WORKING PAPERS

RESEARCH DEPARTMENT

WORKING PAPER NO. 11-18

PRIVATE EQUITY PREMIUM IN A GENERAL EQUILIBRIUM MODEL OF UNINSURABLE INVESTMENT RISK

Francisco Covas
Board of Governors of the Federal Reserve System
and
Shigeru Fujita
Federal Reserve Bank of Philadelphia

May 2011

Supersedes Working Paper 07-30

RESEARCH DEPARTMENT, FEDERAL RESERVE BANK OF PHILADELPHIA

Ten Independence Mall, Philadelphia, PA 19106-1574 • www.philadelphiafed.org/research-and-data/

Private Equity Premium in a General Equilibrium Model of Uninsurable Investment Risk*

Francisco Covas† and Shigeru Fujita‡

May 2011

Abstract

This paper studies the quantitative properties of a general equilibrium model where a continuum of heterogeneous entrepreneurs are subject to aggregate as well as idiosyncratic risks in the presence of a borrowing constraint. The calibrated model matches the highly skewed wealth and income distributions of entrepreneurs. We provide an accurate solution to the model despite the significant nonlinearities that are absent in the economy with uninsurable labor income risk. The model is capable of generating the average private equity premium of roughly 3% and a low risk-free rate. The model also produces procyclicality of the risk-free rate and countercyclicality of the average private equity premium. The countercyclicality of the average equity premium is largely driven by tightening (loosening) of financing constraints during recessions (booms).

JEL codes: E22, G11, M13
Keywords: Uninsurable investment risk, aggregate uncertainty, private equity premium

*We are grateful to two anonymous referees and seminar participants at the NBER Summer Institute 2007, Federal Reserve Bank of Philadelphia, and Society of Computational Economics Meeting 2007 for their useful suggestions and discussions. Any remaining errors are our own. The views expressed here are those of the authors and do not represent those of Federal Reserve Bank of Philadelphia or Federal Reserve System. This paper is available free of charge at www.philadelphiafed.org/research-and-data/publications/working-papers/.

†Division of Monetary Affairs, Board of Governors of the Federal Reserve System. E-mail: `francisco.b.covas@frb.gov`

‡Research Department, Federal Reserve Bank of Philadelphia. E-mail: `shigeru.fujita@phil.frb.org`.

1 Introduction

This paper studies the quantitative properties of a general equilibrium model of uninsurable investment risk. The economy consists of a continuum of heterogeneous entrepreneurs who own and manage their businesses. Entrepreneurs in our economy face aggregate as well as idiosyncratic investment risks and are subject to the borrowing limit, which takes the form of a collateral constraint. The framework is designed to capture essential features of "financing frictions" facing privately held businesses.

As emphasized in Angeletos (2007), the size of privately held businesses is quite large in the U.S., and the roles played by entrepreneurs in growth and business cycles are no doubt important.[1] Reflecting their importance, the recent empirical finance literature has revealed a number of economically intriguing evidence on investment and portfolio decisions of entrepreneurs (e.g., Hamilton (2000), Moskowitz and Vissing-Jørgensen (2002), Heaton and Lucas (2000), Carroll (2002), and Gentry and Hubbard (2004)).

Although the model we study in this paper is too stylized to address all of these empirical findings, this paper takes one of the initial steps to evaluate them from a macroeconomic perspective. Our primary focus is on the model's asset pricing implications. Specifically, we examine the model's ability to generate an empirically plausible size and cyclical pattern of the private equity premium. In doing so, we calibrate the model by matching key cross-sectional characteristics of income and wealth distributions of entrepreneurs, using the evidence in the Survey of Consumer Finances (SCF). The model in the present paper is closely related to the models by Angeletos (2007) and Covas (2006). These two papers, however, focus on whether the presence of the uninsurable risk leads to under- or over-accumulation of the capital stock and do not put much emphasis on cross-sectional heterogeneities of entrepreneurs. This paper in contrast makes an effort to replicate the cross-sectional heterogeneities observed in the data. In particular, our calibrated model matches the highly skewed wealth and income distributions of entrepreneurs. Our calibration also incorporates the fact that entrepreneurs face a large idiosyncratic risk (e.g., Hamilton (2000) and Moskowitz and Vissing-Jørgensen (2002)).

A key deviation of our paper from the previous literature (e.g., Angeletos (2007) and Covas (2006)) is that our model features aggregate risk as well as idiosyncratic risk. Augmenting the model with aggregate uncertainty allows us to examine business cycle dimensions of the model. As far as we know, this paper is the first attempt to assess the role of aggregate uncertainty in this class of models. In addition to investing in their business, the entrepreneurs have access to one-period bonds, which they can use to self-insure against the underlying risks. While they are allowed to borrow from other entrepreneurs by selling the bonds, the borrowing is permitted only up to the fraction of the value of their business. In this environment, the presence of aggregate uncertainty poses an important computational challenge. Since entrepreneurs differ in the size of their firms and their idiosyncratic productivities, some may choose to borrow as much as the borrowing constraint permits, while others may

[1] According to NIPA Table 1.13, the share of the noncorporate sector's value added amounts to 23% of the entire business sector's total value added in 2006. Davis et al. (2006) report that more than two-thirds of nonfarm business employment is accounted for by privately held firms.

choose to invest all of their wealth in the safe asset. The general equilibrium of the model requires that the bond price clear the market every period and that the individual-level decision based on the perceived evolution of the market-clearing bond price be accurate.

A similar computational difficulty is common in models with uninsurable labor income risk. In that environment, solving the individual worker's problem also requires a prediction of the next-period equilibrium price, which, in theory, is a function of the current-period wealth distribution. In practice, however, Krusell and Smith (1998) and other studies, including Young (2005), find that a simple linear forecasting rule that links the aggregate capital stock with the price gives a very accurate prediction.[2] The key to this finding is an approximate linearity of the saving policy function. That is, the marginal propensity to save is approximately constant for a wide range of wealth levels. Consequently, workers with different wealth levels are simply the scaled-up or -down version of the average worker. In this environment, information regarding the wealth distribution other than its mean simply does not have many implications for the equilibrium price.

The environment in our model is sufficiently different from the one assumed in models with uninsurable labor income risk. In our model, entrepreneurs operate a production technology that is decreasing returns to scale in privately held capital. This implies that entrepreneurs who are "poor" (measured by the size of their capital stock) have a better risk-return tradeoff. In particular, entrepreneurs who draw a good idiosyncratic shock (which is assumed to be highly persistent) have a strong incentive to borrow and invest in their private businesses. This incentive of high leverage together with the borrowing constraint makes the saving function of those productive entrepreneurs highly nonlinear across a wide range of their wealth levels. Our model consequently generates a highly skewed wealth distribution that is populated by a large mass of poor entrepreneurs. In contrast, in the models with uninsurable labor income risk, being poor simply means facing the risk of a large drop in consumption at the time of job loss. Workers thus try to avoid running into this situation through self-insurance. As a result, the left tail of the wealth distribution is only thinly populated by unlucky workers whose saving function exhibits a small nonlinearity due to the borrowing constraint.[3] Given that the economic mechanisms in these two classes of models differ significantly with each other, it is unclear ex ante whether the solution method based on the average capital stock performs well in our setup.

As mentioned above, the key equilibrium object in our model is the market-clearing bond price. In calculating the approximate equilibrium, we postulate several specifications for the bond-price forecasting rule and examine their accuracy. We first consider the one often used in the literature, where the average capital stock is linearly linked to the bond price. Second, we add an additional piece of information, more specifically, the dispersion measure of the capital distribution or the fraction of entrepreneurs who are borrowing constrained, to the

[2]This result is often called the approximate aggregation and has also been studied by den Haan (1997) and Krusell and Smith (2006).

[3]Note that, in our model, productive and poor entrepreneurs have a marginal propensity to save that is greater than one due to the incentive to invest in their business, whereas poor workers in the models of labor income risk have a marginal propensity to save that is less than one in order to sustain their consumption level. Poor and unproductive entrepreneurs in our model behave similarly to these workers.

first specification. Last, we consider a forecasting rule in which the entrepreneurs directly take the market-clearing bond price as a state variable and use a linear autoregressive rule to predict the next-period market-clearing price. We find that the performance of the first-moment-only specification is less than satisfactory. Further, the error term is found to exhibit a strong serial correlation, which implies that agents do not fully take advantage of the information available to them. We find that while adding the dispersion measure has little impact on forecast accuracy, incorporating the fraction of the constrained entrepreneurs materially improves accuracy. This last finding suggests that the borrowing constraint has important implications for asset pricing in our model. A clear downside of this specification, however, is the computational cost that accompanies having an additional state variable. In contrast, the parsimonious autoregressive forecasting rule works very well in terms of accuracy as well as eliminating unexploited variations in the bond-price forecasting equation. The performance of this last specification beats the one with the fraction of the constrained entrepreneurs in all of the accuracy measures we look at, even though its computational burden is comparable to the one that uses the average capital stock only. We show that the autoregressive specification continues to perform well in alternative calibrations as well. The reason for this result is not entirely clear, but we suspect that the current-period bond price efficiently summarizes the information in the entire wealth distribution and thus also carries the information necessary to predict the next-period bond price, given that underlying shocks are highly persistent. The robustness of the autoregressive forecasting rule appears to imply that this specification would work well in other applications as well.

We find that the model has promising asset-pricing implications. We first show that the model generates the average equity premium of 2.8% in the steady state. This relatively sizable premium mostly results from the wedge in the Euler equation due to the borrowing constraint. In particular, highly productive and small entrepreneurs have strong incentives to borrow and invest in their business but are financially constrained. The equity premium of these entrepreneurs attributed to the wedge amounts to as much as 20%. The traditional risk-compensation channel contributes to generating only a small premium, even though the underlying risks facing entrepreneurs are calibrated to be quite large. In the presence of aggregate uncertainty, the average private equity premium goes up to roughly 3%. We show that the model produces the procyclical risk-free rate and the countercyclical average equity premium. The latter feature is largely driven by tightening (loosening) of financing constraints during recessions (booms).

This paper is organized as follows. The next section lays out the model. Section 3 presents the calibration. Section 4 examines the properties of the steady-state equilibrium. There we emphasize the importance of nonlinear features in our environment and explain the mechanism through which a relatively large private equity premium and the low risk-free rate arise. The model under aggregate uncertainty is solved in Section 5. We first examine the accuracy of various specifications of bond-price forecasting equations. We then discuss the cyclical properties of the model. The last section concludes the paper by offering some direction for future research. Appendix A presents the construction of the empirical data from the SCF. Appendix B provides the details of the computational algorithm used to solve the model.

2 Entrepreneurial Economy

The economy is inhabited with measure one of infinitely lived entrepreneurs. Each entrepreneur has an ability to operate his/her own technology. This technology is subject to individual-specific shocks that are assumed to be uninsurable. This technology also faces aggregate uncertainty. Other features of the model include that (i) entrepreneurs face an occasionally binding borrowing constraint and (ii) they are also allowed to trade one-period riskless bonds.

2.1 Environment

There is only one consumption good. The utility function of each entrepreneur, $U(\cdot)$, is strictly increasing, strictly concave, obeys the Inada conditions, and is twice continuously differentiable in consumption, c_{it}. The entrepreneur's problem is to maximize the expected lifetime utility derived from consumption:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_{it}), \tag{1}$$

where $0 < \beta < 1$ is the discount factor.

There are two investment opportunities facing each entrepreneur: Investment in his own business or in a one-period risk-free bond that yields a sure return over the period. The idiosyncratic risk cannot be insured by any direct insurance markets. Furthermore, issuance of equity is not allowed. The firm is therefore "privately held" by the entrepreneur.[4] The entrepreneurs instead can borrow funds to finance both consumption and the risky investment by investing a negative amount in the safe asset. However, there is a limit to borrowing, which takes the form of a collateral constraint. The precise specification is described below. The risky technology available to the entrepreneur i is represented by:

$$y_{it} = \theta_t z_{it} f(k_{it}), \tag{2}$$

where θ_t and z_{it}, respectively, represent aggregate and idiosyncratic technology processes and k_{it} is entrepreneur i's capital stock. It is assumed that $f(\cdot)$ is continuously differentiable, strictly increasing, strictly concave with $f(0) = 0$, and satisfying the Inada conditions. The amount of labor input is fixed at a normalized value of one. This assumption is also adopted in the literature (e.g, Angeletos and Calvet (2006), Cagetti and De Nardi (2006), and Caggese

[4]This financing friction is one of the features that make our model distinct from models in the lumpy investment literature. For example, in the model of Khan and Thomas (2008), production units are subject to idiosyncratic and aggregate uncertainty as in our model but are allowed to issue shares, which are held by the representative household.

(2009)). The idea is that "the entrepreneur himself is an unsubstitutable input" and "the labor input takes even longer to reallocate across production sites than does the capital input" (Krusell and Smith (2006)). Further, concavity of the production function can be interpreted as reflecting diminishing returns to span of control. The decreasing-returns-to-scale property is an important nonlinear feature of our model.[5] Quantitatively, it allows us to match the wealth distribution of entrepreneurs, which is highly skewed in the data.

Both idiosyncratic and aggregate productivities follow a first-order Markov process. Entrepreneurial capital depreciates at a fixed rate, δ, and the gross risky investment is given by:

$$i_{it} = k_{it+1} - (1-\delta)k_{it}. \tag{3}$$

Let b_{it+1} denote the resources of the entrepreneur allocated to the risk-free bond, which delivers one unit of the consumption good in the next period. The price of the asset is denoted as q_t. The rate of return is determined in equilibrium such that the bond market clears in each period. Furthermore, it varies over time under the presence of the aggregate shock θ_t. The entrepreneur's budget constraint is written as follows:

$$c_{it} + k_{it+1} + q_t b_{it+1} = x_{it}, \tag{4}$$
$$x_{it+1} = \theta_{t+1} z_{it+1} f(k_{it+1}) + (1-\delta)k_{it+1} + b_{it+1}, \tag{5}$$

where x_{it} denotes the entrepreneur's period-t wealth.

2.2 Individual Problem

The set of decision-relevant state variables are the following four variables: (i) individual wealth x, (ii) the idiosyncratic technology shock z, (iii) the aggregate technology shock θ, and (iv) the type distribution over (z, x), which we denote by Γ. From here on, we adopt the notational convention of dropping the time subscript t and using "prime" to denote one-period-ahead values. We can write the entrepreneur's dynamic decision problem in a recursive manner as follows:

$$\begin{aligned} V(z_i, x_i; \Gamma, \theta) &= \max_{c_i, k_i', b_i'} \; U(c_i) + \beta \mathbb{E} V(z_i', x_i'; \Gamma', \theta'), \\ &\text{subject to} \\ c_i + k_i' + q(\Gamma,\theta) b_i' &= x_i, \\ x_i' &= \theta' z_i' f(k_i') + (1-\delta)k_i' + b_i', \\ \Gamma' &= H(\Gamma, \theta, \theta'), \\ k_i' &\geqslant 0 \quad \text{and} \quad q(\Gamma,\theta) b_i' \geqslant -\kappa k', \end{aligned} \tag{6}$$

[5] This feature is absent in the model of Angeletos (2007), where technology exhibits constant returns to scale in labor input and the capital stock. In this case, the capital income is linear in the capital stock. See Lemma 1 in Angeletos (2007).

where $H(\cdot)$ is the equilibrium transition function for Γ and $\mathbb{E}$ is the conditional expectation operator with respect to z_i' and θ'. The last inequality in (6) specifies the borrowing limit as a collateral constraint: The amount of borrowing $q(\Gamma, \theta)b_i'$ cannot exceed a certain fraction κ of the entrepreneur's capital stock.[6] From the properties of the utility and production functions of the entrepreneur, the optimal levels of consumption and the risky investment are always strictly positive. The only constraint that may be binding is the borrowing constraint. The first-order conditions of problem (6) can be written as:

$$U_c(c_i) = \frac{\beta}{q(\Gamma, \theta)} \mathbb{E} U_c(c_i') + \lambda_i \tag{7}$$

$$U_c(c_i) = \beta \mathbb{E}[(\theta z_i' f_k(k_i') + 1 - \delta) U_c(c_i')] + \lambda_i \kappa \tag{8}$$

$$\lambda_i \left[q(\Gamma, \theta) b_i' - \kappa k' \right] = 0 \tag{9}$$

where λ_i is the nonnegative Lagrange multiplier associated with the entrepreneur's borrowing constraint. Using the following definition of the interest rate on the safe asset $r \equiv 1/q - 1$, Equations (7) and (8) imply that returns to safe and risky investments are related by:

$$\beta(1+r)\mathbb{E} U_c(c_i') + \lambda_i (1-\kappa) = \mathbb{E}[(\theta' z_i' f_k(k_i') + 1 - \delta) U_c(c_i')]. \tag{10}$$

In order to clarify how the excess return on the private business over the safe investment is related to the underlying risks and the borrowing constraint, we rewrite Equation (10) as follows:

$$\mathbb{E}\theta' z_i' f_k(k_i') - \delta - r = \frac{\lambda_i(1-\kappa)}{\beta \mathbb{E} U_c(c_i')} - \frac{\text{cov}[\theta' z_i' f_k(k_i'), U_c(c_i')]}{\mathbb{E} U_c(c_i')}. \tag{11}$$

The left-hand side defines the private equity premium. The right-hand side decomposes it into two parts. The first term corresponds to the wedge associated with the borrowing constraint and the second term corresponds to the premium compensating the idiosyncratic as well as aggregate risks borne by the entrepreneur. Note that $\theta' z_i' f_k(k_i')$ and $U_c(c_i')$ are negatively correlated, the covariance term is negative. Both terms thus contributes positively to the private equity premium of each entrepreneur.[7]

2.3 General Equilibrium

The recursive competitive equilibrium is defined by the value function $V(z_i, x_i; \Gamma, \theta)$; the policy functions $\{k'(z_i, x_i; \Gamma, \theta), b'(z_i, x_i; \Gamma, \theta), c(z_i, x_i; \Gamma, \theta)\}$; the pricing function $q(\Gamma, \theta)$; and a law of motion for the distribution $\Gamma' = H(\Gamma, \theta, \theta')$ such that (i) given the aggregate states

[6] Note that once the exogenous shocks are realized at the beginning of period t, the entrepreneur chooses k' and b', and the bond price is then simultaneously determined in the market. Note that this specification of the borrowing constraint differs from the one in Covas (2006), who assumes that borrowing is allowed up to an exogenously specified fixed amount.

[7] Angeletos and Calvet (2006) study a model similar to ours and derive the analytical solution corresponding to (11) under the conditions that (i) the idiosyncratic shock is i.i.d. normal, (ii) there is no aggregate uncertainty, (iii) agents have CARA preferences, and (iv) there is no borrowing constraint.

$\{\Gamma, \theta\}$, the bond price $q(\Gamma, \theta)$ and the law of motion for the distribution $\Gamma' = H(\Gamma, \theta, \theta')$, the entrepreneur's policy functions solve problem (6), (ii) entrepreneurial capital and bond holdings are given by

$$K = \int_{z_i} \int_{x_i} k'(z_i, x_i; \Gamma, \theta) d\,\Gamma(z_i, x_i), \tag{12}$$

$$B = \int_{z_i} \int_{x_i} b'(z_i, x_i; \Gamma, \theta) d\,\Gamma(z_i, x_i), \tag{13}$$

(iii) the bond market clears $B = 0$, and (iv) H is generated by the optimal policy functions.

3 Calibration

The properties of the model can be evaluated only numerically. This section assigns functional forms and parameter values to find the numerical solution of the model. We choose one period in the model economy to be one year. This section describes the benchmark calibration. We later conduct the sensitivity analysis by changing some of the important parameters.

3.1 Parameters Set Externally

First, the annual depreciation rate δ is chosen to be 8% as in Covas (2006). Next, we adopt a constant relative risk-aversion (CRRA) specification for the utility function of entrepreneurs:

$$U(c_i) = \frac{c_i^{1-\gamma}}{1-\gamma}, \tag{14}$$

where γ is the risk-aversion parameter. Since we lack evidence on γ for entrepreneurs, we simply set γ to 2, a number often used in the representative-agent DSGE framework.

The entrepreneur's risky technology $f(k_i)$ is specified by k_i^{α}. The curvature parameter α is determined in the next subsection. The idiosyncratic productivity process is first-order Markov:

$$\ln z_i' = \rho_z \ln z_i + \sigma_z (1 - \rho_z)^{1/2} \epsilon', \tag{15}$$

where $\epsilon \sim N(0, 1)$. While we have ample evidence on the idiosyncratic labor income process (e.g., Aiyagari (1994), Storesletten et al. (2004) among others), we lack direct information that we can use to calibrate the process for entrepreneurs. It appears, however, relatively uncontroversial to assume some persistence in the process. We set the serial correlation parameter ρ_z at 0.90 in the benchmark calibration as in Covas (2006). Angeletos (2007) and Angeletos and Calvet (2006) assume an i.i.d. shock for analytical tractability, but our focus is on quantitative evaluations of the model, and thus, allowing for a high degree of persistence is important to us. We will later examine how differently the model behaves when we use a lower value for this parameter. The unconditional standard deviation σ_z is

Table 1: Parameter Values: Benchmark Calibration

Discount factor	β	0.92
Risk aversion	γ	2.00
Curvature of production	α	0.70
Depreciation rate	δ	0.08
Serial correlation of productivity risk	ρ_z	0.90
Unconditional standard deviation of productivity risk	σ_z	0.50
Collateral constraint	κ	0.41
Idiosyncratic productivity shock		
Number of states	n_z	7
Discrete states		
$\bar{z} = [0.29; 0.44; 0.66; 1.00; 1.50; 2.26; 3.40]$		
Transition matrix:		
$\Pi_z = \begin{bmatrix} 0.7351 & 0.2321 & 0.0305 & 0.0021 & 0.0001 & 0.0000 & 0.0000 \\ 0.0387 & 0.7453 & 0.1945 & 0.0204 & 0.0011 & 0.0000 & 0.0000 \\ 0.0020 & 0.0778 & 0.7514 & 0.1560 & 0.0123 & 0.0004 & 0.0000 \\ 0.0001 & 0.0061 & 0.1170 & 0.7535 & 0.1170 & 0.0061 & 0.0001 \\ 0.0000 & 0.0004 & 0.0123 & 0.1560 & 0.7514 & 0.0778 & 0.0020 \\ 0.0000 & 0.0000 & 0.0011 & 0.0204 & 0.1945 & 0.7453 & 0.0387 \\ 0.0000 & 0.0000 & 0.0001 & 0.0021 & 0.0305 & 0.2321 & 0.7351 \end{bmatrix}$		
Number of grid points for wealth	n_x	500
Aggregate productivity shock		
Number of states	n_θ	2
Discrete states		
$[\bar{\theta}_1 \; \bar{\theta}_2] = [0.95 \; 1.05]$		
Transition matrix:		
$\Pi_\theta = \begin{bmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{bmatrix}$		
Number of grid points for the bond price:	n_q	40

NOTES: The policy function is defined over a continuum of wealth and the current period bond price. The function values that are not on the grid points are found by piecewise bilinear interpolation. The discretization of the idiosyncratic productivity process follows the method proposed by Rouwenhorst (1995).

determined internally by matching observable cross-sectional moments and thus discussed in the following subsection. Once the numerical values are assigned to ρ_z and σ_z, we use the procedure suggested by Rouwenhorst (1995) to approximate it by a Markov chain with 7 states.[8]

[8]This method is known to be more accurate than the one developed by Tauchen and Hussey (1991), especially when the underlying process is highly persistent, as in our case. See Kopecky and Suen (2010) for

The aggregate technology state switches between two states $\bar{\theta}_1$ and $\bar{\theta}_2$ with $\bar{\theta}_2 > \bar{\theta}_1$ following a Markov chain as in Krusell and Smith (1997, 1998). Accordingly, the transition matrix is specified as:

$$\begin{pmatrix} \pi_{1|1} & \pi_{1|2} \\ \pi_{2|1} & \pi_{2|2} \end{pmatrix}.$$

We set $\pi_{1|1} = \pi_{2|2} = 0.875$, so that the average duration of business cycles is 8 years in the model.

3.2 Parameters Set Internally

The levels of aggregate states $\bar{\theta}_1$ and $\bar{\theta}_2$ are determined so that the volatility of aggregate output in our model economy matches its empirical counterpart, which is calculated as real output of the noncorporate private sector taken from NIPA table 1.13. The standard deviation of logged HP-filtered aggregate output during the post-war period amounts to roughly 3%. By setting $\theta_1 = 0.95$ and $\theta_2 = 1.05$, respectively, we match this volatility.

It remains to determine the parameter κ for the collateral constraint, the curvature parameter of the production function α, the unconditional standard deviation of the idiosyncratic productivity process σ_z, and the discount factor β. Note that while the discussion below relates one parameter to one statistic, they are actually jointly determined.

To determine the values of these parameters, we match the cross-sectional information available through the SCF (Survey of Consumer Finances) using the steady-state version of the model. Specifically, we look at the data in the recent five surveys between 1992 and 2004. First, we define "entrepreneurs" in the SCF as those households that satisfy the following two qualifications; (i) own or share ownership in any privately held businesses, farms, professional practices or partnerships and (ii) have an active management role in any of these businesses. This definition appears appropriate in light of our model. In our sample, entrepreneurial households amount to 8% of the U.S. households on average across the five surveys.[9] While entrepreneurs make up a small fraction of households, they hold a large fraction of wealth, as is established by many previous studies such as those cited in footnote 9.[10]

First, to determine the standard deviation of the idiosyncratic productivity process σ_z, we use information regarding the income distribution of entrepreneurs in the SCF. More specifically, we target the Gini coefficient of the entrepreneurs' income distribution. This procedure yields $\sigma_z = 0.5$. Note that, in models with uninsurable labor income risk, the unconditional standard deviation of 20-40% is considered reasonable (e.g., Aiyagari (1994), Storesletten et al. (2004)). Therefore, our choice is consistent with the idea emphasized by Hamilton (2000) and Moskowitz and Vissing-Jørgensen (2002) that the idiosyncratic risk

comparison of these methods.

[9]This number is roughly comparable to the numbers based on various definitions of entrepreneurs. See, for example, Cagetti and De Nardi (2006), Quadrini (2000) and Gentry and Hubbard (2004) for definitions of entrepreneurs adopted in the literature.

[10]According to Table 1 of Cagetti and De Nardi (2006), entrepreneurs that qualify for our definition hold 33% of total wealth in the 1989 wave of the SCF. Their table also shows that, if more broadly defined, entrepreneurs own up to 53% of total wealth in the U.S.

Table 2: Wealth and Income Distributions of Entrepreneurs

	Gini	Percentile Range						
	Index	0−20	20−40	40−60	60−80	80−100	95−100	99−100
Income								
Data	0.621	0.008	0.043	0.101	0.214	0.635	0.308	0.126
Model	0.632	0.016	0.046	0.092	0.182	0.664	0.347	0.137
Wealth								
Data	0.611	0.012	0.044	0.102	0.209	0.633	0.298	0.095
Model	0.604	0.019	0.051	0.099	0.193	0.639	0.317	0.119

NOTES: Each entry from the second column onward represents the fraction of wealth (or income) owned by those in each percentile range. Empirical statistics are based on the sample of entrepreneurs in the SCF and the averages across five waves (1992, 1995, 1998, 2001, and 2004). See Appendix 1 for the construction of the income and wealth measures.

facing entrepreneurs is larger than the idiosyncratic risk facing workers. The first two rows of Table 2 compare the entrepreneur's income distribution observed in the data and in the model. The first column presents the Gini index, which we target. The remaining columns present the fraction of incomes earned by those within each percentile range. These entries clearly indicate that the income distribution of entrepreneurs is highly skewed. For example, the top 5% of entrepreneurs earn more than 30% of the aggregate income of entrepreneurs. The model matches quite well the entire income distribution. Next consider the span of control parameter α, which has strong influences on the shape of the firm-size distribution and thus the wealth distribution. We determine the parameter value by matching the Gini coefficient of the wealth distribution of entrepreneurs in the SCF. The chosen value $\alpha = 0.70$ is similar to the values used in the related papers (for example, Cagetti and De Nardi (2006) and Terajima (2006) set it to 0.88 and 0.70, respectively). The bottom two rows of Table 2 compare the wealth distributions. Again, the empirical statistics illustrate well that the distribution is highly skewed. The model matches the entire wealth distribution as well as the Gini index.

To choose the value of the collateral constraint parameter κ, we target the average level of the leverage ratio (debt-to-net-wealth ratio) in the SCF. Following this strategy, we obtain $\kappa = 0.41$. Table 3 presents the comparison between the data and the model. In addition to the leverage ratio, we also present the wealth-to-income ratio. While the model predicts a somewhat smaller wealth-to-income ratio than observed in the data, our calibration matches the overall pattern that the relative level of wealth is quite high among entrepreneurs as pointed out by Quadrini (2000). Kitao (2008) also calibrates a parameter that characterizes a collateral constraint similar to ours by targeting the maximum leverage ratio of 0.5. In our model, the corresponding leverage ratio can be defined by $-b'/((1-\delta)k' + b')$. For the constrained entrepreneurs, it is rewritten as $\frac{\kappa}{(1-\delta)q-\kappa}$ with the use of the last equation of problem (6), namely, $qb' = -\kappa k'$. The calibrated value for κ together with the equilibrium

Table 3: Leverage Ratio and Wealth-to-Income Ratio

	Data	Model
Leverage ratio	0.29	0.27
Wealth-to-income ratio	8.20	5.99

NOTES: Empirical statistics are based on the sample of entrepreneurs in the SCF and the averages across five waves (1992, 1995, 1998, 2001, and 2004). See Appendix 1 for the construction of the statistics.

interest rate of the safe asset (which will be discussed in the next paragraph) implies that the leverage ratio of the constrained entrepreneurs in our model equals 0.83. Note that the higher leverage ratio in our calibration means that entrepreneurs in our model face a looser borrowing constraint than Kitao's calibration implies. As we will argue, the tightness of the borrowing constraint is one of the key ingredients that introduce nonlinearity into the model. The looser constraint puts us on the conservative side of the calibration. Last, we assign the discount factor to match the level of the return on the safe asset. The interest rate deviates from the $1 - 1/\beta$ due to the incomplete market assumption. We calculate the empirical measure by taking the one-year constant-maturity Treasury yield reported in the FRB−H.15 table and then subtracting the realized inflation rate calculated by the PCE deflator. The average real interest rate over 1960-2009 was 2.23%. By setting β at 0.92, we obtain the steady-state risk-free rate at 2.44%. We set the steady-state risk-free rate to a level somewhat higher than the historical average, taking into account that the presence of the aggregate risk pushes down the interest rate.

4 Steady-State Equilibrium

In this section, we first study the behavior of the model in the steady-state equilibrium under the benchmark calibration just discussed. We then discuss the behavior of the model under several alternative calibrations.

4.1 Nonlinearity in Saving Functions

Let us first study the characteristics of the entrepreneurs' saving policy functions under the benchmark calibration. Recall from problem (6) that each entrepreneur enters the current period with the sum of his capital stock and bond holding $(1-\delta)k_i + b_i$. After idiosyncratic productivity z_i is realized, he makes the consumption-saving decision based on the total resources available to him x_i. The saving decision also includes the portfolio choice between investment in the safe asset qb' and in his own risky technology k'.

In panel (a) of Figure 1, total saving $k_i' + qb_i' (= x_i - c_i)$ is plotted against the initially

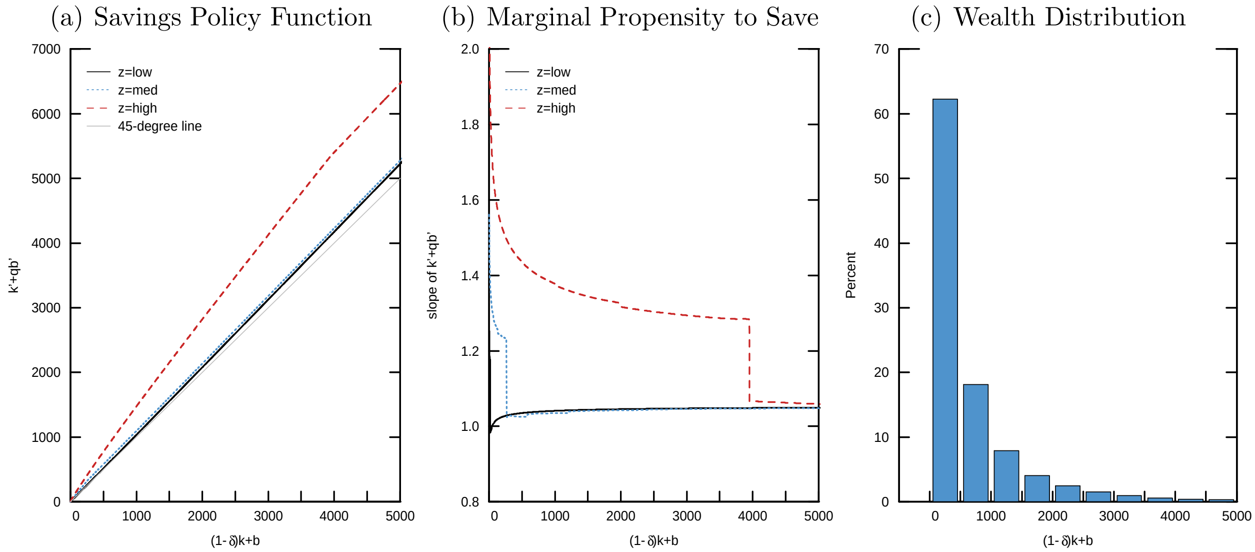

Figure 1: Nonlinearity of Savings Function and Wealth Distribution

NOTES: Saving policy function for each productivity level is computed by assuming that the entrepreneur enters the current period with $(1-\delta)k+b$ (measured along the horizontal axis) and a certain realized level (high, medium, and low) of z in the previous period. The vertical axis measures the expected level of $k'+qb'$ conditional on the previous period-level of z. High and low levels of z are, respectively, located at $\pm 1.9 \times \sigma_z$ from the center (= medium level) of the state space of $\log(z)$.

available resources $(1-\delta)k_i + b_i$. Note that total saving also differs with respect to idiosyncratic productivities z_i. Three lines correspond to the decision rules with three different levels of realized productivities high, medium, and low. The noticeable feature in this figure is that the most productive entrepreneurs save significantly more than the other types of entrepreneurs. Observe also a kink around the initial wealth level around 4,000 in the saving rule of the most productive entrepreneurs. While nonlinearity is less obvious in the policy function of the other two types of entrepreneurs, it indeed exists.

Panel (b) of Figure 1 plots the slopes of the saving rules of the same three types of entrepreneurs. Again, first let us focus on the most productive entrepreneurs. This entrepreneur has a strong incentive to save across a wide range of the initial wealth level. The saving takes the form of investment in his risky technology. Further, the smaller the initial wealth level is, the stronger the incentive to expand the size of the firm. This feature comes from the decreasing returns to scale of the technology. The downward jump in the slope, which corresponds to the kink in panel (a), has to do with the borrowing constraint. Panel (a) of the next figure (Figure 2) presents the underlying portfolio choice of these productive entrepreneurs. As can be seen clearly, these entrepreneurs borrow as much as the borrowing constraint permits until the initial wealth level hits a certain level, at which point the borrowing constraint is no longer binding.

Next, consider the entrepreneurs whose productivity level is in the middle of the state

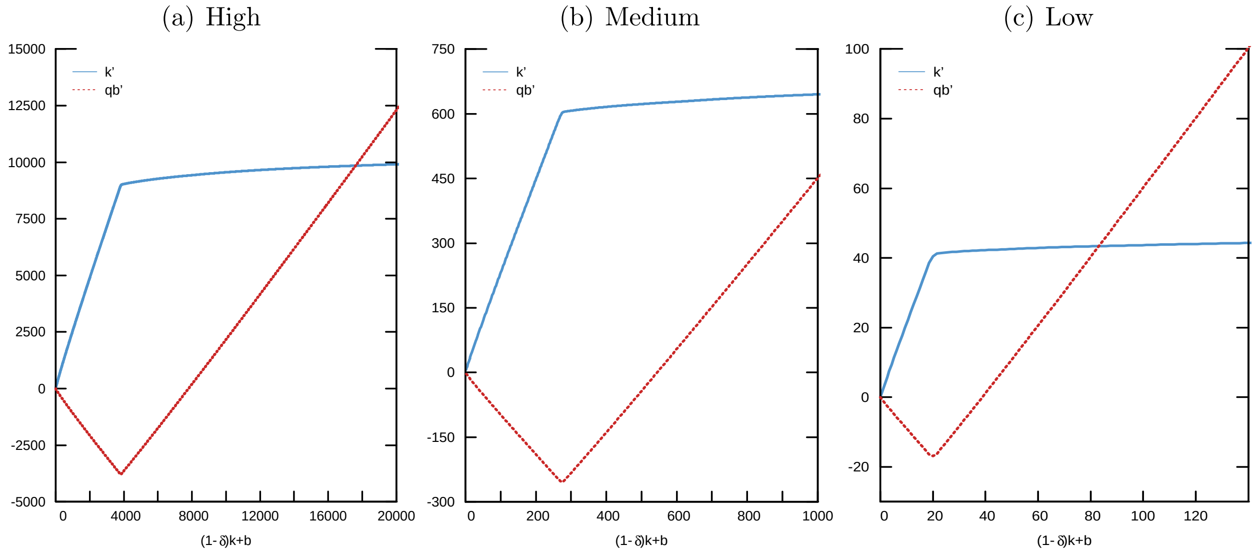

Figure 2: Portfolio Choice of Different Types of Entrepreneurs

NOTES: Each panel plots the portfolio choice of entrepreneurs with each productivity type, high, medium or low. See notes to Figure 1 for explanations about the three productivity levels.

space. While the saving policy function itself did not present nonlinearity clearly, its slope illustrates the nonlinearity of the function more clearly. Qualitatively speaking, the same idea just discussed for the productive entrepreneurs applies: Up to a certain level of initial wealth, their marginal propensity to save declines sharply from a high level and then jumps down to a level close to one, from which point on it is roughly constant. Panel (b) of Figure 2 describes the underlying portfolio decision. Note that this panel zooms in on the initial wealth level that is much smaller than that in the panel to the left. The portfolio choice takes the form of investing in his risky business, selling bonds as much as the constraint permits. Note, however, that the point at which the entrepreneur is no longer constrained comes much sooner than in the case of the most productive entrepreneurs. Last, when idiosyncratic productivity is low, the middle panel of Figure 1 shows that similar nonlinearity exists only for a very small level of initial wealth. Thus, for almost all wealth levels, saving is (almost) linearly related to the initial wealth level.

4.2 Comparison to the Models with Labor Income Risk

Note that the nature of nonlinearity in our model differs significantly from that in the models with uninsurable labor income risk. For instance, consider the model of Krusell and Smith (1998), in which workers face an uninsurable risk of losing his/her job. In this environment, nonlinearity is most visible for those who are the poorest (measured by the wealth level) and are unemployed. The marginal propensity to save for those workers is less than that for the rest of population, since they must cut down their wealth to consume. In our environment, nonlinearity is most noticeable for the poorest and *most productive* entrepreneurs and the marginal propensity to save is *greater* than those for the rest of the population. This feature makes sense, given that the entrepreneurs with a low level of capital have a stronger incentive to borrow and invest the funds in their private businesses because of decreasing returns to scale in technology. The incentive is larger for those who draw higher idiosyncratic shocks.[11]

Another important difference is the shape of the wealth distribution. The last panel of Figure 1 presents the stationary wealth distribution of x_i. By comparing this figure with Figure 1, one can see that a large mass of entrepreneurs lie in the region where the saving rule is nonlinear. Recall also that in the models with uninsurable labor income risk, the lower end of the distribution has a thin mass because of the self-insurance motive to avoid a large drop in consumption at the time of job loss. In our model, in contrast, a large mass of poor entrepreneurs exists in equilibrium because poor entrepreneurs enjoy a large expected return from their risky capital, which counteracts the self-insurance mechanism. Another important result in our model is that a large fraction of entrepreneurs are constrained. The benchmark calibration implies that roughly one-half of entrepreneurs are constrained (the first column of Table 5 summarizes the steady-state values of important variables).

4.3 Risk-Free Rate

The heterogeneous portfolio choice discussed above determines the market-clearing bond price. Recall that we have chosen the entrepreneurs' discount factor at 0.92, which generates the risk-free rate at 2.44% in the steady state. That is, a relatively low risk-free rate arises even when entrepreneurs heavily discount the future. The low risk-free rate arises because, while there is a large mass of borrowers in equilibrium as indicated by the previous discussion, the supply of funds from those who are rich and happen to receive a bad productivity draw is large enough to depress the risk-free rate.

Krusell and Smith (2006) study a two-period version of a model similar to ours, assuming that the idiosyncratic productivity shock is i.i.d. They find that the two-period model is also able to generate a low risk-free rate. However, they emphasize the feature in their setup that poor entrepreneurs have a strong motivation to invest in the safe asset, since they value insurance more (i.e., precautionary savings), lowering the risk-free rate. The mechanism working in our setup is somewhat different. Recall that in our environment and calibration, the least productive entrepreneurs are almost always bond holders, and similarly, productive entrepreneurs are likely to be borrowers. In other words, consideration of realized idiosyncratic productivity levels dominates the portfolio choices of these entrepreneurs given that idiosyncratic risks are highly persistent in our model. Further, entrepreneurs whose productivity is in the middle of the state space become bond holders as they become richer not poorer because the returns to their business decline as the business gets larger. The low risk-free rate in our model mainly results from these rich entrepreneurs who are savers with

[11] Poor and unproductive entrepreneurs in our model behave similarly to the poor unlucky workers in the models with uninsurable labor income risk. One can see this from the black solid line in panel (b) of Figure 1 where the poor and unproductive are cutting down wealth.

Table 4: Distribution of Private Equity Premiums

	Wealth Percentile Range					
z_i	0−20	20−40	40−60	60−80	80−100	95−100
Low	3.25	2.81	2.73	2.66	2.58	2.53
	(0.32)	(0.02)	(0.01)	(0.00)	(0.00)	(0.00)
Middle	8.03	5.58	4.36	3.51	2.79	2.67
	(4.65)	(2.39)	(1.29)	(0.56)	(0.00)	(0.00)
High	26.23	19.37	14.62	10.70	6.04	4.26
	(21.17)	(14.98)	(10.67)	(7.08)	(2.84)	(1.24)

NOTES: See Equation (11) for the definition of the private equity premium. Each entry gives the average private equity premium (%) when an entrepreneur is in the specified idiosyncratic productivity level and wealth percentile range. The premium due to the borrowing constraint is in parenthesis and calculated according to the first term on the right-hand side of (11). See notes to Figure 1 for explanations about the productivity levels.

low expected returns to their business.[12]

4.4 Private Equity Premiums

Next, we examine how the model's properties discussed so far relate to private equity premiums. Note that in the model each entrepreneur's risky investment is directly subject to a large idiosyncratic risk. The risk-averse entrepreneur demands a premium over the safe asset associated with this risk. Angeletos (2007) shows that in his model, this mechanism generates the average private equity premium of 2-8% (depending on the calibrations). Our model, however, differs from this model in several ways. First, the idiosyncratic shock in his model is assumed to be i.i.d., while it is highly persistent in our model. Second, entrepreneurs in our economy are subject to a borrowing constraint. Equation (11) shows that the wedge associated with the borrowing constraint can play an important role in generating the equity premium in our model.

In our benchmark calibration, the economy generates the average equity premium of 2.8% in the steady state. While it still seems lower than the realistic value reported by Moskowitz and Vissing-Jørgensen (2002), it is larger than the equity premium reported in the models with uninsurable labor income risk.[13]

To see the mechanism generating the relatively large average equity premium in our

[12]It is true that poor entrepreneurs indeed have a stronger precautionary saving motive with everything else being constant. However, the effect on the risk-free rate from these poor entrepreneurs is quantitatively small.

[13]It is difficult to obtain an off-the-shelf target for the empirical value for the average private equity premium. Moskowitz and Vissing-Jørgensen (2002), however, report that the median of the distribution of capital gains in private business investment is 6.9% in the 1989 SCF.

Table 5: Effects of Alternative Parameter Values

	Benchmark Calibration	$\kappa = 0.55$	$\rho_z = 0.50$	$\sigma_z = 0.35$
Gini index (Wealth)	0.593	0.594	0.388	0.477
Gini index (Income)	0.627	0.649	0.443	0.492
Leverage ratio	0.273	0.337	0.219	0.243
Wealth-to-income ratio	5.99	6.04	5.37	5.48
Average equity premium (%)	2.70	1.74	2.25	1.70
Risk-free rate (%)	2.44	3.06	4.66	4.82
Fraction of borrowers (%)	64.77	57.16	61.84	63.20
Fraction of constrained borrowers (%)	49.41	32.74	35.34	41.66

model, Table 4 presents private equity premium over wealth and idiosyncratic productivity levels. The table shows a clear pattern that more (less) productive entrepreneurs demand a higher (lower) premium, and smaller (larger) entrepreneurs demand a higher (lower) premium. To gauge the importance of the borrowing constraint, the table further presents the distribution of the wedge associated with the borrowing constraint in parenthesis. One can see that the borrowing constraint plays a dominant role in generating private equity premiums in our economy. As we emphasized before, smaller and more productive entrepreneurs have a stronger incentive to invest in their business, and this incentive interacts with the borrowing constraint. That is, smaller and more productive entrepreneurs are more likely to be constrained by the borrowing limit and thus the wedge created by the borrowing constraint is quantitatively more important for those entrepreneurs.

4.5 Comparative Statics

Table 5 presents how endogenous variables in the steady state respond to several alternative calibrations. The first column presents the results under the benchmark calibration. We focus on changes in the following three parameters: (i) κ: tightness of the collateral constraint, (ii) ρ_z: the persistence parameter of the idiosyncratic productivity process, and (iii) σ_z: its unconditional standard deviation.[14]

First, consider the effects of raising κ from 0.41 to 0.55, allowing entrepreneurs to borrow more in terms of a fraction to their capital stock. The direct effect can be seen as an increase in the average leverage ratio from 27% to 34%. Another direct effect is in the decline in the fraction of the constrained borrowers (last row). The Gini index for wealth and income both increase slightly. This makes intuitive sense because the borrowing constraint works to limit the expansion of the capital stock. Larger inequality in the capital stock translates into a

[14] See Covas (2006) for more extensive comparative statics. This paper uses a different specification for the borrowing constraint than the one used in Covas (2006) but many of the intuitions carry over.

large inequality in output (income). The average equity premium decreases from 2.8% to 1.8% because the wedges associated with the borrowing constraint shrink. Next, the risk-free rate increases by roughly 60 basis points. The higher risk-free rate makes sense given that relaxing the borrowing constraint creates an additional demand for funds from previously constrained entrepreneurs and that it reduces the precautionary demand for the risk-free asset. However, the fraction of borrowers drops from 65% to 57%. This comes from the general equilibrium effect of the higher interest rate that some of the previously borrowing entrepreneurs become suppliers of funds.[15]

Next, consider the effect of lowering ρ_z from 0.9 to 0.5. This case is important in that the earlier studies often assume for analytical reasons that the shock is i.i.d. The lower persistence parameter naturally reduces inequalities of income and thus wealth (i.e., times with good (bad) shocks do not continue as long as in the benchmark calibration). The incentive to borrow when the same good shock is realized becomes weaker: As the persistence of the shock increases, entrepreneurs attempt to exploit the opportunity as much as they can, and this incentive is reduced as the shock becomes less persistent. Both the fraction of borrowers and the fraction of the constrained borrowers consequently decline. Despite the weaker demand for funds, the equilibrium risk-free rate increases significantly from 2.44% to 4.66%. A lower persistence of the shock reduces the traditional precautionary saving motive through the safe asset.[16] Thus, especially when entrepreneurs receive the bad shock, their bond holdings are lower than in the benchmark calibration. This mechanism generates the higher interest rate, even though the demand for borrowing declines. The decline in the average equity premium is consistent with the higher risk-free rate and the lower fraction of the constrained borrowers.

The last column presents the results when σ_z is reduced from 0.5 to 0.35. Again, this directly reduces income and wealth inequalities. Other than this direct effect, there are two effects that drive the results. First, there is a smaller chance that entrepreneurs are constrained by the borrowing limit. We therefore see the fraction of the constrained entrepreneurs going down. Second, lower uncertainty reduces the need for savings through bonds for the precautionary reason to smooth consumption. This increases the risk-free rate as discussed in the previous paragraph. The average equity premium decreases from 2.7% to 1.7% because entrepreneurs demand a lower premium on risky investments given the lower uncertainty and because they are less likely to be hit by the borrowing constraint.

5 Aggregate Uncertainty

This section examines the quantitative properties of the model under the presence of aggregate uncertainty. To solve the model, we need to take a stand on how to capture the information of the wealth distribution Γ. Thus, we first examine the accuracy and efficiency

[15]Note that the effect on the interest rate is more likely to be affected by the change in the behavior of relatively wealthy entrepreneurs, whereas the fraction measure weights all entrepreneurs equally and therefore can decline as a result of the increase in the equilibrium interest rate.

[16]The same result can be found in the model with uninsurable labor income risk (see Aiyagari (1994)).

of several different ways to capture this information. The details of the solution algorithm are presented in Appendix B. We then look at cyclical properties of the model, using the most accurate solution.

5.1 Bond Price Forecasting Rules

The following four specifications are considered for capturing the information of the wealth distribution Γ. In the first specification, it is assumed that the mean of the capital distribution ($\bar{K}$) is the only information that the entrepreneurs use to forecast the bond price. More specifically, $\bar{K}$ is linearly related to q. Solving the individual problem requires to predict the next period bond price q', which is carried out by predicting the mean of the next-period capital distribution:

$$\begin{aligned} \log \bar{K}' &= \phi_{11}(\theta) + \phi_{12}(\theta) \log \bar{K}, \\ q &= \phi_{21}(\theta) + \phi_{22}(\theta) \log \bar{K}, \end{aligned} \tag{16}$$

where ϕs are regression coefficients. Note that the coefficients depend on the aggregate state θ. Specifically, the two different sets of coefficients are used depending on the realization of θ, which takes two discrete values $\bar{\theta}_1$ and $\bar{\theta}_2$. Note also that this specification is exactly the same as the one used by Krusell and Smith (1997), where workers have access to two assets: aggregate capital and risk-free bonds.[17] Krusell and Smith (1997) face the same issue as we do – that they need to solve for the equilibrium bond price that satisfies the market clearing condition. They also assume that the bond price is linearly related to the aggregate capital stock.[18]

The next specification we consider adds dispersion (i.e., standard deviation) of the capital distribution to the information set as follows:

$$\begin{aligned} \log \bar{K}' &= \phi_{11}(\theta) + \phi_{12}(\theta) \log \bar{K} + \phi_{13}(\theta) \log \sigma_k, \\ q &= \phi_{21}(\theta) + \phi_{22}(\theta) \log \bar{K} + \phi_{23}(\theta) \log \sigma_k, \\ \log \sigma_z' &= \phi_{31}(\theta) + \phi_{32}(\theta) \log \bar{K} + \phi_{33}(\theta) \log \sigma_k, \end{aligned} \tag{17}$$

where σ_k is the standard deviation of the distribution of the capital holdings by entrepreneurs. This can be considered a somewhat naive specification given that it is unclear how the second moment helps improve the forecasting performance. We include this specification in our analysis simply because it is a natural extension of the previous one and is often considered in other applications in the literature.

The next specification adds the fraction of the constrained entrepreneurs to the first

[17] The model is an extension of Krusell and Smith (1998) where aggregate capital is the only vehicle for saving available to the workers.

[18] The return on the capital stock of the representative firm can be directly calculated as its marginal product of the production function, once the aggregate capital stock is known.

specification.

$$\begin{aligned}\log \bar{K}' &= \phi_{11}(\theta) + \phi_{12}(\theta)\log \bar{K} + \phi_{13}(\theta)\omega,\\ q &= \phi_{21}(\theta) + \phi_{22}(\theta)\log \bar{K} + \phi_{23}(\theta)\omega,\\ \omega' &= \phi_{31}(\theta) + \phi_{32}(\theta)\log \bar{K} + \phi_{33}(\theta)\omega,\end{aligned} \tag{18}$$

where ω is the fraction of the constrained entrepreneurs. As we saw in the previous section, our economy consists of a large number of entrepreneurs who are willing to invest more in their business but cannot, due to the borrowing constraint. Thus, it seems natural to expect that this statistic has predictive power for the bond price (we will discuss this more specifically later). Note that relative to the specification in (16), solving the model with the ones specified by (17) and (18) requires us to deal with an additional state variable and is thus computationally more expensive.

The last specification adopts the following parsimonious forecasting rule, in which the next-period bond price is assumed to be a linear function of the current-period bond price (conditional on current- and next-period aggregate state).[19]

$$q' = \psi_1(\theta, \theta') + \psi_2(\theta, \theta')q. \tag{19}$$

In this specification, the current-period bond price is directly taken to be a state variable. The entrepreneur uses this forecasting equation to predict the next-period bond price when solving his problem. Note that the bond-price forecast can be conditioned on the four possible cases depending on θ and θ'. The reason is that the current-period bond price is pinned down after the aggregate shock θ is realized. In the previous three cases, this conditioning was not made because the distribution of capital is predetermined at the start of each period (before θ is realized).[20] The main idea behind the previous three specifications is to add moments or some other statistics characterizing the distribution to the information set. However, it is ex-ante unclear what information is most useful for predicting the market-clearing price. But the autoregressive specification subscribes to the idea that q itself can provide the useful information for predicting q'. This procedure is expected to work when the transition function $H(.)$ that links $\Gamma(q)$ and $\Gamma'(q')$ is stable, although this is ultimately a quantitative question. A clear advantage of this specification from the previous two specifications is its efficiency in terms of computational time.[21]

[19] A similar approach has been used in the literature in different contexts. See, for example, Telmer and Zin (2002), Zhang (2005), and Gomes and Michaelides (2006).

[20] Note that if one assumes a specification in which the moments of the distribution of x' are used to predict the bond price, the same conditioning as in (19) is possible, given that the distribution x' reflects the realized value of θ' (see the definition of x' in 6)). We considered a specification in which the first moment of the total wealth is linearly linked to the bond price as in (16) but the same conditioning as in (19) is permitted. We find that performance of that specification is very similar to the one based on (16). This implies that differences in the accuracy, which we discuss shortly, are not due to the conditioning on θ' allowed in the autoregressive specification.

[21] The first-moment-only specification has two aggregate state variables θ and $\bar{K}$ where ones with either the dispersion measure or the fraction measure have three aggregate state variables.

Table 6: Forecasting Rules: Three Distribution-Based Specifications

Specification	$\bar{K}$ only		$\bar{K}$ and σ_k		$\bar{K}$ and ω	
$\theta =$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$
Forecast for	$\log \bar{K}'$					
ϕ_{11}	0.1722	0.2048	0.1629	0.1953	0.2027	0.2410
	(0.0009)	(0.0010)	(0.0010)	(0.0010)	(0.0012)	(0.0013)
ϕ_{12}	0.9725	0.9700	0.9554	0.9500	0.9714	0.9684
	(0.0001)	(0.0002)	(0.0010)	(0.0009)	(0.0001)	(0.0001)
ϕ_{13}			0.0172	0.0199	−0.0449	−0.0516
			(0.0010)	(0.0009)	(0.0015)	(0.0015)
R^2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
S.E.	0.0004	0.0004	0.0004	0.0004	0.0003	0.0003
Max | error |	0.0017	0.0015	0.0016	0.0015	0.0013	0.0010
DW stat.	0.2681	0.2713	0.3774	0.4139	0.4284	0.4448
Forecast for	q					
ϕ_{21}	0.8013	0.7882	0.8121	0.7983	0.7574	0.7405
	(0.0010)	(0.0010)	(0.0010)	(0.0010)	(0.0008)	(0.0009)
ϕ_{22}	0.0274	0.0282	0.0468	0.0492	0.0290	0.0302
	(0.0001)	(0.0002)	(0.0011)	(0.0010)	(0.0001)	(0.0001)
ϕ_{23}			−0.0196	−0.0209	0.0660	0.0693
			(0.0011)	(0.0010)	(0.0010)	(0.0011)
R^2	0.9591	0.9545	0.9662	0.9656	0.9890	0.9879
S.E.	0.0004	0.0004	0.0004	0.0004	0.0002	0.0002
Max | error |	0.0015	0.0015	0.0015	0.0013	0.0007	0.0008
DW stat.	0.1741	0.1808	0.2645	0.2980	0.7948	0.7931
Forecast for			$\log \sigma_k'$		ω'	
ϕ_{31}			0.2846	0.3060	0.1276	0.1187
			(0.0131)	(0.0131)	(0.0061)	(0.0066)
ϕ_{32}			0.1081	0.0691	0.0046	0.0060
			(0.0134)	(0.0122)	(0.0006)	(0.0006)
ϕ_{33}			0.8583	0.8939	0.6937	0.6810
			0.0134	(0.0120)	(0.0075)	(0.0078)
R^2			0.9956	0.9951	0.8564	0.8465
S.E.			0.0048	0.0049	0.0016	0.0016
Max | error |			0.0176	0.0167	0.0051	0.0063
DW stat.			2.0165	1.9794	1.6214	1.5611

NOTES: See equations (16) for the "$\bar{K}$ only" specification, (17) for the "$\bar{K}$ and σ_k" specification, and (18) for the "$\bar{K}$ and ω" specification. Standard errors are in parenthesis.

5.2 Accuracy

Tables 6 and 7 present the converged coefficients and some accuracy measures under the four specifications just described. In assessing the accuracy of each forecasting rule, we consider

Table 7: Autoregressive Forecasting Rules

$(\theta, \theta') =$	$(\bar{\theta}_1, \bar{\theta}_1)$	$(\bar{\theta}_1, \bar{\theta}_2)$	$(\bar{\theta}_2, \bar{\theta}_1)$	$(\bar{\theta}_2, \bar{\theta}_2)$
	(a) Benchmark Calibration			
ψ_1	0.0122	−0.0654	0.0759	0.0130
	(0.0021)	(0.0063)	(0.0057)	(0.0022)
ψ_2	0.9875	1.0590	0.9298	0.9868
	(0.0021)	(0.0064)	(0.0059)	(0.0023)
R^2	0.9943	0.9919	0.9930	0.9930
$S.E.$	0.0002	0.0002	0.0002	0.0002
Max |error|	0.0006	0.0004	0.0005	0.0007
DW stat.	1.936	$n.a.$	$n.a.$	1.744
	(b) $\kappa = 0.55$			
ψ_1	0.0140	−0.0546	0.0768	0.0159
	(0.0018)	(0.0055)	(0.0050)	(0.0019)
ψ_2	0.9854	1.0479	0.9287	0.9838
	(0.0019)	(0.0056)	(0.0052)	(0.0020)
R^2	0.9956	0.9934	0.9948	0.9945
$S.E.$	0.0001	0.0002	0.0001	0.0002
Max |error|	0.0005	0.0004	0.0005	0.0006
DW stat.	1.968	$n.a.$	$n.a.$	1.786
	(c) $\rho_z = 0.50$			
ψ_1	0.0251	−0.0313	0.0823	0.0311
	(0.0013)	(0.0040)	(0.0037)	(0.0014)
ψ_2	0.9736	1.0239	0.9223	0.9676
	(0.0014)	(0.0042)	(0.0039)	(0.0015)
R^2	0.9974	0.9959	0.9972	0.9969
$S.E.$	0.0001	0.0001	0.0001	0.0001
Max |error|	0.0004	0.0004	0.0003	0.0005
DW stat.	2.450	$n.a.$	$n.a.$	2.365
	(d) $\sigma_z = 0.35$			
ψ_1	0.0165	−0.0493	0.0775	0.0195
	(0.0010)	(0.0030)	(0.0028)	(0.0011)
ψ_2	0.9826	1.0423	0.9277	0.9798
	(0.0011)	(0.0032)	(0.0029)	(0.0011)
R^2	0.9986	0.9978	0.9984	0.9982
$S.E.$	0.0001	0.0001	0.0001	0.0001
Max |error|	0.0004	0.0003	0.0004	0.0004
DW stat.	1.925	$n.a.$	$n.a.$	1.712

NOTES: See equation (19) for the specification. The D.W. statistic is not calculated when $(\theta, \theta') = (\bar{\theta}_1, \bar{\theta}_2)$ or $(\bar{\theta}_2, \bar{\theta}_1)$.

the following three metrics.[22] The first is the fit of the regression R^2, as is often used in the literature. While commonly used in the literature, it may not necessarily be the right metric to assess the accuracy of the solution, as pointed out by den Haan (2010). The reason is that an R^2 scales the error term by the variance of the dependent variable and thus can sometimes be misleading. Another problem is that there is no benchmark to determine whether an R^2 is high or low. The standard error of the regression is free from these problems and thus we use it as a second metric. The third metric we use is the largest absolute error over a long time series. Also discussed by den Haan (2010), the R^2 and regression standard error measure only the average accuracy over the long time series and thus can hide large forecasting errors that occur only occasionally. The third metric deals with this concern.

Recall that the forecasting rules differ depending on the value of θ (and θ' in the case of the autoregressive specification) and thus are estimated separately. Also note that we report the Durbin-Watson (DW) statistic to examine the presence of autocorrelation in the error term. The presence of the autocorrelation implies that there remain unexploited predictable variations in the variable to be forecast.

The first two columns of Table 6 present the converged coefficients and the accuracy measures for the first-moment-only specification. The top panel shows that the forecasting equation for the first moment of the capital stock performs very well: The fit of the regression is (almost) perfect; the standard error of the regression is less than 0.04%; the maximum error in predicting the average capital stock is roughly 0.1-0.2%. The ultimate interest, however, is in the bond-price equation, which is presented in the middle panel of the table. The performance of the bond price is less than satisfactory: The fit of the regression is around 0.95; the regression standard error is 4 basis points (i.e., entrepreneurs over- or under-predict the next period bond price by 4 basis points on average); and the maximum error agents make amounts to 15 basis points. Observe also that the DW statistic is substantially smaller than 2, indicating the presence of a strong serial correlation in the error term. Krusell and Smith (1997) provide a useful point of comparison. In their application, forecasting equations for both the average capital stock and the bond price display high accuracy: The fit of both equations is higher than 0.99999999.[23] Compared to the highly accurate solution often obtained in the model of labor income risk, the performance of this specification is not robust in our model and there is certainly room for improvement.

The next two columns present the results when we add the dispersion measure of the capital distribution to the list of the state variables. Forecasting rules for the first and second moments perform reasonably well with the fit exceeding 0.99 for both variables. However, the performance of the bond price equation improves little: the fit increases from 0.96 to 0.97; and the improvement of the regression standard error and maximum error is also minimal. We can thus conclude that the gains are too small to justify the computational cost of this specification. Next, adding the fraction of the constrained entrepreneurs to the model brings material improvement in forecasting the bond price (as shown in the last two

[22] Our economy consists of 100,000 entrepreneurs with 3,000 periods after discarding the first 500 periods.

[23] More precisely speaking, their bond price equation includes the squared term of the mean capital stock. It is unlikely that this term has a large effect on improving the fit. They do not report either the regression standard error or maximum error.

columns of the same table), although the fit of the regression is still not as good as the one reported by Krusell and Smith (1997). The fit of the regression goes up to a level close to 0.99. Significant improvements are also observed in the regression standard errors and the maximum errors. Interestingly, despite the additional forecasting power of this variable, the DW statistic still indicates the presence of a positive autocorrelation in the error terms. Further, a clear downside of this specification is again the significant computational cost that accompanies the assumption that the entrepreneurs compute these additional moments and incorporate them in making their decision.

The performance of the autoregressive forecasting rules is presented in Table 7. The results from the baseline calibration are presented in the top panel. Relative to the first-moment-only specification (which has roughly the same computational cost), the advantage of this specification is clear in every accuracy measure. We can also observe improvements of the accuracy measures, even relative to the last specification that includes a fraction of the constrained borrowers. Further, this specification eliminates the positive serial correlation that was previously prevalent, i.e., the DW statistic is now close to 2.

5.3 Cyclical Properties of the Model

Having established that the autoregressive forecasting rule gives the accurate solution in an efficient manner, we can discuss the cyclicality of the model using the solution based on this forecasting rule. We focus on the model's asset pricing implications. Panel (a) of Table 8 presents the mean, the standard deviation, and the correlation coefficient with aggregate output of the three variables, the risk-free rate, average equity premium, and the fraction of the constrained entrepreneurs.[24] We calculate the statistics conditional on the current period aggregate state ($\bar{\theta}_1$ (low) or $\bar{\theta}_2$ (high)) as well as the unconditional statistics.

Let us first discuss the effects of the presence of aggregate uncertainty on the mean level of these three variables. Relative to the steady-state levels presented in the first column of Table 5, the risk-free rate has a lower mean, while the average equity premium and the fraction of the constrained entrepreneurs have a higher mean. Recall that the risk-free rate is calibrated to be 2.44% in the steady state. It goes down slightly to 2.36% with aggregate uncertainty. Aggregate uncertainty lowers the risk-free rate because entrepreneurs value the risk-free asset more. The fraction of the constrained entrepreneurs goes up by one percentage point relative to the steady-state level. In subsection 4.5, we saw that reducing the size of the idiosyncratic risk lowers the fraction of the constrained entrepreneurs. The opposite idea applies here: A larger risk facing the entrepreneurs makes it more likely that they are hit by the borrowing constraint. The average equity premium also increases from 2.7% to 3%. The additional 30 basis points of the premium comes from two sources. The first is the standard risk-compensation channel for risk-averse entrepreneurs. The second is through larger wedges associated with the borrowing constraint, as reflected in a larger fraction of the constrained entrepreneurs.

[24] The equity premium of each individual entrepreneur is calculated by $E\theta' z_i' f'(k_i') - r$. We take the average across all entrepreneurs.

The standard deviation of the risk-free rate amounts to roughly 40 basis points unconditionally. The next two cells indicate that roughly one-half of this total variation comes from the conditional standard deviations: The risk-free rate fluctuates as the wealth distribution evolves within a given aggregate state (we will discuss the mechanism in the next paragraph). The unconditional standard deviation of the average equity premium also amounts to 40 basis points. The size of the fluctuations is roughly comparable to the volatility of the (public) equity premium in a standard RBC model (see, for example, Table 2 in Guvenen and Kuruscu (2006)). We do not have a tight empirical estimate for the volatility of the private equity premium. But Moskowitz and Vissing-Jørgensen (2002) claim that the volatility of the average private equity return is as large as that of an index of publicly traded equities. Assuming that this observation is plausible, fluctuations of the aggregate private equity premium in our model are too small.

The last three columns present conditional as well as unconditional correlations with aggregate output. The risk-free rate is procyclical unconditionally as expected. This procyclicality comes from the switches of the aggregate state: exogenously higher productivity shifts resources to the private business away from the risk-free bonds, thereby raising the risk-free rate. The risk-free rate is, however, strongly negatively correlated with aggregate output within each state. To understand this, suppose that the economy continues to be in the good aggregate state ($\theta = \bar{\theta}_2$). The capital stock continues to grow along this path. However, since its marginal return declines as the size grows, entrepreneurs shift their portfolio to the bonds, lowering its interest rate. This effectively serves to shield the entrepreneurs against the risk of lower consumption when the aggregate state eventually switches to the bad state. In the case when the bad aggregate state continues, the risk-free rate increases over time even though aggregate output declines, because the marginal product of capital increases along the path.[25]

Our model generates the countercyclical financial constraint, as reflected in the countercyclicality of the fraction of the constrained entrepreneurs. The countercyclicality exists both unconditionally and conditionally. The unconditional countercyclicality results from the increases (declines) in output during good (bad) times relaxing (tightening) the borrowing constraint.[26] This statistic is also negatively related to output within each state. Suppose again that the economy continues to be in a good state. As the good state continues, an entrepreneur, who used to be constrained, can accumulate enough wealth, so that he is no longer constrained. The opposite mechanism works when the bad state continues for several

[25]Note that in the absence of the decreasing returns to scale, the risk-free rate would have been procyclical within the same aggregate state because as the entrepreneurs get richer (poorer), they have lower (greater) demand for the safe asset.

[26]To see this more explicitly, consider the situation in which the aggregate state switches from the bad state to the good state. Since a smaller fraction of the entrepreneurs is constrained in the good state, it must be the case that there exist "marginal" entrepreneurs who were constrained before but no longer constrained after the switch. The demand for investment in their business for these agents increases as the aggregate state switches to the good state. However, the increased "cash flow" (i.e., output) today is more than enough to cover the increased demand for investment again due to declining marginal product of capital. The opposite is true for those who were unconstrained in the boom period but are not constrained when the economy switches to a downturn.

Table 8: Cyclical Properties of Asset Prices Under Various Calibrations

	Mean (%)			Std. Dev.			Cor. with Y		
		$\theta=\bar{\theta}_1$	$\theta=\bar{\theta}_2$		$\theta=\bar{\theta}_1$	$\theta=\bar{\theta}_2$		$\theta=\bar{\theta}_1$	$\theta=\bar{\theta}_2$
(a) Benchmark Calibration									
r	2.36	2.00	2.73	0.43	0.22	0.22	0.35	−0.96	−0.96
$E(R_i')-r$	2.98	3.29	2.66	0.33	0.09	0.08	−0.90	−0.95	−0.92
ω	50.43	51.02	49.82	0.75	0.44	0.43	−0.72	−0.22	−0.25
(b) $\kappa=0.55$									
r	2.97	2.60	3.35	0.44	0.23	0.23	0.35	−0.97	−0.97
$E(R_i')-r$	2.31	2.63	1.98	0.33	0.07	0.06	−0.88	−0.96	−0.91
ω	34.39	35.07	33.67	0.82	0.42	0.43	−0.74	−0.19	−0.24
(c) $\rho=0.50$									
r	4.58	4.20	4.97	0.47	0.26	0.26	0.27	−1.00	−1.00
$E(R_i')-r$	2.15	2.47	1.81	0.33	0.08	0.06	−0.88	−0.99	−0.97
ω	37.88	38.38	37.35	0.56	0.23	0.21	−0.82	−0.53	−0.41
(d) $\sigma_z=0.35$									
r	4.70	4.29	5.13	0.50	0.28	0.27	0.28	−0.99	−0.99
$E(R_i')-r$	1.73	2.17	1.26	0.46	0.07	0.05	−0.83	−0.95	−0.88
ω	43.76	44.40	43.08	0.79	0.44	0.44	−0.83	−0.54	−0.54

NOTES: r: risk-free rate; $E(R_i')-r$ where $R_i'=\theta' z_i' f'(k_i')-r$: average expected excess return on private business; ω: fraction of the constrained entrepreneurs; Y: aggregate output. The first column of each block presents the unconditional statistics. The second and third columns present the same statistics conditional on the current aggregate TFP being $\bar{\theta}_1$ (low) and $\bar{\theta}_2$ (high), respectively.

periods. Recall that in Table 6, we considered the bond-price forecasting rule that features the fraction of the constrained entrepreneurs. In that specification, we find that this variable is positively related to the bond price. This is consistent with our finding here that the risk-free rate is conditionally negatively related to the fraction of the constrained entrepreneurs. Last, the average equity premium is countercyclical both conditionally and unconditionally. It should be clear that this countercyclicality comes from two sources: first, the countercyclical wedges associated with the borrowing constraint; and second, the usual channel that the covariance between the marginal utility of consumption and output is negative (see Equation (11)).

5.4 Sensitivity

We briefly discuss how the dynamic behavior of the model changes under alternative calibration. As before we consider three cases: (i) higher κ (0.41 → 0.55), (ii) lower ρ_z (0.90 → 0.5), and (iii) lower σ_z (0.50 → 0.35). First, panels (b) through (d) in Table 7 present the corresponding autoregressive forecasting rules under these three alternative calibrations.

Across all three cases, we obtain the solutions that are as accurate as the one under the benchmark calibration, implying the robustness of the autoregressive forecasting rule with respect to the alternative calibrations. The robustness with respect to alternative calibration as well as an earlier analysis that compares the different forecasting rules under the benchmark calibration appears to suggest that this specification would work well in other applications as well.

Panels (b) through (d) in Table 8 present the cyclical properties of the model under the alternative calibrations. All three cases imply that aggregate uncertainty lowers the average risk-free rate and raises the average equity premium and the fraction of the constrained entrepreneurs relative to the corresponding steady-state values in Table 5. The correlation pattern is largely intact in relation to the pattern observed in the benchmark calibration. Across all three calibrations considered, the risk-free rate is procyclical (unconditionally), the financing constraint measured by the fraction of the constrained entrepreneurs is countercyclical, and equity premium is countercyclical.

6 Conclusion and Future Research

This paper has studied the quantitative characteristics of a general equilibrium model of uninsurable investment risk. The model is calibrated to match key features of the income and wealth distributions of entrepreneurs. We show that the calibrated model has many features that are different from models with uninsurable labor income risk, whose quantitative features have been extensively studied in the literature. In particular, we have shown that the collateral constraint together with decreasing returns to scale play important roles in generating sizable private equity premiums and a highly skewed wealth distribution.

In solving the model under the presence of aggregate uncertainty, we have examined the accuracy and efficiency of several different bond-price forecasting rules. We find that the parsimonious autoregressive forecasting rule provides a significantly more accurate solution than the popular forecasting rule based on the first moment of the distribution of capital holdings. We also find that it performs better than more computationally demanding forecasting rules that incorporate the dispersion of the capital distribution or the fraction of the constrained entrepreneurs.

We find that the model has some promising asset-pricing implications. The model is capable of generating a low risk-free rate as well as a relatively high average equity premium. It also produces the procyclicality of the interest rate and countercyclicality of the equity premium and financing constraint. However, it is true that the model lacks many features that are important in reality. Further extensions are necessary in order to use the model to understand broader macroeconomic issues. Of first-order importance seems to be augmenting the model with the corporate sector as well as the household sector as is done in Covas (2006) (by suppressing aggregate uncertainty). This extension would allow us to examine broader asset-pricing implications in a more realistic environment (e.g., a joint examination of the private equity premium and public equity premium).[27] The findings in this paper serve us

[27]Other possible extensions include introducing entry into and exit from entrepreneurship, as considered

well to prepare for these ambitious future research projects.

A Construction of Empirical Measures

The statistics used to calibrate the model are constructed using the Survey of Consumer Finances (SCF) for 1992, 1995, 1998, 2001, and 2004. "Entrepreneurs" are identified as households that satisfy the following two criteria: respondent or any member of the family living together (i) owns or shares ownership in any privately held businesses, farms, professional practices or partnerships and (ii) has an active management role in any of these businesses. The series used to calibrate the model are constructed as follows.

- Net Wealth = Assets − Debts
- Asset = Financial Assets + Nonfinancial Assets
- Financial Asset = Checking Accounts + Savings Accounts + MMDA + MMMF + Call Accounts + Certificates of Deposit + Savings Bonds + State and Local Tax-Exempt Bonds + Mortgage-Backed Securities + US Government Bonds + Corporate and Foreign Bonds + Stocks + Stock Mutual Funds + Tax-Free Bond Mutual Funds + Other Bond Mutual Funds + Combination and Other Mutual Funds + IRA Accounts + Thrift Plans + Future Pension Benefit Accounts + Cash Value of Life Insurance + Annuities, Trusts and Other Managed Assets + Other Financial Assets
- Nonfinancial Assets = Primary Residence + Other Residential Real Estate + Nonresidential Real Estate + Vehicles +Actively Managed Businesses + Nonactively Managed Businesses + Other Nonfinancial Assets
- Debts = Mortgages (Primary Residence and Other Residential Properties) + Debt Associated with Nonresidential Properties + Unsecured Lines of Credit + Installment Loans (Vehicles, Education, and Other) + Credit Card Debt + Other Debts
- Income = Wages and Salaries + Business Income + Other Interest Income + Dividends + Net Gains or Losses from the Sale of Stocks, Bonds, or Real Estate + Net Rent, Trusts, or Royalties + Unemployment Benefits + Child Support or Alimony + Welfare + Social Security or Other Pensions + Any Other Sources
- Wealth/Income Ratio = (Net Wealth)/Income
- Leverage Ratio = Debts/Net Wealth

Note that the model abstracts from many of the components in the above data construction. Given that there is little guidance regarding how we associate the concepts in

by Quadrini (2000) and Cagetti and De Nardi (2006), and "real" frictions, such as capital adjustment costs or irreversibility of investment.

the model and in the observed data, all balance-sheet and income components are included. To account for outliers in the micro data, we use a Stata module, Winsor, by which we replace the top and bottom 0.5% of observations by the next value, counting inward from the extremes. However, this treatment has little impact on the final results.

B Solution Algorithm

This subsection describes the numerical procedures used to compute equilibrium in our economy with aggregate shocks. We first describe the algorithm in which each entrepreneur uses the first moment of capital only to predict the next-period bond price. Two steps are involved. In the first step, the numerical procedure solves the individual problem taking the forecasting rules for the first moment of capital and the bond price as given. In the second step, it updates the forecasting rules and goes back to the first step. This iteration continues until the forecasting rules converge. The algorithm starts by guessing the coefficients of the following forecasting rules for the first moment of capital and the bond price:

$$\log \bar{K}' = \begin{cases} \phi_{11}(\bar{\theta}_1) + \phi_{21}(\bar{\theta}_1) \log \bar{K}, \\ \phi_{11}(\bar{\theta}_2) + \phi_{21}(\bar{\theta}_2) \log \bar{K}, \end{cases} \tag{20}$$

$$q' = \begin{cases} \phi_{21}(\bar{\theta}_1) + \phi_{21}(\bar{\theta}_1) \log \bar{K}', \\ \phi_{21}(\bar{\theta}_2) + \phi_{22}(\bar{\theta}_2) \log \bar{K}'. \end{cases} \tag{21}$$

As mentioned in the main text, note that the two sets of forecasting rules are used depending on the realized level of θ. Using these rules, the algorithm solves the entrepreneurial problem by solving for a fixed point in the consumption function. The policy function $c(z, x; \theta, \bar{K})$ is approximated with a piecewise bilinear interpolant of the state variables, $(x, \bar{K})$. The variable x is discretized in non-uniformly spaced grid points with 500 nodes. More grid points are allocated to lower values of wealth, given the nonlinearity in the consumption function owing to the presence of borrowing constraints. The variable $\bar{K}$ is discretized in a uniformly spaced grid points with 40 nodes. The idiosyncratic productivity process, z, is assumed to follow a Markov chain with seven states. The discretization of the exogenous stochastic process follows the numerical method proposed by Rouwenhorst (1995). The aggregate technology process, θ, follows a Markov chain with two states and the discretization is described in the main text.

Given an initial guess c_0, use expressions (4), (7), and (8) with $\lambda = 0$ to find (c_1, k_1', b_1') at each grid point. After computing the solution at each grid point, check whether the choice of bond holdings violates the borrowing constraint. In the cases where the borrowing constraint is violated, i.e., $b_1' < -\kappa k_1'$, set $b_1' = -\kappa k_1'$ and determine (c_1, k_1') using (4) and (8) at those grid points. Use c_1 as the new initial guess and iterate on this procedure until $\sup |\ln c_1 - \ln c_0|$ over all grid points is less than some convergence parameter. We set $\epsilon_1 = 0.0000001$.

In the second step of the procedure the algorithm updates the coefficients in (20) and (21).

It generates a large panel that consists of 100,000 entrepreneurs over 3,500 periods. In each period, the equilibrium bond price q is determined such that the aggregate bond holdings are zero. Determining the equilibrium bond price uses the bisection algorithm. It guesses the bond price q by assuming that the equilibrium bond price lies in the interval $[q_l, q_u]$. Given this interval, let the equilibrium bond price equal $\frac{1}{2}[q_l + q_u]$ and solve the entrepreneur's problem. Then, compute the sample average of bond holdings. If it is positive, then set $q_l = q$ and repeat the above steps. Otherwise, set $q_u = q$ and repeat until $|E(b')| < \epsilon_1$. This process is repeated for 3,500 periods. After discarding the first 500 periods, we run the regressions using 3,000 observations of $\bar{K}$ and q. This entire process is repeated until the difference of all coefficients between the two iterations is less than some convergence parameter. We set $\epsilon_2 = 0.0001$.

To solve the economy with two moments of the distribution, we augment equations (20)–(21) with the second moment of the capital distribution as in (17). This case adds one more state variable to the problem. The same procedure is used to solve the case in which the fraction of the constrained entrepreneurs in lieu of the second moment of the capital distribution is added to the set of the state variables.

Lastly, when the forecasting rule of the bond price follows a first-order autoregressive process, the set of state variables is now $\{z, x; \theta, q\}$. Note that the current bond price is a state variable of the problem and replaces $\bar{K}$. In order to solve the entrepreneur's problem we need to evaluate the term $E[c(z', x'; \theta', q'|z, \theta)]$. The algorithm starts by guessing the coefficients of the forecasting rule presented in equation (19), which can be rewritten more explicitly as follows:

$$q' = \begin{cases} \psi_1(\bar{\theta}_1, \bar{\theta}_1) + \psi_2(\bar{\theta}_1, \bar{\theta}_1)q, \\ \psi_1(\bar{\theta}_1, \bar{\theta}_2) + \psi_2(\bar{\theta}_1, \bar{\theta}_2)q, \\ \psi_1(\bar{\theta}_2, \bar{\theta}_1) + \psi_2(\bar{\theta}_2, \bar{\theta}_1)q, \\ \psi_1(\bar{\theta}_2, \bar{\theta}_2) + \psi_2(\bar{\theta}_2, \bar{\theta}_2)q. \end{cases} \tag{22}$$

As mentioned in the main text, the forecasting rule can be conditioned on θ' as well as θ. This is because the equilibrium bond price is determined after the aggregate productivity level is realized. The remaining steps of the algorithm follow the same steps as before.

References

Aiyagari, R., "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics* 109 (1994), 659–684.

Angeletos, G.-M., "Idiosyncratic Investment Risk and Aggregate Savings," *Review of Economic Dynamics* 10 (2007), 1–30.

Angeletos, G.-M. and L.-E. Calvet, "Idiosyncratic Production Risk, Growth and the Business Cycle," *Journal of Monetary Economics* 53 (2006), 1095–1115.

CAGETTI, M. AND M. DE NARDI, "Entrepreneurship, Frictions, and Wealth," *Journal of Political Economy* 114 (2006), 835–870.

CAGGESE, A., "Entrepreneurial Risk, Investment and Innovation," Unpublished Manuscript, 2009.

CARROLL, C. D., "Portfolios of the Rich," in L. Guiso, M. Haliassos and T. Jappelli, eds., *Household Portfolios: Theory and Evidence* (Cambridge, Massachusetts: MIT Press, 2002), 299–339.

COVAS, F., "Unisured Idiosyncratic Production Risk with Borrowing Constraints," *Journal of Economic Dynamics and Control* 30 (2006), 2167–2190.

DAVIS, S., J. HALTIWANGER, R. JARMIN AND J. MIRANDA, "Volatility and Dispersion in Business Growth Rates: Publicly Traded and Privately Held Firms," *NBER Macroeconomics Annual* (2006).

DEN HAAN, W., "Solving Dynamic Models with Aggregate Shocks and Heterogeneous Agents," *Macroeconomic Dynamics* 1 (1997), 355–386.

———, "Assessing the Accuracy of the Aggregate Law of Motion in Models with Heterogeneous Agents," *Journal of Economic Dynamics and Control* 34 (2010), 79–99.

GENTRY, W. AND G. HUBBARD, "Entrepreneurship and Household Saving," *Advances in Economic Analysis & Policy* 4 (2004), 1–52.

GOMES, F. AND A. MICHAELIDES, "Asset Pricing with Limited Risk Sharing and Heterogeneous Agents," *Review of Financial Studies* (2006).

GUVENEN, F. AND B. KURUSCU, "Does Market Incompleteness Matter for Asset Prices?," *Journal of the European Economic Association Papers and Preceedings* 6 (2006), 484–492.

HAMILTON, B., "Does Entrepreneurship Pay? An Empirical Analysis of the Returns to Self-Employment," *Journal of Political Economy* 108 (2000), 604–631.

HEATON, J. AND D. LUCAS, "Portfolio Choice and Asset Prices: The Importance of Entrepreneurial Risk," *Journal of Finance* 55 (2000), 1163–1198.

KHAN, A. AND J. THOMAS, "Idiosyncratic Shocks and the Role of Nonconvexities in Plant and Aggregate Investment Dynamics," *Econometrica* 76 (2008), 395–436.

KITAO, S., "Entrepreneurship, Taxation and Capital Investment," *Review of Economic Dynamics* 11 (2008), 44–69.

KOPECKY, K. AND R. SUEN, "Finite State Markov-Chain Approximations to Highly Persistent Processes," *Review of Economic Dynamics* 13 (Jun 2010), 701–714.

Krusell, P. and A. Smith, "Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns," *Macroeconomic Dynamics* 1 (1997), 387–422.

———, "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy* 106 (1998), 867–896.

———, "Quantitative Macroeconomic Models with Heterogeneous Agents," in R. Blundell, W. Newey and T. Persson, eds., *Advances in Economics and Econometrics: Theory and Applications* (Cambridge University Press, 2006).

Moskowitz, T. and A. Vissing-Jørgensen, "The Returns to Entrepreneurial Investment: A Private Equity Premium Puzzle?," *American Economic Review* 92 (2002), 745–778.

Quadrini, V., "Entrepreneurship, Saving, and Social Mobility," *Review of Economic Dynamics* 3 (2000), 1–40.

Rouwenhorst, G., "Asset Pricing Implications of Equilibrium Business Cycle Models," in T. Cooley, ed., *Frontiers of Business Cycle Research* (Princeton University Press, 1995).

Storesletten, K., C. Telmer and A. Yaron, "Cyclical Dynamics in Idiosyncratic Labor Market Risk," *Journal of Political Economy* 112 (2004), 695–717.

Tauchen, G. and R. Hussey, "Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models," *Econometrica* 59 (1991), 371–396.

Telmer, C. and S. Zin, "Prices as Factors: Approximate Aggregation with Incomplete Markets," *Journal of Economic Dynamics and Control* 26 (2002), 1127–1157.

Terajima, Y., "Education and Self-Employment: Changes in Earnings and Wealth Inequality," Bank of Canada Working Paper No. 2006-40, 2006.

Young, E., "Approximate Aggregation," Mimeo, 2005.

Zhang, L., "The Value Premium," *Journal of Finance* 60 (2005), 67–103.