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Abstract

Stollery (1998) studied a polluting oil extracting economy governed by the constant utility criterion. The pollution caused the growth of temperature, negatively affecting production and utility. Stollery provided a closed form solution for the case with the Cobb-Douglas production function and temperature affecting only production. This paper offers a closed form solution to a non-trivial example of this economy with utility affected by temperature.

Key words: essential nonrenewable resource; polluting economy; sustainable development; special function representation

JEL : O13; Q32; Q38

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1. Introduction

A social planner in Stollery's (1998) problem followed a constant utility criterion where utility and production were negatively affected by irreversible global warming resulting from oil use. Stollery obtained the closed form solutions, considering the case with temperature affecting only production for the extended Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974) under the Hartwick investment rule (Hartwick, 1977).[1] Stollery did not consider the case where temperature affected utility, noting that "exactly the same energy path results from temperature effects in a standard constant elasticity utility function" (Stollery, 1998, p. 734).

However, the case with utility affected by global warming raises some interesting and important questions since the negative effect of temperature can represent an aggregate damage resulting from economic activity. For example, Bazhanov (2009b) analyzed a solution to this problem for an imperfect economy that has been extracting the resource for a period of time before considering the goal of sustainable development in the form of the constant utility criterion. This approach showed that the economy can be sustainable or unsustainable depending on the parameters of the hazard function and on the technology and the initial endowments of the economy. The result implied the necessity of a more general notion of sustainability (semisustainability) that can provide an opportunity for an economy to decline asymptotically to a sufficient survival level instead of collapsing in finite time after a period of overconsumption. A similar problem arises in the cases when utility is positively affected by the remaining resource stock, e.g., when the stock has an amenity value (Krautkraemer, 1985; Schubert and d'Autume, 2008).

In Bazhanov (2009b), the problem was studied numerically using a differential equation for capital, which implied all other paths in the economy including the path of the hazard function. The current paper provides the closed form

[1] Stollery showed that the Hartwick rule is still optimal in this problem.

solution for a specific case in this economy. Unlike Bazhanov (2009b), where the problem was formulated for an imperfect economy with the given state of the oil extraction industry (the initial rate of extraction), this paper offers a conventionally specified example for the given initial assets of capital and the resource reserve. The uncertainty of policy recommendations in this case is discussed using a numerical example, which resembles the current state of the world's oil extracting industry.

2. The model

Stollery provided a closed form solution to an oil-burning DHSS economy[2] with the production function negatively affected by growing temperature and with an isoelastic utility that depended only on consumption. The current paper considers the case with utility alone affected by the hazard function T. A social planner chooses the path of per capita consumption c (by choosing a saving rule) and the path of the per capita resource extraction r (by choosing a tax) to maximize the constant over time level of per capita utility $\overline{u}$:

$$u(c,T) = \left(cT^{-1}\right)^{(1-\eta)}/(1-\eta) = \overline{u} = const\,[c(t),r(t)] \to \max_{c(t),r(t)}\,. \qquad (1)$$

The balance equation and the production function are

$$q(t) = c(t) + \dot{k}(t) = k^{\alpha}(t)r^{\beta}(t), \qquad (2)$$

where q and k are per capita output and capital; $\alpha, \beta \in (0,1)$, $\alpha+\beta < 1, \alpha > \beta$.[3] The optimal investment rule is $\dot{k} := dk/dt = rq_r = \beta q$, where $q_r := \partial q/\partial r$. Stollery assumed that technical change compensated for the effect of growing

[2] The Cobb-Douglas production function, which is used in the DHSS model, has become one of the most popular tools in Resource Economics both for theoretical studies (e.g., Dasgupta and Heal, 1979; Asheim, 2005, Hamilton and Withagen, 2007) and for practical applications, e.g., for the global climate change assessment (Nordhaus and Boyer, 2000).

[3] The share of labor in this problem is $1-\alpha-\beta$. The Solow (1974) condition $\alpha > \beta$ guarantees the convergence of the integral $\int_0^\infty r dt$.

population, so there are no explicit technical advances in the model, and population is constant.[4] The hazard T grows with the resource extraction:

$$T(t) = T\left[r(t)\right] = T_0\left[\Theta\int_0^t r(\xi)d\xi + 1\right]^{\varphi} = T_0[\Theta(s_0 - s(t)) + 1]^{\varphi}, \quad (3)$$

where $\Theta, \varphi \geqslant 0$ are the parameters, s_0 is the initial per capita oil stock, and $s = s(t)$ is the current oil stock, implying $r = -\dot{s}$. Function T can vary from constant to polynomial depending on the value of φ.[5]

The constant utility criterion requires the following paths of per capita output q, consumption c, and, differentiating q, the path of the growth rate $\dot{q}/q$:

$$q(t) = q_0\left\{\Theta[s_0 - s(t)] + 1\right\}^{\varphi}, \quad (4)$$

$$c(t) = c_0\left\{\Theta[s_0 - s(t)] + 1\right\}^{\varphi}, \quad (5)$$

$$\frac{\dot{q}(t)}{q(t)} = \frac{\varphi\Theta r(t)}{\Theta\left[s_0 - s(t)\right] + 1}. \quad (6)$$

The rate of growth for $\varphi\Theta > 0$ is positive, declining starting from

$$\frac{\dot{q}(0)}{q(0)} = \varphi\Theta r_0, \quad (7)$$

and approaching zero with $t \to \infty$. The optimal initial consumption is $c_0 = (1 - \beta)q_0$, where $q_0 = k_0^{\alpha} r_0^{\beta}$, and the value of the initial rate of extraction $r_0 = r_0(k_0, s_0, \varphi, \Theta)$ is linked to the initial stocks and the intensity of the hazard via the efficiency condition $s_0 = \int_0^{\infty} r(t)dt$. [6]

In contrast to the Solow-Hartwick case ($\varphi\Theta = 0$), per capita output and consumption grow here under the same Hartwick investment rule when $\varphi\Theta > 0$. The growth is limited by

$$q_{\infty} = q_0\left\{\Theta s_0 + 1\right\}^{\varphi} \quad \text{and} \quad (8)$$

$$c_{\infty} = c_0\left\{\Theta s_0 + 1\right\}^{\varphi} \quad (9)$$

[4] A plausible alternative to this assumption can be a TFP (Total Factor Productivity) compensating for capital decay. In more details see Bazhanov (2009a).

[5] The specifications of utility and temperature functions are different here from the ones considered by Stollery; in particular, Stollery related temperature to the remaining resource stock ($T'(s(t)) < 0$); here temperature depends on the extracted (burned) resource stock ($T'(s_0 - s(t)) > 0$). The other differences are discussed in Bazhanov (2009b).

[6] See, e.g., formulas (40) and (41).

respectively. The limit for temperature growth is $T_\infty = T_0 (s_0\Theta + 1)^\varphi$. The only source of output and consumption growth is a redistribution of the resource among generations. A social planner imposes a positive declining tax on extraction, resulting in a lower rate of initial extraction (implying lower c_0 and q_0) and a slower decline in the rates of extraction. Namely, from the specification of the production function, the equation for the rate of output growth is $\dot{q}/q = \alpha\dot{k}/k + \beta\dot{r}/r$. Then, given $\dot{k}_0 = \beta q_0$ and using (7), the initial rate of change in the rate of extraction is $\dot{r}_0/r_0 = [\varphi\Theta r_0 - \alpha\beta q_0/k_0]/\beta$ or

$$\frac{\dot{r}_0}{r_0} = \frac{r_0}{\beta}\left[\varphi\Theta - \alpha\beta k_0^{\alpha-1} r_0^{\beta-1}\right]$$

yielding the following condition:

$$\dot{r}_0 \gtreqless 0 \quad \text{iff} \quad \varphi\Theta \gtreqless \frac{\alpha\beta}{k_0^{1-\alpha} r_0^{1-\beta}}. \tag{10}$$

This condition, however, does not directly imply that a growing initial extraction can be optimal in this problem because r_0 declines with the growth of $\varphi\Theta$. A specific example of the optimal $\dot{r}_0 > 0$ is provided in Section 4, condition (42).

The growth of output and the declining to zero flow of the resource imply an unbounded growth of capital in this problem.

The constant-utility criterion in the form of (1) seemingly implies that the optimal initial values of $q_0 = c_0/(1-\beta)$ and c_0 should depend on the preference parameter η and the initial temperature T_0, namely, $cT^{-1} = \widehat{u} = const = [\overline{u}(1-\eta)]^{1/(1-\eta)}$ yielding $c_0 = T_0 [\overline{u}(1-\eta)]^{1/(1-\eta)}$. However, since η is a constant here, the optimal policy in this framework maximizes $\widehat{u}$ and the corresponding value of $\overline{u}(\widehat{u}, \eta)$ regardless of the preference parameter. As to the value of T_0 as a "preindustrial level" of temperature, the normalization $T_0 = 1$ can be used assuming that the hazard does not affect utility when the resource is not being extracted. Hence, problem (1) is equivalent to the problem of finding

$$\widehat{u} = const\,[c(t), r(t)] = \max_{c(t),r(t)} c(t) T\,[r(t)]^{-1} \tag{11}$$

with $T_0 = 1$ in formula (3).

For obtaining the optimality conditions, Stollery used the approach of Leonard and Long (1992, pp. 300-304), which reformulates problem (11) into the following equivalent problem:

$$\text{maximize} \quad V(t) \equiv \int_t^{\infty} \widehat{u}\delta e^{-\delta\tau} d\tau \text{ for } t = 0 \ (V(0) = \widehat{u} = const) \tag{12}$$

for an arbitrary constant δ subject to (omitting the dependence on time)

$$\dot{k} = q - c, \quad \dot{s} = -r, \text{ and } \widehat{u} = u(c, T). \tag{13}$$

The Hamiltonian of this problem is

$$H = \widehat{u}\delta e^{-\delta t} + \mu_k(q - c) - \mu_s r. \tag{14}$$

The utility constraint yields the Lagrangian to be maximized:

$$L = H + \lambda(u - \widehat{u}).$$

In the general case, when both production and utility are affected by the hazard, the Pontryagin-type necessary conditions for the state variables k and s are[7]

$$L_c = \lambda u_c - \mu_k = 0, \tag{15}$$

$$L_r = \mu_k q_r - \mu_s = 0, \tag{16}$$

$$\dot{\mu}_k = -\frac{\partial L}{\partial k} = -\mu_k q_k, \tag{17}$$

$$\dot{\mu}_s = -\frac{\partial L}{\partial s} = -\left(\mu_k q_T + \lambda u_T\right) T_{s_0 - s} \cdot \partial(s_0 - s)/\partial s, \tag{18}$$

$$\int_0^{\infty} L_{\widehat{u}} dt = \int_0^{\infty} \left(\delta e^{-\delta t} - \lambda\right) dt = 1 - \int_0^{\infty} \lambda dt = 0. \tag{19}$$

Eq. (18) with μ_k from Eq. (15) results in

$$\dot{\mu}_s = T_{s_0 - s} \lambda u_c \left(q_T + \frac{u_T}{u_c} \right). \tag{20}$$

[7] Here μ_k and μ_s are indexed dual variables unlike u_c, q_k, and q_r, which are the partial derivatives of u and q.

The time derivative of Eq. (16) is $\dot{\mu}_s = \dot{\mu}_k q_r + \mu_k \dot{q}_r$, which, divided by q_r and combined with Eq. (20), gives

$$\frac{\dot{q}_r}{q_r}\mu_k + \dot{\mu}_k = \frac{T_{s_0-s}\lambda u_c}{q_r}\left(q_T + \frac{u_T}{u_c}\right).$$

Substitution for $\dot{\mu}_k$ from Eq. (17) with μ_k from Eq. (15) yields

$$\frac{\dot{q}_r}{q_r} = q_k + \frac{T_{s_0-s}}{q_r}\left(q_T + \frac{u_T}{u_c}\right), \tag{21}$$

which is the Hotelling rule with the negative additional term $\tau(t) := T_{s_0-s}\left(q_T + u_T/u_c\right)/q_r$ resulting from the effects of the externality. The fact that $\tau(t) \neq 0$ in the presence of the hazard factor ($\varphi\Theta > 0$) implies that the optimal paths in this problem can only be asymptotically efficient because the standard Hotelling rule ($\tau = 0$) as a necessary efficiency condition[8] is satisfied only with $t \to \infty$ due to exhaustion of the resource.

Stollery obtained the optimality of the Hartwick rule from the Hamilton-Jacobi-Bellman equation[9] for the problem (12), (13) instead of using necessary conditions (15) – (19). Namely, the Hamilton-Jacobi-Bellman equation establishes the following link between the maximized Hamiltonian and value function:

$$-\partial V^*/\partial t = H^*. \tag{22}$$

An autonomous infinite-horizon problem such as (12), (13) has the property: $V(t) = V(0)e^{-\delta t}$.[10] Then $-\partial V^*/\partial t = V(0)\delta e^{-\delta t} = \widehat{u}\delta e^{-\delta t}$ and $H^* = \widehat{u}\delta e^{-\delta t} + V_k\dot{k} + V_s\dot{s}$ (μ_k^* and μ_s^* are the shadow prices of capital and the resource stock) yielding $\mu_k^*\dot{k} + \mu_s^*\dot{s} = 0$, which means that the investment $\mu_k^*\dot{k}$ must be equal to the resource rent $\mu_s^* r$ under optimal prices (Hartwick, 1977).

In this framework, the equations for the optimal paths in the economy can be derived from

a) the condition $cT^{-1} = \widehat{u} = const$,

b) the Hartwick rule, which provides the maximum level of $\widehat{u}$,

[8] See, e.g., Dasgupta and Heal (1979).
[9] See, e.g., Leonard and Long (1992, p. 182-183).
[10] Leonard and Long (1992, Theorem 9.4.1, p. 293).

c) the balance equation (2) specifying the production function, and

d) the Hotelling rule (21), which gives the optimal tax on extraction.

3. Optimal paths

The optimal path of output (4) can be written as $q(t) = q_0 \left[\Theta \int_0^t r(\xi)d\xi + 1\right]^{\varphi}$. Raising to the power $1/\varphi$ yields $q^{1/\varphi} = q_0^{1/\varphi} \left[\Theta \int_0^t r(\xi)d\xi + 1\right]$ (restriction $\varphi \neq 0$ will be lifted below). Time derivative, substituting for $r = q^{1/\beta}k^{-\alpha/\beta}$, is $q^{1/\varphi-1}\dot{q}/\varphi = q_0^{1/\varphi}\Theta r = q_0^{1/\varphi}\Theta q^{1/\beta}k^{-\alpha/\beta}$. This equation with the optimal saving rule gives a system of the two differential equations in q and k :

$$\begin{aligned} q^{1/\varphi-1-1/\beta}dq/dt &= \varphi q_0^{1/\varphi}\Theta k^{-\alpha/\beta}, \\ dk/dt &= \beta q. \end{aligned}$$

Following Schubert and d'Autume (2008), the system can be solved by eliminating time ($dt = dk/(\beta q)$): $q^{1/\varphi-1/\beta}dq = A_1 k^{-\alpha/\beta}dk$, where $A_1 = \varphi q_0^{1/\varphi}\Theta/\beta > 0$. Integration gives $q^{1+1/\varphi-1/\beta}/(1 + 1/\varphi - 1/\beta) = A_1 k^{1-\alpha/\beta}/(1 - \alpha/\beta) + C_1$ or $q^a = A_2 k^{1-\alpha/\beta} + C_2$, where $a = 1 + 1/\varphi - 1/\beta = [\varphi(\beta - 1) + \beta]/(\varphi\beta)$ and $A_2 = aA_1/(1 - \alpha/\beta)$. Note that $a \gtreqless 0$ and $A_2 \lesseqgtr 0$ when $\varphi \lesseqgtr \beta/(1 - \beta)$. Calibration at $t = 0$ gives $C_2 = q_0^a(1 - B_1 k_0^{1-\alpha/\beta})$, where $B_1 = A_2 q_0^{-a} = q_0^{1/\beta-1}\Theta\left[\varphi(1/\beta - 1) - 1\right]/(\alpha - \beta)$ ($B_1 \gtreqless 0$ when $A_2 \gtreqless 0$). Then

$$q = q_0 \left(B_1 k^{-(\alpha/\beta-1)} + C_3\right)^b, \tag{23}$$

where $C_3 = 1 - B_1 k_0^{1-\alpha/\beta}$ and $b = 1/a = \varphi\beta/[\varphi(\beta - 1) + \beta]$. Henceforth, the restriction $\varphi \neq 0$ is not relevant; the case with $\varphi \to \beta/(1 - \beta)$ causing $b \to \infty$ is considered below.

Eq. (23) shows that $q(t) \to q^* = q_0 C_3^b$, and the saving rule implies that $c(t) \to c^* = q_0(1 - \beta)C_3^b$ as $k(t) \to \infty$ with $t \to \infty$. The obtained expression for q combined with the optimal saving rule gives a differential equation in capital, and then the dynamics of the economy is defined by the following system:

$$\dot{k} = \dot{k}_0 \left(B_1 k^{-(\alpha/\beta-1)} + C_3\right)^b, \tag{24}$$

$$r(t) = q(t)^{1/\beta}k(t)^{-\alpha/\beta}, \tag{25}$$

where $\dot{k}_0 = \dot{k}(0) = \beta q_0$.

The case with $\varphi = 0$ implies $b = 0$, yielding linear capital $k(t) = \beta q_0 t + k_0$, which coincides with the Solow-Hartwick case and with Stollery's solution for $\gamma = 0$. The extraction in this case is $r(t) = r_0 \left(r_1 t + 1\right)^{-\alpha/\beta}$, where $r_1 = \beta q_0 / k_0$.

For the case with $\varphi \to \beta/(1-\beta)$, implying $a \to 0$ or $b \to \infty$, Eq. (24) can be rewritten as follows: $\dot{k} = \dot{k}_0 C_3^b \left(1 + k^{-(\alpha/\beta-1)} B_1/C_3\right)^b$ or $\dot{k} = \dot{k}_0 C_3^b \left(1 - k^{-(\alpha/\beta-1)} a B_5\right)^{1/a}$, where $B_5 := q_0^{-a} A_1 / \left[(\frac{\alpha}{\beta} - 1) C_3\right]$. Note that $C_3 \to 1$ when $a \to 0$, but $C_3^b = (1 + a B_6 k_0^{-(\alpha/\beta-1)})^{1/a}$, where $B_6 := q_0^{-a} A_1 / (\frac{\alpha}{\beta} - 1) > 0$, so $\lim_{a \to 0} C_3^b = \exp\left[B_6 k_0^{-(\alpha/\beta-1)}\right]$ and $\lim_{a \to 0} B_5 = B_6$. Then, Eq. (24) takes the form

$$\dot{k} = \dot{k}_0 e^{B_6 k_0^{-(\alpha/\beta-1)}} e^{-B_6 k^{-(\alpha/\beta-1)}}. \tag{26}$$

Eq. (24) is integrable in quadratures:

$$\int_{k_0}^{k(t)} \frac{d\varkappa}{\left(B_1 \varkappa^{-(\alpha/\beta-1)} + C_3\right)^b} = \dot{k}_0 t + Const; \tag{27}$$

however, in the general case, $k(t)$ obtained from this equation cannot be expressed in elementary functions.[11] The following section provides a nontrivial ($\varphi \neq 0$) example with the closed form solution to Eq. (24).

A differential equation for the tax on extraction $\Upsilon(t)$ can be obtained from Eq. (21) using the fact that the resource price with no imperfections $q_r - \Upsilon$ should satisfy the standard Hotelling rule ($\tau \equiv 0$):

$$\frac{d(q_r - \Upsilon)/dt}{q_r - \Upsilon} = q_k.$$

[11] The LHS of Eq. (27) can be expressed using special functions. For example, in the case with $a = 0$, this equation takes the form of a nonlinear equation in k : $\digamma(k) = \dot{k}_0 \exp(B_6 k_0^{-(\alpha/\beta-1)}) t + Const$, where $\digamma(k) = D_1(k) \left[\delta \left(\delta B_6 k^\delta/(1+\delta) + 1\right) \cdot WhittakerM(D_2, D_3, B_6 k^\delta) + (1+\delta) \cdot WhittakerM\left(D_2 + 1, D_3, B_6 k^\delta\right)\right]$, with $\delta := 1 - \alpha/\beta$, $D_1(k) := B_6^{-(1+3\delta)/(2\delta)} k^{(1-3\delta)/2} \exp\left(-B_6 k^\delta/2\right)/(1 + 2\delta)$, $D_2 := (2-\delta)/(2\delta)$, $D_3 := (1+2\delta)/(2\delta)$. Here $WhittakerM(\cdot)$ is the Whittaker M special function, which is available, e.g., in Maple or Mathematica software. When $a = 0$ and $\alpha = 2\beta$, the LHS of Eq. (27) is $\digamma(k) = k \exp(-B_6/k) - \mathrm{Ei}(1, B_6/k)$, where $\mathrm{Ei}(\cdot)$ is the Exponential integral special function, which is also available in computational software.

This equation gives the dynamic condition for the tax depending on the path of the Hotelling rule modifier $\tau(t)$:

$$\dot{\Upsilon} - \Upsilon q_k - \tau q_r = 0. \tag{28}$$

Eq. (28) has a solution[12]

$$\Upsilon(t) = \Upsilon_0 \left(k(t)/k_0\right)^{\alpha/\beta} - q_r(t) \left[\left(q(t)/q_0\right)^{1/\beta-1} - 1\right], \tag{29}$$

where $\Upsilon_0 = \Upsilon(0)$ is the initial condition.

The value of Υ_0 can be expressed from the formula for the initial resource price with no imperfections: $q_r(0) - \Upsilon(0) = \beta q_0^0/r_0^0$, where $q_0^0 = k_0^\alpha \left(r_0^0\right)^\beta$ and r_0^0 satisfies Eq. (32). Namely,

$$\Upsilon_0 = \beta \left\{k_0^\alpha r_0^{\beta-1} - k_0/\left[s_0(\alpha-\beta)\right]\right\}. \tag{30}$$

In the case with $\Theta = 0$, there is no tax since $r_0 = r_0^0$. Eq. (30) yields the optimal initial tax that is required to obtain the optimal initial rate of extraction r_0, defined in the conventional approach from the efficiency condition given k_0, s_0, φ, and Θ. Eq. (30) can be inverted to show the link between the value of Υ_0 and the resulting value of $r_0 : r_0^{\beta-1} = \left[\Upsilon_0 s_0(\alpha-\beta) + \beta k_0\right] / \left[\beta k_0^\alpha s_0(\alpha-\beta)\right]$ or

$$r_0 = \left\{\frac{(\alpha-\beta)s_0/k_0^{1-\alpha}}{1 + (\alpha-\beta)s_0\Upsilon_0/(\beta k_0)}\right\}^{\frac{1}{1-\beta}}, \tag{31}$$

which coincides with the r_0^0 in the Solow-Hartwick case ($\Upsilon_0 = 0$)

$$r_0^0 = \left[(\alpha-\beta)s_0/k_0^{1-\alpha}\right]^{\frac{1}{1-\beta}} \tag{32}$$

and monotonically declines to zero with $\Upsilon_0 \to \infty$.

4. An example of a closed form solution

Let $\varphi = \beta$ and $\alpha = 2\beta$. Then $b = a = 1$, $A_2 = aA_1/(1-\alpha/\beta) = -A_1 = -q_0^{1/\beta}\Theta < 0$, $B_1 = A_2 q_0^{-a} = -q_0^{1/\beta-1}\Theta < 0$, and $C_3 = 1 + \Theta k_0^{1-\alpha} r_0^{1-\beta} > 0$. In this case, Eq. (24) becomes

$$\dot{k} = \frac{B_4}{k} + C_4, \tag{33}$$

[12] See Bazhanov (2009a, Appendix 1).

where $B_4 := \dot{k}_0 B_1 < 0$ and $C_4 := \dot{k}_0 C_3 > 0$. Eq. (33) in quadratures, denoting $D_0 := B_4/C_4 = B_1/C_3 < 0$, is

$$\int \frac{k}{k + D_0} dk = C_4 t + \widehat{C},$$

where $\widehat{C}$ is the constant of integration. Integration of the LHS yields

$$k - D_0 \ln (k + D_0) = C_4 t + \widehat{C}. \tag{34}$$

After denoting $x := \ln (k + D_0)$, the last equation becomes $e^x - D_0(x + 1) = C_4 t + \widehat{C}$ or $e^x = D_0 (x - p)$, where $p := -C_4 t/D_0 - 1 - \widehat{C}/D_0$. Multiplication of both sides by $-e^{-x+p}/D_0$ results in the equation $-e^p/D_0 = e^{(p-x)} (p - x)$, which, by the definition of the Lambert W function,[13] has the solution

$$p - x = W (-e^p/D_0).$$

Then, $k + D_0 = \exp [p - W (-e^p/D_0)] = -D_0 [-e^p/D_0] e^{-W(-e^p/D_0)}$. The definition of the Lambert W function implies that $W(z) = ze^{-W(z)}$, transforming the last equation as follows: $k + D_0 = -D_0 W (-e^p/D_0)$ or

$$k(t) = -D_0 \left\{ 1 + W \left(-\frac{e^{-\frac{C_4}{D_0} t - 1 - \frac{\widehat{C}}{D_0}}}{D_0} \right) \right\}, \tag{35}$$

where $\widehat{C}$, defined from Eq. (34) at $t = 0$, is $\widehat{C} = k_0 - D_0 \ln (k_0 + D_0)$. After substitution for $\widehat{C}$ into Eq. (35), it becomes

$$k(t) = D_0 \left\{ -1 - W \left[e^{-\frac{C_4}{D_0} t} \cdot e^{-\frac{k_0}{D_0} - 1} \cdot \left(-\frac{k_0}{D_0} - 1 \right) \right] \right\}, \tag{36}$$

which must be equal to k_0 at $t = 0$. Indeed, by the definition, $W [e^z z] = z$; therefore, the RHS of Eq. (36) at $t = 0$ equals $D_0 \left\{ -1 - \left[-\frac{k_0}{D_0} - 1 \right] \right\} = k_0$. Since $-C_4/D_0 < 0$, $D_0 < 0$, and W is monotonically growing function with

[13] Lambert W function is the solution to the equation $ye^y = z$, namely, $y = W(z)$. The derivative (for $z \neq -1/e$) of W is $dW/dz = W(z)/ [z (1 + W(z))]$, and the antiderivative of $W(z)$ (using the substitution $w = W(z) \Rightarrow z = we^w$) is $\int W(z)dz = z [W(z) - 1 + 1/W(z)] + C$. In more details about the Lambert W function see, e.g., Corless et al. (1996).

$\lim_{z\to\infty} W(z) = \infty$, capital is growing with no limit along a path that grows faster than a linear function (Fig. 6). Formula (36) can be considered as a closed form solution for $k(t)$ (using an alternative definition of this notion) since function W is uniquely defined for $k_0 \geqslant 0$,[14] and numerical implementations of this function are available in major computational software.

Substitution for $k(t)$ into Eq. (24) gives the closed form path of investments, which, using the optimal saving rule $\dot{k} = \beta q$, implies the paths of output, consumption, and extraction:

$$q(t) = q_0 \left\{ \frac{B_1}{D_0} \left(-1 - W\left[e^{-\frac{C_4}{D_0}t} \cdot e^{-\frac{k_0}{D_0}-1} \cdot \left(-\frac{k_0}{D_0} - 1 \right) \right] \right)^{-1} + C_3 \right\}, \quad (37)$$

$$c(t) = c_0 \left\{ \frac{B_1}{D_0} \left(-1 - W\left[e^{-\frac{C_4}{D_0}t} \cdot e^{-\frac{k_0}{D_0}-1} \cdot \left(-\frac{k_0}{D_0} - 1 \right) \right] \right)^{-1} + C_3 \right\}, \quad (38)$$

$$r(t) = q(t)^{1/\beta} k(t)^{-\alpha/\beta}, \quad (39)$$

where $c_0 = (1-\beta)q_0$. On the other hand, the constant utility criterion implies that $c(t) = c_0[\Theta(s_0 - s(t)) + 1]^{\varphi}$, yielding (for $\varphi = \beta$) the equation for the path of the current reserve:

$$s(t) = s_0 - \frac{1}{\Theta}\left(\left\{ B_1 k(t)^{-1} + C_3 \right\}^{1/\beta} - 1 \right).$$

In the limit, this formula becomes $s_\infty = \lim_{t\to\infty} s(t) = s_0 - \frac{1}{\Theta}\left(C_3^{1/\beta} - 1 \right) = s_0 - \left[\left(1 + \Theta k_0^{1-\alpha} r_0^{1-\beta} \right)^{1/\beta} - 1 \right] / \Theta$. Then, the efficiency condition $s_0 = \int_0^\infty r dt$ results in the following relationship between k_0, s_0, and r_0 :

$$r_0 = \left\{ \left[(s_0\Theta + 1)^{\beta} - 1 \right] / \left(k_0^{1-\alpha}\Theta \right) \right\}^{1/(1-\beta)}. \quad (40)$$

Note that r_0 is a decreasing function of Θ in this case (Fig. 1), which means that the greater the intensity of the hazard the larger the amount of the resource

[14] Function $W(z)$ is uniquely defined for $z \geqslant -1/e$ implying that k_0 should satisfy the condition $e^{-\frac{C_4}{D_0}t} \cdot e^{-\frac{k_0}{D_0}-1} \cdot \left(-\frac{k_0}{D_0} - 1\right) \geqslant -1/e$ for any $t \geqslant 0$. This inequality holds when $-\frac{k_0}{D_0} - 1 \geqslant 0$ or $k_0 \geqslant -D_0$ ($-D_0$ is a positive number here: $-D_0 = -B_1/C_3 = k_0^{\alpha/\beta-\alpha} r_0^{1-\beta}\Theta / \left[1 + \Theta k_0^{1-\alpha} r_0^{1-\beta}\right]$). Then, after dividing both sides by $k_0 / \left(1 + \Theta k_0^{1-\alpha} r_0^{1-\beta}\right)$, the condition of the uniqueness of the representation via the Lambert W function for the case with $\alpha = 2\beta$ is $1 + \Theta k_0^{1-\alpha} r_0^{1-\beta} \geqslant \Theta k_0^{1-\alpha} r_0^{1-\beta}$, which is always true.

that should be left for the future in order to offset the hazard with the growth of consumption according to the criterion. At the same time, reallocation of the resource to the future flattens the path of temperature until the rates of growth of temperature and consumption completely compensate for each other.

The initial tax on extraction should be higher (Fig. 2) for a larger Θ, and the initial level of consumption should be lower as a result of a lower r_0. Letting $\Theta \to 0$ and using the L'Hôpital's rule, condition (40) becomes

$$r_0^0 = \lim_{\Theta \to 0} r_0 = \left(\beta s_0 / k_0^{1-\alpha}\right)^{\frac{1}{1-\beta}} \tag{41}$$

coinciding with the expression (32) in the Solow-Hartwick case for $\alpha = 2\beta$.

Formula (40) specifies condition (10) of the optimality of the growing initial extraction depending on the hazard factors. Namely, the second inequality of this condition becomes $\Theta \gtreqless \alpha k_0^{1-\alpha} \Theta / \left\{ k_0^{1-\alpha} \left[(s_0\Theta + 1)^\beta - 1 \right] \right\}$ or $1 \gtreqless \alpha / \left[(s_0\Theta + 1)^\beta - 1 \right]$, yielding

$$\dot{r}_0 \gtreqless 0 \quad \text{iff} \quad \Theta \gtreqless \frac{(1+\alpha)^{1/\beta} - 1}{s_0}. \tag{42}$$

According to this condition, the optimal pattern of extraction can be hump-shaped even in the case with a small intensity of the hazard Θ when the initial reserve s_0 is large. An example of the optimal hump-shaped extraction path is provided in the next section (Fig. 4, solid line).

Formula (30) for $\alpha = 2\beta$ becomes

$$\Upsilon_0 = \beta k_0^\alpha r_0^{\beta-1} - k_0/s_0. \tag{43}$$

The explicit dependence of the initial tax on the hazard parameter Θ results from combining Eqs. (40) and (43):

$$\Upsilon_0(\Theta) = \frac{\beta k_0 \Theta}{(s_0\Theta + 1)^\beta - 1} - \frac{k_0}{s_0}. \tag{44}$$

Expressed in the terms of the resource price, the initial tax is $\Upsilon_0(\Theta)/q_r(0) = 1 - k_0^{1-\alpha} / \left(s_0 \beta r_0(\Theta)^{\beta-1}\right)$ or

$$\Upsilon_0(\Theta)/q_r(0) = 1 - \frac{(s_0\Theta + 1)^\beta - 1}{\beta s_0 \Theta}.$$

This equation shows (using the L'Hôpital's rule) that $\Upsilon_0(\Theta)/q_r(0)$ asymptotically approaches unity with $\Theta \to \infty$ starting from zero when $\Theta = 0$ (Fig. 2).

It can be easily shown that the asymptotes[15] for $q(t)$ and $c(t)$ given by Eqs. (37) and (38) coincide with q_∞ and c_∞ given by formulas (8) and (9) for $\varphi = \beta$. For example, $q_\infty = q_0 C_3 = q_0 \left(1 + \Theta k_0^{1-\alpha} r_0^{1-\beta}\right)$, which after substitution for r_0 from formula (40) yields formula (8) with $\varphi = \beta$.

Given Eq. (29) and the other paths (36) – (39), the path of the tax in terms of the resource price Υ/q_r is $\Upsilon/q_r = \Upsilon_0 k^{\alpha/\beta} / \left(k_0^{\alpha/\beta} \beta k^\alpha r^{\beta-1}\right) - (q/q_0)^{1/\beta-1} + 1$, which can be rewritten as follows:

$$\frac{\Upsilon(t)}{q_r(t)} = q(t)^{1/\beta-1} \left[\frac{\Upsilon_0}{\beta k_0^{\alpha/\beta}} - \frac{1}{q_0^{1/\beta-1}}\right] + 1. \tag{45}$$

The boundedness of output in this problem implies the value of the asymptote for Υ/q_r, using Eqs. (8) and (43):

$$\begin{aligned}
\frac{\Upsilon_\infty}{q_{r\infty}} &= {q_\infty}^{1/\beta-1} \left[\frac{\Upsilon_0}{\beta k_0^{\alpha/\beta}} - \frac{1}{q_0^{1/\beta-1}}\right] + 1 \\
&= \frac{\Upsilon_0 q_0^{1/\beta-1} \left(s_0\Theta + 1\right)^{1-\beta}}{\beta k_0^{\alpha/\beta}} - \left(s_0\Theta + 1\right)^{1-\beta} + 1 \\
&= \frac{\beta r_0^{-1} q_0^{1/\beta} \left(s_0\Theta + 1\right)^{1-\beta} - q_0^{1/\beta-1} \left(s_0\Theta + 1\right)^{1-\beta} k_0/s_0}{\beta k_0^{\alpha/\beta}} - \left(s_0\Theta + 1\right)^{1-\beta} + 1 \\
&= 1 - \frac{k_0^{1-\alpha/\beta}}{\beta s_0} q_0^{1/\beta-1} \left(s_0\Theta + 1\right)^{1-\beta} = 1 - \frac{k_0^{1-\alpha}}{\beta s_0} r_0^{1-\beta} \left(s_0\Theta + 1\right)^{1-\beta},
\end{aligned}$$

which after substitution for r_0 from Eq. (40) becomes

$$\frac{\Upsilon_\infty}{q_{r\infty}} = 1 - \frac{1}{\beta s_0 \Theta} \left[\left(s_0\Theta + 1\right) - \left(s_0\Theta + 1\right)^{1-\beta}\right]. \tag{46}$$

Using the properties of the Lambert W function, it can be shown that for any parameters of the problem the tax becomes negative in the long run when $\Theta > 0$. Namely, from Eq. (46), the condition that $\Upsilon_\infty/q_{r\infty} < 0$ for $s_0\Theta > 0$ is $\left(s_0\Theta + 1\right) - \left(s_0\Theta + 1\right)^{1-\beta} > \beta s_0 \Theta$. The substitutions $p := s_0\Theta + 1$ and $-v := 1 - \beta + 1/(p-1)$ (or $\beta = v + 1 + 1/(p-1)$) transform this condition into the

[15] The asymptotes follows from the fact that $\lim_{z\to\infty} W(z) = \infty$.

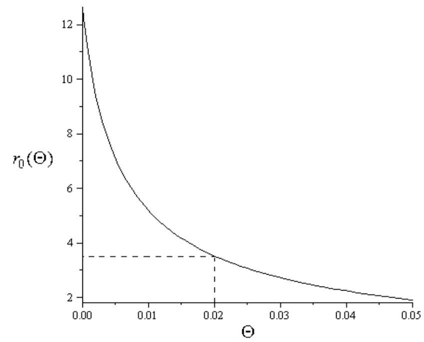

Figure 1: The dependence of the initial rate of extraction r_0 on the hazard factor Θ.

following form:[16] $p^{-v}p^{-1/(p-1)} < -v(p-1)$, which can be rewritten as

$$vp^v < -\frac{p^{-1/(p-1)}}{p-1} \quad \text{or} \quad e^{v\ln p}v\ln p < -\frac{p^{-1/(p-1)}}{p-1}\ln p.$$

The definition of the Lambert W function yields

$$v\ln p < W(-\frac{p^{-1/(p-1)}}{p-1}\ln p) \quad \text{or} \quad v < \frac{W(-\frac{p^{-1/(p-1)}}{p-1}\ln p)}{\ln p}.$$

The last inequality in the original variables is

$$\beta < 1 + \frac{W\left[-\frac{(s_0\Theta+1)^{-1/s_0\Theta}}{s_0\Theta}\ln(s_0\Theta+1)\right]}{\ln(s_0\Theta+1)} + \frac{1}{s_0\Theta}. \tag{47}$$

Since $W(ze^z) = z$ (denoting $z := -\frac{1}{s_0\Theta}\ln(s_0\Theta+1)$), the numerator of the first fraction in (47) is

$$W\left[-\frac{1}{s_0\Theta}\ln(s_0\Theta+1)(s_0\Theta+1)^{-1/s_0\Theta}\right] = -\frac{1}{s_0\Theta}\ln(s_0\Theta+1),$$

and then condition (47) becomes $\beta < 1$,which is always true in this problem. Hence, if $\Theta > 0$, there exists $\bar{t} > 0$ such that $\Upsilon(t) < 0$ for any $t \geqslant \bar{t}$ and for any values of the parameters in this problem (see, e.g., Fig. 3).

[16] The same inequality can be obtained from Eq. (45) as a condition of the existence of the moment of time where Υ/q_r becomes negative.

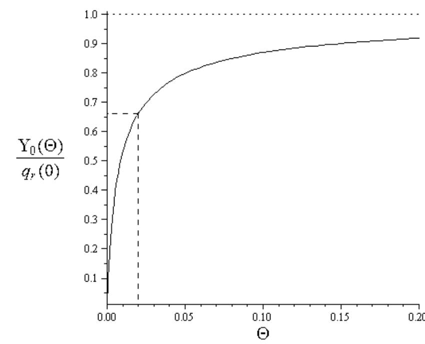

Figure 2: The dependence of the initial tax in terms of the initial resource price $\Upsilon_0(\Theta)/q_r(0)$ on the hazard factor Θ.

5. Numerical example

Let the shares of capital and the resource are $\alpha = 0.3$ [17] and $\beta = 0.15$; the hazard function parameters: $\varphi = \beta$, $\Theta = 0.02$, $T_0 = 1$; the initial stocks of the economy: $s_0 = 371$ bln t, $k_0 = 14.35$.[18] Formula (40) yields the optimal initial rate of extraction $r_0 = 3.524$ bln t/year (cf. $r_0 = 12.61$ bln t/year for $\Theta = 0$, Fig. 1). This reduced initial extraction results from the tax $\Upsilon_0 = 0.0756$ (or in the terms of the resource price $\Upsilon_0/q_r(0) = 0.66$) applied at $t = 0$ and estimated by formula (44).

The externality causes the following deviation from the standard Hotelling rule at $t = 0 : \tau(0) = T_{s_0 - s}(0)u_T(0)/\left(u_c(0)q_r(0)\right) = T_0\varphi\Theta\cdot(-c_0/T_0^2)/\left(\beta k_0^\alpha r_0^{\beta-1}/T_0\right) = -(1-\beta)k_0^\alpha r_0^\beta \Theta/\left(k_0^\alpha r_0^{\beta-1}\right) = -(1-\beta)\Theta r_0$ or $\tau(0) = -0.0599$. Eq. (28) yields

[17] See, e.g., Nordhaus and Boyer (2000).

[18] To make the example more illustrative, these initial values imply the rate of extraction r_0 that is close to the current world oil rate of extraction given s_0 as the world oil reserve estimate. Namely, the world rate of crude oil extraction on January 1, 2010 is 70502.6 [1000 b/day]/7.3 [b/t] $\times$ 365 $\times$ 10^{-6} = 3.525 [bln t/year] (World Oil, 2009). CERA (2006) claimed that actual world oil reserve in 2006 was three times larger (about 512 bln t) than the conventional estimate. I take here s_0 = 2$\times$ 185.5 = 371 [bln t] = 2$\times$1,354,182,395 [1000 b]/7.3 [b/t] $\times$ 10^{-6}, where 185.5 [bln t] is the conventional estimate (World Oil, 2009). One ton of crude oil equals here 7.3 barrels.

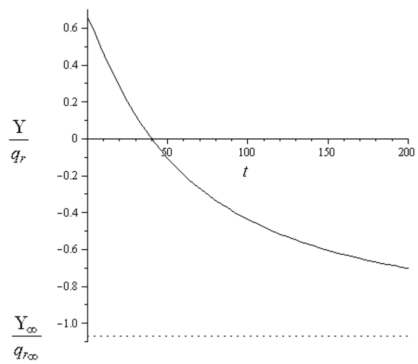

Figure 3: The path of the tax in terms of the resource price $\Upsilon(t)/q_r(t)$. In the Solow-Hartwick case the tax is zero.

$\dot{\Upsilon}(0) = \Upsilon_0 q_k(0) + \tau(0) q_r(0)$. The initial values of marginal productivities $q_k(0) = \alpha k_0^{\alpha-1} r_0^{\beta} = 0.056$ and $q_r(0) = \beta k_0^{\alpha} r_0^{\beta-1} = 0.11$ result in $\dot{\Upsilon}(0) = -0.0026$ showing that the tax is declining at $t = 0$.

The path of the tax in terms of the resource price (45) is depicted in Fig. 3. The tax becomes negative after 40 years and approaches the negative asymptote $\Upsilon_\infty / q_{r\infty} = -1.07$.

Capital for $\Theta = 0.02$ (Fig. 6, solid line) grows faster than a linear function.[19] Linear capital in the Solow-Hartwick case (Fig. 6, circles) has a steeper slope ($\dot{k}_0^0 = 0.488$) due to the higher rate of extraction at $t = 0$. The optimal paths of per capita consumption for the cases with $\Theta = 0.02$ and $\Theta = 0$ are in Fig. 7.

The tax imposed by the planner for $\Theta = 0.02$ results in a hump-shaped optimal path of extraction (Fig. 4, solid line). The path in circles in Fig. 4 corresponds to the case with $\Theta = 0$ (Solow-Hartwick case, no tax). Note that, in the conventional approach,[20] a relatively small uncertainty in the hazard parameter leads to a large uncertainty in the short-run resource policy (Figs. 2 – 5). The model shows that if the planner is unaware of the externality

[19] This linear function $k_0 + \dot{k}_0 t$ with k_0 and $\dot{k}_0 = 0.403$ for $\Theta = 0.02$ is depicted as a dotted line in Fig. 6

[20] I mean here the approach where r_0 is to be derived as an optimal or equilibrium value.

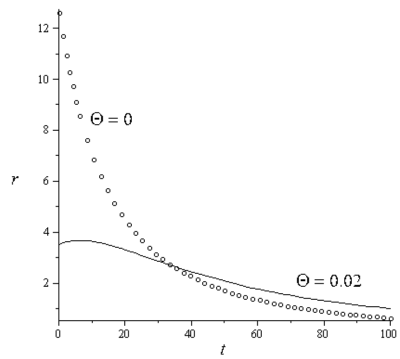

Figure 4: The optimal paths of extraction: for $\Theta = 0.02$ - solid line, for $\Theta = 0$ (Solow-Hartwick case) - circles.

or is going to neglect its effect and implement an economic program with the maximum constant per capita consumption over time, she should apply the policies that will result in the current rate of extraction $r_0 = 12.61$ bln t/year, which is 3.6 higher than in the case with $\Theta = 0.02$.

The uncertainty of the conventional approach in defining r_0 with respect to the imprecision of the reserve estimate s_0 is illustrated in Figs. 5a and 5b, where s_0^1 is a conventional world oil reserve estimate (World Oil, 2009), s_0^3 is the estimate of CERA (2006), and $s_0^2 = 2 \times s_0^1$ is the estimate that is used in this example as s_0 and is somewhere between s_0^1 and s_0^3. Note that a small hazard factor (Fig. 5b) results in higher uncertainty than the large one since, for the larger values of Θ, the initial rate of extraction should be essentially lower, reducing the uncertainty of this value.

For a small extracting firm that has just discovered or obtained an oil field at an auction, an approach that provides the initial rate of extraction as a policy recommendation could be possible when the oil-extracting capital is available in required quantities and the elasticity of the demand for the resource is high. However, for a large incumbent firm that has been extracting the resource for a period of time and is going to reestimate the optimal path, this approach can be questionable due to the high volatility of this recommendation with respect

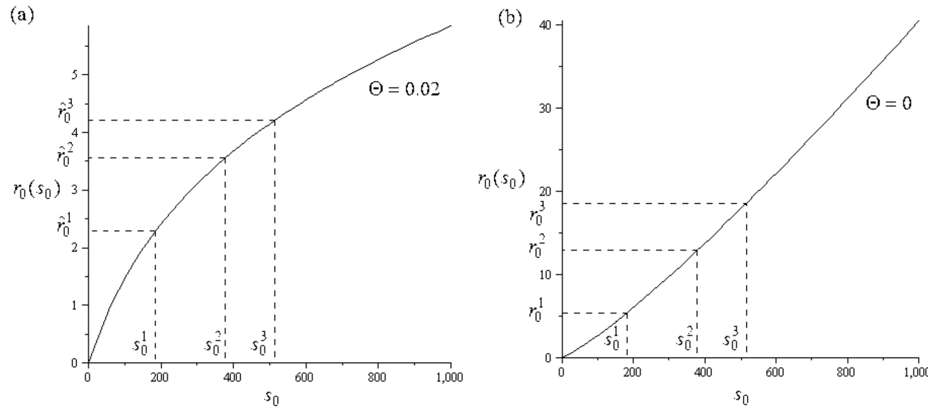

Figure 5: The dependence of the initial rate of extraction r_0 [bln t/year] on the reserve s_0 for different values of the hazard factors Θ : (a) $\Theta = 0.02$; (b) $\Theta = 0$; $s_0^1 = 185.5$ bln t – the current world oil reserve according to World Oil (2009); $s_0^2 = 2 \times 185.5$ bln t; $s_0^3 = 512.3$ bln t – CERA (2006) world oil reserve estimate.

to uncertainties.

For example, if the extraction was started under the constant consumption criterion ($\Theta = 0$) and with the initial reserve estimate $s_0 = s_0^1$, the initial rate of extraction would be $r_0^1 = 5.58$ bln t/year (Fig. 5b). The announcement similar to CERA (2006) about the larger actual reserve $s_0 = s_0^3$ would cause the immediate jump in the rate of extraction up to the new value $r_0^3 = 18.44$ bln t/year as a result of the reestimation of the optimal path using the same approach. Then, if the social planner takes into account information about the hazards of the extraction of the resource and decides to follow the constant utility path with $\Theta = 0.02$, the imposed tax should instantly cut down the rate of extraction to the new initial value $\hat{r}_0^3 = 4.2$ bln t/year (Fig. 5a).

The effect of deviation of the optimal path from the initially estimated path recalculated at a later date is called "dynamic inconsistency" in the literature.[21] In this case, inconsistency takes the form of considerable discontinuous jumps in

[21] For example, Newbery (1981) considered various reasons for dynamic inconsistency in oil markets including the changes in the market structure.

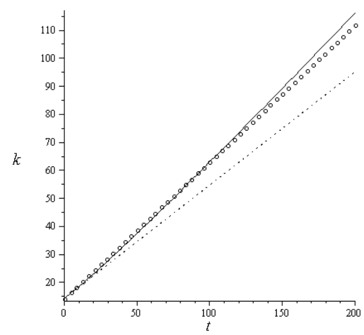

Figure 6: The optimal paths of capital: for $\Theta = 0.02$ - solid line, the linear path for $\Theta = 0$ (Solow-Hartwick case) - circles, the path $k_0 + \dot{k}_0 t$ with k_0 and $\dot{k}_0$ for $\Theta = 0.02$ - dotted line.

resource policies that can lead to socioeconomic and environmental damage;[22] some of these jumps can be unrealizable in practice.

Hence, the approaches that result in the paths that are discontinuous with respect to the initial state of economy could be appropriate only for small firms entering the market or for theoretical studies where the questions of the transition to an optimal state are not important (Bazhanov, 2010). In many cases, reestimation of the optimal path requires a solution that is linked to the initial conditions, including the initial state of the extracting industry (Bazhanov, 2009a, Section 9; Bazhanov, 2009b).[23]

6. Concluding remarks

This paper has offered an example of the closed form solution for the problem of irreversible global warming under the constant utility criterion (Stollery, 1998) with utility negatively affected by the hazard factor. The solution was expressed via the Lambert W special function, which has convenient analytical properties

[22] One can recall the consequences of the oil embargo in 1973.

[23] Pezzey (2004, formula (15), p. 477) offered an example of solving the problem of dynamic inconsistency by specifying the discount factor in the utilitarian criterion for given technological parameters and the current state of economy.

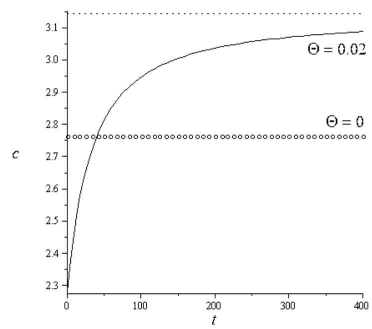

Figure 7: The optimal paths of per capita consumption: for $\Theta = 0.02$ - solid line with the asymptote (dotted line), the constant path for $\Theta = 0$ (Solow-Hartwick case) - circles.

for any parameters in this problem. For example, using the properties of this function, it was shown that the declining tax in this problem becomes negative in the long run.

The main qualitative distinctions of this problem from the Solow-Hartwick case with no hazard are:

(a) output and consumption are growing and asymptotically approaching positive constants;

(b) the initial rate of the resource extraction is lower, implying (for the same initial capital) lower levels of initial output and consumption;

(c) the economy is efficient only asymptotically with exhaustion of the polluting resource;

(d) the optimal path of the resource extraction can be hump-shaped;

(e) capital is growing faster than a linear function.

The example has shown that the initial rate of extraction and the initial tax, provided in the conventional approach as policy recommendations, can be significantly uncertain due to the uncertainties in the initial reserve and in the intensity of the hazard. The uncertainty is considerably higher in the case of the low values of the intensity of the hazard (Figs. 1, 2, 4, and 5).

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