# Algorithmic Social Sciences Research Unit
## ASSRU

Department of Economics
University of Trento
Via Inama 5
381 22 Trento Italy

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# Introduction to the Phillips Machine & the Analogue Computing Tradition in Economics*

## K. Vela Velupillai

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* In recent years, troubled by the claims of many in macroeconomic theory on the importance, indeed the necessity, of numerical algorithms to solve analytically intractable problems, I have felt the need to return to the noble tradition of *analog* computing. There is no better way to do this than through a thorough understanding of the meaning and aims with which Phillips constructed his hydraulic *analog* computing machine. Conversations with my friend and colleague Stefano Zambelli and our graduate students, Kao Selda and V. Ragupathy have, as always, been edifying. Alas, they are not responsible for the remaining infelicities. This draft paper was prepared for presentation at the conference to celebrate the 60th anniversary of the Phillips Machine, 7/8 December, 2010. *Analog* and *analogue* have been used in slightly different senses that should be clear from the context!

# Abstract

In this paper I try to argue for the desirability of analog computation in economics from a variety of perspectives, using the example of the Phillips Machine. Ultimately, a case is made for the underpinning of both analog and digital computing theory in constructive mathematics. Some conceptual confusion in the meaning of analog computing and its non-reliance on the theory of numerical analysis is also discussed.

# 1 Notes on the Phillips Machine as an Analog Computer

"If a single group of equations can be written which defines the *assumed performance* for two separate systems (each of which within itself represents an orderly or definable behavior), one system may be called the complete *analogue* of the other."

[26], p. 557; italics added

In the case of the Phillips Machine, a paradigmatic analog computing device, this reasonable definition leads to a peculiar dissonance. Phillips observes, quite correctly and perceptively, that:

"The hydraulic model [the Phillips Machine] will give solutions for non-linear systems as easily as for linear ones. It is not even necessary for the relationships to be in analytic form; *so long as the curves can be drawn the machine will record the correct solutions*, within the limits of its accuracy[1]. In giving the equivalent mathematical model, however, the usual linearity assumption will be made, in view of the difficulty of working with non-linear differential or difference equations."

[21], pp. 287-8; italics added

Why is it necessary to give 'the equivalent mathematical model'? Obviously the definition of analogue, given in [26], above, is predicated upon the construction (or, is it the existence) of 'a group of equations' defining 'the assumed performance for two separate system'. But is it not possible to define 'analogue computation'[2] in the same sense in which the intuitive notion of *effective calculability* was, eventually, encapsulated in the definition of a Turing Machine? If the analogue computing machine, such as the Phillips Machine, was constructed – from an engineering point of view – on the basis of economic principles, whether in mathematical form or not, then its functioning should mirror the performance of the economy of which it is a 'model'. The intermediary stage of constructing 'a group of equations' raises a whole host of foundational questions – in addition to the serious problem of the meaning of 'model' to be used here – that have not been faced, squarely or otherwise, in the analogue tradition of computation, except in relation to recursion theory[3].

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[1] By this Phillips must mean engineering precision limits to accuracy, as well as, the usual constraints due to the need to respect the laws of the natural sciences in the manufacturing of machine components. 'Limits of its accuracy' has nothing to do with usual finite precision constraints of a digital computer, nor with the problems of numerical analysis (discretizations, etc.).

[2] I think the inadequacy of the definition in [26] is primarily due to the absence of the qualifying notion of 'computation'. We are, at least in the present context, concerned essentially with computation by analogue machines – just as we would be concerned with computation by digital machines, if we were using Turing Machines, or its approximations.

[3] See the companion piece to this one, [33].

The 'foundational questions' I am referring to are mostly mathematical and metamathematical in nature. What kind of mathematics should be used in the setting up of the 'group of equations'? In the case of the digital computer there is, by now, a clear, rigorous, answer to a similar question: recursion theory or constructive mathematics. In the case of analogue computation it may appear as if one can rely on standard mathematics for setting up the relevant 'group of equations'. This, however, is an untenable answer, but I will not go into the philosophical or epistemological foundations of the mathematical foundations of analogue computation in this paper (although, as mentioned in the previous footnote, I have tried to do so in the companion piece to this paper).

More importantly, a characteristically perceptive observation made by Richard Goodwin, in his review of Alan Tustin's pioneering servomechanism-based approach to modelling analogue computation in economics, is directly relevant for the issues I am trying to raise:

> "After an introductory defense of using [the very powerful analytical tools of servomechanism engineering] .. in economics, Professor Tustin launches into a lucid exposition of the basic concepts and methods of what has come to be known as the operational calculus. ... It is here that he employs *graphical analysis* with outstanding success. This emphasis on graphics has a special justification in that *economics*, like engineering, *must work with raw material given in the form of empirical curves*."
> [11], p.209; italics added.

This is a refreshingly original view and in stark contrast to the indiscriminate assumptions of economic data being defined on, and generated from, uncountable, uncomputable, undecidable domains. The versatility and power of the Phillips Machine, as an analog computer, is precisely the feature that is emphasised here by Goodwin, reflecting the method's importance in Tustin's parallel – servomechanism-based – approach to anlogue modelling and analog computation in economics.

The epistemology and philosophy of analogue modelling, in its almost pure mathematical senses, is a question that has to be addressed separately. As far as I am concerned, the Phillips Machine is an analogue computing device – although I am well aware that there was more to its construction than as a repository of mechanisms to facilitate computation.

But why did Phillips decide to build his machine with – mainly – hydraulic components and on hydraulic principles?[4] After all, Phillips himself was a trained, experienced, electrical engineer and he could have built a servomechanism-type of analog computer, if the sole purpose was pure computation. To place the

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[4]In a typically hair-splitting, pointless,discussion about Coddington's use of the phrase 'hydraulic Keynesianism', Patinkin ([20], footnote 8) makes all sorts of allusions and invokes utterly senseless references and correspondence with all and sundry - without taking the trouble, so far as I can infer, to actually read the Phillips paper of 1950. The most astonishing absurdity is a reference to Shackle's phrase *'reduction of economics to hydraulics'* ([24], p.189), and the quite unbelievable conjectures about how he (Shackle) came to construct it.

above notes, and this question, in the context of the way Phillips himself admitted were the inspirations for the construction of a *hydraulic* analogue computing device, it may be useful to recall what he wrote on this point:

> "Fundamentally, the problem is to design and build a machine the operations of which can be described by a particular system of equations which it may be found useful to set up as the hypotheses of a mathematical model, in other words, a calculating machine for solving differential equations. Since, however, the machines are intended for exposition rather than accurate calculation, a second requirement is that the whole of the operations should be clearly visible and comprehensible to the onlooker. For this reason hydraulic methods have been used in preference to electronic ones which might have given greater accuracy and flexibility, the machines being made of transparent plastic ('Perspex') tanks and tubes, through which is pumped coloured water.
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> Both of the models mentioned above deal with macroeconomic theory in terms of money flows; but they are based on an analogy given by Professor Boulding to show how the production flow, stocks and price of a commodity may react on one another."
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> [21], pp. 283-4; italics added.

Just for the record, I copy here, as Figure 1, the relevant figure and section in Boulding's vintage text on *Economic Analysis* (third edition, [5], p. 107[5]). His co-constructor, Walter Newlyn, in his interesting chapter on *The Origin of the Machine* ([19], pp. 31-2) is more specific:

> "[Phillips'] Figure 3 [in '*Saving and Investment : Rate of Interest and Level of Income*'] shows Boulding's supply and demand hydraulic analogy (1948, p.117) modified for the inclusion of stocks and the interconnections between stocks and flows linked mechanically"[6]

Even a cursory comparison of FIG 2 in [21], p.285 with the figure in Boulding would show that the former is inspired by the latter, in the precise senses

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[5] Phillips refers to p. 117 in an earlier edition of the Boulding book (p. 284).

[6] Incidentally, there is a curious unscholarly 'aside' in this otherwise interesting narrative of 'The Origin of the Machine in a Personal Context', when Newlyn suggests that the 'inclusion' of the accelerator 'in the Mark II machines is the element which gave it the dynamic feature which stimulated the work of Ricard Goodwin – a far cry from being a teaching aid.' This unfortunate inaccuracy is triply wrong, First of all, Phillips was inspired by the article on the 'accelerator', by Goodwin, in the Alvin Hansen *Festschrift* ([8]; [21], footnote 1, p.298). Secondly, Goodwin had developed the 'flexible accelerator' model, that which was published as the lead article in the first number of the first issue of the 1951 volume of the **Econometrica**, much earlier than the construction of even the prototype Phillips Machine, in 1949 (cf. [9]). Thirdly, Goodwin had gone much beyond the classic 'flexible accelerator' in his remarkable review of the book by Hicks on the Trade Cycle ([10]). No models by Phillips – including those in his celebrated EJ articles of the 1950s ([22], [23]) – ever managed to encapsulate the richness of Goodwin's nonlinear elements. On the other hand, the Phillips Machine could easily implement the full nonlinear Goodwin model quite easily!

## Production, Consumption, and Stocks

We must get a picture of the production and consumption of any given commodity—let us say, wheat—as a process something like the flow of water into and out of a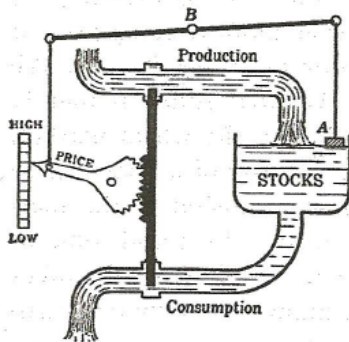 tank. The quantity of water in the tank corresponds to the quantity of the commodity in existence at a moment of time —1,000,000,000 bushels of wheat, say, in the various storehouses of the world. From the production of farms wheat is constantly being added to the total stock; this corresponds to the flow of water into the tank. The flow is not of course regular, for at harvest time it is very large and at other times very small. However, we may call the rate of flow, averaged over a number of years, the *rate of production*, measured not in bushels but in *bushels per year*.

FIG. 9. How Price Regulates Production and Consumption.

Wheat is also continually flowing out into consumption, corresponding to the flow of water out of the tank. This flow may also not be quite regular, though it will be more regular than the flow of production. The average rate of flow may be called the *rate of consumption*, measured also in bushels per year. Fig. 9 may help to visualize this process.

Figure 1: Boulding's Hydraulic Analogy

in which Phillips and Newlyn suggest. Any perceptive reader of Boulding's suggestive analogy will realize that he is describing non-steady dynamics, an euphemism, surely, for nonlinear dynamics. It is strange, then, given the aims with which Phillips constructed an analog *hydraulic* computing machine, that he chose to build a model represented by 'a particular system of equations' that were linear.

I should like to end this Introductory section with a few personal remarks on the role of analogue computation and the Phillips Machine in my own intellectual history. I was privileged to have been an engineering student, in the department of mechanical engineering, at Kyoto University, in the transition period when massive digital computing facilities were becoming available and old-fashioned wind tunnels were being replaced by simulations of theoretically derived differential equations on these modern computers. But we were fortunate to have the benefit of the analogue computing tradition being practised and demonstrated in the analysis and studies of nonlinear dynamics by one of the great masters of this method and tool: Chihiro Hayashi ([13]). His lectures on electric circuit theory, with especial emphasis on studying, via analogue computation, the dynamics of the van der Pol equation – in both forced and unforced forms – was my introduction to this method and its versatility and to the famous equation itself. Graphical techniques dominated the way Hayashi approached the study of the intractable forced van der Pol equation – presaging and preparing me for instruction by Richard Goodwin, a few years later, at Cambridge, when this very equation was now given an extraordinary economic interpretation in terms of a Keynesian trade cycle model. Goodwin continued the graphical, geometric, tradition of Hayashi, but also gradually began to move towards representations of the van der Pol equations in digital computing machines.

Remarkably, when I switched to economics, first in the department of economics at the University of Lund, I came under the wise influence of Professor Björn Thalberg, whose work at that time – and for many years earlier and later – was based on the contributions by Goodwin and Phillips to the modelling of a Keynesian vision of the macroeconomy ([28], [29] and [30]). Thalberg had himself attended the very first series of lectures given by Goodwin, at Cambridge, utilising the Cambridge Phillips Machine, in 1952.

Many years later, my external examiner, for the Cambridge PhD thesis that I worked on under Goodwin's enlightened guidance, was Sir Roy Allen, whose successful textbooks of the 1950s, 1960s and 1970s were among the core 'modern' texts, in addition to Samuelson's *Foundations* and Patinkin's *Money, Interest and Prices*, through which I learned macroeconomics in Thalberg's graduate courses, in the early 1970s. No macroeconomic textbook emphasised a servomechanism-based modelling approach to macroeconomics, using, in particular, the Phillips and Goodwin dynamic models, than Roy Allen's lucid textbooks of that vintage period (see, for example, [2] and [3]).

That my formative years as a macroeconomist were at the feet of Thalberg and Goodwin gave me, what I consider to be an immense advantage over many of the 'modernists': to learn and understand a vision of the macroeconomy in which computation was given crucial interpretive, investigative and theoretical roles. These were the roles that Phillips emphasised, at least in those classic early works and constructions, and Goodwin grappled with all his intellectual life.

## 2    A Perspective on the Analogue Computing Tradition in Economics

"The [hydraulic] mechanism just described is the physical analogue of the ideal economic market. The elements which contribute to the determination of prices are represented each with its appropriate rôle and open to the scrutiny of the eye. We are thus enabled not only to obtain a clear and analytical *picture* of the interdependence of the many elements in the causation of prices, but also to employ the mechanism as an instrument of investigation and by it, study some complicated variations which could scarcely be successfully followed without its aid."

([7], p.44, italics in original)

Calculating, estimating, comparing, constructing and reasoning with numerical ratios, averages, series, tables areas, volumes and so on – in short, 'analyzing data', whether natural or artificial – underpinned much inference and some deduction is the way our classical and physiocratic predecessors came to policy precepts. However, with the exception of Charles Babbage and, possibly, Jevons, till Irving Fisher ([7]), in 1891, constructed his 'remarkable hydraulic [analogue computing] apparatus for calculating equilibrium prices' ([6], p.57,) resorting to *actually constructed machine models of computing* in economics seems to have remained an isolated example. Fisher's own description – quoted above – of the functioning of his hydraulic analogue computing machine clarifies an important feature of such computations: *their independence from any intermediation via numerical analysis.*

There were, of course, the famous computing machine *metaphors* used by Walras, Pareto – and, then, inspired by Barone, in the important 'Socialist Calculation Debate', most comprehensively summarised, both critically and constructively by Hayek ([14] & [15]). Lange, returning to the theme over thirty years later, in his *Dobb Festschrift* article on *The Computer and the Market* ([17]), muddied the issue by unscholarly and unsubstantiable claims for the possibilities of a digital computer (having, in the meanwhile, also forgotten that the initial discussions were with reference to *analog computing machines* and, in particular, the metaphor of *the market as an analogue computer*). None of the participants had any technical knowledge of the mathematical underpinnings of computing, in a sense understandably so, since the mathematical foundations of computing were being placed on a rigorous basis just during those very years that the debate was at its height.

*Analogue computing techniques* in economics had the proverbial still birth. There was a flurry of activities in the late 1940s and early 1950s, quite apart from Phillips, at the hands of Richard Goodwin, Herbert.A. Simon, Robert H.Strotz, Otto Smith, Arnold Tustin, Roy Allen, Oscar Lange and a few others. As we know, now, Phillips built his famous *MONIAC* hydraulic national income machine at the end of the 40s and it was used at many Universities - and even at the Central Bank of Guatemala - for teaching purposes and even

as late as the early 70s Richard Goodwin, at Cambridge University, taught me elementary principles of coupled market dynamics using such a machine. Strotz and his associates, at Northwestern University, built electro-analogue machines to study inventory dynamics and nonlinear business cycle theories of the Hicks-Goodwin varieties. Otto Smith and R.M. Saunders, at the University of California at Berkeley, built an electro-analogue machine to study and simulate a Kalecki-type business cycle model. Roy Allen's successful textbooks on Macro-economics and Mathematical Economics of the 50s - extending into the late 60s - contained pedagogical circuit devices modelling business cycle theories (cf:[2] especially chapter 9; and [3], especially chapter 18). Arnold Tustin's highly imaginative, but failed textbook attempt to familiarise economists with the use of servomechanism theory to build analogue machines as models of economic dynamics ([31]) and Oscar Lange's attractive, elementary, expository book with a similar purpose ([18]) also suffered the fate of 'stillbirth', at the dawn of the digital computing age.

Humphreys ([16]) refers to nonlinear business cycle theories as examples of computational 'studies' that straddle 'the pre-computational era and the era of computational economics', claiming that 'there is no sharp divide between 'the two eras'. This claim can be substantiated by a more finessed study of the particular example of a canonical nonlinear business cycle equation, using – as was, indeed, actually done – analogue computing machines as in the 'pre-computational era' and comparing it with its study using a digital computing machine of the 'era of computational economics'.

The example I have chosen here encapsulates a noble tradition of *computation in economics* in every sense of this concept, to study a precisely specified mathematical system on both analogue and digital computers. It is, in a precise sense, also a substitute for an analytical study (because such a study is provably 'unlikely' to succeed in any meaningful way). Moreover, it can be viewed as an explicit example of an epistemological tool to interpret the results (most of which were unexpected). Finally, to gain insight into the link between a computing machine and *its* theory and the theory of nonlinear dynamical systems. The latter point is turning out to be the most significant from the point of view of the epistemology of computation, since the interaction can only be explored by representing the one system by the other – and, therefore, even an exploration into a new domain: studying the repertoire of digital machine behaviour with analogue computing machines, and *vice versa*.

Consider, therefore, the following equation, representing a classical Keynesian nonlinear multiplier-accelerator model of the dynamics of national income, $y$:

$$\epsilon \dot{y}(t) + (1 - \alpha) y(t) = \phi [\dot{y}(t - \theta)] + \beta(t) + l(t) \tag{1}$$

Now, there are *at least* six different ways to investigate solutions to this nonlinear difference-differential equation:

- In old fashioned analytical modes;

- Using Non-standard analysis;

- Graphically, i.e., in terms of the geometry of dynamic behaviour, as usually done in the qualitative theory of differential equations;

- By the method of equivalent linearization;

- Using an electro-analogue computer;

- Using digital computers;

It is, of course, only the last two alternatives that are of relevance in this discussion. Assuming, for example, $\beta(t) + l(t)$ a constant[7] and reinterpreting $y(t)$ as a deviation from the unstable equilibrium of (1) ($\frac{\beta(t)+l(t)}{(1-\alpha)}$), one obtains a mixed nonlinear difference-differential equation:

$$\epsilon \dot{y}(t+\theta) + (1-\alpha) y(t+\theta) = \phi[\dot{y}(t)] \tag{2}$$

In the first case, expanding (2) by a Taylor series approximation and *retaining only the first two terms*, one obtained the famous (unforced) *Rayleigh (- van der Pol) – type* equation:

$$\ddot{y} + \left[ \chi \left( \frac{\dot{x}}{\dot{x}} \right) \right] \dot{x} + x = 0 \tag{3}$$

With this approximated reformulation began an 'industry' in the endogenous theory of the business cycle, where the cardinal desideratum was the existence of a unique, stable, limit cycle, independent of initial conditions. All four desiderata were violated when the approximations were more precise – in a purely technical sense – and the analysis proceeded via studies by means of analogue and digital computing machines. Even more interestingly, the insights obtained from an analogue computing machine study provided hints in setting up a computing study of (1) by means of digital computing machines.

Now, using an *electro-analog computer*, it was found, in [27], that the approximation of (1) retaining the first four terms of a Taylor series expansion, generated twenty-five limit cycles, and *a potential for a countable infinity of limit cycles* with further higher order terms included in the approximations. Moreover, in its original formulation, one of the desired criteria for the nonlinear formulation of the endogenous model of the business cycle, was to generate self-sustaining fluctuations, *independent of initial conditions*. This latter property was lost when the approximation was made more precise.

Next, coupling two equations of type (3), via the *Phillips Electro-Mechanical-Hydraulic Analogue Computing Machine* ([12]), Goodwin and Phillips were able to generate – *unexpectedly* – the *quasi-periodic paradox* (cf., [1]). Neither Goodwin, nor Phillips, who did the coupled-dynamics computation on the *Phillips*

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[7] If $\beta(t) + l(t)$ was not assumed a constant, the obdurate *forced* version of (3) would have to be confronted, wihtout any hope of a disciplined solution even with the help of computing machines, whether analog or digital.

*Machine*, had any clue – theoretical or otherwise – about interpreting and encapsulating this outcome in any economic theoretical formalization. The key point is that they were surprised by the outcome and did not know how to interpret it when it emerged. This is where the richness of the epistemology of computation manifests itself most dramatically. There was no macrodynamic theory to which they could relate the observed behaviour, which was contrary to expected behaviour.

Finally,– Zambelli ([34] – repeated the exercise in [27], but this time on a *digital* computer. His results came as a surprise to him: although we can confirm the results in [27], the outcomes are richer and more varied and we would have no idea which way to proceed, if one was wedded to an equilibrium norm to which the results have to conform.

It goes without saying that one of the key differences between analogue and digital computing is that in the latter the intermediation between the continuous and the discrete is achieved by means of numerical procedures; this intermediation is circumvented in the analogue tradition, as pointed out above. In this sense, there *is* a sharp difference between 'the pre-computational era and the era of computational economics'. Much of what is routinely referred to as computational economics in the modern era is simply variations on the theme of numerical analysis, without any anchoring in the mathematical theory of the computer, whether digital or analogue.

There is no better way, at least in my opinion, to end this section than to recall Richard Stone's extraordinarily perceptive reflections on the power and possibilities of analog computing in the kind of multisectoral macrodynamics he was then embarking on, summarised in his talk at a Conference on Automatic Control, as far back as 1951 ([25], pp. 82-3)[8]:

> "Analogue machines are not unknown to economists, but in the past they have been used for demonstration purposes but not for computing. Examples of such machines are the hydrostatic model of general economic exchange described by Irving Fisher in 1892 and subsequently built, and the Phillips-Newlyn Hydraulic model for demonstrating the interdependence of the main variables in general aggregative analysis which was designed a few years ago and which has now been adopted for expository purposes in a number of universities in this country. The use of electric analogues has been suggested recently by various authors, one of these instruments being concerned with the investigation of inventory oscillations and another with the investigation of equilibrium among spatially separated markets. ....
>
> It is impossible to say without detailed investigation whether such an approach to economic computing problems have any ad-

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[8]I have suppressed the many references that were in the full quote. Alan Tustin's talk, immediately after Richard Stone, and published back-to-back with Stone's paper ([32]), is also worth a perusal and some reflection, since, by that time, his own important book ([31]) on a servomechanism-based approach to analog computation was close to completion.

vantage over the usual digital methods. Its utility would probably depend on whether a more detailed investigation of economic responses revealed patterns for which a good electric analogue could be devised. Since the machine would be designed to perform calculations arising from a specified system it should be easily modifiable, since otherwise it would be too rigid for the changing circumstances of economic life. It goes without saying that should the design of such a machine be contemplated it would be necessary to introduce many complications which have not been elaborated here. In particular there would be a need to consider (i) stocks as well as flows, (ii) prices and quantities separately instead of simply their product ...., (iii) exogenous factors .... (iv) expectations .... and finally (v) the specification of ..errors. "

Those who are familiar with the Phillips Machine, and its architecture, will know that from the outset Phillips was concerned with issues (i), (ii) and (iv). Knowing his general intellectual interests and subsequent work, there is no doubt at all in my mind that (iii) and (v), too, would have been incorporated in any modern reconstruction of the Phillips Machine.

## 3    Analogue Reflections in a Digital World

"[I]t is, therefore, desirable to form some view of the degree of complexity that may be expected in a 'scheme of dependence' (or a 'model' as the economist calls it) such as would be adequate as a basis for prediction and the analysis of requirements for stabilisation.

...

My own conclusion is that the representation by a reasonably accurate analogue of any of the schemes of dependence that have so far been proposed by economists presents no inherently insuperable problems, though it provides a sufficient spice of difficulty to make the problem interesting. I think it is for the economists to say whether such a project, if it were realised, would be of value. If it were I am sure that there are many engineers skilled in the field of analogue devises who would be glad to help transform the suggestion into reality."

[32], pp. 85, 89

In the intervening six decades, or so, since Tustin's 'conclusion' and 'suggestion', the economist has abandoned the engineer and pawned the subject to the mathematician – and not just any mathematician, but to the kinds whose foundations are most alien to computation. How can the economist reclaim the analogue vision, to return to thinking in terms of 'schemes of dependence', and to construct machine models that place at the centre of concern 'the analysis of requirements for stabilisation', without ideological anchorings in a political economy of nihilism?

Paradoxically, my own diagnosis of the malaise in current mathematical economics is rooted in a critique of its recent obsession with computation and the halo surrounding numerical analysis. Computable General Equilibrium theory lies at the core of the frontier research in macroeconomics in its guise as Recursive Macroeconomics, whose 'scheme of dependences' are formalised as Stochastic Dynamic General Equilibrium schemes. Another strand of popular research at the frontiers is the so-called agent-based economic 'scheme of dependences', again underpinned by – and in – atheoretical computation schemes, but, in reality, disciplined by (unconsciously) recursion theory. Varieties of experimental economics, algorithmic game theory and a loose amalgam of subjects under the umbrella phrase 'computational economics' are all trying to do the impossible – i.e., to compute the uncomputable, decide the undecidable and complete the incompleatable.

Why?

Only because there is an unbridgeable gap between the mathematics in which the above economic sub-disciplines are theorized and that in which their computation lives are implemented. If you theorise in terms of real analysis, founded on set theory plus the axiom of choice, but compute with the aid of a digital computer, then the mathematical foundations of the latter – recursion theory or constructive mathematics – creates the inevitable dissonance.

Surely there are two obvious ways to overcome this dissonance. One, to return economic theory to its algorithmic roots and theorise in terms of constructive or computable mathematics, ab initio. Two, to retain the rich harvest of results that have been obtained in economic theory, in the almost two and a half centuries of sustained effort by a galaxy of economic theorists, by means of returning to the tradition of formulating 'schemes of dependence', free of allegiance to any kind of mathematics, and build analogue devices to represent them in action.

This is the message I infer from the pioneering work, and aims, that went into the construction, and operation, of the Phillips Machine.

There are eminent applied mathematicians and computer scientists who have been seeking a model of computation that can resolve a different dissonance: that between computability theory and numerical analysis. Smale and his co-workers are a distinguished example of this group. In their defining work, [4], they emphasise the following distinctions between computability theory and numerical analysis:

> "There is a substantial conflict between theoretical computer science and numerical analysis. These two subjects with common goals have grown apart. For example, computer scientists are uneasy with calculus, whereas numerical analysis thrives on it. On the other hand numerical analysts see *no use for the Turing machine.*
>
> The conflict has at its roots another age-old *conflict,* that *between the continuous and the discrete.* Computer science is oriented by the digital nature of machines and by its discrete foundations given by Turing machines. For numerical analysis, systems of equations and

differential equations are central and this discipline depends heavily on the continuous nature of the real numbers. ...

Use of Turing machines yields a unifying concept of the algorithm well formalized. ....

The situation in numerical analysis is quite the opposite. Algorithms are primarily a means to solve practical problems. *There is not even a formal definition of algorithm in the subject.* ....

A major obstacle to reconciling scientific computation and computer science is the present view of the machine, that is, the digital computer. As long as the computer is seen simply as a finite or discrete subject, it will be difficult to systematize numerical analysis. We believe that the Turing machine as a foundation for real number algorithms can only obscure concepts.

Towards resolving the problem we have posed, we are led to expanding the theoretical model of the machine to allow real numbers as inputs."

[4], p.23; italics added.

Unfortunately, they fail to point out that also in constructive mathematics 'there is not even a formal definition of algorithm in the subject'; yet it can act as the mathematical foundation for the digital computer without any of the conundrums that an intermediary role for numerical analysis. But most paradoxically, constructive mathematics can also act as an adequate mathematical foundation for analog computation!

I think the 'conflict' Smale and others see between computer science and numerical analysis, between the continuous and the discrete, between the finite and the infinite, are, in my opinion artificial phantoms, created by the unholy alliance between the metaphysics of the real number system and the spectre of numerical analysis.

There is no better way to summarise the main theme in this paper than in terms of the hilarious wisdom of Terry Pratchett in **Making Money, p. 63** (italics added):

"The Glooper, as it is affectionately known, is what I call a quote 'analogy machine' unquote. *It solves problems not by considering then as numerical exercise but by actually duplicating them in a form we can manipulate*: in this case, the flow of money and its effects within our society becomes water flowing through a glass matrix, the Glooper."

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