Algorithmic Social Sciences Research Unit

ASSRU

Department of Economics
University of Trento
Via Inama 5
381 22 Trento Italy

Discussion Paper Series

10 – 11

COUPLED DYNAMICS IN A PHILLIPS MACHINE MODEL OF THE MACROECONOMY♣

Stefano Zambelli

December 2010

♣ This paper has been written thanks to the valuable suggestions and the inspirations of my teacher, colleague and friend Vela Velupillai. Just as an example on how he has been influential in shaping this article, let me just say that the title is his suggestion.

Abstract

In this paper it is claimed that the Phillips machine is a nonlinear mechanism. Phillips (1950) presented his machine as being described with a linear differential equation. In this paper it is claimed that the machine is better described with a system of nonlinear difference.-differential equations. This system itself is approximated by a set of nonlinear differential equations. This differential equations are of the Hicks-Goodwin flexible multiplier accelerator type. Consistently with this conjecture and following Phillips' suggestion we have presented a 'digital' simulation of two coupled would-be Phillips machines.

JEL Codes: A10, B16, B41, C63, F44

Keywords: computable economics; flexible-accelerator, coupled dynamical systems, Goodwin, Phillips machine, MONIAC.

1. Introduction

Phillips (1950, p. 283) stated that his analog machine was *"... an attempt to develop some mechanical models which may help non-mathematicians by enabling them see the quantitative changes that occur in an inter-related system of variables following initial changes in one or more of them"*.

It was much more than just that. From the theoretical as well as practical point of view simulations are important non only as '*pedagogical*' devices, but are extremely useful in order to study problems with tools that go beyond what one – mathematician or not - can achieve with the abstractions that are possible with the aid of paper and pencil (i.e. mathematical symbolic manipulations). Mathematics – any type one can endorse to – has the limitation that it has to be, in a way or the other, tractable (at least from a symbolic point of view). Mathematics or meta-mathematics allow clarity of the methods and of the aims, but it might force the scholars to leave out details or to 'modify' the problems so as to fit the tight jacket of the mathematical notation.

The Phillips machine is a beautiful machine which should have been studied by Phillips himself for what it actually delivered as results. It should have been a type of laboratory to be used to run experiments and not only as a pedagogical tool for *'non-mathematicians'*. In fact, and most likely, the more famous Phillips curve is a result coming out of attempts to have the MONIAC to provide 'answers' to questions posed to it.

Clearly, the Phillips Machine was a remarkable project and was extremely advanced. At the time in which Phillips was constructing the machine others were building the very first electronic digital computers. It was the time in which one of the first computers, the ENIAC, was operating with vacuum tubes and its memory could store only 20 numbers with 10 digits.

The idea of discussing economic problems by making an analog(y) with other systems is of course not new. Following the history of economic thought one can find many examples. Irving Fischer, a great *'mathematical'* economist used the analogies of rational mechanics and that of hydraulic systems. In fact he did construct an analog hydraulic machine. And also Joseph Schumpeter, one of the greatest *'non-mathematical'* economist of the 20th century, used a large number of analogies with the aim of trying to encapsulate his ideas on business cycles. These systems went from 'hydraulic pendula', 'sound waves' and so on. He was demanding to the "mathematically" oriented economists to develop models to tackle his 'economics'. The exchange he had with Frisch (1933) is a demonstration of this aim and does contain a description of his many conceptual analog models[a].

2. The Linearity Dogma

The construction of a machine to be used to study the economy and to study stabilization policy is no doubts a great achievement, but Phillips, with the exceptions of a few (among them Goodwin and Hicks), did fell into what Samuelson (1974, p. 10) called a dogma. Samuelson writes "… *Thus, by 1940, Metzler and I as graduate students at Harvard* ***fell into the dogma*** *that all economic business-cycle models should have* damped *roots. We accepted Frisch's criticism of the Kalecki procedure of imposing constraints on his parameter-estimating equations so that roots would be neither damped nor undamped ... what was so bad about the dogma? Well, it slowed down our recognition of the importance of* ***non-linear*** *auto-relaxation models of the van der Pol-Rayleigh type, with their characteristic amplitude features lacked by linear systems. And, in my case, it led to suppressing development of the Harrod-Domar exponential growth aspects that kept thrusting themselves on anyone who worked with accelerator-multiplier systems (emphasis added)".*

[a] Please note that both Fisher and Schumpeter have been founders of the Econometric Society. They were searching for ways to model their ideas with the use of mathematical tools. I think that it can be claimed that both Fisher and Schumpeter they were not at all 'modifying' their theories so as to adjust the available mathematical tools.

Velupillai (1992) has made the strong point, that is easily supported with a careful reading of the original article, that Frisch's (1933) *"Propagation Mechanism"* was as explicitly stated by Frisch himself a non-linear one. It is in the attempt of allowing for a solution that Frisch made the functional forms linear by '*first approximations'*. These '*first approximations'* where never removed and many of the students of the business cycles by following Frisch's approach fell into the dogma of linearity. The mathematical business cycle models by Kalecki (1935), Samuelson (1939), Metzler (1941) are all linear and the (damped or undamped) cycles are possible thanks to leads and lags of the variables. Kaldor (1940) did present a non-linear model of the cycle using diagrams, but did not have a mathematical formulation for it. Literal economists, like Schumpeter, had to rely on their written explanations to be able to grasp the non-linear elements of their theories. As Samuelson pointed out most mathematically oriented economists did fell into the dogma of linearity. His *multiplier-accelerator* model is an example of it.

Economics is filled with nonlinear relations. One can claim that there is no actual economic phenomena or magnitude which is **not** nonlinear[b]. The very existence of physical constraints should be enough to convince any reader.

[b] One should not even feel the need to explain. But given that economic theory is filled with linear relations we are now so accustomed to think linearly that I feel obliged to spend a few words on it. Just think in terms of physical constraint. A machine which functions by transforming factors of production into the final output will be naturally bound to a maximum speed, beyond which it will either break or produce less. The consumption capacity of individuals is limited by satiation, i.e. per each unit of time there is so much that one individual can consume. The speed in which goods can be transferred from one place to another is limited by a maximum speed. Any aggregate economy cannot use more natural resources, land and labour than it actually has.

Phillips did fell on the linear dogma in several ways.

First, there is no doubt, at least to the present writer, that the MONIAC was and is a nonlinear machine. It is nonlinear simply because there is a maximum speed in which a flow of water can fall. This is so in general, i.e. when the water is not constrained inside a pipe, but it is even more so when constrained inside a pipe. If water falls from one tank to another located below through a pipe, the speed in which the water would flow would depend on the pressure (quantity of water in the tanks) and would change as the water decreases from one tank and increases in the other. Moreover the pipes do bend and each bending is a form of nonlinearity which introduces at the bending regions different frictions. Furthermore there are 'ceiling' and 'floors' that limit the operation of the system.

Second, the mathematical presentation of the model made by Phillips is constrained to linear functional forms. It is clear that he was mostly inspired by Metzler's (1941) linear leads and lags model[c].

Third, it will be claimed below that the different mathematical models presented in his Phillips (1950) article are all difference-differential linear equations. The difficulty in solving a difference-differential equations model is that for its solution it needs as 'initial condition' the values associated to the interval $[(t_0 - \varepsilon), t_0]$ where ε is the length of the lag. Given the difficulties to deal with the specification of this function – which would be most likely nonlinear – further simplifying assumptions are made. One of them is to consider the lag ε

[c] When Phillips refers to Hicks' work he refers to his assumption about leads and lags and the distinction between liquid and working stocks (see Phillips, 1950, p. 289). When he refers to Goodwin's work he refers to support a linear relation between income and transactions money balances (Phillips 1950 p. 289) or in the case of the use of a *linear* accelerator (Phillips 1950, p. 298).

constant and the other is to assume that any function mapping the values from $[(t_0 - \varepsilon), t_0]$ to $[t_0, (t_0 + \varepsilon)]$ is a constant function (i.e. linear).

Frisch (1933) did make assumptions like the one described above. His system was formed by the three equations below[d] :

(1) $$y(t) = mx(t) + \mu\dot{x}(t)$$

(2’) $$\dot{x}(t) = c - \lambda(rx(t) + sz(t))$$

(3) $$z(t) = \int_0^{\varepsilon} D(\tau) y(t-\tau) d\tau$$

The first two are linear and the functional linearity of the third one depends on $D(\tau)$. Frisch did simplify by defining $D(\tau) = 1/\varepsilon$. Hence (3) can be substituted with

(4) $$\dot{z}(t) = \frac{y(t) - y(t-\varepsilon)}{\varepsilon}$$

This means that there is a constant lag with value ε.

I have written elsewhere that Frisch never solved, contrary to what generally believed, the above system of equation. Its solution would have required either a digital or analogue computation. In Zambelli (2007) I provide the digital solution.

[d] For a detailed presentation of the above see Frisch (1933) or Zambelli (2007). As mentioned above the system is a linear system, but in the first part of the article Frisch did explicitly refer to nonlinear functional forms.

3. The Phillips linear difference-differential equations

Phillips (1950) article contains the description of analogue machines which are graphically described in four figures. One is an actual photograph of the first machine and the other three are drawings of their functioning. As mentioned above an inspection of the three drawings would show that the machine could not work in a linear fashion: it could do so only when ‘forced’ to do so.

Phillips describes the Phillips machine for different cases:

a) *The Multiplier with Constant Rate of Interest* (Phillips 1950, pp. 294-6);
b) *The Multiplier with Constant Quantity of Money* (Phillips 1950, pp. 296-7);
c) *The* (Linear) *Accelerator* (Phillips 1950, pp. 297-8).

For the above cases the mathematical description is highly linear and strong assumptions are present.

3.1. *a) The Multiplier with Constant Rate of Interest (Phillips 1950, pp.294-6). A comment.*

The model described by Phillips here implies a constant period of production P, and constant consumption. Given that $I(t) \equiv S(t)$ is an accounting identity in order for equation (I) in Phillips (1950, 294) to hold it must be that the saving value is lagged, say with lag *L*, $I(t) - S(t-L) = \frac{dM_1}{dt}$, and hence we have that $E(t) - Y(t-L) = I(t) - S(t-L) = P\frac{dY(t)}{dt} = \frac{dM_1}{dt}$. These relations imply a difference-differential system of equations or a nonlinear system (or both). Moreover an inspection of the equations will also determine that $C(t) = C(t-L) = \overline{C}$. This requires a very special and specific dynamical path and very specific functional forms. In fact, the description provided at the top of page 296 is relative to a mixed system of difference-differential equations. An inspection of the drawing of the machine, Fig. 3 (Phillips, 1950 p. 290) will confirm that there is a lag between the S and I. Clearly whether water molecules become consumption or saving would depend on the different valves and the liquid in the different tanks, but saving is ‘transformed’ in investment (or income

Y is transformed into demand E) only after it has gone through the pipes going from the junction Y-C-S down to the junction C-I-E and this takes time.

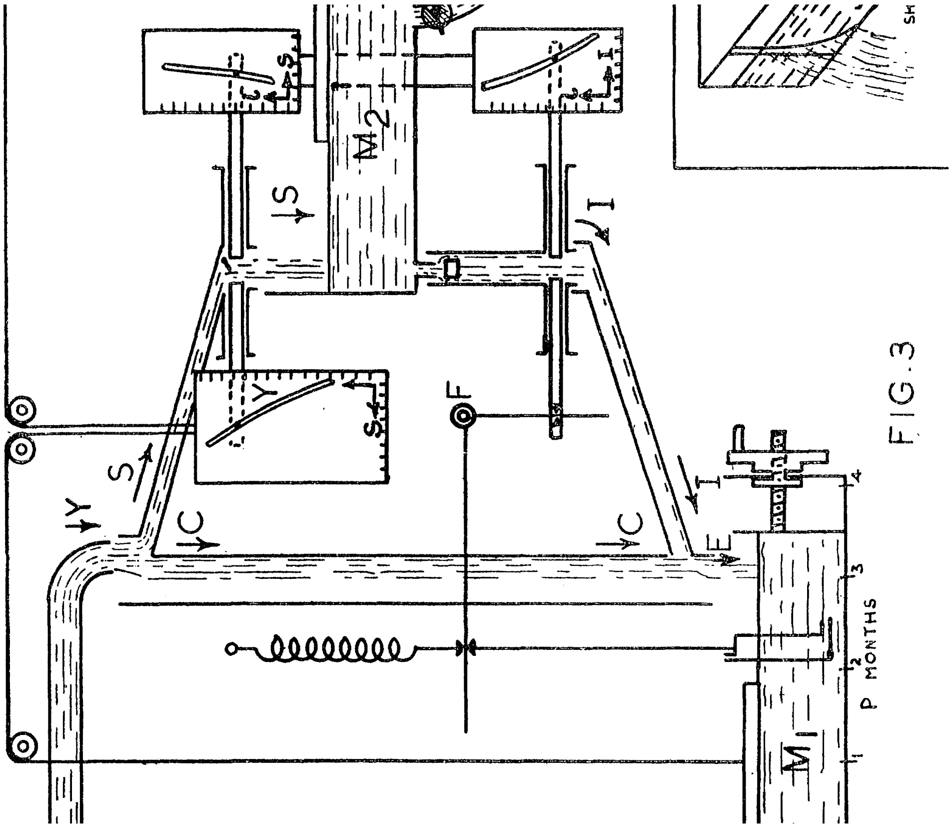


FIG. 3

3.2. *b) The Multiplier with Constant Quantity of Money (Phillips 1950, pp. 296-7). A comment.*

The equations of this session are based on the equations of session of 3.1 and by extension either the actual system must be nonlinear or it must be a difference-differential system of equations. Furthermore, now like consumption also the quantity of money held for transaction purposes is assumed to be constant. This is adds a further constraint to the model. In order to fulfill these constraints these other variables would have to follow nonlinear functional rules. That is a variation of . This implies that other elements would have to be free to vary.

3.3. *c) The (Linear) Accelerator (Phillips 1950, pp. 297-8). A comment.*

The description of the accelerator of this section is too linear. Investment is given by (Phillips, 1950, p. 298)

(5) $$I(t) = P\frac{dY(t)}{dt} + \sigma Y(t)$$

With the proper change of symbolic notation the above is the same accelerator function as in the above equation (1), which is the linear Frisch (1933) accelerator.

Differentiation of (5) leads to the second order linear differential equation (in Phillips 1950, p.298 is equation 15)

(6) $$\gamma P\frac{d^2Y(t)}{dt^2} + \left(\gamma\sigma + P - \beta\right)\frac{dY(t)}{dt} + \sigma Y(t) = 0$$

The contributions by Hicks and by Goodwin have been that (5) is actually non linear in the sense that there are boundaries (a ceiling and a floor) which limit the speed of accumulation and decumulation of capital. This imply, almost tautologically, that the model is nonlinear. If this is so for their paper and pencil ‘theoretical’ model it must be even more so for the Phillips machine where the physical constraints are set by physical boundaries such as fixed maximum volume, gravitational force and so on.

4. The flexible nonlinear accelerator

Equations (1) and (5) above are special cases of a more general accelerator relation

(7) $$I(t) = \dot{K}(t) = F(Y(t), \dot{Y}(t))$$

With the standard meaning for the symbols, I(t) is aggregate investment, *K(t)* is aggregate capital and *Y(t)* is aggregate output.

A reasonable functional form which is consistent with Goodwin's flexible accelerator idea may be given by[e]:

(8) $$I(t) = \dot{K}(t) = \varphi(vY(t) - K(t))$$

$$\frac{\delta\varphi(vY(t)-K(t))}{\delta(vY(t)-K(t))} > 0 \text{ and } \frac{\delta\varphi(0)}{\delta(vY(t)-K(t))} = v$$

$$\lim_{vY(t)-K(t)\to+\infty} \varphi(v(Y(t)-K(t)) = k^{**} \;\; ; \;\; \lim_{vY(t)-K(t)\to-\infty} \varphi(v(Y(t)-K(t)) = k^{*}$$

[e] For further elaborations and explanations see Zambelli (2011). The functional form presented in the text has an argument which is different from both Hicks (1950) and Goodwin (1951) accelerator models. The seminal works by Frisch (1933), Hicks (1950) and Goodwin (1951) contain a minor infelicity. The idea of the accelerator is the idea that investment is determined by the difference between the *desired* level of capital and the *actual* level of capital. The infelicity is given by the fact that a dynamic equation for K(t) is not made explicit (although mentioned in the textual description) and is actually removed. Any attempt to reinsert will show that in all the three models while Y(t) stays in between boundaries (like the constant in the computation of the primitive function of an integral) K(t) will evolve without boundaries. Here this is amended and the difference between desired capital, vY(t), and actual capital, K(t), are the determinants of investment decisions. This explains the formulation of the text.

The idea behind the above functional form is similar to that of the flexible accelerator of the original article by Goodwin (1951) where vY(t) is the desired capital level and K(t) the existing one. In normal conditions, i.e., near normal production capacity exploitation, the net investment adjusts fast to the production needs so that around the equilibrium condition, i.e., *vY(t)-K(t)* = 0, a linear accelerator holds. In fact $\delta\varphi(0)/\delta Y(t)$ is equal to the constant capital output ratio *v*. When current demand is inadequate with respect to production capacity, because it is either too high or too low, the investments levels tend to either *k*** or *k** : the production or destruction of capital goods per time unit cannot go above or below these physical limits. It is the opinion of this writer that the water flowing inside the Phillips machine has to follow similar constraints.

Like in Phillips (1950) we can work assuming two different lags: the lag between consumption decision and its actual realization ε (Robertson lag); the lag between the moment in which decision of investment is made and its realization $\vartheta+\varepsilon$ (the Lundberg lag).

The total demand at time $t+\vartheta+\varepsilon$ is given by

$$Y((t+\vartheta)+\varepsilon) = C((t+\vartheta)+\varepsilon) + I((t+\vartheta)+\varepsilon) \tag{9}$$

where ε is the Robertson lag and $\vartheta+\varepsilon$ the Lundberg lag.

The lag between the moment in which income is earned and it is spent may be described as follows:

$$C((t+\vartheta)+\varepsilon) = C_0 + Y(t+\vartheta) \tag{10}$$

The fact that it takes *time to build* and consequently there is a lag between the moment at which an investment decision is made and the capital goods are actually delivered may be described as follows:

$$I((t+\vartheta)+\varepsilon) = \dot{K}(t) = \varphi(vY(t) - K(t)) \tag{11}$$

Substituting equations, (10) and (11) into (9) we have the law of motion of our economy which is described by a mixed nonlinear difference-differential equation.

This differential equation can be ‘solved’ either through digital computation or through an analog machine like the Phillips machine. It is my conjecture that the Phillips machine must have had constraints represented by the above.

Following the method used by Goodwin (1951) in order to maintain the structure of the model as simple as possible the mixed difference differential equation can, with some losses in precision, be approximated by a second order differential equation. After some simple truncations of Taylor series expansions we obtain

(12) $$\varepsilon\vartheta Y(t) = -(\varepsilon + (1-c)\vartheta)\dot{Y}(t) - (1-c)Y(t) + C_0 + \varphi(vY(t) - K(t))$$

The state space representation of equation (10) is given by:

(13a) $$\dot{Y}(t) = Z(t)$$

(13b) $$\dot{Z}(t) = b\left[Co - (l\text{-}c)\,Y(t) - aZ(t) + \varphi(vY(t)\text{-}K(t))\right]$$

(13c) $$\dot{K}(t) = \varphi(vY(t) - K(t))$$

$$b = \frac{1}{\varepsilon\theta}; \quad a = \varepsilon + \vartheta(1-c)$$

One can check that the above model can account for cyclical behavior. For a wide range of the parameter values, the dynamical system evolves towards a limit cycle. In fact the analysis of the Jacobian shows that the equilibrium point is a repellor and that the variables evolve inside a closed compact set.

5. Coupling or coupled Phillips machines: the two national systems case.

Phillips (1950, p. 305) concluded his article by writing “*It is possible to connect together two of the models shown in Fig. 4, to deal with the multiplier relationship between the incomes of two countries, or of one country and the rest of the world. To connect more than two would be difficult, since each country would have to have a propensity to import function for each other country. The easiest method of interconnection would be to assume a fixed rate of exchange, and run the imports flow of one model into the export tube of the other*”.

This coupling of machines is definitely worthwhile if the machine is not a linear machine. In the case of a linear machine the results are relatively simple and analytical solutions can be computed.

This is not so in the case of a nonlinear machine. From what written above it should be clear that it is my conjecture that the Phillips machine was and is a nonlinear one. The model written in section 4 is most likely the closest to the Phillips described case of the Phillips machine operating with fixed interest rates. This being the case a digital coupling of two digital would-be Phillips machines has been made in Zambelli (2011). As described above in the Phillips' (1950) quotation the countries are linked through trade where the exchange rate is fixed and where it is the case that "*the imports flow of one model (run) into the export tube of the other* ".

A natural way in which to introduce coupling is by considering the fact that countries are linked through trade.

The flow accounting descriptions of economy 1 and 2 are given by:

(14) $$M_1(t+\vartheta_1+\varepsilon_1)+Y_1(t+\vartheta_1+\varepsilon_1)\equiv C_1(t+\vartheta_1+\varepsilon_1)+I_1(t+\vartheta_1+\varepsilon_1)+X_1(t+\vartheta_1+\varepsilon_1)$$

$$M_2(t+\vartheta_2+\varepsilon_2)+Y_2(t+\vartheta_2+\varepsilon_2)\equiv C_2(t+\vartheta_2+\varepsilon_2)+I_2(t+\vartheta_2+\varepsilon_2)+X_2(t+\vartheta_2+\varepsilon_2)$$

Following traditional lines one can assume that the demand for foreign goods is demand for final goods which is exposed to the same lag as consumption. Therefore:

(15) $$M_1(t+\vartheta_1+\varepsilon_1)=m_1Y_1(t+\vartheta_1)$$
$$M_2(t+\vartheta_2+\varepsilon_2)=m_2Y_2(t+\vartheta_2)$$

Obviously the export of one country is the import of the other so that:

(16) $$X_1(t+\vartheta_1+\varepsilon_1)=m_2Y_2(t+\vartheta_1+\varepsilon_1-\varepsilon_2)$$
$$X_2(t+\vartheta_2+\varepsilon_2)=m_1Y_1(t+\vartheta_2+\varepsilon_2-\varepsilon_1)$$

Maintaining the model described by system (11), enlarged by (12) (13) and (14), and proceeding through a Taylor series expansion approximation we obtain:

(17a) $$\dot{Y}_1(t) = Z_1(t)$$

(17b) $$\dot{Z}_1(t) = \omega\left(\Omega_1(t) + m_1 \frac{(\varepsilon_1 - \varepsilon_2)}{\varepsilon_1} \Omega_2(t) \right)$$

(17c) $$\dot{K}_1(t) = \varphi(v_1 Y_1(t) - K_1(t))$$

(17d) $$\dot{Y}_2(t) = Z_2(t)$$

(17e) $$\dot{Z}_2(t) = \omega\left(\Omega_2(t) + m_2 \frac{(\varepsilon_2 - \varepsilon_1)}{\varepsilon_2} \Omega_1(t) \right)$$

(17f) $$\dot{K}_2(t) = \varphi(v_2 Y_2(t) - K_2(t))$$

where:

$$\Omega_1(t) = \frac{C_{01} - e_1 Y_1(t) - a_1 Z_1(t) + m_2\big(Y_2(t) + f_1 Z_2(t)\big) + \varphi_1(v_1 Y_1(t) - K_1(t))}{\vartheta_1 \varepsilon_1}$$

$$\Omega_2(t) = \frac{C_{02} - e_2 Y_2(t) - a_2 Z_2(t) + m_1\big(Y_1(t) + f_2 Z_1(t)\big) + \varphi_2(v_2 Y_2(t) - K_2(t))}{\vartheta_2 \varepsilon_2}$$

$$\omega = \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 \varepsilon_2 + m_1 m_2 (\varepsilon_1 - \varepsilon_2)};$$

$f_1 = \vartheta_1 + (\varepsilon_1 - \varepsilon_2)$; $f_2 = \vartheta_2 + (\varepsilon_2 - \varepsilon_1)$;

$e_1 = (1 + m_1 - c_1)$; $e_2 = (1 + m_2 - c_2)$

The above is a six dimensional dynamic system composed of two coupled oscillators (economies). For certain sets of the parameters the two economies will be highly synchronized while for others highly a-synchronized.

Figure 2.

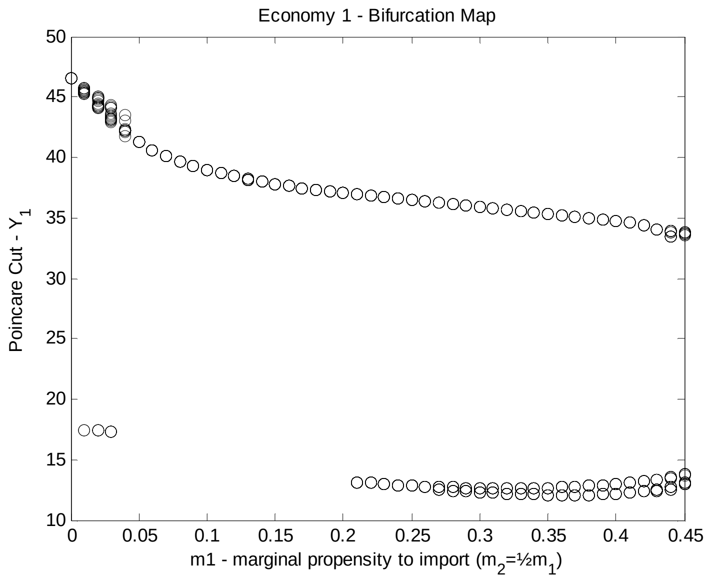


Figure 3.

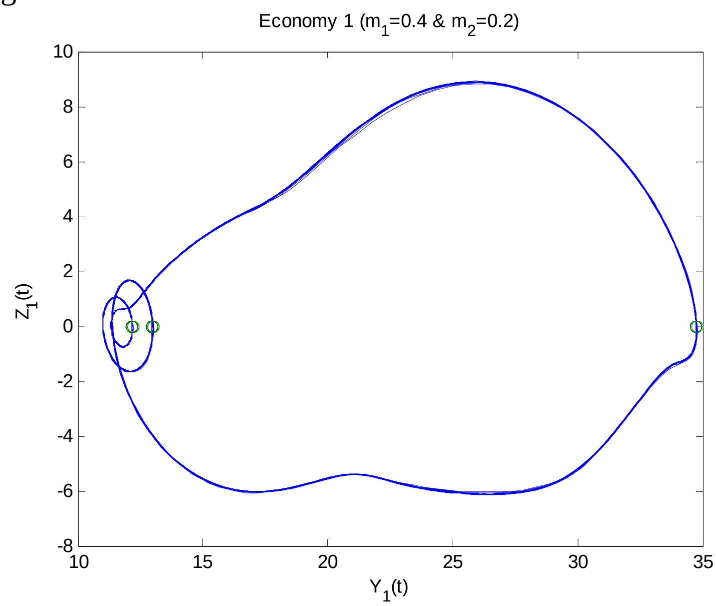


Figure 4.

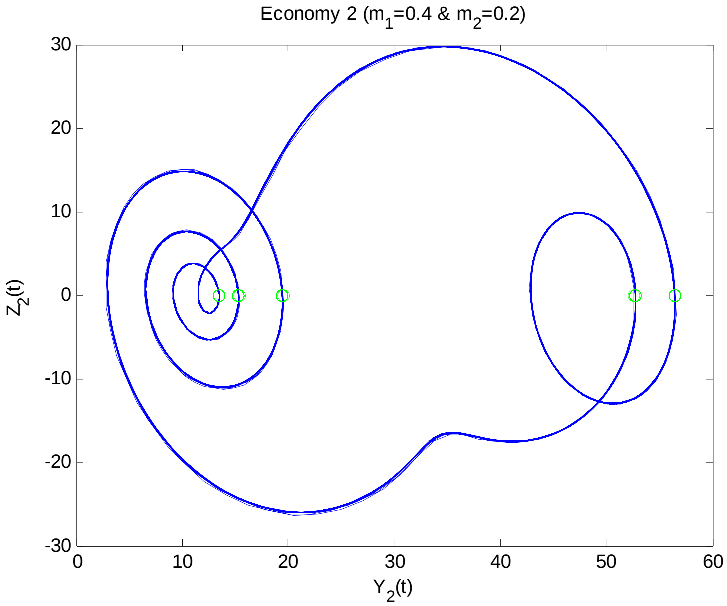
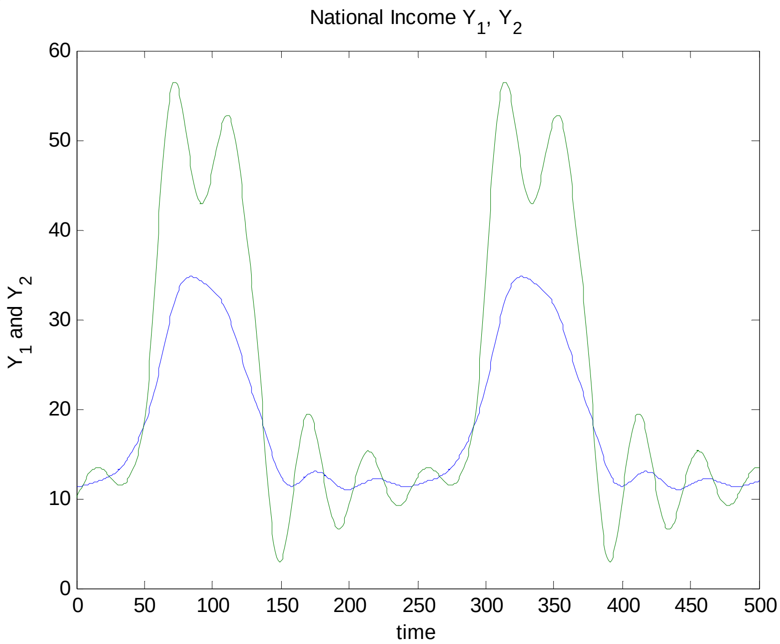

Economy 2 (m1=0.4 & m2=0.2)
30
20
10
0
-10
-20
-30
Z2(t)
0
10
20
30
40
50
60
Y2(t)

Figure 5.


National Income Y1, Y2
60
50
40
30
20
10
0
Y1 and Y2
0
50
100
150
200
250
300
350
400
450
500
time

6. Concluding Note: digital and analogical computing

In the above I have made a conjecture that the Phillips MONIAC Machine is best modeled as a nonlinear system of difference-differential equations (eqs. 9, 10, 11) which is in turn approximated with the nonlinear differential equation 12 which has state space representation eqs. 13.

It would be interesting to investigate on whether the above approximated digital simulations do, for properly chosen parameters, shadow the original Phillips analogue machine.

Here I have gone further and I have tried to simulate, as suggested at the end of his article by Phillips (1950 p.306) himself, the behavior of two coupled virtual Phillips machines. To my knowledge I think that his has not been done before. It would be interesting to do it.

The richness and the specific dynamic behavior of coupled dynamical systems depend on several factors. The theoretical system (15) for two countries, which is generalized to the n-countries in Zambelli (2011) are Taylor series approximations of the original theoretical model. Moreover in order to conduct digital computations the approximated systems are further approximated with difference equations; different approximating algorithms: Euler, Runge-Kutta and so on.

Before concluding let me simply point out that here I have made the conjecture that the mathematical description that Phillips (1950) has made of the functioning of his machine, being linear, is inappropriate. Here I have proposed the Goodwin flexible-accelerator model as being a better mathematical formulation of the MONIAC. But also this formulation is inappropriate. As explained above a better mathematical formulation seems to be in terms of a difference-differential nonlinear systems of equations. The computation of the dynamics of such a system is best made with an analogue computer like the Phillips MONIAC or with an analogue electrical system like the one constructed by Strotz et al. (1953).

This analogue systems are important to study complex dynamical systems without relying to ‘linear’ first approximations. Frisch (1933) achievements have been remarkable, but we have yet not gone back and have not removed his ‘first approximations’ (see Velupillai, 1992). To do that we might need the use of (analog) computing machines and, perhaps, we need to study the Phillips MONIAC machine itself.

In the concluding section of his *Dynamical Coupling with Especial Reference to Markets Having Production Lags,* Goodwin (1947, 204) wrote that there are *"... complications arising with general dynamic interdependence. To go from two identical markets to n nonidentical ones will require the prolonged services of yet unborn calculating machines"*. Whether Goodwin meant digital or analogue calculating machines is not known. Goodwin and Phillips were friends.

7. Appendix

Here are reported the parameter levels associated with the generations of the different graphs.

Figures 2, 3, 4

$\vartheta_1 = 1, \quad \varepsilon_1 = 0.25, \quad C_{01} = 10, \quad c_1 = 0.6, \quad v_1 = 2.0, \quad k_1^* = -3, \quad k_1^{**} = 9,$

$\vartheta_2 = 0.5, \quad \varepsilon_2 = 0.25, \quad C_{02} = 2, \quad c_1 = 0.8, \quad v_2 = 1.4, \quad k_1^* = -2, \quad k_2^{**} = 4,$

8. References

Frisch, R., (1933), Propagation Problems and Impulse Problems in Dynamic Economics, in Essays in Honour of Gustav Cassel, Allen & Unwin, London, pp.171-205.

Goodwin, R. M. (1947) Dynamical Coupling with Especial Reference to Markets Having Production Lags, *Econometrica*. Vol. 15, #3, July, pp. 181-204.

Goodwin, R.,.M., (1951) The Nonlinear Accelerator and the Persistence of Business Cycles, *Econometrica*, vol.19, no.1, pp.1_17.

Hicks, J., 1950, *A Contribution to the Theory of the Trade Cycle*, Oxford University Press, Clarendon Press, Oxford.

Kalecki M. (1935) A Macrodynamic Theory of Business Cycle, *Econometrica*, vol. 3, pp.327-344.

Kaldor, N. (1940) A Model of the Trade Cycle, Economic Journal, vol. 50, pp. 78-92.

Metzler, L.A. (1941) The Nature and Stability of Inventory Cycles, *Review of Economics and Statistics,* Vol. 23, No. 3, August, pp.113-29.

Phillips, A.W. (1950) Mechanical Models in Economic Dynamics, *Economica*, Vol. 17, No. 67, pp. 283-305

Samuelson, P.A., (1939) Interactions Between the Multiplier Analysis and the Principle of Acceleration, *Review of Economics and Statistics*, vol. 21, pp. 75-78.

Samuelson, P. (1974) Remembrances of Frisch, *European Economic Review,* vol. 5, p.7-23.

Strotz, R.H., McAnulty J.C. and Naines, J. B. (1953) Goodwin's Nonlinear Theory of the Business Cycle: An Electro-Analog Solution, *Econometrica*, Vol. 21, No. 3, July, pp. 390-411.

Velupillai, K. (1992) Implicit Nonlinearities in the Economic Dynamics of "Impulse and Propagation", in Velupillai, K. (ed.), Nonlinearities, Disequilibria and Simulation - Proceeding from the Arne Ryde Symposium on Quantitative Methods in the Stabilization of Macrodynamic Systems, Essays in Honour of Bjørn Thalberg, London, Macmillan, pp.57-71.

Zambelli, S., (1992) The Wooden Horse that Wouldn't Rock: Reconsidering Frisch, in Velupillai, K. (ed.), Nonlinearities, Disequilibria and Simulation - Proceeding from the Arne Ryde Symposium on Quantitative Methods in the Stabilization of Macrodynamic Systems, Essays in Honour of Bjørn Thalberg, London, Macmillan, pp.27-54.

Zambelli, S. (2007) A Rocking Horse That Never Rocked: Frisch's "Propagation Problems, *History of Political Economy,* Vol. 31, No. 1, pp.145-66.

Zambelli, S. (2011), Flexible Accelerator Economic Systems as Coupled Oscillators, *Journal of Economic Surveys, XXY,* forthcoming.