_t = \frac{C_t}{S_t}$  denotes the relative price of the option at date  $t$  ( $S_t$  is the price of the underlying asset and  $C_t$  the price of the option at date  $t$ ). Its relative exercise price at  $t+1$ , denoted  $\kappa$ , is equal to  $\kappa = \frac{K}{S_t}$  where  $K$  is the exercise price at  $t+1$ . The relative price of the underlying asset at  $t+1$  is equal to  $\exp(x_{t+1})^1$ . The future cash flow generated by the option at  $t+1$  denoted  $g_{t+1}$ , is therefore  $\exp(x_{t+1}) - \kappa$  if the option is exercised, 0 if not.

In other words, we have:

$$g_{t+1} = \max(\exp(x_{t+1}) - \kappa, 0) = (\exp(x_{t+1}) - \kappa)^+ . \quad (\text{A2.1})$$

The relative price of the option at date  $t$  is then:

$$c_t = E_t(M_{t,t+1}(\exp(x_{t+1}) - \kappa)^+), \quad (\text{A2.2})$$

where  $E_t(\cdot)$  is the conditional expectation calculated using the historical distribution (Gaussian mixture). In the risk-neutral world, this is written:

$$c_t = \exp(-r^f) E_t^Q((\exp(x_{t+1}) - \kappa)^+), \quad (\text{A2.3})$$

where  $r^f$  is the risk-free interest rate between  $t$  and  $t+1$  (known at  $t$ ) and  $E_t^Q(\cdot)$  is the conditional expectation calculated using the risk-neutral distribution. It should be recalled that, under the model's assumptions, the risk-neutral distribution is also a mixture of Gaussian processes.

By calculating the right-hand side of equation (A2.2), we show that the option pricing formula is a linear combination of Black and Scholes-type formulas. This formula depends on the means and variances of the Gaussian distributions used in the Gaussian mixture. In the Gaussian case, *i.e.* if we make the assumption of a single regime, we obtain the traditional Black and Scholes option pricing model (see Box 2). It is also possible to show that this formula can be applied to options with a maturity of over one period.

1 By definition  $x_{t+1} = \ln\left(\frac{S_{t+1}}{S_t}\right)$  where  $S_t$  is the price of the underlying asset at date  $t$ .

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