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Abstract

This paper considers cooperative game theoretic settings in which forming coalitions can act as Stackelberg leaders. We define a value function which modifies the γ-value function by letting members of deviating coalitions "move first" in choosing a coordinated strategy. We accordingly define the ϕ-core, and characterize the ϕ-core allocations of a cartel formation game and of a public goods economy.

Keywords: Core, Cooperative Games, Oligopoly, Public Goods.

1 Introduction

The traditional representation of cooperative games with transferable utility is based on a "characteristic" function, specifying for each coalition the amount of utility that its members can ensure themselves in the underlying normal form game. This formulation is meant to isolate coalitional decisions, abstracting from the strategic complexity of the cooperation process. However, unless the payoffs of the members of a coalition and of its complement are independent (orthogonal games) or opposite (constant sum games), the characteristic function fails to be well defined[1]. Indeed, this is the case of many meaningful strategic situations, in which the payoff of each player may generally depend on the strategies of all

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[1] In Shubik (1982) terminology, the game is not a c-game.

players in the game. In such cases, the characteristic function can still be well defined by introducing some assumptions on the strategies of players in the complementary coalitions (the "outside players").

One way to deal with this problem, first proposed by von Neumann-Morgenstern (1944) and considered by Aumann (1967), is to assume that outside players coordinate their strategies to minimize the aggregate payoff of the forming coalition. A temporal structure is implicitly introduced in the players' choice of strategies. In the so called α-core, the forming coalition acts as a leader, and chooses its best strategies, given the minimizing behaviour of outside players; in the β-core, conversely, it behaves as a follower, and maximizes its payoff given the coordinated strategies of outside players. Since in both cases deviations are very costly, α and β-core are usually very large. Moreover, still fulfilling a rationality requirement in constant sum games, α and β-assumptions do not seem justifiable in most economic settings[2].

An alternative approach proposed by Aumann (1959) extends Nash Equilibrium "passive" expectations to the cooperative framework. The concept of strong equilibrium defined by the author assumes that deviating coalitions take as given the strategies of outside players. Being immune from the deviations of any coalition, thus including the grand coalition and every individual player, strong equilibria are both Nash equilibria and efficient strategies. However, since in games with positive externalities the efficient strategies of excluded players make coalitional deviations "too" profitable, strong equilibria do not exist for many economic problems.

In the contest of some recent economic applications, a different approach has proved useful in ensuring a non-empty core without making use of extreme assumptions on the behaviour of outside players such as the α and β conjectures. This approach, named γ-approach by Chander-Tulkens (1997), assumes that outside players neither jointly minimize the payoff of a deviating coalition (as in the α and β-core), nor keep their strategies fixed (as in the Strong Nash Equilibrium), but they rather maximize their own utility as singletons. Here, the behaviour of deviating players and which of outside players is implicitly assumed to develope in two stages. In the first stage, similarly to the Γ game by Hart and Kurtz (1983),[3] a

[2] Indeed, in costant sum games, the α-core coincides with the *modified characteristic function* proposed by Harsanyi (1959), assigning to each coalition the solution of the variable threats Nash bargaining problem with the respective complementary coalition.

[3] The Γ game is indeed a strategic coalition formation game with fixed payoff division, in which the strategies consist of the choice of a coalition. Despite the different nature of the two games, there is an analogy concerning the coalition structure induced by a deviation from the grand coalition. In the Γ game, any deviation from the the grand coalition's strategy profile induces a coalition structure in which the deviating coalition stay

coalition forms and the excluded players split up as singletons; in the second stage, members of the deviating coalition and excluded players simultaneously choose their strategies in the underlying normal form game, *given* the specific coalition structure originated in the first stage. Consequently, the strategy profile induced by the deviation of a coalition $S \subset N$ is the Nash equilibrium among S and each individual player in $N \backslash S$.

In this paper we modify the γ-assumption by removing this two stage structure and reintroducing the temporal sequence in the choice of players' strategies *in the underlying normal form game,* typical of the α and β-core. We assume that the formation of a coalition and the choice of a coordinated strategy by its members in the underlying game are two simultaneous events, that can be thought of as a unique action. When a set of players form a coalition, at the same time they choose a coordinated strategy, taking as given the (non-cooperative) reaction of the excluded players as singletons. In this respect, deviating coalitions possess a first mover advantage with respect to the outside players. We thus associate with the deviation of every coalition S the Stackelberg equilibrium in which S acts as leader and players in $N \backslash S$ play (individually) as followers.

According to this assumption, we define a modified version of the γ-core, denoted ϕ-core. We then show how some recent applications of the γ-core to oligopolistic markets and public goods production problems are affected by our assumption. For the linear oligopoly case, we prove that, although the γ-core is very large, the only allocation in the ϕ-core is the equal split allocation. For the linear-quadratic oligopoly, conversely, we show that, differently from the γ-core, the ϕ-core is empty. For the case of public goods production, we consider a simple economy with one public and one private good, and we discuss the validity of Chander and Tulkens (1997) result of non-emptiness of the γ-core. We consider the case of symmetric agents, and show that if preferences are linear in the public good, then the allocation the authors propose belongs to the ϕ-core. However, if preferences are strictly concave, the ϕ-core is shown to be empty for the specific case of quadratic utility and quadratic cost.

2 The general set-up

Let $\Gamma = (\{X_i, u_i\}_{i \in N}, \{X_S\}_{S \subseteq N})$ be a strategic form game, where N is the (finite) players set, X_i is the strategy set of player i, and X_S is the strategy set of a coalition of players S.[4] Let $P(N)$ be the set of all possible partitions π of the players set N; let X_π denote the set

together and the outside players split up.

[4]Note that, in general, X_S may not coincide with the set $\prod_{i \in S} X_j$.

$\prod_{T \in \pi} X_T$, for any $\pi \in P(N)$. The set $X \equiv \bigcup_{\pi \in P(N)} X_\pi$ is the set of all possible outcomes (in terms of strategies) of the game Γ. The function $u_i : X \rightarrow R_+$ represents players' preferences. We restrict our attention to transferable utility functions u_i.

Definition 1 *A Nash Equilibrium of* Γ *is a strategy profile* $\overline{x}$ *such that, for all* $i \in N$, $\overline{x}_i \in X_i$ *and, for all* $x_i \in X_i$, $u_i(\overline{x}) \geq u_i(x_i, \overline{x}_{-i})$.

2.1 The value function under the γ-assumption

The γ-assumption postulates that the worth of a coalition is the aggregate utility of its members in the Nash equilibrium between that coalition (acting as a single player) and the outside players (acting as singletons). The value function $v_\gamma(S)$ is defined for all $S \subseteq N$ by:

$$v_\gamma(S) = \sum_{i \in S} u_i \left(\hat{x}_S, \{\hat{x}_j\}_{j \in N \setminus S} \right) \tag{1}$$

where,

$$\hat{x}_S = \underset{x_S \in X_S}{\arg\max} \sum_{i \in S} u_i \left(x_S, \{\hat{x}_j\}_{j \in N \setminus S} \right) \tag{2}$$

and, $\forall j \in N \setminus S$,

$$\hat{x}_j = \underset{x_j \in X_j}{\arg\max} u_j \left(\hat{x}_S, \{\hat{x}_k\}_{k \in (N \setminus S) \setminus \{j\}}, x_j \right). \tag{3}$$

Definition 2 *The joint strategy* $\widehat{x} \in X_N$ *is in the* γ*-core, if there exists no coalition* S *such that* $v_\gamma(S) > \sum_{i \in S} u_i(\widehat{x})$.

2.2 The value function under the ϕ-assumption

The new value function we introduce is based on the assumption that deviating coalitions exploit a first-mover advantage. As under the γ-assumption, when a coalition S forms, players in $N \setminus S$ split up as singletons. Differently from the γ case, the members of S choose a coordinated strategy as leaders, thus anticipating the reaction of the players in $N \setminus S$, who simultaneously choose their best response as singletons. The strategy profile associated to the deviation of a coalition S is the Stackelberg equilibrium of the game in which S is the leader and the players in $N \setminus S$ are, individually, the followers. We denote this strategy profile as a *partial equilibrium* with respect to S. Formally, this is the strategy profile $\tilde{x}(S) = (\tilde{x}_S, x_j(\tilde{x}_S))$ such that

$$\tilde{x}_S = \underset{x_S \in X_S}{\arg\max} \sum_{i \in S} u_i \left(x_S, \{x_j(x_S)\}_{j \in N \setminus S} \right) \tag{4}$$

and, $\forall j \in N \backslash S$,

$$x_j(x_S) = \underset{x_j \in X_j}{\arg\max}\, u_j\left(x_S, \{x_k(x_S)\}_{k \in (N \backslash S) \backslash \{j\}}, \; x_j\right). \tag{5}$$

We first establish sufficient condition for the existence of $\tilde{x}(S)$.

For every coalition $S \subset N$ and strategy profile $x_S \in X_S$, we define the restriction $\Gamma(N \backslash S, x_S)$ of the game Γ to the set of players $N \backslash S$, given the fixed profile x_S.

Proposition 1 *Let Γ be a strategic form game. For every $S \subset N$ and $x_S \in X_S$, let the game $\Gamma(N \backslash S, x_S)$ possess a unique Nash Equilibrium. For every $S \subset N$, let X_S be compact. Let each player's payoff be continuous in every other player's strategy. Then, for every $S \subset N$, there exists a partial equilibrium of Γ with respect to S. Moreover, if payoffs are strictly concave in each players strategy, such a partial equilibrium is unique.*

Proof. By condition (5), the strategy profile $\{x_j(x_S)\}_{j \in N \backslash S}$ is the unique Nash equilibrium of $\Gamma(N \backslash S, x_S)$. By the closedness of the Nash equilibrium correspondence (see, for instance, Fudenberg and Tirole (1991), pag.30), members of S maximize a continuous function over a compact set (condition (4)); thus, by Weiestrass Theorem, a maximum exists. Uniqueness comes as a straightforward consequence of the strict concavity of the leader's maximization problem. ■

We can thus define the value function $v_\phi(S)$ as follows:

$$v_\phi(S) = \sum_{i \in S} u_i\left(\widetilde{x}_S, \{x_j(\widetilde{x}_S)\}_{j \in N \backslash S}\right). \tag{6}$$

Definition 3 *The joint strategy $\widetilde{x} \in X_N$ is in the ϕ-core, if there exists no coalition S such that $v_\phi(S) > \sum_{i \in S} u_i(\widetilde{x})$.*

In the next to sections we apply the concept of ϕ-core to two widely studied economic problems: cartel formation in oligopolies and resource allocation in economies with public goods.

3 Cartel formation in oligopoly

In recent years there has been a renewed interest in the application of cooperative solution concepts to the problem of cartel formation under oligopoly [see, for a survey, Bloch (1997)]. A specific use of the γ-core is contained, for instance, in Rajan (1989). The author shows that in a symmetric Cournot oligopoly with linear demand and quadratic costs, for a number of firms $n \geq 3$, firms never chose to stay separate (i.e., giving rise to the coalition structure $\{1, 1, ..., 1\}$); moreover, it is proved that, for $n \leq 4$, the γ-core is non empty.

In what follows, after a short description of the Cournot setting, we first show that, in a symmetric oligopoly with *linear* demand and *linear* costs, the γ-core strictly includes the equal split allocation for any number of firms. For the same model specification we then prove that the equal split allocation is the *unique* allocation contained in the ϕ-core. Finally, we show that, when costs are *quadratic,* the ϕ-core can be empty.

3.1 The Cournot setting

Let $\pi_i(y, y_i) = p(y)\, y_i - C_i(y_i)$ be the profit function of every firm $i \in N = \{1, 2, ..., n\}$, where y_i is the output of a firm, $y = \sum_{i=1}^{n} y_i$ the total output, $p(y)$ the usual inverse demand function and $C_i(y_i)$ the cost function of every firm. Let also $C_i(.) = C_j(.)$, for every i, j in N.

We introduce the following standard assumptions:

A.1 The function $\pi_i(.)$ is twice continuously differentiable;

A.2 For every firm i, the capacity constraint $\overline{y}_i < \infty$ determines the maximum production level;

A.3 $p''(.)\, y_i + p'(.) < 0$ and $p'(.) - C_i^{''} < 0$.

Consistently with Section 2, we now define the normal form game, denoted as Γ_1, associated to our problem. Each player (firm) strategy set is:

$$X_i = \{y_i \in R_+ : y_i \leq \overline{y}_i\} \equiv Y_i. \tag{7}$$

Players' preferences are linear in profit and, for every coalition S, the strategy set is represented by:

$$X_S \equiv \left\{ (y_S, t_S) : y_S \in \prod_{i \in S} Y_i, \text{ and } t_S = (t_1, ..., t_s), \text{ such that } \sum_{i \in S} t_i = 0 \right\} \tag{8}$$

where t_S is a vector of transfers.

Proposition 2 *There exists a unique Nash equilibrium of the game* Γ_1.

Proof. By A.1, every player's payoff functions is continuous in the strategy profile $y \in Y_N$ and, by A.3, strictly concave on y_i. By A.2, strategy sets are non empty, compact and convex, so that existence of a Nash equilibrium follows. Uniqueness is implied by A.3 as follows. Since, for each firm, $p''y_i + p' < 0$ and $p' - C_i'' < 0$, the function $F(y_i, y) \equiv p'y_i + p - C'$ is decreasing both in y_i and y. In fact, $\frac{\partial F(y_i,y)}{\partial y_i} = p' - C_i'' < 0$ and $\frac{\partial F(y_i,y)}{\partial y} = p''y_i + p' < 0$. Suppose now that there exist two Nash Equilibria y^1 and y^2 of Γ_1. Suppose also, without loss of generality, that $y^1 > y^2$. At a Nash Equilibrium, $p'y_i + p - C_i' = 0$, so that, if $\sum_{i=1}^{n} y_i^1 > \sum_{i=1}^{n} y_i^2$, it follows from A.3 that $y_i^1 < y_i^2$ for every $i = 1, .., n$, leading to a contradiction. ■

3.2 The γ-core

By applying the definition of $v_\gamma(S)$ to the Cournot setting introduced above, we obtain the following expression:

$$v_\gamma(S) = \sum_{i \in S} \left[p(\hat{y}_S, \hat{y}_{-S}) \hat{y}_i - C_i(\hat{y}_i) + \widehat{t}_i \right] \tag{9}$$

where

$$\hat{y}_S = \arg\max_{y_S \in Y_S} \sum_{i \in S} \left[p(y_S, \hat{y}_{-S}) y_i - C_i(y_i) + \widehat{t}_i \right] \tag{10}$$

and where $\widehat{t}_i$ is the equilibrium lump-sum transfer for every $i \in S$, and

$$\hat{y}_j = \arg\max_{y_j \in Y_j} p\left(y_j, \hat{y}_S, \underset{\substack{k \neq j \\ k \notin S}}{\hat{y}_k} \right) y_j - C_i(y_j), \quad \forall j \in N \backslash S. \tag{11}$$

By A.1, we can differentiate $v_\gamma(S)$ and, by symmetry of players, the strategy profile $\hat{y} \in Y_N$ characterizing $v_\gamma(S)$ is such that, for every $i \in S$, $\hat{y}_i$ respects:

$$p(\hat{y}) + p'(\hat{y}) s\hat{y}_i = C_i'(\hat{y}_i), \tag{12}$$

where $s = |S|$, while, for every $j \in N \backslash S$, $\hat{y}_j$ respects:

$$p(\hat{y}) + p'(\hat{y}) \hat{y}_j = C_j'(\hat{y}_j). \tag{13}$$

3.3 The ϕ-core

We now apply our equilibrium concept to the oligopolistic setting described above. According to the general setup, the function $v_\phi(S)$ is as follows:

$$v_\phi(S) = \sum_{i \in S} \left[p\left(\tilde{y}_S, \{y_j(\tilde{y}_S)\}_{j \in N \backslash S} \right) \tilde{y}_i - C_i(\tilde{y}_S) + \tilde{t}_i \right] \tag{14}$$

where

$$\tilde{y}_S = \underset{y_S \in Y^S}{\arg\max} \sum_{i \in S} \left[p\left(y_S, \{y_j(y_S)\}_{j \in N \setminus S} \right) y_i - C_i(y_i) + \tilde{t}_i \right] \tag{15}$$

and $\forall j \in N \setminus S$,

$$y_j(y_S) = \underset{y_j \in Y^j}{\arg\max}\, p\left(y_S, \{y_k(y_S)\}_{k \in (N \setminus S) \setminus \{j\}}\,,\ y_j \right) y_j - C_j(y_j). \tag{16}$$

Note first that, as $\sum_{i \in S} \tilde{t}_i = 0$, the function $v_\phi(S)$ is fully defined by the choice of a vector $\tilde{y}_S$ by the members of S.

Proposition 3 *There exists a unique value $v_\phi(S)$ for every $S \subseteq N$.*

Proof. We apply Proposition 1. By Proposition 2, there exists a unique Nash equilibrium for every restricted game $\Gamma_1(N \setminus S, y_S)$. Continuity of payoffs follows from A.1 and compactness of every strategy set from A.2. Moreover, by A.3 payoffs are strictly concave, so that the value $v_\phi(S)$ is unique. ■

According to the above result, under A.1 and symmetry, the FOCs characterizing $\widetilde{y} \in Y_N$ are, for every $i \in S$:

$$p(\tilde{y}) + p'(\tilde{y})\, s\tilde{y}_i = C_i'(\tilde{y}_i) \tag{17}$$

and, $\forall j \in N \setminus S$,

$$p(\tilde{y}) + p'(\tilde{y})\, y_j(\tilde{y}_S) = C_j'(y_j(\tilde{y}_S)). \tag{18}$$

3.4 The linear case

Having defined the γ and ϕ-core for the Cournot setting, we now study the linear case, i.e. the case in which $p(y) = a - by$, and, for every $i \in N$, $C_i(y_i) = cy_i$, with $a > c \geq 0$ and $b > 0$.

Proposition 4 *Under linearity and symmetry, the γ-core of the game Γ_1 is non empty and strictly includes the equal split allocation.*

Proof. Conditions (12) implies that:

$$v_\gamma(N) = \frac{(a-c)^2}{(2b)^2}$$

and

$$v_\gamma(S) = \frac{(a-c)^2}{b^2(n-s+2)^2}$$

where $s = |S|$ and $n = |N|$. Without loss of generality let us normalize $\frac{(a-c)^2}{b^2} = 1$, so that the equal split allocation gives to each player in N a payoff of $\frac{v_\gamma(N)}{|N|} = \frac{1}{4n}$ and $v_\gamma(S) = \frac{1}{(n-s+2)^2}$.

Consider now the equal split allocation for a coalition S, $\frac{v_\gamma(S)}{|S|} = \frac{1}{s(n-s+2)^2}$. Whatever distribution of the worth $v_\gamma(S)$ may be chosen by S, at least one player in S must get a payoff not greater than $\frac{1}{s(n-s+2)^2}$. This implies that coalition S improves upon the equal split allocation for N if and only if

$$\frac{1}{s(n-s+2)^2} > \frac{1}{4n}.$$

Straightforward calculations show that the above inequality is satisfied respectively for:

$$\begin{aligned} s &> n \\ s &< 2 + \frac{n - \sqrt{n^2+8n}}{2} < 1 \\ s &> 2 + \frac{n + \sqrt{n^2+8n}}{2} > n \end{aligned}$$

and hence, it is never satisfied for $1 < s \leq n$. It follows that the equal split allocation for N, characterized by the strategy vectors $(\widehat{y}, \widehat{t})$, where $\widehat{y}$ respects (12) and $\widehat{y} = (0, 0, ..., 0)$, belongs to the γ-core. To see that this allocation is strictly included in the γ-core, note that, since individual deviations assign to a player just $v_\gamma(\{i\}) = \frac{1}{(n+1)^2} < \frac{v_\gamma(N)}{|N|} = \frac{1}{4n}$, different and unequal allocations belong as well to the γ-core. In particular, any allocation giving to a player i his worth $v_\gamma(\{i\})$, and $\frac{v_\gamma(N\setminus\{i\})}{|N-1|} = \frac{v_\gamma(N)-v_\gamma(\{i\})}{|N-1|}$ to any remaining player, is not objectable. ■

We now characterize the ϕ-core of the game Γ_1 under linearity and symmetry. The next proposition shows that, once deviating coalitions are allowed to exploit a first mover advantage, all allocations but the equal split one are blocked.

Proposition 5 *In a linear symmetric oligopoly the equal-split allocation is the unique allocation belonging to the ϕ-core.*

Proof. As in the proof of Proposition 4, under normalization, we get:

$$v_\phi(N) = \frac{1}{4}$$

and, from condition (17),

$$v_\phi(S) = \frac{1}{4(n-s+1)}.$$

Hence, straightforward calculations show that, for every $S \subset N$, $\frac{v_\phi(S)}{|S|}$ is less than $\frac{v_\phi(N)}{|N|}$ for $1 < s < n$, and equal to $\frac{v_\phi(N)}{|N|}$ either for $s = n$ or $s = 1$. It follows that, since in any deviating coalition $S \subset N$ at least one player gets a payoff less than or equal to $\frac{v_\phi(S)}{|S|}$, no coalition $S \subset N$ can make all its member better off than in the equal split allocation $\frac{v_\phi(N)}{|N|}$, which is then in the ϕ-core. To see that the equal-split is the unique allocation in the ϕ-core, note that any other allocation would require to give to at least one player less than $\frac{v_\phi(N)}{|N|}$. However, such a player could always improve his payoff by deviating and, from the result above, getting a worth equal to $v_\phi(\{i\}) = \frac{1}{4n}$. ■

3.5 The linear-quadratic case

We now consider the case of linear demand function $p(y) = a - y$ and quadratic cost function $C_i(y_i) = \frac{y_i^2}{2}$. As indicated above, we know from Rajan (1989) that, for $n = 2$, $n = 3$ and $n = 4$, the γ-core is non empty. We now show that this result does not hold under the ϕ-core assumption.

By conditions (17) and (18), the following result can be proved.

Proposition 6 *Under linear demand and quadratic costs for every firm, the ϕ-core can be empty.*

Proof. >From first order conditions, it is obtained that:

$$v_\phi(N) = \frac{a^2 n^2}{(1 + 2n)^2}$$

and

$$v_\phi(\{i\}) = \frac{a^2\left(a^2 + 5n - 1\right)}{(n+1)(n+5)^2}.$$

Simple calculations show that, for every $i \in N$, and for $n \geq 2$, $v_\phi(\{i\}) > \frac{v_\phi(N)}{|N|}$. By efficiency of the equal split solution, in any other efficient allocation at least one player would receive a lower utility. This fact together with the above result that any player can improve upon the equal split allocation by deviating as singleton, imply that any efficient allocation can be objected by the deviation of a single player. This, in turn, implies that the ϕ-core is empty. ■

4 The core of a public good economy

In this section we study the ϕ-core of an economy with one private and one public good. We mostly refer to the work on γ-core by Chander and Tulkens (1997) (C-T hereafter), and show that their results carry over to the ϕ-core if and only if preferences are linear in the public good.[5]

4.1 The economy

We consider an economy with one public good q and one private good y. The set of agents is $N = \{1, ..., n\}$; each agent i is endowed with ω_i units of the private good, and produces the public good out of the private good with convex cost $C_i(q_i)$. For every $S \subset N$, we denote by q_S the vector $(q_i)_{i \in S}$, and by Q_S the term $\sum_{i \in S} q_i$; for simplicity, we write q instead of q_N and Q instead of Q_N. Preferences are represented by a quasilinear utility function $u_i(q, y_i) \equiv v_i(Q) + y_i$. We denote by $\pi_i(Q) \equiv \frac{\partial v_i(Q)}{\partial Q}$ the marginal rate of substitution between public and private good for player i, and for all coalitions $S \subseteq N$, we let $\pi_S(Q)$ denote the term $\sum_{i \in S} \pi_i(Q)$.

We make the following assumptions.

A.4: $v_i(Q)$ concave, twice differentiable and such that $\pi_i(Q) > 0$ for all q such that $\sum_{i \in N} C^i(q_i) \leq \sum_{i \in N} \omega_i$.

A.5: $C_i(q_i)$ strictly concave, twice differentiable and such that $C_i'(q_i) \geq 0$ for all $q_i \geq 0$ and $C_i'(q_i) = 0$ for $q_i = 0$.

We associate to this economy the normal form game denoted Γ_2, where strategy sets and preferences are as follows:

$$\begin{aligned} X_i &= \left\{(q_i, y_i) \in R_+^2 : C(q_i) + y_i \leq \omega_i\right\}; \\ X_S &= \left\{(q_S, y_S) \in R_+^{2\#S} : \sum_{i \in S} C_i(q_i) \leq \sum_{i \in S} \omega_i - \sum_{i \in S} y_i\right\}; \\ u_i(x) &= v_i(Q) + y_i. \end{aligned}$$

Proposition 7 ***(Chander-Tulkens)****: There exists a unique Nash Equilibrium of the game* Γ_2.

The Nash Equilibrium $(\bar{q}, \bar{y}) = (\bar{q}_1, ..., \bar{q}_n, \bar{y}_1, .., \bar{y}_n)$ of Γ_2 is characterized by the following FOC's:

$$\pi_i\left(\bar{Q}\right) = C_i'(\bar{q}_i), \quad \forall i \in N. \tag{19}$$

[5] Although C-T's results are obtained for an economy with pollution, they generalize to public goods economies under the assumptions made in this paper.

4.2 The γ-core

Chander an Tulkens propose a specific allocation (q^*, y^*), bearing for an equilibrium interpretation of the economy E, and show by construction that it belongs to the γ-core of the game Γ_2. We report their result in the following Proposition.

Proposition 8 ***(Chander-Tulkens)****: The joint strategy (q^*, y^*) where:*

$$q^* \text{ is such that } \pi_N(Q^*) = C_i'(q_i^*), \text{ for all } i \in N;$$

$$y_i^* = \omega_i - C_i(\bar{q}_i) - \frac{\pi_i(Q^*)}{\pi_N(Q^*)}\left[\sum_{i \in N}(C_i(q_i^*) - C_i(\bar{q}_i))\right]$$

is in the γ-core.

In what follows we will refer to (q^*, y^*) as the C-T allocation.

4.3 The ϕ-core

In this section we analyze the symmetric case (identical players) and we show that under linear preferences, Proposition 8 carries over to the case of ϕ-core. However, we also show that, if preferences are strictly concave, the ϕ-core may be empty.

4.3.1 The function v_ϕ

By definition, any partial equilibrium $[(\tilde{q}_S, \tilde{y}_S), (q_j, y_j)(\tilde{q}_S, \tilde{y}_S)]$ of Γ_2 with respect to S is such that

$$\begin{aligned} \tilde{q}_S &\in \arg\max_{q_S, y_S} \sum_{i \in S} v_i\left(Q_S + \sum_{j \in N \setminus S} q_j(q_S)\right) + \sum_{i \in S} y_i \\ \text{s.t.} \quad \sum_{i \in S}\omega_i &\geq \sum_{i \in S}[C_i(q_i) + y_i] \end{aligned}$$

and, $\forall j \in N \setminus S$

$$\begin{aligned} q_j(q_S) &= \arg\max_{q_j, y_j} v_j\left(Q_S + \sum_{k \in (N \setminus S) \setminus \{j\}} q_j(q_S) + q_j\right) + y_j \\ \text{s.t.} \quad \omega_j &\geq C_j(q_j) + y_j \end{aligned}$$

Proposition 9 *For every $S \subseteq N$, there exists a partial equilibrium of Γ_2 with respect to S. Moreover, all partial equilibria with respect to S are characterized by the same vector $\tilde{q}$.*

Proof. By Proposition 7, the Nash equilibrium of $\Gamma_2 (N \backslash S, q_S)$ exists and is unique for all S and q_S. By continuity of v_i, (A.4), and of $C_i(q_i)$, (A.5), Proposition 1 can be applied here. Moreover, as the maximization problem of S can be written as a function of just q_S, by concavity of v_i and strict convexity of $C_i(q_i)$, Proposition 1 can again be applied to show uniqueness. ■

4.3.2 Some characterization of the partial equilibria of Γ_2

We now analyze in greater detail the partial equilibria of Γ_2.

We first consider the first order condition for every player $j \in N \backslash S$: by symmetry, we can write

$$\pi_j (q_j + (n-s-1)q_j + Q_S) - C'(q_j) = 0. \tag{20}$$

By Assumptions 1 and 2 and applying the implicit function theorem to the mapping $f(q_j, q_S) \equiv \pi_j ((n-s)q_j + Q_S) - C'(q_j)$, we conclude that the function $q_j(q_S)$ is differentiable. Thus, totally differentiating the FOC above, we obtain, in equilibrium, the condition

$$\frac{\partial \pi_j}{\partial q} \left[1 + (n-s) \frac{\partial q_j}{\partial Q_S} \right] - C''(q_j) \frac{\partial q_j}{\partial Q_S} = 0$$

yielding the reaction function

$$\frac{\partial q_j}{\partial Q_S} = \frac{\frac{\partial \pi_j}{\partial Q}}{C''(q_j) - (n-s) \frac{\partial \pi_j}{\partial Q_S}} < 0.$$

The term $\frac{\partial q_j}{\partial Q_S}$ gives us the reaction of player j to changes in the vector q_S as determined by the changes in j's Nash equilibrium strategy in the game $\Gamma_2 (N \backslash S, q_S)$.

Given the reaction function of each outside player j, the maximization problem of coalition S yields the following FOCs:

$$\pi_S \left(\tilde{Q} \right) \left(1 + (n-s) \frac{\partial q_j}{\partial Q_S} \right) = C'(\tilde{q}_i), \quad \forall i \in S. \tag{21}$$

By plugging the expression for $\frac{\partial q_j}{\partial Q_S}$ into (21), we obtain

$$\pi_S (Q_S + (n-s) \cdot q_j(q_S)) (1-k) = C_i'(q_i) \tag{22}$$

where

$$0 < (1-k) = \left((n-s) \frac{\frac{\partial \pi_j}{\partial Q}}{C_i''(q_j) - (n-s) \frac{\partial \pi_j}{\partial Q_S}} + 1 \right) \leq 1. \tag{23}$$

Indeed, the presence of the term $(1-k)$ is the only difference between our optimality conditions and the ones obtained by C-T. Comparing the conditions characterizing v_γ and v_ϕ, it can be easily checked that the aggregate amount of public good induced by the deviation of a coalition S under the γ-assumption is greater than or equal to that induced under the ϕ-assumption.

In order to prepare the analysis of the next section, we establish here some properties of partial equilibria. We will refer to the original concept of partial equilibrium introduced by C-T as to the partial equilibria under the γ-assumption.

Lemma 10 *The aggregate amount of public good produced in the partial equilibrium with respect to S is not greater under the ϕ-assumption than under the γ-assumption.*

Proof. Let $Q^\phi(S)$ and $Q^\gamma(S)$ be the aggregate levels of public goods in the partial equilibrium w.r.t. S under ϕ and γ-assumption, respectively. Suppose that $Q^\phi(S) > Q^\gamma(S)$; then, by FOC (20), for each player $j \in N \backslash S$, $q_j^\phi(S) \leq q_j^\gamma(S)$. Moreover, as $(1-k) \leq 1$, by FOC (22) for every player $i \in S$, $q_i^\phi(S) \leq q_i^\gamma(S)$. The two inequalities imply a contradiction. ■

Lemma (10) and Proposition 5 in Chander-Tulkens (1997) imply that the aggregate amount of public good produced in the partial equilibrium w.r.t. S under the ϕ assumption is not greater than the efficient one.

Lemma 11 *If preferences are linear in the public good, then:*
i) $q_i^\phi(S) \leq q_i^*$, $\forall i \in N$;
ii) $\bar{q}_i \leq q_i^\phi(S)$, $\forall i \in N$;
iii) $\bar{q}_j = q_j^\phi(S)$, $\forall j \in N \backslash S$.

Proof. *i):* By definition of the term $(1-k)$ in condition (23), if preferences are linear then $(1-k) = 1$. By condition (22) this implies the following implications for all $i \in S$:

$$C_i'\left(q_i^\phi(S)\right) = \pi_S < \pi_N = C_i'(q_i^*).$$

Similarly, for all $j \in N \backslash S$, condition (20) implies:

$$C_j'\left(q_j^\phi(S)\right) = \pi_j < \pi_N = C_j'\left(q_j^*\right).$$

The two implications, together with strict convexity of $C_i(.)$ for every $i \in N$, imply the result.

ii) and iii): By conditions (22) and (19), for all $i \in S$:

$$C_i'(\bar{q}_i) = \pi_i < \pi_S = C_i'\left(q_i^\phi(S)\right).$$

By conditions (20) and (19), for all $j \in N \backslash S$:

$$C_j'(\bar{q}_j) = \pi_j = C_j'\left(q_j^{\phi}(S)\right).$$

Again by convexity of cost functions, the results follow. ■

4.3.3 The robustness of Chander-Tulkens result under linear preferences

We are now able to show that under linear preferences for the public good, Proposition 8 by C-T generalizes to the ϕ-core.

Proposition 12 *If preferences are linear, then the C-T allocation* (q^*, y^*) *belongs to the* ϕ*-core.*

Proof. The proof of Proposition 2 in Chander-Tulkens (1997) can be directly applied using Lemma (11). Indeed, Lemma (11) establishes all the properties that are needed in the proof of that proposition. ■

4.3.4 The ϕ-instability of Chander-Tulkens allocation under non-linear preferences

Under non linear preferences, C-T's result requires an additional assumption (Assumption 1" in their paper) concerning the marginal rate of substitution characterizing respectively a Nash and an efficient allocation. Under this assumption, and using a few properties both of Nash and partial equilibrium allocations under the γ-assumption, the authors prove Proposition 8 also for the non linear case. Using the notation introduced in the previous sections, such properties are that $q_i^{\gamma}(S) \geq \bar{q}_i$, for all $i \in S$, and that $q_j^{\gamma}(S) \leq \bar{q}_j$, for all $j \in N \backslash S$.
It is easy to check that the first property does not longer hold under the ϕ-assumption: indeed, in C-T's paper this property is proved through the following chain of implications:

$$C_i'(q_i^{\gamma}(S)) = \pi_S(Q^{\gamma}(S)) \geq \pi_S(Q^*) \geq \pi_j\left(\bar{Q}\right) = C_i'(\bar{q}_i),$$

where the inequality $\pi_S(Q^*) \geq \pi_j\left(\bar{Q}\right)$ is indeed Assumption 1".
Under ϕ-assumption, the above chain of implications would write

$$C_i'\left(q_i^{\phi}(S)\right) = \pi_S\left(Q^{\phi}(S)\right)(1-k) \geq \pi_S(Q^*) \geq \pi_j\left(\bar{Q}\right) = C_i'(\bar{q}_i)$$

which, as $(1-k) < 1$ by non-linearity of preferences, may well not be true. Actually, as Example 1 below shows, linearity turns out to be a necessary condition for C-T result to carry over under ϕ-assumption. Indeed, as it is proved in Proposition (13), in Example 1 the

ϕ-core is empty.

Example 1. Let preference be described by the utility function

$$u_i(q, x_i) = \left(Q - \alpha Q^2\right) + y_i$$

and let costs be described by the function

$$C(q) = \frac{q^2}{2}.$$

It can be easily checked that Assumption 1" in Chander-Tulkens (1997) is satisfied if $\alpha \geq \frac{1}{2}$. We consider the deviation of a single player i, producing a zero amount of public good. By showing that, given the reactions of the other players, this strategy represents for him an improvement upon the allocation proposed by C-T, we show that he can improve upon it under the ϕ-assumption, as zero production is always a feasible strategy for him. The reaction of the other $(n-1)$ players to the "no production" strategy of i is obtained by the FOC

$$1 - 2\alpha q_j (n-1) = q_j$$

yielding

$$q_j = \frac{1}{1 + 2\alpha(n-1)}$$

and

$$Q = \frac{n-1}{1 + 2\alpha(n-1)}.$$

By using Samuelson's efficiency condition

$$n(1 - 2\alpha Q^*) = \frac{Q^*}{n}$$

we obtain the efficient level of public good

$$Q^* = \frac{n^2}{1 + 2n^2\alpha}.$$

We are then able to compare the utility (u_i^*) received by i in the C-T allocation with the utility u_i^0 that i receives through a (zero production) deviation:

$$\begin{aligned} u_i^* &= \frac{n^2}{1+2n^2\alpha} - \alpha\left[\frac{n^2}{1+2n^2\alpha}\right]^2 - \frac{1}{2}\left(\frac{n}{1+2n^2\alpha}\right)^2; \\ u_i^0 &= \frac{n-1}{1+2\alpha(n-1)} - \alpha\left[\frac{n-1}{1+2\alpha(n-1)}\right]^2. \end{aligned}$$

By straightforward calculations, it turns out that, for $n \geq 2$ and $\alpha \geq 0.5$, $\left(u_i^0 - u_i^*\right)$ is always positive; hence, every player can individually improve upon the C-T allocation, which, therefore, is not in the ϕ-core. We report in the table below a few numerical values for $\left(u_i^0 - u_i^*\right)$.

$$\begin{array}{ll} n=2,\ \alpha=0.5 & \left(u_i^0 - u_i^*\right) = 0.224 \\ n=10,\ \alpha=0.5 & \left(u_i^0 - u_i^*\right) = 0.8 \\ n=50,\ \alpha=0.5 & \left(u_i^0 - u_i^*\right) = 0.96 \\ n=100,\ \alpha=0.5 & \left(u_i^0 - u_i^*\right) = 0.98 \end{array}$$

Proposition 13 *Let costs and preference be as in Example 1. Then the ϕ-core of the associated cooperative game is empty.*

Proof. It is shown in Example 1 that any player could improve upon C-T's solution by exploiting a first mover advantage. By efficiency of that solution, for any other efficient solution (q, y), at least one player i would receive a lower utility than in (q^*, y^*). But as any player can improve upon (q^*, y^*) by deviating as singleton, than player i can improve upon (q, y) in the same way. ■

5 Concluding remarks

This paper has presented a new solution concept for cooperative games. Our concept modifies the γ-core by introducing a temporal structure in the choices of strategies in the underlying normal form game which is similar to the one adopted in the α-core. At the same time, it is maintained the γ-assumption that outside players react to a forming coalition by splitting up into singletons. This approach is meant to account for those cases in which coalitions can break an agreement and, in so doing, force the outside players to react to their new strategy. In this paper we have focused our attention on two applications: Cournot oligopolies and public good provision. Our results on cartel formation show that, in a linear symmetric oligopoly, considering the ϕ-core restricts the set of core outcomes to the equal split allocation. Moreover, differently from the γ-core, under quadratic costs the ϕ-core may be empty. In the second application, Chander and Tulkens (1997) results are shown to be robust against the temporal structure assumed in the ϕ-core if and only if preferences are linear in the public good. In the case of non linear preferences, conversely, whenever a coalition can exploit a first mover advantage, the γ-assumption on coalition formation is no longer sufficient to yield a non-empty core.

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