DCPT-04/11
SPhT-T04/037
hep-th/0404014

Integrable quantum field theory with boundaries: the exact g-function

Patrick Dorey[1,2], Davide Fioravanti[3], Chaiho Rim[4] and Roberto Tateo[1,5]

[1] *Dept. of Mathematical Sciences, University of Durham, Durham DH1 3LE, United Kingdom*

[2] *Service de Physique Théorique, CEA-Saclay, F-91191 Gif-sur-Yvette Cedex, France*

[3] *Dept. of Mathematics, University of York, York YO10 5DD, United Kingdom*

[4] *Dept. of Physics, Chonbuk National University, Chonju 561-756, Korea*

[5] *Dip. di Fisica Teorica and INFN, Università di Torino, Via P. Giuria 1, 10125 Torino, Italy*

E-mails:
p.e.dorey@durham.ac.uk, df14@york.ac.uk, rim@chonbuk.ac.kr, tateo@to.infn.it

Abstract

The g-function was introduced by Affleck and Ludwig in the context of critical quantum systems with boundaries. In the framework of the thermodynamic Bethe ansatz (TBA) method for relativistic scattering theories, all attempts to write an exact integral equation for the off-critical version of this quantity have, up to now, been unsuccesful. We tackle this problem by using an n-particle cluster expansion, close in spirit to form-factor calculations of correlators on the plane. The leading contribution already disagrees with all previous proposals, but a study of this and subsequent terms allows us to deduce an exact infrared expansion for g, written purely in terms of TBA pseudoenergies. Although we only treat the thermally-perturbed Ising and the scaling Lee-Yang models in detail, we propose a general formula for g which should be valid for any model with entirely diagonal scattering.

Keywords: Boundary problems; Conformal field theory; Integrability; Thermodynamic Bethe ansatz
PACS: 05.20.-y; 11.25.Hf; 11.55.Ds; 68.35.Rh

1 Introduction

The study of two-dimensional conformal field theories with boundaries [1] and their integrable perturbations [2, 3, 4] is of interest both in condensed matter physics [5] and in string theory [6]. An important quantity emerging from the definition of the cylinder partition function for these field theories is the g-function, the 'ground-state degeneracy' or 'boundary entropy', which for models critical in the bulk was introduced some years ago by Affleck and Ludwig [7]. While many interesting questions remain in these cases [8, 9], in this paper we shall deal with the further issues which arise for off-critical, massive, systems.

The g-function for massive field theories can be defined as follows [10, 11, 12, 13]. There are two possible Hamiltonian descriptions of the cylinder partition function. In the so-called L-channel representation the rôle of time is taken by L, the circumference of the circle:

$$Z_{\alpha\beta} = \mathrm{Tr}_{\mathcal{H}_{(\alpha,\beta)}} e^{-LH^{\mathrm{strip}}_{\alpha\beta}(M,R)} = \sum_{n=0}^{\infty} \; e^{-LE^{\mathrm{strip}}_n(M,R)} \; . \tag{1.1}$$

In this formula, $H^{\mathrm{strip}}_{\alpha\beta}$ propagates states in $\mathcal{H}_{(\alpha,\beta)}$, the Hilbert space for an interval of length R with boundary conditions α and β imposed at the two ends, $E^{\mathrm{strip}}_n \in \mathrm{spec}(H^{\mathrm{strip}}_{\alpha\beta})$, and M is the mass of the lightest particle in the theory. In the R-channel representation the rôle of time is instead taken by R, the length of the cylinder:

$$Z_{\alpha\beta} = \langle\, \alpha |\, e^{-RH^{\mathrm{circ}}(M,L)} \,|\beta\,\rangle = \sum_{n=0}^{\infty} \; \mathcal{G}^{(n)}_{\alpha}(l)\mathcal{G}^{(n)}_{\beta}(l)\, e^{-RE^{\mathrm{circ}}_n(M,L)}, \quad (l = ML) \tag{1.2}$$

where $E^{\mathrm{circ}}_n \in \mathrm{spec}(H^{\mathrm{circ}})$ and

$$\mathcal{G}^{(n)}_{\alpha}(l) = \tfrac{\langle\, \alpha|\psi_n\,\rangle}{\langle\, \psi_n|\psi_n\,\rangle^{1/2}} \; . \tag{1.3}$$

In equation (1.2), the boundary states $|\alpha\rangle$, $|\beta\rangle$ and the eigenbasis $\{|\psi_n\rangle\}$ of the Hamiltonian H^{circ} have been used. These are defined on a circle of circumference L and propagate along the 'time' direction R. At large l, the function $\ln \mathcal{G}^{(0)}_{\alpha}(l)$ grows linearly:

$$\ln \mathcal{G}^{(0)}_{\alpha}(l) \sim -f_{\alpha}L \; , \tag{1.4}$$

where the constant f_α contributes to the constant (boundary) part of the ground-state energy on the strip (see eq. (A.5)). The standard g-function is then defined as

$$\ln g_{\alpha}(l) = \ln \mathcal{G}^{(0)}_{\alpha}(l) + f_{\alpha}L \; . \tag{1.5}$$

In theories with only massive excitations in the bulk, $\ln g_{\alpha}(l)$ tends exponentially to zero at large l.

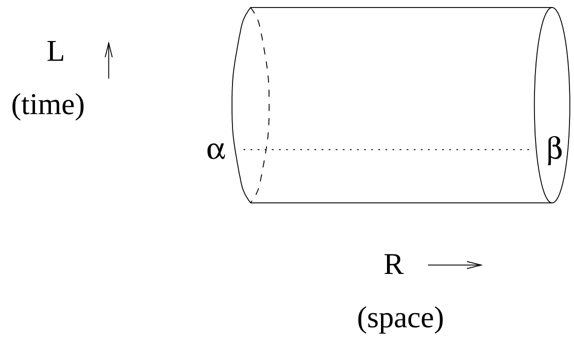

Figure 1: The L-channel decomposition; states $|\chi_n\rangle$ live on the dotted line segment along the cylinder.

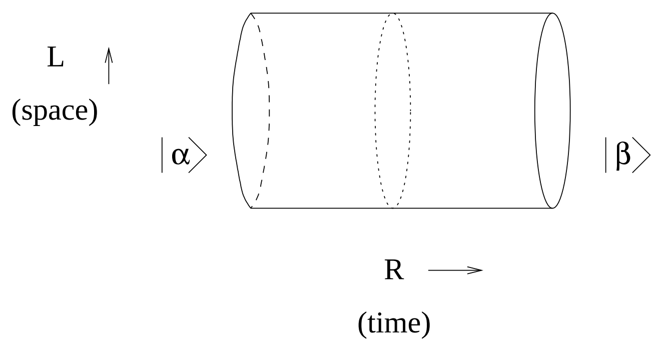

Figure 2: The R-channel decomposition; states $|\psi_n\rangle$ live on the dotted circle around the cylinder.

The two decompositions are illustrated in figures 1 and 2.

The equality of (1.1) and (1.2) results in the following important identity:

$$\sum_{n=0}^{\infty} e^{-LE_n^{\rm strip}(M,R)} = \sum_{n=0}^{\infty} \mathcal{G}_\alpha^{(n)}(l)\mathcal{G}_\beta^{(n)}(l)\, e^{-RE_n^{\rm circ}(M,L)}. \tag{1.6}$$

The purpose of this paper is to develop an exact expression for the ground-state function $\ln \mathcal{G}_\alpha^{(0)}(l)$ through the large-R limit of (1.6), with boundary conditions $\beta = \alpha$. As it stands, the fact that $E_0^{\rm circ}(M,L)$ is negative makes the RHS of (1.6) diverge as $R \to \infty$; however, rearranging gives

$$\begin{aligned} 2\ln \mathcal{G}_\alpha^{(0)}(l) &= RE_0^{\rm circ}(M,L) - LE_0^{\rm strip}(M,R) \qquad (1.7)\\ &+ \ln\left(1+\sum_{n=1}^{\infty} e^{-L(E_n^{\rm strip}(M,R)-E_0^{\rm strip}(M,R))}\right) + O(e^{-R(E_1^{\rm circ}-E_0^{\rm circ})})\ . \end{aligned}$$

We shall restrict our attention to massive theories with non-degenerate ground state on the plane. For these models the non-zero mass gap gives the final term the leading behaviour

$$O(e^{-R(E_1^{\rm circ}(M,L)-E_0^{\rm circ}(M,L))}) \sim O(e^{-RM}) \tag{1.8}$$

in the domain $R \gg L \gg 0$. In this same domain, $E_0^{\rm strip}(M,R)$ tends to its limiting form as

$$E_0^{\rm strip}(M,R) = \mathcal{E} M^2 R + 2 f_\alpha + O(e^{-RM}) \tag{1.9}$$

where $\mathcal{E}$ and f_α are the extensive bulk and boundary free energies, as in (A.5). These constraints are crucial for the validity of the perturbative treatment to be introduced in the following sections: the higher corrections have a clear dependence on R and do not contribute to the g-function. Discarding these exponentially-suppressed terms and using the definition (1.5), we finally obtain

$$2 \ln g_\alpha^{(0)}(l) \sim R\Big(E_0^{\rm circ}(M,L) - \mathcal{E} M^2 L\Big) + \ln\left(1 + \sum_{n=1}^{\infty} e^{-L(E_n^{\rm strip}(M,R) - E_0^{\rm strip}(M,R))}\right). \tag{1.10}$$

Having taken R to be large, the cluster expansion involves letting L tend to infinity as well, so that an expansion of the RHS of (1.10) can be developed in terms of one-, two- and so on particle contributions, which themselves can be consistently estimated using the Bethe-ansatz approximated levels (A.6), (A.7). Note that this differs from the strategy adopted in [10], where a saddle-point evaluation of the dominant contributions at finite L was made instead.

The rest of this paper is organised as follows. In section 2 the cluster method is exemplified by studying the free fermion theory associated to the thermally-perturbed Ising model. The resulting integral expression for $\ln g_\alpha(l)$ turns out to be in full agreement with previous results of [10, 11]. In section 3 two previous proposals [10, 14] for $\ln g_\alpha(l)$ are described and in section 4 the scattering data for the scaling Lee-Yang model, our working interactive example, are summarised. The ultraviolet result obtained from the conformal perturbation theory and the boundary truncated conformal space approximation (BTCSA) [12, 13, 9] is compared with infrared numerical results from the Bethe Ansatz, and the equivalence between the two functions is confirmed by a large overlap at intermediate scales.

This agreement motivates the search for an exact analytic expression. This is the main objective of sections 5.1, 5.2 and 5.3 where the large strip-width ($R \to \infty$) limit is explicitly taken and sums over the quantum numbers are transformed into integrals in rapidity variables. This analysis leads to the final exact expansion for $\ln g_\alpha(l)$ given in equation (5.24). This and its generalization (5.30) constitute the main results of the paper. In section 5.3 we also briefly comment on the similarity between our results and one recently obtained by Woynarovich in [15]. Section 6 contains our conclusions. Finally, in Appendix A we summarise the main equations used to develop our programme: the thermodynamic Bethe ansatz [16] and the Bethe quantisation conditions. In Appendix B the reflection factors for the Ising model are recalled and explicit expressions for the boundary entropy for free-free and fixed-fixed conditions are presented.

2 A simple example: the Ising model

We start with the study of the free Majorana fermion theory corresponding to the thermally-perturbed Ising model on a strip. The partition function is (cf. [11]) * :

$$Z_{\alpha\alpha} = e^{-LE_0^{\mathrm{strip}}(M,R)} \prod_{j>0} \left(1 + e^{-l\cosh\theta_j}\right) \ , \quad (l = ML) \ , \tag{2.1}$$

or

$$\begin{aligned} \ln Z_{\alpha\alpha} &= -LE_0^{\mathrm{strip}}(M,R) + \sum_{j>0} \ln(1 + e^{-l\cosh\theta_j}) \\ &= -LE_0^{\mathrm{strip}}(M,R) + \frac{1}{2}\sum_{j=-\infty}^{\infty} \ln(1 + e^{-l\cosh\theta_j}) - \frac{1}{2}\ln(1 + e^{-l}) \, . \end{aligned} \tag{2.2}$$

Due to the singular behaviour for the Ising model of the bulk and linear terms $\mathcal{E}$ and f_α defined in (A.2) and (A.5) below, it is convenient, exceptionally for this case, to work with subtracted energies tending exponentially to zero at large scales:

$$E_0^{\mathrm{strip}}(M,R)\Big|_{R\gg 1} \sim 0 \tag{2.3}$$

and consistently to set

$$E_0^{\mathrm{circ}}(M,L) = -\int_{\mathbb{R}} \frac{d\theta}{2\pi} \, M\cosh\theta \, \ln\left(1 + e^{-l\cosh\theta}\right) \tag{2.4}$$

(cf. (A.2)). Starting from the quantization condition

$$r\sinh\theta_j - i\ln R_\alpha(\theta_j) = \pi j \ , \quad (r = MR) \tag{2.5}$$

with integer j and $R_\alpha(\theta_j)$ as defined in (B.1), writing equation (2.5) with $j \to j+1$ and subtracting (2.5) from the result, we find in the large R limit

$$\frac{\Delta\theta_j}{\pi}\left(r\cosh(\theta_j) - i\frac{d}{d\theta}\ln R_\alpha(\theta_j)\right) + O((\Delta\theta_j)^2) = 1 \, . \tag{2.6}$$

Substituting this into (2.2),

$$\ln Z_{\alpha\alpha} \sim \frac{1}{2}\int_{\mathbb{R}} d\theta \left(\frac{r}{\pi}\cosh(\theta) + \phi_\alpha(\theta) - \delta(\theta)\right) \ln(1 + e^{-l\cosh\theta}) \, , \tag{2.7}$$

where $\phi_\alpha(\theta)$ is given in (B.2). In the latter equation we recognize a part corresponding to $RE_0^{\mathrm{circ}}(M,L)$, and, considering also (2.3), we arrive at the exact result

$$2\ln g_\alpha(l) = \lim_{R\to\infty} (\ln Z_{\alpha\alpha}(L,R) + RE_0^{\mathrm{circ}}(M,L)) = \frac{1}{2}\int_{\mathbb{R}} d\theta \, (\phi_\alpha(\theta) - \delta(\theta)) \ln(1 + e^{-l\cosh\theta}) \, . \tag{2.8}$$

*Notice that the zero momentum ($\theta_j = 0 \leftrightarrow j = 0$) particle state is forbidden.

This coincides with the result found in [10, 11] using a different technique. The match confirms the correctness of our method, at least in this case, and motivates its study in more complicated models.

In figures 3 and 4 the integration of (2.8) for free-free and fixed-fixed boundary conditions, corresponding to $k = 1$ and $k = -\infty$ in (B.1) and (B.2)†, is compared with numerical results obtained by estimating the large-R partition function (2.1) using the Bethe ansatz quantized energy levels (2.5) directly, and then extracting the boundary entropy through the relation

$$2 \ln g_\alpha(l) \sim \left(\ln Z_{\alpha\alpha}(L, R) + R E_0^{\rm circ}(M, L) \right)\Big|_{r \gg 1} . \tag{2.9}$$

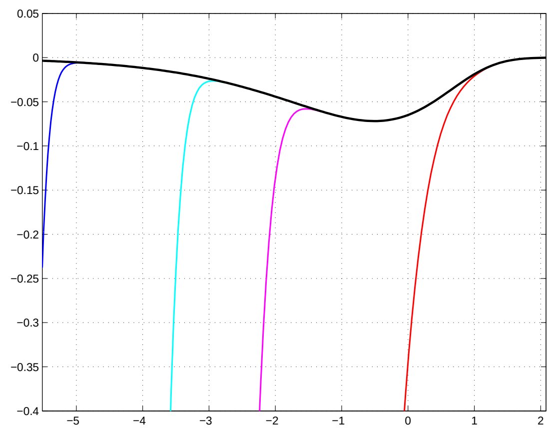

Figure 3: 2 ln $g_{\bf fixed}$ vs. ln(l) for Ising with fixed boundary conditions. From the bottom, the lines represent 5, 100, 500, and 5000 particle contributions. The maximum quantum number is 80 and $r = 10$. The top line is the exact result.

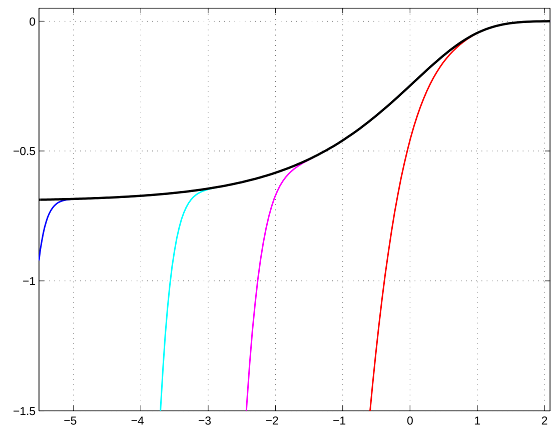

Figure 4: 2 ln $g_{\bf free}$ vs. ln(l) for Ising with free boundary conditions. From the bottom, the lines represent 5, 100, 500, and 5000 particle contributions. The maximum quantum number is 80 and $r = 10$. The top line is the exact result.

For interacting models a compact expression such as (2.1) is not available, and one is forced to build the partition function using the LHS of (1.6) directly. In figures 5 and 6 we test this more general way to estimate a g-function. A similar idea was first applied to the scaling Lee-Yang model in [12, 13]; however in that case the energy levels were estimated using the BTCSA method [12], rather than the Bethe ansatz.

†By studying the monodromies of the integral (2.8), we have also found more explicit expressions for $\ln g_{\bf fixed}(l)$ and $\ln g_{\bf free}(l)$; these are given in Appendix B.

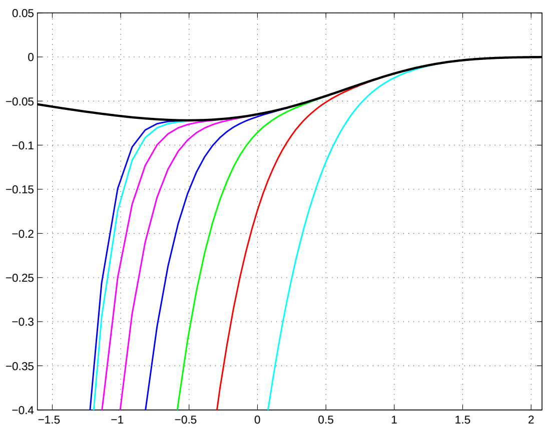

Figure 5: 2 ln $g_{\mathbf{fixed}}$ vs. $\ln(l)$ for Ising with fixed boundary conditions. From the bottom, the lines represent cluster contributions of $1, 2, \ldots, 8$ particles. The maximum quantum number 80 and $r = 10$. The top line is the exact result.

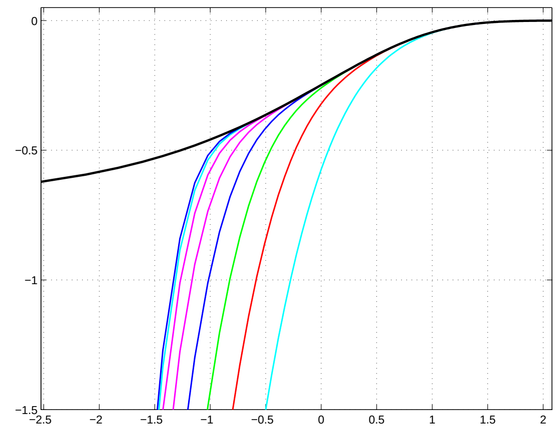

Figure 6: 2 ln $g_{\mathbf{free}}$ vs. $\ln(l)$ for Ising with free boundary conditions. From the bottom, the lines represent cluster contributions of $1, 2, \ldots, 8$ particles. The maximum quantum number 80 and $r = 10$. The top line is the exact result.

We see that it is hard to get a good estimate of the ultraviolet value of $\ln g$ from this form of the cluster expansion. In section 4 we shall solve this numerical problem for the case of the scaling Lee-Yang model by matching this numerical Bethe ansatz calculation with the ultraviolet perturbed CFT results of [13], while in section 5 we shall develop a more analytical treatment.

3 Earlier proposals for g

Consider a 1+1 dimensional integrable field theory with entirely diagonal scattering and N particle species. According to the proposal of [10] the boundary entropy should be given by an expression of the form

$$\ln g_\alpha(l) = \frac{1}{4} \sum_{a=1}^{N} \int_{\mathbb{R}} d\theta \, \Theta_a(\theta) \ln(1 + e^{-\varepsilon_a(\theta)}) \,, \tag{3.1}$$

where the function $\varepsilon_a(\theta)$ is the solution of the periodic-boundary-conditions TBA (A.1), and

$$\Theta_a(\theta) = \left(\phi_\alpha^{(a)}(\theta) \; - \; 2\phi_{aa}(2\theta) \; - \; \delta(\theta) \right) \tag{3.2}$$

with

$$\phi_\alpha^{(a)}(\theta) = -\frac{i}{\pi} \frac{d}{d\theta} \ln R_\alpha^{(a)}(\theta) \;, \quad \phi_{ab}(\theta) = -\frac{i}{2\pi} \frac{d}{d\theta} \ln S_{ab}(\theta) \;. \tag{3.3}$$

(Note, the normalisations of $\phi^{(a)}_\alpha(\theta)$ and $\phi_{ab}(\theta)$ differ from those in [13, 14] by factors of π and 2π respectively. This change is merely to simplify some later formulae.)

However, the detailed analysis of [13] showed that, for non-zero values of the lightest bulk mass M, the resulting l-dependence was incorrect, both in the total change in $g_\alpha(l)$ between UV and IR, and in the behaviour of the small-l series expansion. On the other hand, the predictions of (3.1) and (3.2) for dependence of $g_\alpha(l)$ on the boundary parameters at fixed l, and also for the ratios of g-functions $g_\alpha(l)/g_\beta(l)$, *were* in very good agreement with conformal perturbation theory and the BTCSA. This suggested that the formulae should be modified by some boundary condition independent extra terms, but provided little clue as to what those extra terms should be.

Subsequently, it was proposed in [14] that (3.2) should be replaced by

$$\Theta_a(\theta) = \Big(\,\phi^{(a)}_\alpha(\theta) \;-\; 2\phi_{aa}(2\theta) \;-\; \phi_{aa}(\theta)\Big)\;. \tag{3.4}$$

However, using results tabulated in [13] it can be checked that this modification does not cure the problems arising in the bulk-massive case.

4 The scaling Lee-Yang model

The spectrum in the bulk scaling Lee-Yang theory consists of a single particle species, with two-particle bulk scattering amplitude [17, 18]

$$S(\theta) = -(1)(2)\;,\quad (x) = \frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi x}{6}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi x}{6}\right)}\;. \tag{4.1}$$

When a boundary is present, two different types of boundary conditions arise, which were labelled $\mathbb{1}$ and $\Phi(h)$ in [12]. The corresponding reflection factors are

$$R_{\Phi(h)}(\theta) = R_b(\theta)\;,\qquad R_{\mathbb{1}}(\theta) = R_0(\theta)\;, \tag{4.2}$$

where

$$h \sim -|h_c|\,\sin((b+.5)\pi/5)M^{6/5}\;,\quad h_c = -0.6852899\ldots \tag{4.3}$$

is the coupling of the boundary field and

$$R_b(\theta) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{4}{2}\right)^{-1}\left(S(\theta+i\pi\tfrac{b+3}{6})S(\theta-i\pi\tfrac{b+3}{6})\right)^{-1}\;. \tag{4.4}$$

We first use the Bethe ansatz equation together with (1.7) to obtain the g-function up to six-particle contributions. The results are shown in figure 7 for the boundary condition $\mathbb{1}$, and are compared with the ultraviolet expansion obtained from (boundary) conformal perturbation theory and the BTCSA [13]:

$$2\ln g_{\mathbb{1}}(l) = \frac{1}{2}\ln\left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right) + 2\,\frac{f_{\mathbb{1}}}{M}\,l + 2\sum_{n=1}^{4} d_n\left(\frac{l}{\kappa}\right)^{\frac{12}{5}n} + O(l^{12})\,, \tag{4.5}$$

where $f_{\mathbb{1}} = \frac{1}{4}(\sqrt{3}-1)M$, $d_1 \approx -0.25312$, $d_2 \approx 0.0775$, $d_3 \approx -0.0360$, $d_4 \approx 0.0195$, and $\kappa \approx 2.6429$.

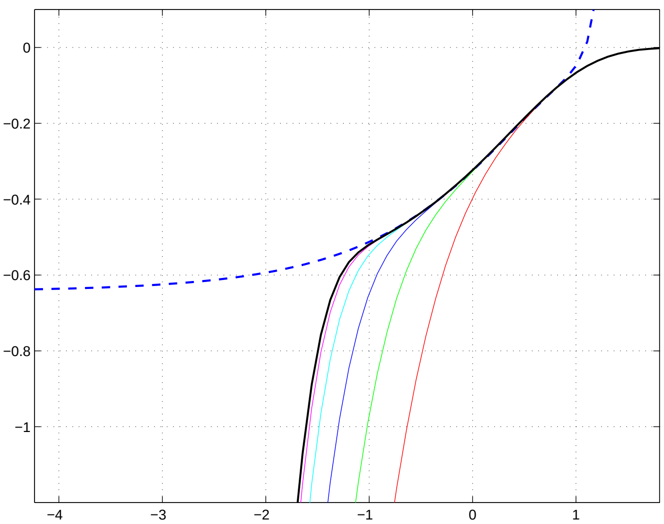

Figure 7: $2 \ln g_{\mathbb{1}}$ vs. $\ln(l)$ with $(\mathbb{1}, \mathbb{1})$ boundary conditions. The dotted line is the 4^{th} order result of [13] and the solid lines are BA results with $r = 8$. The solid lines, from the bottom, represent the $1, 2, \cdots, 6$ particle contributions respectively. The maximum quantum number used was 80 (57 for the 6-particle contribution).

As can be seen from the figure, the results from the Bethe ansatz and from perturbed conformal field theory overlap over a significant range of scales. This supports our hypothesis that the two approaches are describing the same function $g_{\mathbb{1}}(l)$, expanded about either the IR or the UV.

5 Infrared expansion for the Lee-Yang model

The purpose of this section is to develop an analytic technique to check the earlier proposals described in section 3 and at the same time to give hints about the appropriate modifications. The idea, successfully applied above to the Ising model, is to start from the Bethe ansatz and to set up a cluster expansion by transforming the sums into integrals as $R \to \infty$. The method is quite powerful, and already at first order it confirms the questions raised in [13] about the proposals described in section 3. To simplify the discussion we shall only treat the $(\mathbb{1}, \mathbb{1})$ boundary conditions directly. However, this restriction is of no real significance since the results in [4, 13, 19] show that

$$\mathcal{G}^{(0)}_{\Phi(h)}(l) = Y\Big(i\pi \frac{b+3}{6}\Big)\mathcal{G}^{(0)}_{\mathbb{1}}(l) \ , \quad Y(\theta) = e^{\varepsilon(\theta)} \ , \tag{5.1}$$

where $\varepsilon(\theta)$ is the solution of the ground state TBA equation with periodic boundary conditions.

5.1 The one particle contribution

We start from the large-R equation (1.10), truncated at the one-particle level:

$$2\ln g_{\mathbb{1}} \sim R\Big(E_0^{\rm circ}(M,L) - \mathcal{E}M^2L\Big) + \ln\Big(1 + \sum_{n_1>0} e^{-l\cosh\theta_1(n_1)} + \dots\Big)\,, \tag{5.2}$$

where the one-particle Bethe ansatz essentially coincides with that for a free particle,

$$r\sinh\theta_1 - i\ln R_{\mathbb{1}}(\theta_1) = \pi n_1\,, \tag{5.3}$$

with integer n_1. Performing the continuous limit as in section 2 we find

$$P_1 = \sum_{n_1>0} e^{-l\cosh\theta_1} = \frac{1}{2}\Big(\sum_{n_1=-\infty}^{\infty} e^{-l\cosh\theta_1} - e^{-l}\Big) \longrightarrow \frac{1}{2}\int_{\mathbb{R}} d\theta\,(J^{(1)}(\theta) - \delta(\theta))\,e^{-l\cosh\theta} \tag{5.4}$$

where the Jacobian for the change of variable $n_1 \to \theta_1 \equiv \theta$ is

$$J^{(1)}(\theta) = \frac{r}{\pi}\cosh\theta + \phi_{\mathbb{1}}(\theta)\,. \tag{5.5}$$

The cosh term cancels the leading part of the term linear in R on the RHS of (5.2), leaving the first contribution to $\ln g_{\mathbb{1}}$ as

$$2\ln g_{\mathbb{1}} = \frac{1}{2}\int_{\mathbb{R}} d\theta\,(\phi_{\mathbb{1}}(\theta) - \delta(\theta))\,e^{-l\cosh\theta} + \dots\,. \tag{5.6}$$

Comparing this result with the proposals of section 3, we conclude that both are incorrect: in particular no $\phi(2\theta)$ or $\phi(\theta)$ terms are involved in the leading large l asymptotic.

Next, we want to use this result to gain a hint as to how the earlier proposals should be modified. However, the task to totally or even partially re-sum the cluster expansion directly is, in principle, very hard. Our work is driven by the extra assumption that the final result should depend, just like the earlier 'partially correct' proposals (3.2) and (3.4), on the bare single-particle energies only through the TBA pseudoenergies $\varepsilon(\theta)$. As will be reported in more detail below, the consistency of this assumption was checked carefully up to four particles and confirmed, by a more superficial inspection, to all orders.

The attempt to find an exact expression for $\ln g_{\mathbb{1}}$, therefore, naturally starts from

$$\ln g_{\mathbb{1}} = [\ln g_{\mathbb{1}}]_D^{(1)} + \dots \tag{5.7}$$

where we have defined the 'dressed' version of (5.6) to be

$$2[\ln g_{\mathbb{1}}]_D^{(1)} = \frac{1}{2}\int_{\mathbb{R}} d\theta\,(\phi_{\mathbb{1}}(\theta) - \delta(\theta))\ln(1 + e^{-\varepsilon(\theta)}). \tag{5.8}$$

Figure 8 gives some initial numerical support for the conjecture.

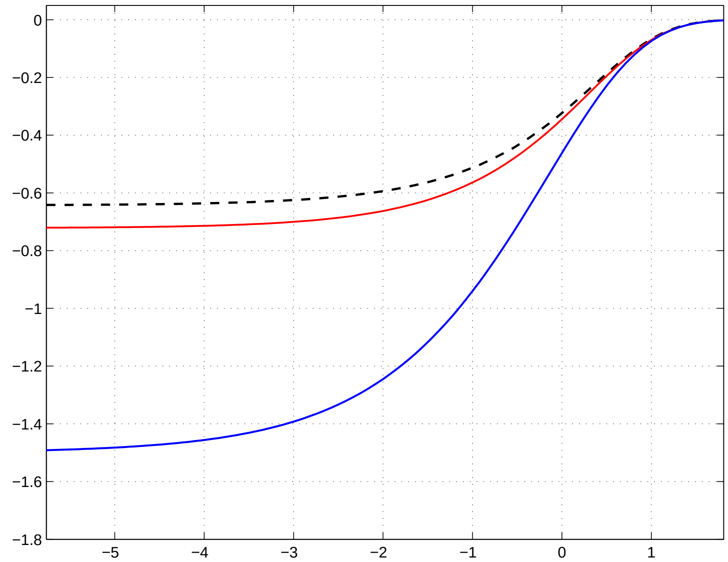

Figure 8: $2 \ln g_{\mathbb{1}}$ vs. $\ln(l)$. The top dotted line is the 'exact' result obtained by combining the CFT and BA results from figure 7. The solid lines are obtained from (5.6) (bottom line) and (5.7) (middle line).

Notice now that $[\ln g_{\mathbb{1}}]_D^{(1)}$ also contains $n(>1)$-particle contributions. To see this at second order, we expand

$$\ln(1+e^{-\varepsilon(\theta)}) \quad \text{as} \quad e^{-\varepsilon(\theta)} - \frac{e^{-2\varepsilon(\theta)}}{2} + \dots\,, \tag{5.9}$$

and use the exponential of equation (A.1) expanded in terms of the bare particle energy $e(\theta) = M\cosh\theta$

$$e^{-\varepsilon(\theta)} = e^{-l\cosh\theta}\left(1 + \int_{\mathbb{R}} d\theta' \phi(\theta-\theta') e^{-l\cosh\theta'}\right) + \dots \tag{5.10}$$

to see that

$$2[\ln g_{\mathbb{1}}]_D^{(1)} = \frac{1}{2}\int_{\mathbb{R}} d\theta\ (\phi_{\mathbb{1}}(\theta) - \delta(\theta))\, e^{-l\cosh\theta} - \frac{1}{2}\int_{\mathbb{R}} d\theta\, \phi(\theta) e^{-l\cosh\theta - l} + \frac{1}{4}e^{-2l}$$
$$- \frac{1}{4}\int_{\mathbb{R}} d\theta\, \phi_{\mathbb{1}}(\theta) e^{-2l\cosh\theta} + \frac{1}{2}\int_{\mathbb{R}^2} d\theta_1 d\theta_2\, \phi_{\mathbb{1}}(\theta_1)\phi(\theta_1-\theta_2) e^{-l\cosh\theta_1 - l\cosh\theta_2} + \dots \tag{5.11}$$

The aim of the analysis in the next sections is to justify the replacement of $e^{-l\cosh\theta}$ by $\ln(1+e^{-\varepsilon(\theta)})$ and is also to find some hints as to the origin and form of the further correction terms in (5.7).

5.2 Two and three particle contributions

We start again from (1.10), this time keeping both one and two particle contributions:

$$
\begin{aligned}
2\ln g_{\mathbb{1}} &= R\Big(E_0^{\text{circ}}(M,L) - \mathcal{E}M^2 L\Big) \\
&+ \ln\Big(1 + \sum_{n_1>0} e^{-l\cosh\theta_1} + \sum_{n_1>0}\sum_{n_2>n_1} e^{-l\cosh\theta_1 - l\cosh\theta_2} + \dots\Big)\,, \quad (5.12)
\end{aligned}
$$

where the two-particle-state momenta (θ_1, θ_2) are related to their quantum numbers (n_1, n_2) via the Bethe ansatz equations

$$
\begin{aligned}
\frac{r}{\pi}\sinh\theta_1 - \frac{i}{\pi}\ln R_{\mathbb{1}}(\theta_1) - \frac{i}{2\pi}\ln S(\theta_1-\theta_2)S(\theta_1+\theta_2) &= n_1\,; \\
\frac{r}{\pi}\sinh\theta_2 - \frac{i}{\pi}\ln R_{\mathbb{1}}(\theta_2) - \frac{i}{2\pi}\ln S(\theta_2-\theta_1)S(\theta_2+\theta_1) &= n_2\,. \quad (5.13)
\end{aligned}
$$

The new piece in (5.12) can be written as

$$
\begin{aligned}
P_2 &= \sum_{n_1>0}\sum_{n_2>n_1} e^{-l\cosh\theta_1 - l\cosh\theta_2} \equiv \frac{1}{8}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} e^{-l\cosh\theta_1 - l\cosh\theta_2} \\
&- \frac{1}{4}\sum_{n_1=-\infty}^{\infty} e^{-l\cosh\theta_1 - l} - \frac{1}{4}\sum_{n_1=-\infty}^{\infty} e^{-2l\cosh\theta_1} + \frac{3}{8}e^{-2l}\,. \quad (5.14)
\end{aligned}
$$

As $R\to\infty$ the continuous limit can be taken as:

$$
\begin{aligned}
\sum_{n_1>0}\sum_{n_2>n_1} e^{-l\cosh\theta_1 - l\cosh\theta_2} &\longrightarrow \frac{1}{8}\int_{\mathbb{R}^2} d\theta_1 d\theta_2 J_1^{(2)}(\theta_1,\theta_2) e^{-l\cosh\theta_1 - l\cosh\theta_2} \\
&-\frac{1}{4}\int_{\mathbb{R}} d\theta J_2^{(2)}(\theta) e^{-l\cosh\theta - l} - \frac{1}{4}\int_{\mathbb{R}} d\theta J_3^{(2)}(\theta) e^{-2l\cosh\theta} + \frac{3}{8}e^{-2l} \quad (5.15)
\end{aligned}
$$

The Jacobians $J_1^{(2)}(\theta_1,\theta_2)$, $J_2^{(2)}(\theta)$ and $J_3^{(2)}(\theta)$ can be calculated from the Bethe ansatz equations (5.13) as before. Notice that the correct subtractions of the excluded contributions (those excluded by the statistics) are crucial to get the corresponding Jacobians $J_2^{(2)}(\theta)$ and $J_3^{(2)}(\theta)$: one has to take the derivatives only after the forbidden values of the quantum numbers n_1 and n_2 are fixed. Although we have performed the calculation in full and checked the consistent cancellations of the r (strip size) dependent parts, for brevity we shall concentrate on the subleading, r-independent, parts $j_1^{(2)}$, $j_2^{(2)}$, $j_3^{(2)}$:

$$
\begin{aligned}
j_1^{(2)}(\theta_1,\theta_2) &= \phi_{\mathbb{1}}(\theta_1)\phi_{\mathbb{1}}(\theta_2) + 2\phi(\theta_1-\theta_2)\phi_{\mathbb{1}}(\theta_2) + 2\phi(\theta_1-\theta_2)\phi_{\mathbb{1}}(\theta_1) \\
&+ 4\phi(\theta_1+\theta_2)\phi(\theta_1-\theta_2)\,, \\
j_2^{(2)}(\theta) &= \phi_{\mathbb{1}}(\theta) + 2\phi(\theta)\,, \quad j_3^{(2)}(\theta) = \phi_{\mathbb{1}}(\theta) + 2\phi(2\theta)\,. \quad (5.16)
\end{aligned}
$$

Expanding the logarithm in (5.12), we have at second order

$$\ln(1+P_1+P_2+\ldots) = P_1 + \left(P_2 - \frac{P_1^2}{2}\right) + \ldots . \tag{5.17}$$

The up-to second order $2\ln g_{\mathbb{1}}$ contains seven distinct contributions, the first five coinciding with those written explicitly on the RHS of (5.11) and corresponding to the up-to-two particle expansion of $2[\ln g_{\mathbb{1}}]_D^{(1)}$. (This confirms the correctness of the dressed formulae (5.7, 5.8) up to this point). There are also two genuinely new terms, and we find:

$$\begin{aligned} 2\ln g_{\mathbb{1}} &= 2[\ln g_{\mathbb{1}}]_D^{(1)} + \frac{1}{2}\int_{\mathbb{R}^2} d\theta_1 d\theta_2\, \phi(\theta_1+\theta_2)\phi(\theta_1-\theta_2) e^{-l\cosh\theta_1 - l\cosh\theta_2} \\ &- \frac{1}{2}\int_{\mathbb{R}} d\theta\, \phi(2\theta) e^{-2l\cosh\theta} + \ldots , \end{aligned} \tag{5.18}$$

The final step is to iterate the dressing procedure, though in a modified form, by replacing $l\cosh\theta$ with $\varepsilon(\theta)$ and writing $\ln g_{\mathbb{1}} = [\ln g_{\mathbb{1}}]_D^{(1)} + [\ln g_{\mathbb{1}}]_D^{(2)} + \ldots$ with

$$2[\ln g_{\mathbb{1}}]_D^{(2)} = \frac{1}{2}\int_{\mathbb{R}^2} d\theta_1 d\theta_2\, \phi(\theta_1+\theta_2)\phi(\theta_1-\theta_2) e^{-\varepsilon(\theta_1)-\varepsilon(\theta_2)} - \frac{1}{2}\int_{\mathbb{R}} d\theta\, \phi(2\theta) e^{-2\varepsilon(\theta)} . \tag{5.19}$$

Again this dressing prescription can be justified retrospectively by testing at third and higher order. The third-order result turns out to support the assumption, and gives a genuinely new type of correction to $2\ln g_{\mathbb{1}}$, independent of $\phi_{\mathbb{1}}(\theta)$:

$$\begin{aligned} 2\ln g_{\mathbb{1}} &= 2[\ln g_{\mathbb{1}}]_D^{(1)} + 2[\ln g_{\mathbb{1}}]_D^{(2)} + \frac{2}{3}\int_{\mathbb{R}} d\theta\, \phi(2\theta) e^{-3l\cosh\theta} \\ &+ \frac{1}{3}\int_{\mathbb{R}^3} d\theta_1 d\theta_2 d\theta_3\, \phi(\theta_1+\theta_2)\phi(\theta_2-\theta_3)\phi(\theta_3-\theta_1) e^{-l\cosh\theta_1 - l\cosh\theta_2 - l\cosh\theta_3} \\ &- \int_{\mathbb{R}^2} d\theta_1 d\theta_2\, \phi(\theta_1+\theta_2)\phi(\theta_1-\theta_2) e^{-l\cosh\theta_1 - 2l\cosh\theta_2} + \ldots . \end{aligned} \tag{5.20}$$

5.3 The exact result

To go further in the expansion becomes increasingly difficult, due to higher number of Jacobians and the huge number of terms contributing to a single Jacobian. However we managed to complete the analysis up to four particles and to perform a more superficial inspection at higher orders. The following results were deduced: at each order there is always a new contribution of the form

$$C_n = \frac{1}{n}\int_{\mathbb{R}^n} d\theta_1 \ldots d\theta_n\, \phi(\theta_1+\theta_2)\phi(\theta_2-\theta_3)\ldots\phi(\theta_n-\theta_1) e^{-l\cosh\theta_1} \ldots e^{-l\cosh\theta_n} , \tag{5.21}$$

when the lower order terms $C_2, \ldots, C_{n-1}$ are corrected according to the rule

$$e^{-l\cosh\theta} \to \frac{1}{1+e^{\varepsilon(\theta)}} . \tag{5.22}$$

An additional term contains a single integration over the function $\phi(2\theta)$, as already seen in (5.18) and (5.20). To treat these terms is trickier, but after a few attempts we convinced ourselves that the correct procedure is always to replace $l\cosh\theta$ with $\varepsilon(\theta)$, and then to resum, as:

$$\int_{\mathbb{R}} d\theta\,\phi(2\theta)\left(-\frac{1}{2}e^{-2l\cosh\theta}+\dots\right)\to-\int_{\mathbb{R}} d\theta\,\phi(2\theta)\Big(\ln(1+e^{-\varepsilon(\theta)})-\frac{1}{1+e^{\varepsilon(\theta)}}\Big)\,. \quad (5.23)$$

We finally arrive at

$$\begin{aligned} 2\ln g_{\mathbb{1}}(l) &= \frac{1}{2}\int_{\mathbb{R}} d\theta\,(\phi_{\mathbb{1}}(\theta)-\delta(\theta)-2\phi(2\theta))\ln(1+e^{-\varepsilon(\theta)}) \\ &+ \sum_{n=1}^{\infty}\frac{1}{n}\int_{\mathbb{R}^n}\frac{d\theta_1}{1+e^{\varepsilon(\theta_1)}}\cdots\frac{d\theta_n}{1+e^{\varepsilon(\theta_n)}}\,\phi(\theta_1+\theta_2)\phi(\theta_2-\theta_3)\dots\phi(\theta_n-\theta_{n+1})\,, \end{aligned} \quad (5.24)$$

with $\theta_{n+1}=\theta_1$ ‡. It is not hard to check that this formula is consistent with the one-, two-, and three- particle results described above. Another simple (but nontrivial) check can be performed in the ultraviolet, as

$$e^{\varepsilon(\theta)}\to e^{\varepsilon_0}=\frac{\sqrt{5}+1}{2}\,,\qquad \int_{\mathbb{R}} d\theta\,\phi_{\mathbb{1}}(\theta)=-2\,, \quad (5.25)$$

$$\int_{\mathbb{R}} d\theta\,\phi(\theta)=-1\,,\qquad \int_{\mathbb{R}^n} d\theta_1\dots d\theta_n\,\phi(\theta_1+\theta_2)\phi(\theta_2-\theta_3)\dots\phi(\theta_n-\theta_1)=\frac{(-1)^n}{2}\,, \quad (5.26)$$

and (5.24) reduces to

$$\begin{aligned} -\ln(1+e^{-\varepsilon_0}) &+ \sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\Big(\frac{1}{1+e^{\varepsilon_0}}\Big)^n = -\ln(1+e^{-\varepsilon_0})-\frac{1}{2}\ln\Big(1+\frac{1}{1+e^{\varepsilon_0}}\Big) \\ &= \frac{1}{2}\ln\Big(\frac{\sqrt{5}-1}{2\sqrt{5}}\Big)\,, \end{aligned} \quad (5.27)$$

which is the expected $l=0$ value for $2\ln g_{\mathbb{1}}$ given in (4.5).

The first term on the RHS of (5.24) coincides with the proposal of [10], repeated in equations (3.1) and (3.2) above, while the remaining parts constitute a boundary-condition independent correction, which only comes nontrivially into play when the bulk theory is massive. As mentioned at the end of section 3, this was only to be expected given the results of [13], but it is nevertheless satisfying that the already-verified portions of the earlier results have been recovered by this rather different route.

In figure 9 the result from equation (5.24) is compared with the 'exact' result obtained in section 4 by combining UV perturbed CFT results with the IR cluster expansion from the Bethe ansatz. We see that keeping only the first three terms in the series

‡Notice the sequence of signs $+,-,-,\dots,-$ in the arguments of the ϕs in (5.24), that the number of ϕ factors in the $n^{\rm th}$ correction term is n and that according to (5.23) at $n=1$ only $\phi(2\theta)$ survives.

already gives a very good agreement with the exact result: in the UV the agreement is within about 0.3%.

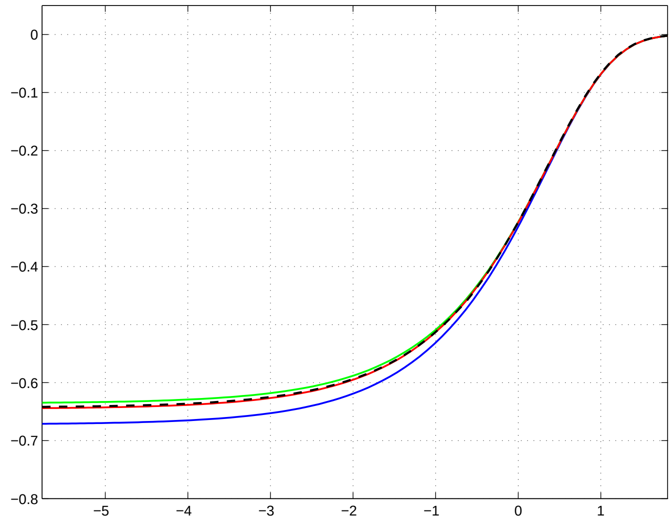

Figure 9: 2 ln $g_{\mathbb{1}}$ vs. ln(l). The dotted line corresponds to the 'exact' contribution. The bottom solid line represents the RHS of (5.24) with the sum truncated at the first term. The solid line just above the dotted line truncates the sum at the doubly-integrated term, and the line just below the dotted line is the total contribution up to the triple integral.

We also used (5.24) to make a numerical estimate of the coefficients of the ultraviolet expansion of $\ln g_{\mathbb{1}}(l)$:

$$\ln g_{\mathbb{1}}(l) \sim -0.3214826953191634 + 0.483692443734693\, x^{\frac{5}{12}} - 0.253117570\, x \\ + 0.0781176\, x^2 - 0.037284\, x^3 + 0.02042\, x^4 - 0.0120\, x^5 + 0.0074\, x^6 + \ldots \quad (5.28)$$

with $x = (l/\kappa)^{12/5}$. This can be compared with the result of [13]:

$$\ln g_{\mathbb{1}}(l) \sim -0.3214826953191634 + 0.4836924437346968\, x^{\frac{5}{12}} - 0.253117581\, x \\ + 0.0775\, x^2 - 0.036\, x^3 + 0.0195\, x^4 + \ldots \quad (5.29)$$

(Exact expressions for the first three coefficients in (5.29) were found in [13]; here we only quote sufficiently-many digits to enable the numerical errors in (5.28) to be assessed.)

By inspection, it is straightforward to generalize (5.24) to more general theories with N particle species and entirely diagonal scattering and reflection matrices. The proposal is

$$\begin{aligned} 2\ln g_\alpha(l) \;=\;& \frac{1}{2}\sum_{a=1}^{N}\int_{\mathbb{R}} d\theta\, \Big(\phi_\alpha^{(a)}(\theta) - \delta(\theta) - 2\phi_{aa}(2\theta)\Big) \ln(1+e^{-\varepsilon_a(\theta)}) \\ &+ \sum_{n=1}^{\infty}\sum_{a_1\ldots a_n=1}^{N} \frac{1}{n}\int_{\mathbb{R}^n} \frac{d\theta_1}{1+e^{\varepsilon_{a_1}(\theta_1)}}\cdots\frac{d\theta_n}{1+e^{\varepsilon_{a_n}(\theta_n)}} \times \\ &\quad \big(\phi_{a_1a_2}(\theta_1+\theta_2)\phi_{a_2a_3}(\theta_2-\theta_3)\ldots\phi_{a_na_{n+1}}(\theta_n-\theta_{n+1})\big)\;, \end{aligned} \quad (5.30)$$

where $\theta_{n+1} = \theta_1$, $a_{n+1} = a_1$, while $\phi_{ab}(\theta)$ and $\phi_\alpha^{(a)}$ are defined in section 3. Note, though, that in some circumstances extra terms may be needed for the correct analytic continuation of the integrals, as discussed in section 4.3.1 of [13].

Before concluding this section we would like to mention that while this project was in progress and some of the analytic results already obtained as they are written here, a preprint by Woynarovich appeared [15]. Our result (5.24) is similar in form to the expression proposed by Woynarovich for the $O(1)$ corrections to the free energy for a one dimensional Bose gas with repulsive δ-function interaction. However, there is also a major difference. The string of kernels

$$\phi(\theta_1 + \theta_2)\phi(\theta_2 - \theta_3)\dots\phi(\theta_n - \theta_{n+1})\,, \tag{5.31}$$

in our (5.24) is replaced by a string of the form

$$\psi(\theta_1, \theta_2)\psi(\theta_2, \theta_3)\dots\psi(\theta_n, \theta_{n+1})\,, \tag{5.32}$$

with $2\psi(k_1, k_2) = \phi(k_1 + k_2) + \phi(k_1 - k_2) \equiv \overline{K}(k_1, k_2)$ in eq. (3.28) of [15] §. That terms of the type (5.31) appear in our formulae and not the expression (5.32) is unmistakably emerging from the Jacobians for the change of variable $\{n_i\} \to \{\theta_i\}$ and from their definitions as determinants. Woynarovich obtained his result by a calculation of the next-to-leading contributions to the free energy, evaluating corrections to the standard saddle point result. Such a direct computation would be a highly desirable alternative to the more indirect approach taken in this paper. Unfortunately, as stated in the paragraph after eq. (5.8) in section V of [15], for the field theory case the result of [15] is divergent in the ultraviolet, $R = 1/T \to 0$, limit. This rules out the possibility of its consistent agreement with a g-function defined in (perturbed) conformal field theory, of the sort studied in [10, 13, 14] and this paper. Nevertheless, the mathematical similarity between the final outcomes is striking and deserves further investigation. To make a more precise comparison note that

$$2\log g_{\mathbb{1}}^{(\text{this paper})} \leftrightarrow \left[-T^{-1}(\Delta F + \phi_0 + \phi_L) + \overline{\Delta S}\ \right]^{(\text{ref. [15]})} \tag{5.33}$$

and that $-T^{-1}(\Delta F + \phi_0 + \phi_L)$ matches the first, single-integral, term on the RHS of (5.24). $\overline{\Delta S}$ should then be compared with the infinite series in (5.24). In eq. (5.8) of [15] Woynoarovich notes that his $\overline{\Delta S}$ can be written as a sum of two contributions: an UV convergent part corresponding to 1/2 of our infinite series, plus an UV divergent term which has no counterpart in (5.24). Thus, in spite of the apparent similarities between our results and those of [15], there are also important discrepancies, which we are currently unable to resolve physically.

§Notice that the kernels in the two cases are actually different, but the derivation in [15] is quite general, and the result is independent of the precise functional form of the kernel.

6 Conclusions

This paper concerned the off-critical version of the boundary entropy g as defined in field theory via the identity (1.6). It was shown numerically that the asymptotic infrared expansion for $\ln g$, obtained using the Bethe ansatz, matches UV results from conformal perturbation theory and the BTCSA at intermediate scales. This was a crucial step in the analysis, because it meant that these two alternative definitions are equivalent, and it opened up the interesting possibility of deriving an exact expression for the conformally-perturbed g-function by using the Bethe ansatz technique. The first step toward this was to give the exact prescription, as the width of the strip R tends to infinity, to transform sums over the quantum numbers into integrals on rapidity variables. The idea is that the subleading R-independent terms in this expansion should build up to form the boundary entropy g. A careful inspection of this expansion, motivated by the plausible assumption that the final result should depend on the bare single-particle energies only through their dressed versions, i.e. through the TBA pseudoenergies $\varepsilon(\theta)$, led to a partial resummation with corrections written purely in terms of $\varepsilon(\theta)$. The expressions for $\ln g$ written in (5.24) and in (5.30) are the main new results of the paper. Equation (5.24) was carefully checked against results obtained using a combination of conformal perturbation theory and the Bethe ansatz. The agreement was extremely good, and showed that the series is rapidly convergent even in the ultraviolet region (see equations (5.27), (5.28) and (5.29)). Although it relied at various points on conjectures, we would also like to stress that our derivation avoided some of the pitfalls that potentially afflict more direct computations of the g-function: by working in the $l \to \infty$ limit, we always dealt with states in which all constituent particles were well-separated, and so the accuracy of the Bethe ansatz wavefunctions for high particle density was not an issue.

There are many open problems related to this project, the first being that the method proposed, for all its virtues, is not particularly elegant and a direct approach would be desirable for the generalization to more complicated models. It would also be interesting to study the corresponding quantities in theories with non-diagonal scattering and in systems with massless excitations in the bulk [20].

Note added:

There is a numerical error in the third term of the expansion for $\ln g_{\mathbb{1}}$ given in eq. (5.29) above, which was pointed out to us by Aliosha Zamolodchikov. This is due to an inaccuracy in Mathematica's evaluation of the generalised hypergeometric function ${}_3F_2$, which arises in formula (3.14) of ref. [13] for the associated quantity I_2. In fact, Aliosha Zamolodchikov has found a simplified expression for I_2, as follows:

$$\begin{aligned} I_2 &= \frac{5}{8}\log 5 - \frac{5\sqrt{5}}{8}\log\frac{\sqrt{5}+1}{\sqrt{5}-1} + \frac{\pi}{4}\cot\frac{2\pi}{5} \\ &= -0.08393791256821845466150\ldots \end{aligned} \tag{6.1}$$

(this contrasts with the value $-0.083937990\ldots$ quoted in ref. [13]). Accordingly, the prediction (5.29) above should be corrected to

$$\begin{aligned}\ln g_{\mathbb{1}}(l) \sim & -0.3214826953191634 + 0.4836924437346968\, x^{\frac{5}{12}} \\ & - 0.253117570093371858\, x + 0.0775\, x^2 - 0.036\, x^3 + 0.0195\, x^4 + \dots \end{aligned} \tag{6.2}$$

with an even better match to (5.28). We would like to thank Aliosha Zamolodchikov for discussions of this point.

Acknowledgements

We are grateful to Z. Bajnok, L. Palla, G.M.T. Watts and Al.B. Zamolodchikov for many useful discussions. We would also like to thank F. Woynarovich for sending us an early version of [15].

DF thanks the Leverhulme Trust (grant F/00224/G) for a fellowship, and RT thanks the EPSRC for an Advanced Fellowship.

This work was partly supported by the EC network "EUCLID", contract number HPRN-CT-2002-00325, and partly by a NATO grant PST.CLG.980424. PED was also supported in part by JSPS/Royal Society grant and by the Daiwa Foundation, and the work by CR was partly supported by the Korea Research Foundation 2002-070-C00025.

PED, PED-DF-RT, and CR, thank SPhT Saclay, APCTP Korea, and the Universities of Durham and Torino, respectively, for hospitality while this project was in progress.

A TBA and BA in purely elastic scattering models

In this appendix we summarise the equations relevant to our analysis.

A.1 Periodic boundary conditions:

The thermodynamic Bethe ansatz equations are [16]

$$\epsilon_a(\theta) = M_a L \cosh\theta - \sum_{b=1}^{N} \int_{\mathbb{R}} d\theta'\, \phi_{ab}(\theta - \theta')\, L_b(\theta')\,, \qquad (a = 1, \dots, N). \tag{A.1}$$

The ground state energy on a circle is expressed in terms of the functions $L_a(\theta) = \ln(1 + e^{-\varepsilon_a(\theta)})$ as

$$E_0^{\rm circ}(M, L) = -\sum_{a=1}^{N} \int_{\mathbb{R}} \frac{d\theta}{2\pi}\, M_a \cosh\theta\, L_a(\theta) + \mathcal{E} M_1^2\, L \tag{A.2}$$

where $\mathcal{E} M_1^2\, L$ is the bulk contribution to the energy and

$$\phi_{ab}(\theta) = -\frac{i}{2\pi}\frac{d}{d\theta} \ln S_{ab}(\theta)\,. \tag{A.3}$$

A.2 (α, β) boundary conditions:

The (R-channel) thermodynamic Bethe ansatz equations are [10]

$$\epsilon_a(\theta) = 2M_a R\cosh\theta - \ln\Big(R_\alpha^{(a)}(i\tfrac{\pi}{2}-\theta)R_\beta^{(a)}(i\tfrac{\pi}{2}+\theta)\Big) - \sum_{b=1}^{N}\int_{\mathbb{R}} d\theta'\phi_{ab}(\theta-\theta')\,L_b(\theta') \quad (A.4)$$

where $a = 1, \ldots, N$; the ground state energy on an interval of length R is then

$$E_0^{\text{strip}}(M,R) = -\sum_{a=1}^{N}\int_{\mathbb{R}}\frac{d\theta}{4\pi}M_a\cosh\theta\,L_a(\theta) + \mathcal{E}M_1^2\,R + f_\alpha + f_\beta\ , \qquad (A.5)$$

where the constant $\mathcal{E}$ is the same as in (A.2), f_α and f_β are R-independent contributions to the energy from the boundaries and $\{R_\alpha^{(a)}(\theta), R_\beta^{(a)}(\theta)\}$ are the reflection amplitudes corresponding to the two boundary conditions α and β. Generalisations of these equations govern the excited state energies $E_n^{\text{strip}}(M,R)$ [12]¶, but in the large R limit we are interested in, they reduce to simple (Bethe Ansatz) forms. Suppose that the n^{th} excited state is made up of $m = \sum_{a=1}^{N} m^{(a)}$ particles, $m^{(a)}$ being the number of particles of type a. Then

$$E_n^{\text{strip}}(M,R) - E_0^{\text{strip}}(M,R) = \sum_{a=1}^{N}\sum_{i=1}^{m^{(a)}} M_a\cosh\theta_i^{(a)} + O(e^{-RM})\,, \qquad (A.6)$$

where sums on the RHS with $m^{(a)} = 0$ are understood to be omitted, and the sets of numbers $\{\theta_i^{(a)}\}$ satisfy the Bethe ansatz equations

$$\begin{aligned} 2\pi n_i^{(a)} &= 2M_a R\sinh\theta_i^{(a)} - i\ln\Big(R_\alpha^{(a)}(\theta_i^{(a)})R_\beta^{(a)}(\theta_i^{(a)})\Big) \\ &- \sum_{b=1}^{N}\sum_{j\neq i} i\ln\Big(-S_{ab}(\theta_i^{(a)}+\theta_j^{(b)})\Big) - \sum_{b=1}^{N}\sum_{j\neq i} i\ln\Big(-S_{ab}(\theta_i^{(a)}-\theta_j^{(b)})\Big)\ . \quad (A.7)\end{aligned}$$

It is to be noted that the logarithmic branches in the Bethe ansatz equations cause some difficulties in the numerics. We impose the branch cut at $-\pi$ so that the function $-i\ln(RR)$ and any of the terms $-i\ln(-S)$ in (A.7) take values in the range $(-\pi, \pi]$. This choice renders the Bethe ansatz (A.7) fully anti-symmetric (a change of sign in any of the quantum numbers $n_i^{(a)} \to -n_i^{(a)}$ corresponds to a change $\theta_i^{(a)} \to -\theta_i^{(a)}$) and one can consistently restrict $\{n_i^{(a)}\}$ to strictly positive integers only.

¶Sometimes such generalisations are required even to describe the ground state correctly [12], but these cases will not concern us here.

B Some exact results for the boundary Ising model

The boundary scattering matrix for the free Majorana fermion is [3]

$$R_k(\theta) = i \tanh\left(\frac{i\pi}{4} - \frac{\theta}{2}\right) \frac{k - i\sinh\theta}{k + i\sinh\theta}\,, \tag{B.1}$$

where $k = 1 - \frac{h^2}{2M}$ and h is the boundary magnetic field. Therefore, we have

$$\phi_k(\theta) = \frac{1}{\pi}\left(\frac{1}{\cosh\theta} - \frac{4k\cosh\theta}{\cosh(2\theta) + 2k^2 - 1}\right)\,. \tag{B.2}$$

In this appendix we would like to report exact expressions for

$$\ln g_{\mathbf{free}} \equiv \ln g_{k=1}(l) = -\frac{1}{4}\int_{\mathbb{R}} d\theta \left(\delta(\theta) + \frac{1}{\pi\cosh\theta}\right)\ln(1 + e^{-l\cosh\theta}) \tag{B.3}$$

and

$$\ln g_{\mathbf{fixed}} \equiv \ln g_{k=-\infty}(l) = -\frac{1}{4}\int_{\mathbb{R}} d\theta \left(\delta(\theta) - \frac{1}{\pi\cosh\theta}\right)\ln(1 + e^{-l\cosh\theta})\,. \tag{B.4}$$

These are obtained from the following identity, which can be deduced by studying the monodromies of the integral (2.8) along the lines sketched in [21]:

$$\int_{\mathbb{R}} d\theta \frac{1}{\pi\cosh\theta}\ln\left(1 + e^{-x\cosh\theta}\right) = \ln(2) - \frac{x}{\pi}(1 + \ln(\pi/x) - \gamma_E) - 2S(x) \tag{B.5}$$

where $\gamma_E = 0.57721566\ldots$ is the Euler-Mascheroni constant and

$$S(x) = \sum_{n=1}^{\infty}\left(\ln\left[\frac{x + (2n-1)\pi - \sqrt{(2n-1)^2\pi^2 + x^2}}{x - (2n-1)\pi + \sqrt{(2n-1)^2\pi^2 + x^2}}\right] + \frac{x}{(2n-1)\pi}\right)\,. \tag{B.6}$$

The idea of the derivation is to determine the positions of the singularities in (B.6) using the pinched-contour argument of [21], adding counterterms to make the infinite sum $S(x)$ convergent. The remaining parts of (B.5) were then fixed by studying the $x \to 0$ limit. (Alternatively, (B.5) can be proved using the Bessel-function technique of [22, 23] and an identity due to Schlömilch [24].) From (B.5), we have

$$4\ln g_{\mathbf{free/fixed}}(l) = -\ln(1 + e^{-l}) \mp \left(\ln(2) - \frac{l}{\pi}(1 + \ln(\pi/l) - \gamma_E) - 2S(l)\right)\,. \tag{B.7}$$

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