RATIONALIZABLE IMPLEMENTATION

By

Dirk Bergemann and Stephen Morris

May 2009

COWLES FOUNDATION DISCUSSION PAPER NO. 1697

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

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Dirk Bergemann† Stephen Morris‡

First Version: October 2008
This Version: May 2009

Abstract

This note studies (full) implementation of social choice functions under complete information in (correlated) rationalizable strategies. The monotonicity condition shown by Maskin (1999) to be necessary for Nash implementation is also necessary under the more stringent solution concept. We show that it is also sufficient under a mild "no worst alternative" condition. In particular, no economic condition is required.

KEYWORDS: Implementation, Complete Information, Rationalizability, Maskin Monotonicity.
JEL CLASSIFICATION: C79, D82

*Conversations with Olivier Tercieux led as to examine the implications of our earlier work on robust implementation for rationalizable implementation. We are grateful to Navin Kartik and Roberto Serrano for comments on an earlier draft. This research is supported by NSF Grant #SES-0518929.

†Department of Economics, Yale University, New Haven, CT 06511, dirk.bergemann@yale.edu
‡Department of Economics, Princeton University, Princeton, NJ 08544, smorris@princeton.edu

1 Introduction

This note studies (full) implementation of social choice functions under complete information in (correlated) rationalizable strategies. The monotonicity condition shown by Maskin (1999) to be necessary for Nash implementation is also necessary under the more stringent solution concept. We show that it is also sufficient under a "no worst alternative" (NWA) condition. In particular, no economic condition is required.

We are able to obtain this strong result because - like much of the classical implementation literature - we allow infinite mechanisms (including "integer games"); and - unlike the classical implementation literature - we allow for stochastic mechanisms. Recent papers by Bochet (2007) and Benoit and Ok (2008) have shown extremely permissive results for Nash implementation allowing stochastic mechanisms.[1] We also allowed for stochastic mechanisms in our earlier work on robust implementation in general environments (Bergemann and Morris (2008)); the results in this note are obtained by adapting arguments in that work.[2]

Another approach in the classical implementation literature is to restrict attention to finite mechanisms (for finite environments), but to restrict attention to *pure* Nash equilibria in the definition of implementation. In this case, "modulo" games can play the same role as integer games. We consider rationalizable implementation in finite mechanisms. With finite mechanisms rationalizable strategies are exactly those surviving iterated deletion of strictly dominated strategies. With finite mechanisms (our results would hold more generally with "regular mechanisms" where best responses are well-defined), we identify necessary conditions for rationalizable implementation allowing stochastic mechanisms. In particular, a strict (and thus stronger) version of Maskin monotonicity is required. And strict (and thus stronger) versions of the necessary conditions of Moore and Repullo (1990) are also necessary. In this sense, there is no gain to allowing stochastic mechanisms in the absence of infinite mechanisms. These results parallel our robust implementation results in Bergemann and Morris (2008): the restriction to finite (or regular) mechanisms leads to stronger - strict - versions of the necessary conditions for unrestricted mechanisms.

Abreu and Matsushima (1992a) have shown that a mild economic condition is sufficient for virtual implementation in rationalizable strategies in finite mechanisms. We show that a mild condition, which we refer to the *common most preferred outcome property* is necessary. This establishes at least some economic condition is required in Abreu and Matsushima (1992a).

[1] Their results are discussed in section 3. Serrano and Vohra (2007) have shown how allowing stochastic implementing mechanisms can give rise to weak sufficient conditions for Bayesian implementation in *mixed strategy* Bayes Nash equilibrium.

[2] In Bergemann and Morris (2009a), we discuss analogous results for implementation in rationalizable strategies in standard incomplete information environments.

These results narrow (small) open questions in the literature. The existing literature shows that Maskin monotonicity is necessary for Nash implementation in any mechanism (even if stochastic mechanisms are allowed). Abreu and Matsushima (1992a) shows that if implementation is made easier by (i) requiring only virtual implementation; and (ii) imposing a weak domain restriction ruling out identical preferences; then implementation is always possible even if it is made harder by (iii) requiring finite mechanisms; and (iv) requiring the stronger solution concept of rationalizability. Our result shows that it is possible to exactly implement a social choice function, in rationalizable strategies, even if domain restriction (ii) fails, as long as infinite, stochastic, mechanisms are allowed.

2 Setup

The *environment* consists of a collection of I agents (we also write I for the set of agents); a finite set of possible states Θ; a countable set of pure allocations Z (we write $Y \equiv \Delta(Z)$ for the set of lotteries on Z); and, for each agent i, a von Neumann-Morgenstern utility function $u_i : Z \times \Theta \to \mathbb{R}$, extended to lotteries as $u_i : Y \times \Theta \to \mathbb{R}$ with

$$u_i(y, \theta) = \sum_{z \in Z} y_z u_i(z, \theta).$$

A *mechanism* $\mathcal{M}$ is given by $\mathcal{M} = \left((M_i)_{i=1}^{I}, g\right)$, where each M_i is countable, $M = M_1 \times \cdots \times M_I$ and $g : M \to Y$.

The environment and the mechanism together describe a game of complete information for each $\theta \in \Theta$. We will use (correlated) rationalizability as a solution concept.[3] Our formal definition will coincide with the standard definition with finite or compact message spaces. But we will also allow infinite, non-compact, message spaces; in this case, our definition is equivalent to one introduced in Lipman (1994). Let a message set profile $S = (S_1, ..., S_I)$, where each

$$S_i \in 2^{M_i}, \tag{1}$$

and we write $\mathcal{S}$ for the collection of message set profiles. The collection $\mathcal{S}$ is a lattice with the natural ordering of set inclusion: $S \leq S'$ if $S_i \subseteq S'_i$ for all i. The largest element is $\overline{S} = (M_1, ..., M_I)$. The smallest element is $\underline{S} = (\varnothing, \varnothing, ..., \varnothing)$.

[3] The original definition of rationalizability of Bernheim (1984) and Pearce (1984) required agents' conjectures over their opponents' play to be independent. We follow the convention of some of the recent literature (e.g., Osborne and Rubinstein (1994)) in using "rationalizability" for the correlated version of rationalizability (see Brandenburger and Dekel (1987) for an early definition and discussion).

We define an operator $b^\theta : \mathcal{S} \to \mathcal{S}$ to iteratively eliminate never best responses with $b^\theta = \left(b_1^\theta, ...b_i^\theta, ..., b_I^\theta\right)$ and b_i^θ is defined by:

$$b_i^\theta (S) = \left\{ m_i \in M_i \left| \begin{array}{l} \text{there exists } \lambda_i \in \Delta (M_{-i}) \text{ such that} \\ (1)\ \lambda_i (m_{-i}) > 0 \Rightarrow m_j \in S_j \text{ for each } j \neq i, \\ (2)\ m_i \in \underset{m_i' \in M_i}{\arg\max} \sum\limits_{m_{-i} \in M_{-i}} u_i \left(g \left(m_i', m_{-i} \right), \theta \right) \lambda_i (m_{-i}), \end{array} \right. \right\}. \qquad (2)$$

We observe that b^θ is increasing by definition: i.e., $S \leq S' \Rightarrow b^\theta (S) \leq b^\theta (S')$. By Tarski's fixed point theorem, there is a largest fixed point of b^θ, which we label $S^{\mathcal{M},\theta}$. Thus (i) $b^\theta \left(S^{\mathcal{M},\theta}\right) = S^{\mathcal{M},\theta}$ and (ii) $b^\theta (S) = S \Rightarrow S \leq S^{\mathcal{M},\theta}$. If $m_i \in S_i^{\mathcal{M},\theta}$, we say that message m_i is rationalizable in the complete information game parameterized by θ.

We can also construct the fixed point $S^{\mathcal{M},\theta}$ by starting with $\overline{S}$ - the largest element of the lattice - and iteratively applying the operator b^θ. If the message sets are finite (as discussed in sections 4 and 5), we have

$$S_i^{\mathcal{M},\theta} \triangleq \bigcap_{n \geq 0} b_i^\theta \left(\left[b^\theta \right]^n \left(\overline{S} \right) \right).$$

In this case, the solution concept is equivalent to iterated deletion of strictly dominated strategies (see Brandenburger and Dekel (1987)).

But if the mechanism $\mathcal{M}$ is infinite, transfinite induction may be necessary to reach the fixed point.[4] It is useful to define

$$S_{i,k}^{\mathcal{M},\theta} \triangleq b_i^\theta \left(\left[b^\theta \right]^{k-1} \left(\overline{S} \right) \right),$$

again using transfinite induction if necessary. Thus $S_i^{\mathcal{M},\theta}$ are the set of messages surviving (transfinite) iterated deletion of never best responses. It is possible to show formally that $S_i^{\mathcal{M},\theta}$ is the set of messages that agent i might send consistent with common certainty of rationality and the fact that payoffs are given by θ (Lipman (1994)).

Now a *social choice function* (SCF) f is given by $f : \Theta \to Y$. Mechanism $\mathcal{M}$ *implements f in rationalizable strategies* if there exists $\mathcal{M}$ such that, for all θ, $S^{\mathcal{M},\theta} \neq \varnothing$ and $m \in S^{\mathcal{M},\theta} \Rightarrow g(m) = f(\theta)$. SCF f is *implementable in rationalizable strategies* if there exists $\mathcal{M}$ such that $\mathcal{M}$ implements f in rationalizable strategies. Mechanism $\mathcal{M}$ *ε-implements f in rationalizable strategies* if there exists $\mathcal{M}$ such that, for all θ, $S^{\mathcal{M},\theta} \neq \varnothing$ and $m \in S^{\mathcal{M},\theta} \Rightarrow \|g(m) - f(\theta)\| \leq \varepsilon$. SCF f is *ε-implementable in rationalizable strategies* if there exists a mechanism $\mathcal{M}$ such that $\mathcal{M}$ ε-implements f in rationalizable strategies. SCF f is *virtually implementable in rationalizable strategies* if, for each $\varepsilon > 0$, f is ε-implementable in rationalizable strategies.

[4] Lipman (1994) contains a formal description of the transfinite induction required. As he notes "we remove strategies which are never a best reply, taking limits where needed".

3 Exact Implementation in General Mechanisms

Restricted to social choice functions, Maskin monotonicity states:

Definition 1 (Maskin monotonicity)
Social choice function f satisfies Maskin monotonicity if

1. $f(\theta) = f(\theta')$ *whenever* $u_i(f(\theta), \theta) \geq u_i(y, \theta) \Rightarrow u_i(f(\theta), \theta') \geq u_i(y, \theta')$ *for all i and y; or, equivalently,*

2. $f(\theta) \neq f(\theta')$ *implies there exist i and $y \in Y$ with* $u_i(f(\theta), \theta) \geq u_i(y, \theta)$ *and* $u_i(y, \theta') > u_i(f(\theta), \theta')$.

This condition, which is necessary for Nash implementation, is also necessary for rationalizable implementation. The latter condition states that there exists at least one agent who, if the true state were θ' and she expected other agents to claim the state is θ, could be offered a reward y that would give her a strict incentive to "report" the deviation of other agents, where the reward y would not tempt her if the true state was in fact θ.

Proposition 1 *If f is implementable in rationalizable strategies, then f is Maskin monotonic.*

Proof. Let $\mathcal{M}$ be a mechanism that implements f in rationalizable strategies. Thus, for each θ, there exists $m_i^*(\theta) \in S_i^{\mathcal{M},\theta}$ for each i satisfying $g(m^*(\theta)) = f(\theta)$. Suppose $f(\theta') \neq f(\theta)$. Then there exists k (perhaps transfinite) such that

$$m_i^*(\theta') \in S_{i,k}^{\mathcal{M},\theta}$$

for each i but

$$m_i^*(\theta') \notin S_{i,k+1}^{\mathcal{M},\theta}(\theta)$$

for some i. To see why, observe first that $m_i^*(\theta') \in S_{i,0}^{\mathcal{M},\theta} = M_i$ for all i. Now if $m_i^*(\theta') \in S_i^{\mathcal{M},\theta}$ for all i, we would have $m^*(\theta') \in S^{\mathcal{M},\theta}$ and thus $f(\theta') = f(\theta)$, a contradiction.

Thus there exists $\widetilde{m}_i$ such that

$$u_i(g(\widetilde{m}_i, m_{-i}^*(\theta')), \theta) > u_i(g(m_i^*(\theta'), m_{-i}^*(\theta')), \theta) = u_i(f(\theta'), \theta).$$

Suppose that $u_i(g(\widetilde{m}_i, m_{-i}^*(\theta')), \theta') > u_i(g(m^*(\theta')), \theta') = u_i(f(\theta'), \theta')$. This contradicts $m_i^*(\theta') \in S_i^{\mathcal{M},\theta'}$. So we must have $u_i(g(\widetilde{m}_i, m_{-i}^*(\theta')), \theta') \leq u_i(f(\theta'), \theta')$.
Writing $y \triangleq g(\widetilde{m}_i, m_{-i}^*(\theta'))$, we have $u_i(y, \theta) > u_i(f(\theta'), \theta)$ and $u_i(f(\theta'), \theta') \geq u_i(y, \theta')$. ■

Oury and Tercieux (2009) have shown that Maskin monotonicity is a necessary condition for "continuous" partial implementation of a social choice function, where "continuous" means the small mechanism must work for types that are close to the complete information types in the product topology. An alternative way to prove their result would be show that "continuous" partial implementation implies full implementation in rationalizable strategies and thus (by proposition 1) the necessity of Maskin monotonicity.

Proposition 1 parallels part (1) of Theorem 1 in Bergemann and Morris (2008). We need an extra condition for the sufficiency result.

Definition 2
Social choice function f satisfies "no worst alternative" (NWA) if, for each i and θ, there exists $\underline{y}_i(\theta)$ such that

$$u_i(f(\theta),\theta) > u_i\left(\underline{y}_i(\theta),\theta\right). \tag{3}$$

Property NWA requires that an agent never get his worst outcome under the social choice function. The NWA property appears in Cabrales and Serrano (2008) as a sufficient condition to guarantee implementation in best-response dynamics. Given a set of allocations $\left\{\underline{y}_i(\theta)\right\}_{\theta\in\Theta}$ it is useful to define the average allocation $\underline{y}_i$ of this set by setting

$$\underline{y}_i \triangleq \frac{1}{\#\Theta}\sum_{\theta\in\Theta}\underline{y}_i(\theta). \tag{4}$$

We now construct an auxiliary set of allocations, denoted by $\{z_i(\theta,\theta')\}_{\theta,\theta'}$, which use the existence of the allocations $\left\{\underline{y}_i(\theta)\right\}_{\theta\in\Theta}$. The allocations $\{z_i(\theta,\theta')\}_{\theta,\theta'}$ are going to appear in the canonical mechanism to be defined shortly where they guarantee the existence of better response for agent i should the remaining agents choose to misreport the true state. In particular, the following Lemma establishes that for agent i the allocation $z_i(\theta,\theta')$ represents an improvement if the true state is θ but the other agents misreport it to be θ'. It also establishes that $z_i(\theta,\theta')$ would not constitute an improvement relative to $f(\theta')$ if the true state were indeed θ'.

Lemma 1
If social choice function f satisfies "no worst alternative" (NWA) then there exists $\{z_i(\theta,\theta')\}_{\theta,\theta'}$ such that for all θ,θ' with $\theta\neq\theta'$:

$$u_i(f(\theta'),\theta') > u_i(z_i(\theta,\theta'),\theta'), \tag{5}$$

and

$$u_i(z_i(\theta,\theta'),\theta) > u_i(z_i(\theta',\theta'),\theta). \tag{6}$$

Proof. We begin with the allocations $\left\{\underline{y}_i\left(\theta\right)\right\}_{\theta\in\Theta}$ given by Definition 2. First note that we can assume without loss of generality that, for all θ and θ', we have

$$u_i\left(f\left(\theta\right),\theta'\right) \geq u_i\left(\underline{y}_i\left(\theta'\right),\theta'\right) \tag{7}$$

and

$$u_i\left(\underline{y}_i\left(\theta\right),\theta'\right) \geq u_i\left(\underline{y}_i\left(\theta'\right),\theta'\right). \tag{8}$$

If this were not true, we could define the set

$$\underline{Y}_i \triangleq \left\{f\left(\theta\right)\right\}_{\theta\in\Theta} \cup \left\{\underline{y}_i\left(\theta\right)\right\}_{\theta\in\Theta}$$

and define a new allocation $\underline{\underline{y}}_i\left(\theta\right)$ by setting

$$\underline{\underline{y}}_i\left(\theta\right) \in \underset{y\in\underline{Y}_i}{\arg\min} u_i\left(y,\theta\right). \tag{9}$$

Now, we have for all $\theta\in\Theta$:

$$u_i\left(f\left(\theta\right),\theta\right) > u_i\left(\underline{y}_i\left(\theta\right),\theta\right) \geq u_i\left(\underline{\underline{y}}_i\left(\theta\right),\theta\right),$$

where the strict inequality follows from NWA and the weak inequality follows from construction of (9). Without loss of generality we can therefore take the allocation $\underline{y}_i\left(\theta\right)$ to coincide with $\underline{\underline{y}}_i\left(\theta\right)$, and it follows that the inequalities (7) and (8) are satisfied.

Based on the allocations $\left\{\underline{y}_i\left(\theta\right)\right\}_{\theta\in\Theta}$ we define a new set of allocations for all θ:

$$z_i\left(\theta',\theta'\right) \triangleq \left(1-\varepsilon\right)\underline{y}_i\left(\theta'\right) + \varepsilon\underline{y}_i,$$

with $\underline{y}_i$ as defined in (4), and for all θ,θ' with $\theta\neq\theta'$:

$$z_i\left(\theta,\theta'\right) \triangleq \left(1-\varepsilon\right)\underline{y}_i\left(\theta'\right) + \varepsilon\left(\underline{y}_i + \frac{1}{\#\Theta}\left(f\left(\theta\right) - \underline{y}_i\left(\theta\right)\right)\right). \tag{10}$$

By NWA, the weak inequalities (8) and the finiteness of the state space Θ, we can find a sufficiently small, but positive, $\varepsilon > 0$ such that for all θ and θ':

$$u_i\left(f\left(\theta'\right),\theta'\right) > u_i\left(z_i\left(\theta,\theta'\right),\theta'\right). \tag{11}$$

Now we observe that that the only difference between $z_i\left(\theta',\theta'\right)$ and $z_i\left(\theta,\theta'\right)$ is the fact that the lottery $\underline{y}_i\left(\theta\right)$ is replaced by the lottery $f\left(\theta\right)$. But now by NWA, this is clearly increasing the expected utility of agent i in state θ, and hence we have for all θ,θ' with $\theta\neq\theta'$:

$$u_i\left(z_i\left(\theta,\theta'\right),\theta\right) > u_i\left(z_i\left(\theta',\theta'\right),\theta\right), \tag{12}$$

which establishes the second strict inequality (6). ■

We establish the sufficient conditions for implementation in rationalizable strategies by means of a canonical mechanism. The canonical mechanism shares many basic features with the implementation mechanism suggested by Maskin and Sjostrom (2004) to establish complete information implementation in the presence of mixed strategies, and is modification of the original mechanism suggested by Maskin (1999). The aforementioned allocations $\left\{z_i\left(\theta,\theta'\right)\right\}_{\theta,\theta'}$ appear in the mechanism if agent i reports a state θ different from the reported state θ' by all the other agents. In this case, the allocation $z_i\left(\theta',\theta'\right)$ is chosen with positive probability, yet this probability can lowered by a suitable message of agent i and be replaced by favorable allocation $z_i\left(\theta,\theta'\right)$.

Proposition 2 *If $I \geq 3$ and f satisfies Maskin monotonicity and NWA, then f is implementable in rationalizable strategies.*

Proof. We prove the proposition by constructing an implementing mechanism $\mathcal{M} = (M, g)$.

Each agent i sends a message $m_i = \left(m_i^1, m_i^2, m_i^3, m_i^4\right)$, where $m_i^1 \in \Theta$, $m_i^2 \in \mathbb{Z}_+$, $m_i^3 : \Theta \to Y, m_i^4 \in Y$. The third component of the message profile will allow agent i to suggest an allocation $m_i^3(\theta)$ contingent on all the other agents $j \neq i$ reporting $m_j^1 = \theta$. The outcome function will make use of the "uniformly worse outcome" for agent i defined earlier by $\underline{y}_i$. Note that for each $\theta \in \Theta$ - by Lemma 1 - there exists $y \in Y$, such that

$$u_i(y,\theta) > u_i(z_i(\theta,\theta),\theta). \tag{13}$$

Now the outcome $g(m)$ is determined by the following rules:

Rule 1: If $m_i^1 = \theta$ and $m_i^2 = 1$ for all i, pick $f(\theta)$.
Rule 2: If there exists $i \in I$ such that $\left(m_j^1, m_j^2\right) = (\theta, 1)$ for all $j \neq i$ and $\left(m_i^1, m_i^2\right) \neq (\theta, 1)$, then we go to two subrules:
(i): if $u_i(f(\theta),\theta) \geq u_i\left(m_i^3(\theta),\theta\right)$, pick $m_i^3(\theta)$ with probability $1 - \frac{1}{m_i^2+1}$ and $z_i(\theta,\theta)$ with probability $\frac{1}{m_i^2+1}$;
(ii): if $u_i(f(\theta),\theta) < u_i\left(m_i^3(\theta),\theta\right)$, pick $z_i(\theta,\theta)$ with probability 1. The allocation $z_i(\theta,\theta)$ was established in Lemma 1.
Rule 3: In all other cases, we identify a pivotal agent i by requiring that $m_i^2 \geq m_j^2$ for all $j \in I$ and that if $m_i^2 = m_j^2$, then $i < j$. The rule then requires that with probability $\left(1 - \frac{1}{m_i^2+1}\right)$ we pick m_i^4, and with probability $\frac{1}{I}\left(\frac{1}{m_i^2+1}\right)$ we pick $\underline{y}_i$.

Claim 1. It is never a best reply for agent i to send a message with $m_i^2 > 1$ (i.e., $m_i \in b_i^\theta\left(\overline{S}\right) \Rightarrow m_i^2 = 1$).

Proof for Claim 1. Suppose $m_i = \left(m_i^1, m_i^2, m_i^3, m_i^4\right) \in S_i^{\mathcal{M},\theta}$ and $m_i^2 > 1$. Then Rule 2 or 3 will be triggered with probability one. But in this case, whatever agent i's beliefs $\lambda_i\left(m_{-i}\right)$ about the other agents' messages, his payoff can be increased by modifying m_i appropriately, in particular by increasing the integer choice from m_i^2. Given the message m_i with $m_i^2 > 1$, the set of messages of the remaining agents in which Rule 2 is triggered is defined by:

$$M_{-i}^2 \triangleq \left\{m_{-i} \in M_{-i} \left| m_j^1 = \theta' \text{ and } m_j^2 = 1 \text{ for some } \theta' \text{ and for all } j \neq i \right.\right\}, \tag{14}$$

and the set of messages of the remaining agents in which Rule 3 is triggered is the complement set, defined by:

$$M_{-i}^3 \triangleq M_{-i} \setminus M_{-i}^2. \tag{15}$$

Suppose first that agent i has a belief $\lambda_i \in \Delta\left(M_{-i}\right)$ under which Rule 3 is triggered with positive probability, so that $\lambda_i\left(M_{-i}^3\right) > 0$. Then agent i can become the pivotal player and realize his most preferred allocation at profile θ with an arbitrarily large probability, though strictly below 1, through the choice a larger integer $\widehat{m}_i^2 > m_i^2$ and the choice of an allocation $\widehat{m}_i^4$. Recall from (13) that there exists $\widehat{m}_i^4 \in \Theta$ with $u_i\left(\widehat{m}_i^4, \theta\right) > u_i\left(\overline{y}_i, \theta\right)$. Now agent i can always improve his expected utility conditional on Rule 3 by choosing an even larger integer.

Now suppose that agent i believes that Rule 2 will be triggered with positive probability, so that $\lambda_i\left(M_{-i}^2\right) > 0$. We first observe that the choice of $\widehat{m}_i^4$ does not effect the outcome of mechanism conditional on Rule 2. Now we know that there exists $\widehat{m}_i^3$ such that:

$$\sum_{\theta' \in \Theta}\left(\sum_{\left\{m_{-i} \in M_{-i}^2 \left| m_j^1 = \theta' \text{for all } j \neq i\right.\right\}} \lambda_i\left(m_{-i}\right)\right)\left(u_i\left(\widehat{m}_i^3\left(\theta'\right), \theta\right) - u_i\left(z_i\left(\theta', \theta'\right), \theta\right)\right) > 0. \tag{16}$$

It follows that the choice of a large integer $\widehat{m}_i^2 > m_i^2$ together with the existence of an allocation $\widehat{m}_i^3\left(\theta\right)$ satisfying (16) strictly improves the expected utility of agent i in case that Rule 2 is triggered, which yields the desired contradiction.

Claim 2. $\left(\theta, 1, m_i^3, m_i^4\right) \in S_i^{\mathcal{M},\theta}$ for all i, θ, m_i^3, m_i^4.

Proof of Claim 2. Suppose that player i in state θ puts probability 1 on each other agent j sending a message of the form $\left(\theta, 1, m_j^3, m_j^4\right)$. If player i announces a message of the form $\left(\theta, 1, m_i^3, m_i^4\right)$, he gets payoff $u_i\left(f\left(\theta\right), \theta\right)$. If he announces a message not of this form, the outcome is determined by Rule 2. But his payoff from invoking Rule 2 is bounded above by $u_i\left(f\left(\theta\right), \theta\right)$.

Claim 3. If $m_i = \left(\theta', 1, m_i^3, m_i^4\right) \in S_i^{\mathcal{M},\theta}$, then $f\left(\theta'\right) = f(\theta)$.

Proof of Claim 3. Suppose $m_i = \left(\theta', 1, m_i^3, m_i^4\right) \in S_i^{\mathcal{M},\theta}$. Given the message m_i, we can define the set of messages of the remaining agents which trigger Rule 1, 2 or 3, respectively. In particular, we define:

$$M_{-i}^1(m_i) \triangleq \left\{ m_{-i} \in M_{-i} \left| \left(m_j^1, m_j^2\right) = \left(m_i^1, m_i^2\right) \text{ for all } j \neq i \right. \right\}, \tag{17}$$

and $M_{-i}^2(m_i)$ and $M_{-i}^3(m_i)$ are as defined earlier in (14) and (15), respectively. Now consider a given belief $\lambda_i(m_{-i})$ of agent i. If $\lambda_i\left(\left\{m_{-i} \in M_{-i}^1(m_i)\right\}\right) = 0$, then Rule 2 or 3 will be triggered with probability one and by exactly the argument of Claim 1, the message m_i cannot be a best reply by agent i. Suppose now that the belief $\lambda_i(m_{-i})$ of agent i is such that:

$$0 < \lambda_i\left(\left\{m_{-i} \in M_{-i}^1(m_i)\right\}\right) < 1. \tag{18}$$

While we still argue that agent i can strictly increase his expected utility by selecting an integer $\widehat{m}_i^2 > 1$, we observe that a complication arises as with $\lambda_i(m_{-i})$ given by (18), a choice of $\widehat{m}_i^2 > 1$ leads from an allocation determined by Rule 1 to an allocation determined by Rule 2, and hence the realization of an unfavorable allocation $\underline{y}_i$ with positive probability. But now we observe that by selecting $\widehat{m}_i^3\left(\theta'\right) = f\left(\theta'\right)$, by selecting a favorable allocation $\widehat{m}_i^4$ and by choosing an integer $\widehat{m}_i^2$ sufficiently large, the small loss in Rule 2 can always be offset by a gain in Rule 3 relative to the allocation achieved under $g(m_i, m_{-i})$.

So if $m_i = \left(\theta', 1, m_i^3, m_i^4\right) \in S_i^{\mathcal{M},\theta}$, it follows player i must be convinced that each other player must be choosing a message of the form $\left(\theta', 1, m_j^3, m_j^4\right)$, and hence

$$\lambda_i\left(\left\{m_{-i} \in M_{-i}^1(m_i)\right\}\right) = 1.$$

Thus there must exist a message of the form $\left(\theta', 1, m_j^3, m_j^4\right) \in S_j^{\mathcal{M},\theta}$ for all j. But suppose that for some j, there existed y such that $u_j\left(f\left(\theta'\right), \theta'\right) \geq u_j\left(y, \theta'\right)$ but $u_j(y, \theta) > u_j\left(f\left(\theta'\right), \theta\right)$. Then $\left(\theta', 2, y, y\right)$ would be a better response for player j than $\left(\theta', 1, m_j^3, m_j^4\right)$, a contradiction. Thus for all j, $u_j\left(f\left(\theta'\right), \theta'\right) \geq u_j\left(y, \theta'\right) \Rightarrow u_j\left(f\left(\theta'\right), \theta\right) \geq u_j(y, \theta)$. Thus by Maskin monotonicity, $f\left(\theta'\right) = f(\theta)$.

Completion of proof. Claims 1, 2 and 3 together imply that $m_i \in S_i^{\mathcal{M},\theta} \Rightarrow m_i^2 = 1$ and $f\left(m_i^1\right) = f(\theta)$. Thus $m \in S^{\mathcal{M},\theta} \Rightarrow g(m) = f(\theta)$. ■

This proposition parallels part 2 of Theorem 1 in Bergemann and Morris (2008). The mechanism $\mathcal{M}$ used here allows each agent to propose a menu of choices $m_i^3 = \left\{m_i^3(\theta)\right\}_{\theta \in \Theta}$. The menu m_i^3 gives agent i the opportunity to select an appropriate allocation in case that Rule 2 is triggered. Maskin (1999) and Maskin and Sjostrom (2004) have used the menu structure to establish complete information implementation in the presence of mixed strategies. In our sufficiency argument, the

NWA property replaces the no veto property which commonly appears in the sufficiency argument for implementation in Nash equilibrium. Yet, in terms of the proof, the role of the NWA property is quite distinct from the no veto property. The NWA property guarantees that in the augmented mechanism, any report in state θ in which an agent expresses his disagreement with the remaining agents (i.e. $m_i^2 > 1$) cannot be a rationalizable report. By contrast, the no veto property guaranteed that if an agent were to express his disagreement, then further disagreement by other agents would only be possible in equilibrium if it would lead to the same equilibrium allocation as prescribed $f(\theta)$.

Bochet (2007) and Benoit and Ok (2008) report sufficient conditions for implementation in Nash equilibrium strategies using stochastic mechanisms.[5] Restricted to our environment (with social choice *functions* and an outcome space given by the set of lotteries over a finite set of outcomes), their results can be stated as follows. The *top strict difference* condition is satisfied if whenever there exist i and $z^* \in Z$ with $u_j(z^*, \theta) \geq u_j(z, \theta)$ for all $j \neq i$ and all $z \in Z$, there exist agents k and l with $u_k(z^*, \theta) > u_k(z, \theta)$ and $u_l(z^*, \theta) > u_l(z, \theta)$ for all $z \neq z^*$. Bochet (2007) shows that if the top strict difference condition holds and $I \geq 3$, then f is Nash implementable if and only if f is Maskin monotonic (on lotteries). The *top coincidence* condition is satisfied for every i, if there exists at most one $z^* \in Z$ satisfying $u_j(z^*, \theta) \geq u_j(z, \theta)$ for all $j \neq i$ and $z \in Z$. Social choice function f is *weakly unanimous* if $\{z^*\} = \bigcap_{i=1}^{I} \{z' : u_i(z', \theta) \geq u_i(z, \theta) \text{ for all } z \in Z\} \Rightarrow f(\theta) = z^*$. Benoit and Ok (2008) shows that if the top coincidence condition holds and f is weakly unanimous, then f is implementable in pure Nash if and only if f is Maskin monotonic (on deterministic outcomes). Our NWA condition does not imply nor is it implied by any of these extra conditions required for sufficiency.

4 Exact Implementation in Finite Mechanisms

We do not have sufficient conditions for rationalizable implementation in finite mechanisms. However, we can report stronger necessary conditions than those for Nash implementation using infinite mechanisms.

Cabrales and Serrano (2008) introduce a "quasimonotonicity" condition which is necessary and almost sufficient for implementation in adaptive best response dynamics.

[5] While we assume agents have expected utility preferences over lotteries, Bochet (2007) and Benoit and Ok (2008) make weaker monotonicity assumptions on the extension of ordinal preferences to lotteries.

Definition 3 (Quasimonotonicity)
Social choice function f satisfies quasimonotonicity if

1. $f(\theta) = f(\theta')$ *whenever* $u_i(f(\theta), \theta) > u_i(y, \theta) \Rightarrow u_i(f(\theta), \theta') > u_i(y, \theta')$ *for all i and y; or, equivalently,*

2. $f(\theta) \neq f(\theta')$ *implies there exist i and $y \in Y$ with* $u_i(f(\theta), \theta) > u_i(y, \theta)$ *and* $u_i(y, \theta') \geq u_i(f(\theta), \theta')$.

The premise of quasimonotonicity is nestedness of the strict lower contour sets and the premise of Maskin monotonicity is nestedness of the (weak) lower contour sets. Thus neither implies the other. We will identify a "strict Maskin monotonicity" condition whose premise is that strict lower contour sets are contained in (weak) lower contour sets. Thus strict Maskin monotonicity is - at least in principle - a strengthening of both Maskin monotonicity and quasimonotonicity:

Definition 4 (Strict Maskin Monotonicity)
Social choice function f satisfies strict Maskin monotonicity if

1. $f(\theta) = f(\theta')$ *whenever* $u_i(f(\theta), \theta) > u_i(y, \theta) \Rightarrow u_i(f(\theta), \theta') \geq u_i(y, \theta')$ *for all i and y; or, equivalently,*

2. $f(\theta) \neq f(\theta')$ *implies there exist i and $y \in Y$ with* $u_i(f(\theta), \theta) > u_i(y, \theta)$ *and* $u_i(y, \theta') > u_i(f(\theta), \theta')$.

In our lottery setting, all three versions of monotonicity will be equal under weak conditions:

Definition 5 (No Best Alternative)
Social choice function f satisfies "no best alternative" (NBA) if, for each i and θ, there exists y^* such that $u_i(f(\theta), \theta) < u_i(y^*, \theta)$.

Property NBA is analogous to NWA, requiring that an agent never gets his favorite alternative under the social choice function.

Lemma 2

1. *Maskin monotonicity and NWA imply strict Maskin monotonicity;*

2. *Quasimonotonicity and NBA imply strict Maskin monotonicity.*

Proof. (1.) Suppose f is Maskin monotonic. Suppose $f(\theta') \neq f(\theta)$. Then there exist i and y such that $u_i(f(\theta),\theta) \geq u_i(y,\theta)$ and $u_i(f(\theta),\theta') < u_i(y,\theta')$. By NWA, there exists y^* with $u_i(f(\theta),\theta) > u_i(y^*,\theta)$. Now let $\widetilde{y} = \varepsilon y^* + (1-\varepsilon)y$. For small enough ε, $u_i(f(\theta),\theta) > u_i(\widetilde{y},\theta)$ and $u_i(f(\theta),\theta') < u_i(\widetilde{y},\theta')$. Thus strict Maskin monotonicity is satisfied.

(2.) Suppose f is quasimonotonic. Suppose $f(\theta') \neq f(\theta)$. Then there exist i and y such that $u_i(f(\theta),\theta) > u_i(y,\theta)$ and $u_i(f(\theta),\theta') \leq u_i(y,\theta')$. By NBA, there exists y^* with $u_i(f(\theta),\theta') < u_i(y^*,\theta')$. Now let $\widetilde{y} = \varepsilon y^* + (1-\varepsilon)y$. For small enough ε, $u_i(f(\theta),\theta) > u_i(\widetilde{y},\theta)$ and $u_i(f(\theta),\theta') < u_i(\widetilde{y},\theta')$. Thus strict Maskin monotonicity is satisfied. ■

Now we have:

Proposition 3 *If f is finitely implementable in rationalizable strategies, then f satisfies strict Maskin monotonicity.*

Proof. Let $\mathcal{M}$ be any mechanism that implements f in rationalizable strategies. Thus, for each θ, there exists $m_i^*(\theta) \in S_i^{\mathcal{M},\theta}$ for each i satisfying

$$g(m^*(\theta)) = f(\theta).$$

Suppose $f(\theta') \neq f(\theta)$. Then we claim that there exists k such that $m_i^*(\theta') \in S_{i,k}^{\mathcal{M},\theta}$ for each i, but $m_i^*(\theta') \notin S_{i,k+1}^{\mathcal{M},\theta}$ for some i. To see why, observe first that $m_i^*(\theta') \in S_{i,0}^{\mathcal{M},\theta} = M_i$ for all i. Now if $m_i^*(\theta') \in S_i^{\mathcal{M},\theta}$ for all i, we would have $m^*(\theta') \in S^{\mathcal{M},\theta}$ and thus $f(\theta') = f(\theta)$, a contradiction. Thus there exists $\widetilde{m}_i$ such that

$$u_i\left(g\left(\widetilde{m}_i, m_{-i}^*(\theta')\right),\theta\right) > u_i\left(g\left(m_i^*(\theta'), m_{-i}^*(\theta')\right),\theta\right) = u_i\left(f(\theta'),\theta\right). \tag{19}$$

Suppose that $u_i\left(g\left(\widetilde{m}_i, m_{-i}^*(\theta')\right),\theta'\right) \geq u_i\left(g\left(m^*(\theta')\right),\theta'\right) = u_i\left(f(\theta'),\theta'\right)$; this would imply $\widetilde{m}_i \in S_i^{\mathcal{M},\theta'}$. But from (19), we know that $g\left(\widetilde{m}_i, m_{-i}^*(\theta')\right) \neq g\left(m_i^*(\theta'), m_{-i}^*(\theta')\right)$, a contradiction to the hypothesis of finite rationalizable implementation; thus $u_i\left(g\left(\widetilde{m}_i, m_{-i}^*(\theta')\right),\theta'\right) < u_i\left(f(\theta'),\theta'\right)$. Now writing $y \triangleq g\left(\widetilde{m}_i, m_{-i}^*(\theta')\right)$, we have $u_i(y,\theta) > u_i\left(f(\theta'),\theta\right)$ and $u_i\left(f(\theta'),\theta'\right) > u_i(y,\theta')$.
■

The following strengthening of Maskin monotonicity was introduced by Moore and Repullo (1990).

Definition 6 (Condition μ)
Social choice function f satisfies condition μ if there exists finite $B \subseteq Y$, and, for each i and θ, $C_i(\theta) \subseteq B$ with

$$u_i(f(\theta), \theta) \geq u_i(y, \theta) \tag{20}$$

for all $y \in C_i(\theta)$, such that for all $\theta^* \in \Theta$, the following three conditions are satisfied:

1. *If $u_i(f(\theta), \theta^*) \geq u_i(y, \theta^*)$ for all $y \in C_i(\theta)$ and i, then $f(\theta^*) = f(\theta)$;*
2. *If $u_i(y^*, \theta^*) \geq u_i(y, \theta^*)$ for some i and $y^* \in B$, for all $y \in C_i(\theta)$ and $u_j(y^*, \theta^*) \geq u_j(y, \theta^*)$ for all $y \in B$ and $j \neq i$, then $f(\theta^*) = y^*$;*
3. *If $y^{**} \in B$ and $u_i(y^{**}, \theta^*) \geq u_i(y, \theta^*)$ for all $y \in B$ and i, then $f(\theta^*) = y^{**}$.*

Social choice function f satisfies condition "strict μ" if the above conditions hold replacing the weak inequality in (20) with the strict inequality.

Proposition 4 *If $I \geq 3$ and f is finitely implementable in rationalizable strategies, then f satisfies condition strict μ.*

Proof. Suppose $\mathcal{M}$ finitely implements f in rationalizable strategies. We first construct sets B and $C_i(\theta)$ that will be used in the constructive proof of the proposition. Let $B = \{y \in Y \,|g(m)\}$. Fix θ. There exists $m^*(\theta) \in S^{\mathcal{M},\theta}$ such that $g(m^*(\theta)) = f(\theta)$. Let

$$C_i(\theta) = \left\{y \in B \,\middle|\, y \neq f(\theta) \text{ and } y = g\left(m_i, m_{-i}^*(\theta)\right) \text{ for some } m_i \in M_i\right\}.$$

Observe that $y \in C_i(\theta) \Rightarrow u_i(f(\theta), \theta) > u_i(y, \theta)$. (We observe that this strictness does not arise in the case of Moore and Repullo (1990)). To see why, suppose that $y \in C_i(\theta)$ and $u_i(f(\theta), \theta) \leq u_i(y, \theta)$. Since

$$m_i^*(\theta) \in \underset{m_i}{\arg\max}\ u_i\left(g\left(m_i, m_{-i}^*(\theta)\right), \theta\right),$$

we have that $u_i(f(\theta), \theta) = u_i(y, \theta)$. By definition of C_i, $y \neq f(\theta)$ and there exists $\widehat{m}_i$ such that

$$y = g\left(\widehat{m}_i, m_{-i}^*(\theta)\right).$$

But this implies $\widehat{m}_i \in S_i^{\mathcal{M},\theta}$, so $\left(\widehat{m}_i, m_{-i}^*(\theta)\right) \in S^{\mathcal{M},\theta}$, and thus $y = g\left(\widehat{m}_i, m_{-i}^*(\theta)\right) = f(\theta)$, a contradiction.

Now fix any θ^*.

1. If $u_i(f(\theta), \theta^*) \geq u_i(y, \theta^*)$ for all $y \in C_i(\theta)$ and i, then $m^*(\theta) \in S^{\mathcal{M},\theta^*}$ and so $f(\theta^*) = f(\theta)$. This verifies (1).

2. If $u_i(y^*, \theta^*) \geq u_i(y, \theta^*)$ for some i and $y^* \in B$ for all $y \in C_i(\theta)$, then there exists $\widehat{m}_i$ such that $y^* = g\left(\widehat{m}_i,\, m^*_{-i}(\theta)\right)$ and

$$\widehat{m}_i \in \underset{m_i}{\arg\max}\;\; u_i\left(g\left(m_i, m^*_{-i}(\theta)\right), \theta^*\right).$$

If in addition, $u_j(y^*, \theta^*) \geq u_j(y, \theta^*)$ for all $y \in B$ and $j \neq i$, then

$$m^*_j(\theta) \in \underset{m_j}{\arg\max}\;\; u_j\left(g\left(\widehat{m}_i, m_j, m^*_{-i,j}(\theta)\right), \theta^*\right).$$

So $\left(\widehat{m}_i,\, m^*_{-i}(\theta)\right) \in S^{\mathcal{M},\theta^*}$ and $f(\theta^*) = y^*$. This verifies (2).

3. If $y^{**} \in B$, there exists $\widehat{m} \in M$ with $g(\widehat{m}) = y^{**}$. If, in addition, $u_i(y^{**}, \theta^*) \geq u_i(y, \theta^*)$ for all $y \in B$ and i, then $\widehat{m} \in S^{\mathcal{M},\theta^*}$ and thus $g(\widehat{m}) = y^{**} = f(\theta^*)$. This verifies (3). ■

The proof of Proposition 4 closely follows the proof of Moore and Repullo (1990), except for an extra step where we establish the strictness of equality (20). Otherwise, the argument is the same except that we have a social choice function rather than a correspondence and use rationalizable strategies rather than Nash equilibrium as a solution concept.

Definition 7 (Condition $\mu 2$)
Suppose $I = 2$. Social choice function f satisfies condition $\mu 2$ if μ holds and in addition, for any $\theta, \theta' \in \Theta$, there exists $y^* \in C_1(\theta) \cap C_2(\theta')$ such that for all $\theta^* \in \Theta$,

$$u_1(f(\theta), \theta^*) \geq u_1(y, \theta^*),\ \forall y \in C_1(\theta) \text{ and } u_2\left(f(\theta'), \theta^*\right) \geq u_2(y, \theta^*),\ \forall y \in C_2(\theta') \;\Rightarrow\; f(\theta^*) = y^*.$$

Social choice function f satisfies condition "strict $\mu 2$" if the above conditions hold replacing the weak inequality in (20) with the strict inequality.

Proposition 5 *If $I = 2$ and f is finitely implementable in rationalizable strategies, then f satisfies condition strict $\mu 2$.*

Proof. Let $y^* = g\left(m^*_1(\theta), m^*_2(\theta')\right)$. If

$$u_1(f(\theta), \theta^*) \geq u_1(y, \theta^*) \text{ for all } y \in C_1(\theta),$$

and

$$\text{and } u_2\left(f(\theta'), \theta^*\right) \geq u_2(y, \theta^*) \text{ for all } y \in C_2(\theta')$$

then $\left(m_1^*(\theta), m_2^*(\theta')\right) \in S^{\mathcal{M},\theta^*}$ and thus $y^* = g\left(m_1^*(\theta), m_2^*(\theta')\right) = f(\theta^*)$. ■

See Dutta and Sen (1991), Sjostrom (1991), Danilov (1992) and Maskin and Sjostrom (2004) for related characterizations of necessary and sufficient conditions for Nash implementation.

5 Virtual Implementation in Finite Mechanisms

Preferences are *non-trivial* if, for each i and θ, there exist $\underline{z}$ and $\overline{z}$ such that

$$u_i(\overline{z}, \theta) > u_i(\underline{z}, \theta).$$

A set of agents $X \subseteq \{1, ..., I\}$ have *identical preferences* at θ if there exists $v : Z \to \mathbb{R}$ such that, for each $i \in X$, $u_i(\cdot, \theta)$ is an affine transformation of $v(\cdot)$. The economic assumption of Abreu and Matsushima (1992a) implies that no pair of agents have identical preferences at any θ and preferences are non-trivial. It also makes the slightly stronger economic requirement that each agent can be made better off without making any other agent better off.

Definition 8 (AM Economic Condition)
The AM economic condition is satisfied if, for each i and θ, there exist $\underline{z}$ and $\overline{z}$ such that $u_i(\overline{z}, \theta) > u_i(\underline{z}, \theta)$, and $u_j(\overline{z}, \theta) \leq u_j(\underline{z}, \theta)$, for all $j \neq i$.

Theorem (Abreu and Matsushima (1992)) *If the AM economic assumption is satisfied, all social choice functions are finitely virtually implementable in rationalizable strategies.*

We show that this assumption cannot be weakened too much. In particular, we will show that a necessary condition for finite virtual implementation in rationalizable strategies is that f satisfies the *common most preferred outcome property*. This property requires that agents' utility when they have common preferences is at least as high as their commonly most preferred outcome from the domain of f.

Definition 9 (Common Most Preferred Outcome)
Social choice function f satisfies the common most preferred outcome property if identical preferences at θ^* imply

$$u_i(f(\theta^*), \theta^*) = \max_{\theta} u_i(f(\theta), \theta^*), \text{ for all } i.$$

Proposition 6 *If f is finitely virtually implementable in rationalizable strategies, then f satisfies the common most preferred outcome property.*

Proof. Fix any mechanism $\mathcal{M}$. Let $Y^{\mathcal{M}}$ be the range of g, i.e.,

$$Y^{\mathcal{M}} = \{y \,|g\,(m) = y \text{ for some } m \in M\}\,.$$

Suppose there are identical preferences which are represented by $v\,(\cdot)$ at θ^*. Now let $Y^{\mathcal{M},\theta^*}$ be the collection of most preferred outcomes in $Y^{\mathcal{M}}$ in state θ^* and let $M_i^{\mathcal{M},\theta^*}$ be the set of messages of agent i that might give rise to those most preferred outcomes, i.e.,

$$Y^{\mathcal{M},\theta^*} = \underset{y \in Y^{\mathcal{M}}}{\arg\max}\; v\,(y)$$

and

$$M_i^{\mathcal{M},\theta^*} = \left\{ m_i \in M_i \,\middle|\, g\,(m_i, m_{-i}) \in Y^{\mathcal{M},\theta^*} \text{ for some } m_{-i} \in M_{-i} \right\}.$$

Now $M_i^{\mathcal{M},\theta^*} \subseteq S_i^{\mathcal{M},\theta^*}$. This can be shown by induction: suppose $g\,(m) \in Y^{\mathcal{M},\theta^*}$; $m \in S_0^{\mathcal{M},\theta^*} = M$; suppose $m \in S_k^{\mathcal{M},\theta^*}$; then $m_{-i} \in S_{-i,k}^{\mathcal{M},\theta^*}$ and $v\,(g\,(m_i, m_{-i})) \geq v\,(g\,(m_i', m_{-i}))$ for all $m_i' \Rightarrow m_i \in S_{i,k+1}^{\mathcal{M},\theta^*}$.

Now suppose $\mathcal{M}$ that ε-implements f in rationalizable strategies. Now $y^* \in Y^{\mathcal{M},\theta^*} \Rightarrow \|y^* - f\,(\theta^*)\| \leq \varepsilon$. Also, for any θ, there exists $y \in Y^{\mathcal{M}}$ such that $\|y - f\,(\theta)\| \leq \varepsilon$. By assumption, $v\,(y^*) \geq v\,(y)$.

Thus if f is finitely virtually implementable in rationalizable strategies, it must be that for each $\varepsilon > 0$ and θ, there exist y^*, y such that $\|y^* - f\,(\theta^*)\| \leq \varepsilon$, $\|y - f\,(\theta)\| \leq \varepsilon$ and $v\,(y^*) \geq v\,(y)$. We conclude that $v\,(f\,(\theta^*)) \geq v\,(f\,(\theta))$. ■

Presumably, the logic of this example could be extended to show that the virtual implementation results under incomplete information of Abreu and Matsushima (1992b), Bergemann and Morris (2009b) and Artemov, Kunimoto, and Serrano (2008) will not work with identical preferences and thus their weak economic assumptions cannot be completely dispensed with.

6 Discussion

6.1 Social Choice Correspondences

We reported results for social choice functions only. The results do not extend to social choice correspondences in a natural way. In particular, we will show that Maskin monotonicity is not a necessary condition for implementation in rationalizable strategies (according to at least one natural definition of these terms).

It is not obvious how to extend our definition of implementation of an SCF in rationalizable strategies to implementation of an SCC in rationalizable strategies. We describe one approach. A *(pure outcome) social choice correspondence* (SCC) is a mapping $F : \Theta \to 2^Z / \varnothing$. A social choice correspondence F is implementable in rationalizable strategies if there exists a mechanism $\mathcal{M}$ with $g\left[S^{\mathcal{M},\theta}\right] = F(\theta)$ for all $\theta \in \Theta$. A SCC F is *Maskin monotonic* if: whenever $z^* \in F(\theta)$ and $u_i(z^*, \theta) \geq u_i(z, \theta) \Rightarrow u_i(z^*, \theta') \geq u_i(z, \theta')$ for all i and z; then $z^* \in F(\theta')$. Note that this definition is given in terms of pure outcomes.

Now consider the following example. There are 2 agents; $\Theta = \{\alpha, \beta\}$; $Z = \{a, b, c, d\}$; payoffs are given by the following table:

	a	b	c	d
α	$1+\varepsilon, 0$	$0, 1+\varepsilon$	$1, 1$	$1+2\varepsilon, 1+2\varepsilon$
β	$1+\varepsilon, 0$	$0, \varepsilon$	$1, 1$	$1+2\varepsilon, 1+2\varepsilon$

The social choice correspondence is $F^*(\alpha) = \{a, b, c, d\}$ and $F^*(\beta) = \{d\}$.

Now F^* is not Maskin monotonic. To see why, note that $a \in F^*(\alpha)$ and that $u_i(a, \alpha) \geq u_i(z, \alpha) \Rightarrow u_i(a, \beta) \geq u_i(z, \beta)$ for all i and z. So Maskin monotonicity would require $a \in F^*(\beta)$.

But F^* is implementable in rationalizable strategies. Consider the mechanism $\mathcal{M}$ with $M_i = \left\{m_i^1, m_i^2, m_i^3\right\}$ and deterministic g given by the following matrix:

	m_2^1	m_2^2	m_2^3
m_1^1	a	b	c
m_1^2	b	a	c
m_1^3	c	c	d

Now, for each i, $S_{i,k}^{\mathcal{M},\alpha} = M_i$ for all k and thus $S_i^{\mathcal{M},\alpha} = M_i$. Thus $g[S^{\mathcal{M},\alpha}] = \{a, b, c, d\} = F^*(\alpha)$. But in state β, we have

$$\begin{aligned}
S_{1,0}^{\mathcal{M},\beta} &= \left\{m_1^1, m_1^2, m_1^3\right\} \text{ and } S_{2,0}^{\mathcal{M},\beta} = \left\{m_2^1, m_2^2, m_2^3\right\}, \\
S_{1,1}^{\mathcal{M},\beta} &= \left\{m_1^1, m_1^2, m_1^3\right\} \text{ and } S_{2,1}^{\mathcal{M},\beta} = \left\{m_2^3\right\}, \\
S_{1,2}^{\mathcal{M},\beta} &= \left\{m_1^3\right\} \text{ and } S_{2,2}^{\mathcal{M},\beta} = \left\{m_2^3\right\},
\end{aligned}$$

and thus $g[S^{\mathcal{M},\beta}] = \{d\} = F^*(\beta)$.

6.2 Domain Restrictions

In this note, the outcome space is a lottery space and we restricted preferences to be expected utility preferences. These assumptions rule out domain restrictions often assumed in the literature. For example, it is often assumed that for every profile of strict orders on outcomes, there is a θ where agents have that ordinal ranking of outcomes. If we imposed this assumption on the lottery space, we would have a contradiction with the expected utility assumption. If we imposed this restriction only on pure outcomes Z, it is not clear if existing results would go through.

Börgers (1995) proved a striking result about implementation in rationalizable strategies using finite mechanisms: he showed that if every unanimous strict ranking of outcomes was in the domain of preferences, then any rationalizable social choice function was dictatorial. Moore and Repullo (1990) show the related result that, with two players, condition $\mu 2$ implies that any efficient social choice rule where the domain includes all strict orderings is dictatorial. It is not clear if and how these results can be mapped into our setting for the reasons given above.

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