Algorithmic Social Sciences Research Unit

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Discussion Paper Series

11 – 11

CONTINUITY, DISCONTINUITY AND DYNAMICS IN MATHEMATICS & ECONOMICS RECONSIDERING ROSSER'S VISIONS

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March 2011

“"*As with the dualism of 'wave-particles,' we may live in a world that is simultaneously continuous and discontinuous. Which it is at any time may depend on whether we need to contemplate the forest or the trees.*"♣

♣ Rosser (2000), p.6.

Abstract

Barkley Rosser has been a pioneer in arguing the case for the mathematics of discontinuity, broadly conceived, to be placed at the foundations of modelling economic dynamics. In this paper we reconsider this vision from the broad perspective of a variety of different kinds of mathematics and suggest a broadening of Rosser's methodology to the study of economic dynamics.

Key Words: Continuity, Discontinuity, Economic Dynamics, Relaxation Oscillations
JEL Codes: B23, B31, B41, C02, C60.

§1. A Preamble[2]

> "By one of the many roughnesses or even superficialities forced upon us by the nature of the task which this volume is to fulfil, we may characterize this as a difference between microscopic and macroscopic points of view: there is as little contradiction between them as there is between calling the contour of a forest discontinuous for some and smooth for other purposes."
>
> Schumpeter, 1939, p. 227.[3]

Barkley Rosser, for almost thirty years, has been tirelessly pursuing an interesting and innovative research program on developing advanced applied mathematical theoretical technologies to provide a vision for complex economic dynamics, at macro, micro and methodological frontiers.

Our own knowledge of his impressive contributions along these lines go 'back to the beginning', so to speak, of his published work, particularly his two early capital theoretic contributions. At the tail end of the capital theory controversies – indeed, during the brief afterglow of that debate – Rosser gave the twin-problems of reswitching and capital-reversal an unusual interpretation in terms of a *cusp catastrophe* (Rosser, 1978, 1983). Rosser was typically candid about the 'eccentric' nature of *his* reswiching model (see, especially, § III, in Rosser, 1983, pp. 184-190). However, our interpretation of the 'eccentricities' in the paper are of a different methodological and economic theoretic nature. The puritan's model of reswitching and capital reversal, i.e., Sraffa's own, has no scope for formal dynamics[4] or optimization – whether of a static or dynamic nature – in it! But that belongs to another story than the one we try to tell here.

Here we shall, instead, concentrate on 'reconsidering' Rosser's visions on the mathematics and economics of continuity, discontinuity and dynamics.

In the next section we discuss and try to suggest a view of 'The Mathematics of (Dis)-Continuity', that may suggest a broadening of the Rosser vision in new mathematical directions that we think retain, and enhance, the complex dynamic backdrop and provide it

[2] Part of the motivation for writing this paper arose out of thoroughly ignorant comments by several of our colleagues about the role of continuity in constructive and computable analysis.

[3] Perhaps the inspiration for the opening quote by Rosser, in this paper, was this typically Schumpeterian 'vision', which also appears as the last quote of Rosser, op.cit. p. 223-4!

[4] Of the kind that is underpinned by formal dynamical systems theory, although we are of the view that this framework does not exhaust what dynamics is, or should mean, in economic analysis.

with a deeper and wider perspective from the point of view of computational economic dynamics.

In section III, we give an example of how an apparently 'discontinuous' dynamic phenomenon in macroeconomic dynamics that can be given a new interpretation and observational validity in terms of *non-standard analysis*.

The brief concluding section summarizes our discussion from methodological and epistemological perspectives and suggests ways in which we should go beyond the frontiers to which Barkley Rosser's work, on the issues mentioned above, has brought us.

§2. The *Pure* Mathematics of Continuity – and Discontinuity

> “”Somehow it is appropriate if ironic that sharply divergent opinions exist in the mathematical House of Discontinuity with respect to the appropriate method for analyzing discontinuous phenomena.”
>
> Rosser, 2000, p.9

Invariably, the 'Mathematical House' defines 'discontinuity' in terms of one or another so-called 'rigorous' characterization of 'continuity', usually some variant or generalization of what we think of as the notoriously unintuitive (or counter-intuitive) 'ϵ-δ' definition. Economists who tackle 'discontinuity in economic theory and economic discontinuities', to use the title of chapter 1 of Rosser (2000), are, usually, '*the slaves of some defunct*' mathematician (*pace* Keynes). To liberate ourselves from this bondage to some defunct mathematician, it will be necessary to understand how the mathematical house came to accept, adopt and even sanctify the particular definition(s) of continuity that prevail in the accepted mathematical culture of modern analysis. For, to the best of our knowledge, all mathematical houses began, first, by defining a particular kind of continuity, as encapsulating the corresponding intuitive notion, and then defining discontinuity negatively – as the absence of that property (or properties) characterizing the defined notion of continuity.

There are at least four kinds of mathematical analysis: standard real analysis, non-standard analysis, constructive analysis and computable analysis[5]. Within these broad strands, there

[5] We would like to add '*smooth infinitesimal analysis*' (Bell, 1998) as a fifth kind of mathematical analysis. However, for the limited purposes of this essay – primarily directed at the mathematically trained economist – such an addition would, perhaps, be an intolerable burden, given the general lack of knowledge of all but standard real analysis even by competent mathematical economists.

are further subdivisions – for example, constructive analysis could be of the pure Brouwerian variety, the Bishop adaptations of the original Brouwer intuitionistic kind of constructive mathematics, Russian constructivism, etc. The Bishop version tries to find common ground with standard real analysis. Russian constructivism is an amalgam of the Brouwerian version and computable mathematics, and so on.

Now, typical rigorous definitions of a continuous function in each of these kinds of mathematical analysis are as follows:

Standard Analysis

A real valued function f is *continuous* at the point x_0 in its domain *if*, $\forall\ \varepsilon > 0, \exists\ \delta > 0$ *s.t if* $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$; and f is *continuous on an interval* if it is continuous every point of the interval.

Boas, 1996, p. 85

Constructive Analysis

A real valued function f defined on a compact interval I is continuous on I *if* $\forall\ \varepsilon > 0$ *there exists* $\omega(\varepsilon)$ *s.t* $|f(x) - f(y)| \leq \varepsilon$ whenever $x,\ y \in I$ and $|x - y| \leq \omega(\varepsilon)$. The operation $\varepsilon \to \omega(\varepsilon)$ is called a *modulus of continuity* for f.

Bishop, 1967, p. 34.

It will be noted that the quantifier symbol for 'there exists' is used in the 'standard' definition but not in the constructive case. This is, of course, because the logical meaning of the concept in the two kinds of mathematics is very different: *tertium non datur* is valid in the former case, not so in the latter case. Equally, for the case of the '*modulus of continuity*'.

Non-Standard Analysis

A function f is *continuous* at a standard point x_0, $a < x_0 < b$, *iff* $f(x) \simeq f(x_0), \forall\ x \simeq x_0$.
A function f is said to be *continuous* if it is continuous at *every* real number x.
If $a - b$ is an *infinitesimal*, then b is said to be *infinitely close* to a and denoted by $a \simeq b$.

Robinson, 1996, pp. 56, 66.

Computable Analysis

A *recursive function* $f(x)$ is *recursively continuous* at x_1 if there is a *recursive function* $c_1(k)$ s.t, $\forall x,\ x - x_1 = o(c_1(k)) \rightarrow f(x) - f(x_1) = o(k)$.

$x = o(k)$ means, for some integer k and rational x, $10^k x = 0$; *hence*, $x = o(k)$ is equivalent to the *recursive relation* $|x| < 1/10^k$.

Goodstein, 1961, pp. 8, 39.

These canonical definitions, representing the different kinds of mathematical analysis do not even begin to exhaust the possibilities for variations on the theme of continuity of functions[6]. This being so, what, then, of *discontinuity*? Assuming we are in the world of the standard real analyst, fortified by the Dirichlet-Kuratowski definition of a function as a graph, consider the following diagram, where ABCD is a trapezoid symmetrical about the line, LL (see, Thurston, 1989):

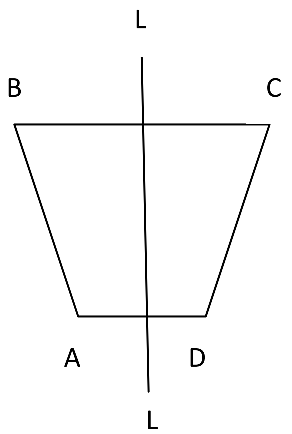

Fig. 1

Now, let S consist of the points of AB a rational distance from A together with the points of CD an irrational distance from D. Next, if LL is, first, considered as the y-axis, then S is the graph of a *discontinuous* function; if, however, LL is considered the x-axis, then S is the graph of a *continuous* function.

Hugh Thurston suggests, in the original paper in which this example appeared, that the *counter-intuitiveness* of the result is the absence of the compactness property of the domain.

[6] In particular, we regret very much that space does not permit an inclusion of the definition of *continuity* along the lines of *smooth infinitesimal analysis*, especially because it is underpinned by the axioms and rules of inference of *free intuitionistic logic*. Aboe all (Bel, op.cit, p. 104):

> "[I]n smooth infinitesimal analysis, the law of excluded middle fails 'just enough' for variables so as to ensure that all maps on R are continuous, but not so much as to affect the propositional logic of closed sentences."

But we like to think the fault lies in Dirichlet-Kuratowski definition of a function as a graph, in the first place[7].

Now, *will the real* <u>*continuous function*</u> *please stand up* (*pace* Keith Devlin, http://www.maa.org/devlin/devlin_5_00.html)?).

Surely, each of the four definitions has a claim to represent an intuition about the concepts 'continuity' and 'function'? Should we, as economists, choose the one that would not allow this kind of schizophrenic monster to emerge in an economic formalism? What exactly is – should be – the role of intuition is forging precise, formal, mathematical definitions?

Consider, now, an example constructed by Barkley Rosser, *Snr*., in the opening pages of his classic book on <u>Logic</u> (Rosser, *Snr*., 1978). In this example of an 'imaginary interview' between a modern mathematician, say Professor X, and Descartes, the dialogue is about coming to an understanding and an agreement about replacing the latter's vague and intuitive notion of continuity, with the former's (ostensibly) more rigorous 'ϵ-δ' definition of the identically worded concept. The dialogue ends with agreements and understandings, but with a final question posed by Descartes to Professor X (ibid, p.2):

> "I have here an important concept which I call continuity. At present my notion of it is rather vague, not sufficiently vague that I cannot decide which curves are continuous, but too vague to permit of careful proofs. You are proposing a precise definition of this same notion. However, since my definition is too vague to be the basis for a careful proof, how are we going to verify that my vague definition and your precise definition are definitions of the same thing."

This eternal problem of replacing vague, intuitive, notions with rigorous mathematical characterizations was felicitously solved in one famous case, where also Rosser,*Snr*., was a pioneer: *The Church-Turing Thesis* (Kleene, 1952, p. 317) on the characterization of the vague and intuitive notion of *effective calculability* with the precise notion of *Turing Machine Computability* or Church's *λ-Calculus* (to the development of which Rosser,*Snr*., made

[7] We should, in a more comprehensive context, have mentioned the recent work by Velleman (1997) and the lucid discussions in a series of papers by Gámez-Merino, et.al., (2011) on 'characterising continuity'. The key question posed is whether the well known 'fact' that continuous functions transform compact or connected sets into compact or connected sets, respectively can be the criterion for characterising continuity.

fundamental contributions, as a student of Alonzo Church), or of any of a few other formally equivalent notions[8].

It has, however, never been squarely faced in the case of continuity; and, hence, obviously, never has a 'Continuity Thesis' been suggested to underpin any formal characterization. In juxtaposing the emergence of the concept of the Church-Turing Thesis, out of the confluence of ideas that led to its formal statement in the 1930s, Robin Gandy made a most pertinent observation:

> "During the early part of this century it was realized that intuitive notions of 'limit' and 'continuous' could be applied in contexts quite different from those in which they had first been given precise definitions (viz. the theory of functions of real and complex variables). It seemed, and in standard texts such as Bourbaki's 'Éléments de Mathématique' still seems, that these notions could be *definitively* characterized by introducing axioms for a 'topological space' and then defining the notion in terms of the primitive terms which occur in the axioms. However, cases have arisen in which there is a clear intuitive notion of continuity which cannot be so defined. For example, the notion of continuity which Kreisel .. uses in introducing his 'continuous functional of finite type', and certain refinements of it, are not topological. But there is a generalization, 'filter space', in which Kreisel's notion of continuity can be simply and directly expressed. Despite much evidence from examples, and from the confluence of definitions – in terms of open sets, neighbourhoods, and operations of closure – 'topological space' is not *the* definitive concept."
>
> Gandy, 1994; p. 73; underlined emphases added.

The same, of course, applies to the *function* concept, not only in the pure mathematical senses of alternative definitions as in the four examples for continuous functions given above, but also as in the example of the controversy surrounding Dirac's δ-function and its alleged illegitimacy as a standard function. It was only after the generalization of the function concept by Laurent Schwartz and others that the δ-function was legitimized as a proper mathematical construct by the mathematicians. Dirac's honed mathematical physics intuition ran far ahead of the internal, almost incestuous, development of the function concept.

Surely, there is a lesson to be learned here, for the mathematical economist?

In general, the formal definition of discontinuity is simply the absence of the property of continuity[9], but it is when discontinuities over a segment or an interval have to be considered

[8] However, one of us – Velupillai – has serious misgivings about the way Church originally suggested this 'equivalence' via an attribution to Turing (Chruch, 1937). There are intricate questions of what Turing meant by the difference between 'machine computable' and effectively calculable by 'a human calculator'. An elegant exposition of scepticism on this fundamental question can found in Hodges (2008).

that some of the more interesting conundrums begin to appear. The classic distinction between a discontinuity of the first and second kind (cf. Rudin, 1976, p. 94) is almost invariably presented in terms of the following example:

Define

$$f(x) = \begin{cases} 1 & x: rational \\ 0 & x: irrational \end{cases}$$

Then f has a discontinuity of the second kind at every point x, since neither the right-hand limit, $f\left(x^{+}\right)$ nor the left-hand limit, $f\left(x^{-}\right)$, of f at x, exists.

However, in the variant of computable analysis (recursive analysis) for which a definition of continuity we have defined above (Goodstein, 1961), the value of a recursively continuous function for a recursive real argument is constructed, not defined. Hence the above example 'finds no place in recursive analysis' (ibid, p. 43).

Most famously, Markov proved, in 1954 (Markov, 1958, p. 192), the famous theorem on the *impossibility of discontinuity of constructive functions of a real variable*. Since this was a result developed within Russian constructivism it may well be more useful to state it in terms of *recursive functions*. A fairly mundane formal statement of the theorem – in terms of recursive functions - is given as an exercise in Beeson's encyclopaedic text on The Foundations of Constructive Mathematics (Beeson, 1985, p. 66, exercise 4):

Definition: Recursive Discontinuity

A function f from the *reals* to the *reals* is said to have a *recursive discontinuity* at x if there is a positive rational number ξ and a recursive sequence x_n of *recursive reals* converging (*recursively*) to x, such that $f(x_n)$ is bounded away from $f(x)$ by ξ.

Markov's Theorem

A recursive function of a real variable cannot have a recursive discontinuity.

The proof is fairly easy, using the *Unsolvability of the Halting Problem for Turing Machines* that an effective operation cannot have a recursive discontinuity (ibid).

A perceptive reader would immediately see the amalgam of constructive mathematics and computable analysis – a feature of *Russian Constructivism* – especially in the proof strategy,

[9] The classic text by Rudin (1976, p.94) is a prime and honest example of standard real analysis:

> "If *x* is a point in the domain of definition of the function *f* at which f is not continuous, we say that *f* is discontinuous at *x*, or that *f* has a discontinuity at *x*."

where the *Unsolvability of the Halting Problem for Turing Machines* is used. This means, of course, an appeal is made to the *Church-Turing Thesis* – something that is avoided in pure Constructive Mathematics (Brouwer, Bishop, Bridges, Richman, etc.).

In terms of pure computable analysis the related result is that *computable implies continuity* (although the contra-positive is not true[10]; cf. Weihrauch, 2000, p. 71). The equivalent in the Brouwerian version of constructive mathematics is, of course, his famous original theorem that *every function defined on a closed unit interval is uniformly continuous*.

In fact, in this case of pure constructive mathematics – i.e., Brouwer's original version, underpinned by *intuitionistic logic* – there is no place for discontinuity at all! Brouwer, in fact, introduced his axioms for intuitionism mainly to try to regain the core results about continuity. Bishop, on the other hand, with his attempt to develop a constructive mathematics consistent, as much as possible, with standard mathematics, avoided the axioms of intuitionism. He simply modified the definition of continuous function on the real numbers to mean uniform continuity – as did Goodstein, even at the elementary analysis level (cf. Palmgren, 2005; Goodstein, 1948).

Weyl's pungent summary of the Constructive approach to *continuity*, given in his Zurich Lectures of 1920, may help focus the way one should view even *discontinuity* (van Stigt, 1990, p. 379; italics added):

> "It is clear that one cannot explain the concept 'continuous function in a bounded interval' without including 'uniform continuity' and 'boundedness in the definition. Above all, *there cannot be any function in a continuum other than continuous functions*. When the Old Analysis introduced '*discontinuous functions*' it showed most *clearly how far it had departed from a clear understanding of the essence of the continuum*. What is nowadays called a discontinuous function is in reality no than a number of functions in separate continua …"

Those of us who do our formal economic modelling with the mathematics of the constructivists – of any variety – or of the recursion theorist do not pay much attention to the conundrums posed by non-continuities (to retain a primitive intuition which seems to have become clouded after the subversion of it in formalising continuity in standard mathematics!).

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[10] In other words, there are continuous functions that are uncomputable. However we have long conjectured that the contra-positive is true in terms of uniform continuity. Using the equivalence we conjecture between Brouwer's *Fan Theorem* and the idea of the '*finiteness property*' in computable analysis (see Weihrauch, 2000, p. 27), it should be possible to prove the contra-positive.

Therefore, the question that puzzles us is quite simple to pose: why should economists confine themselves to standard real analysis for modelling economic phenomena? The domain of real analysis is manifestly unsuited for economic modelling – especially in the age of the digital computer. Should we, then, be surprised that the paradoxes of discontinuity are ubiquitous in models of economic dynamics?

But, then, the paradoxes of uncomputabilities, undecidabilities and incompleteness would have to be faced, as squarely as Barkley Rosser has courageously faced the conundrums posed by discontinuities.

§3. Taming Relaxation Oscillations – with Ducks.

> "It is no exaggeration to say that bifurcation theory *is* the mathematics of discontinuity."
>
> Rosser, 2000, p. 12; italics in the original.

In chapter 2 of his admirable summary 'manifesto' for a General Theory of Economic Discontinuities', Rosser points out (ibid, p. 19; bold italics added:

> "Among the elementary catastrophes the two simplest, the fold and the cusp, have been applied the most to economic problems. Figure 2.4 (Fig.2, here) depicts the fold catastrophe with one control variable and one state variable. Two values of the control variable constitute the catastrophe or bifurcation set, the points where ***discontninuous*** behaviour in the state variable can occur, even though the control variable may be smoothly varying. [Fig. 2] also shows a *hysteresis cycle* as the control variable oscillates and discontinuous jumps and drops of the state variable occur at the bifurcation points.
>
> ….[T]he state variable would drop to the lower branch as soon as it lies under the upper branch and vice versa. The middle branch represents an ***unstable equilibrium and hence unattainable except by infinitesimal accident***."

Are unstable equilibria 'unattainable except by infinitesimal accident'? Rosser has, as usual, perceptively juxtaposed two important concepts of relevance for the economic modelling enterprise: the unlikelihood of a persisting *unstable* configuration in state space and the role of '*infinitesimals*'.

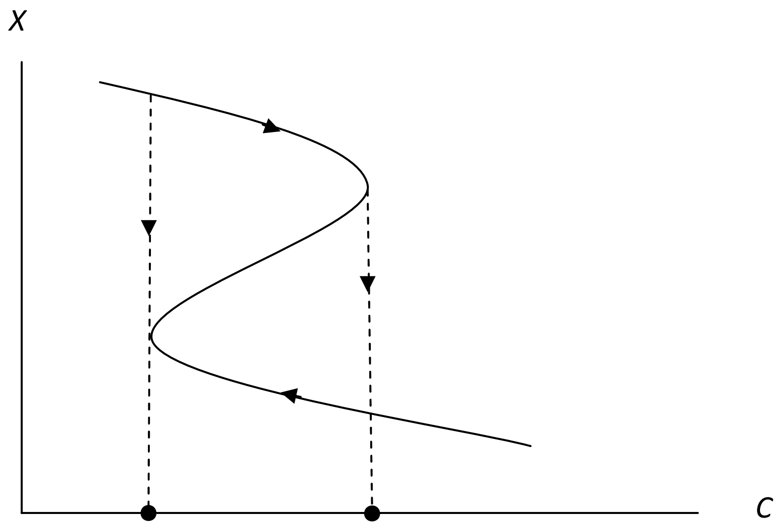

Fig. 2

What would constitute an 'infinitesimal accident'? We shall interpret 'infinitesimal' in its precise non-standard sense and demonstrate the clear possibility of an 'attainable unstable equilibrium'.

Recent results and work in experimental economics (Plott & Smith, 1999) pose fundamental challenges to the traditional economic theorist's view, almost bordering on a *dogma*[11], that *unstable systems are unobservable and unattainable on ordinary scales*. The classic view in economics was articulated a long time ago, by no less an authority than Paul Samuelson:

> "The plausibility of … a stability hypothesis is suggested by the consideration that positions of unstable equilibrium, even if they exist, are transient, nonpersistent states, and hence on the crudest probability calculation would be observed less frequently than stable states. *How many times has the reader seen an egg standing upon its end?*"
>
> Samuelson, 1947, p. 5; italics added.

[11] Later, Samuelson himself raised doubts on adhering to the 'stability dogma' in his homage to Ragnar Frisch (Samuelson, 1974, p.7), crucially in the context of a statement about the van der Pol-Rayleigh systems of nonlinear differential equations. In dynamical systems theory, Ralph Abraham & Jerrold Marsden referred to the 'dogma of stability' in their classic text on the Foundations of Mechanics (Abraham & Marsden, 1978, p. xx).

The *cubic characteristic* of, in particular, the (unforced) van der Pol equation, generates a geometry that is identical to the *fold*. The van der Pol equation, and its integrated form as the Rayleigh equation, played an important role in the nonlinear (endogenous) theory of the business cycle in the `Golden Age' of Keynesian dominance[12]. The full economic background to its use in business cycle theory, and the mathematical underpinnings, are extensively discussed in a series of three papers and we refer the interested reader to them for further information.

Our interest here is to point out how, using *nonstandard analysis*, unusual phase portraits were discovered for this fascinating equation[13]. In particular, to show – only as an example – how the bifurcation of multi-phase dynamical systems (cf. Day, 1994, Ch. 6), in particular of the Liénard-van der Pol variety, can show *the attainability and persistence of unstable limit cycles*. These cycles, emerging from a bifurcation point off the 'cubic characteristic' of, say, a van der Pol equation, at the 'infinitely fast' transition to the stable manifold, are now called 'Canards'[14] in 'honour' of the geometric appearance of the phase portraits, which resemble a 'Duck's Head'. The 'Duck's Head' manifests a persistent cycle and resides in the unstable manifold of the state space.

Zvonkin and Shubin, in their detailed and rigorous analysis of the issue here, summarised admirably the nature of the discovery (Zvonkin & Shubin, 1984, p.69; italics added):

> "Ducks are certain singular solutions of equations with a small parameter, which are studied in the theory of relaxation oscillations. These solutions were *first found for the van der Pol equation*, and their form resembled that of a flying duck. *Duck theory* is, in the authors' opinion, *the most striking application of the techniques of non-standard analysis.*

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[12] Its first appearances in the business cycle literature were in unfortunately neglected papers by Hamburger (1930, 1931) as equation # 7, on p.5, in the former and in footnote 7, p.6 in the latter) in the formal form:

$$\frac{d^2y}{dt^2} - \alpha\left(1 - y^2\right)\frac{dy}{dt} + \omega^2 y = 0$$

[13] We hasten to add that, ex post, standard analysis has been able to re-absorb the new discoveries into its fold. The point remains, however, that the original discovery came about by a fertile use of nonstandard analysis.

[14] *Canard Solutions* were discovered by Georges Reeb's school in Strasbourg in 1977. They appeared during a Hopf bifurcation of a one-parameter family of dynamical systems, particularly the van der Pol equation. A rich variety of results, in an expository manner, is available in Nonstandard Analysis in Practice edited by Francine and Marc Diener (Diener & Diener, 1995), who have themselves pioneered the study of 'Canards'.

......

> It was not by chance that ducks were discovered with the help of non-standard analysis and in connection with it. We think that the language of non-standard analysis will make it easy for a wide circle of mathematicians to become acquainted with the theory of ducks and the theory of relaxation oscillations in general."

Relaxation oscillations encompass two-phase dynamics in the sense that there is an interaction between slow and fast variables in the system, rather like one set of markets clearing `infinitely fast', and another set clearing relatively slowly. The problem, of course, is that `infinitely fast' is a meaningless concept in standard analysis, but an eminently sensible notion in nonstandard analysis; analogously, the `infinitesimal' is a fully viable concept in nonstandard analysis, but not so in standard analysis. Hence, Rosser's perceptive caveat that they are 'infinitesimal accidents' – which they have to be in standard real analysis.

Consider, now, the following variant of the van der Pol equation (or consider it as a special case of the more general Liénard equation)[15]:

$$\varepsilon \frac{d^2x}{dt^2} + \left(x + x^2\right)\frac{dx}{dt} + x + \alpha = 0$$

This can be represented as the equivalent phase-plane system:

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \varepsilon^{-1}\left(y - f\left(x\right)\right)$$

$$\frac{dy}{dt} = -\left(x + \alpha\right)$$

with the 'characteristic' given by:

$$f\left(x\right) = \frac{1}{3}x^3 + \frac{1}{2}x^2$$

The phase-plane dynamics depicted in the diagrams below are for the following numerical values of α and ε (where the red curves are, in both cases, the graphs of the 'cubic characteristic'):

1. For Figure 3,: α=.5;ε=1000;

2. Figure 4 (the `Duck Headed vdP' dynamics) is obtained for: α=.001012345;ε=1000;

[15] This example is clearly and fully analysed in MacGillivray, et.al., (1994). We have used *Matlab* in our replication of the examples.

3. Finally, Figure 5, the `Unheaded Duck is obtained for: α=.00001025;ε=1000;

Figure 3

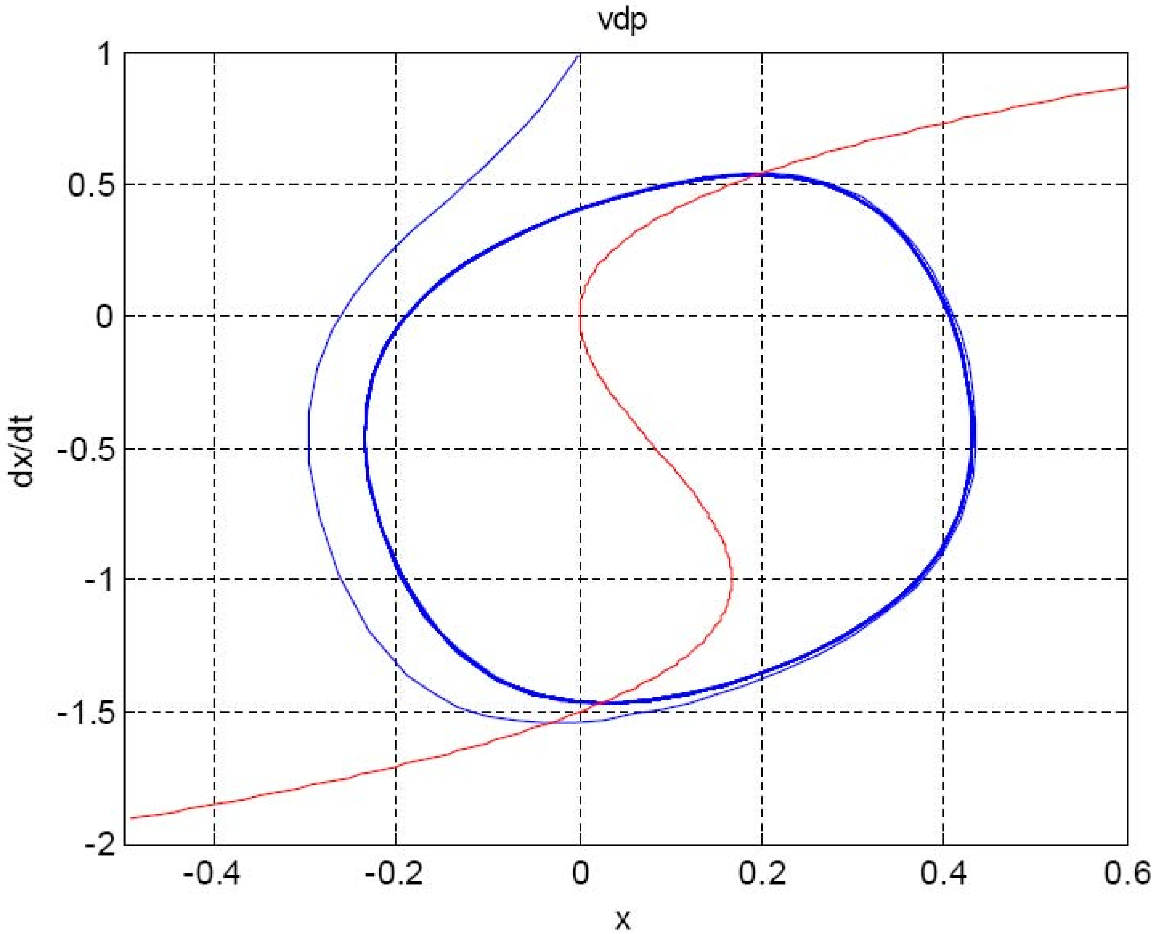


Figure 4

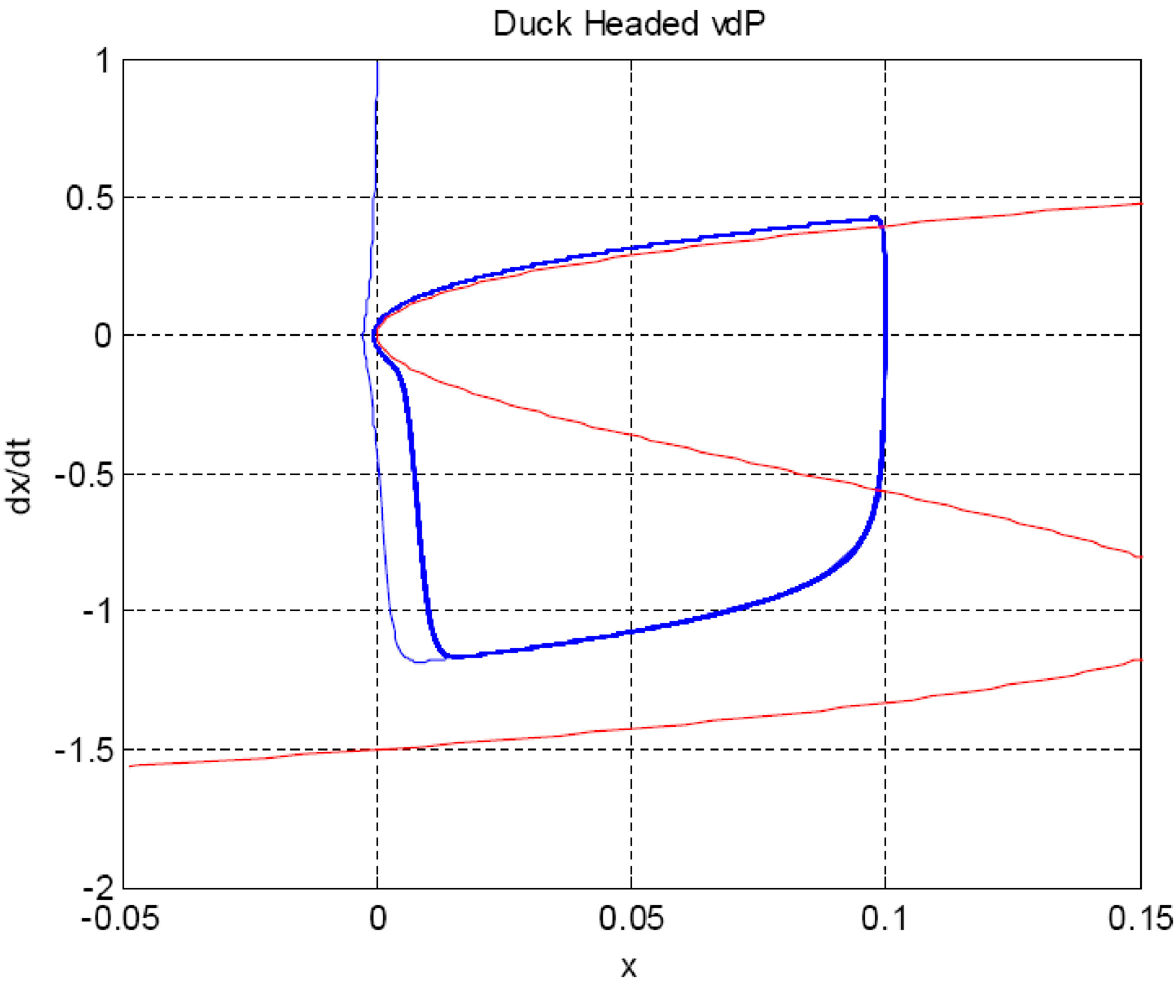
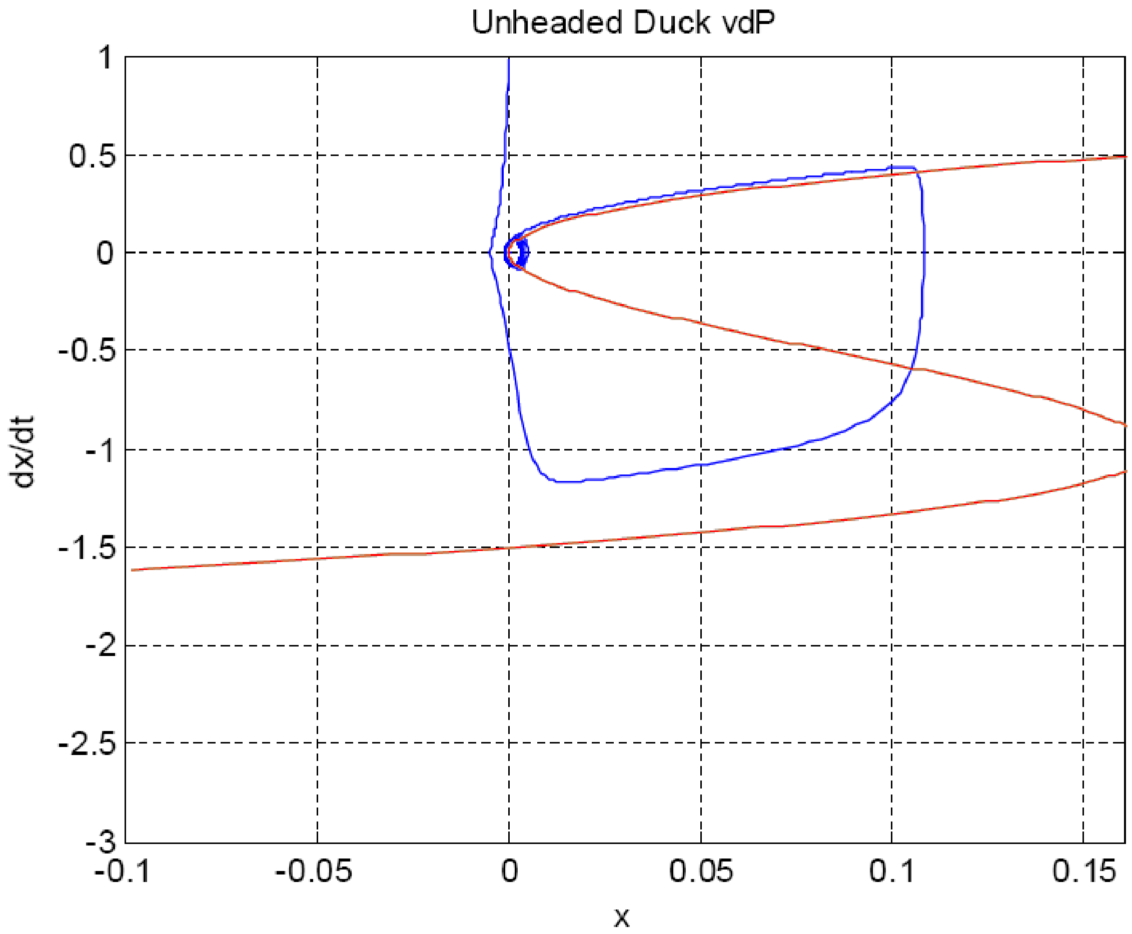


Figure 5



The proof of existence of `*Unheaded Ducks*;, i.e., counter-intuitive cycles being attracted to unstable manifolds, for the van der Pol system is extremely simple - provided one learns a bit of nonstandard analysis - or, at least, nonstandard terminology. Let us simply state it, in as heuristic and intuitive way as possible, to illustrate what we mean; the interested reader can get a clear idea from the exceptionally clear and detailed article by Zvonkin and Shubin (op.cit.). The only thing to keep in mind is that *α in an infinitesimal in the sense of nonstandard analysis*. Then[16] (referring to the last two phase-portraits and using *f* generically for the 'cubic characteristic'):

[16] This is only a sufficient condition and the `admissible curve' is simply a formalization of the traditional `cubic characteristic' for the van der Pol equation. We conjecture that `Duck Cycles' can be shown to exist even without a `cubic characteristic'; say, for example, with a `characteristic' of the form: $\tau\left(e^{x}-2\right)$. Such a form would have only one isolated maximum or minimum.

Definition: *An admissible form for the characteristic, f(x)*

$f(x)$ has an admissible form on a closed interval, say $[\beta_1, \beta_2]$, if:

$(1).f(x) \in [\beta_1, \beta_2]$ is *standard* and $\mathbb{C}^2$;

$(2).f(x) \in [\beta_1, \beta_2]$ has exactly two isolated extremum points, say a minimum at x_0, and a maximum at x_1, and $\beta_1 < x_1 < x_0 < \beta_2$, so that: $f'(x) > 0$ on $[\beta_1, x_1)$ and $(x_0, \beta_2]$ and $f'(x) < 0$ on (x_1, x_0);

$(3).f(\beta_1) < f(x_0)$ and $f(\beta_2) > f(x_1)$;

Theorem

Suppose $f(x)$ has an *admissible form* on $[\beta_1, \beta_2]$, and if $x_\beta \in \Re$ & $x_\beta \in (x_2, x_0)$, *then* $\exists$ a value of the infinitesimal α for which the van der Pol system has a *Duck Cycle* s.t x_β is the value on the x-coordinate corresponding to the '*beak*' of the '*Duck*'.

The point of the exercise is that a knowledge of the possibilities for exploring a dynamical system with parameters and variables taking actual *infinitesima*l and *infinite* values is indispensable - not just for reasons of pure mathematical aesthetics; but also for eminent economic reasons, where financial market variables move `infinitely' fast, at least relative to `real' variables; and reactions in market sentiments to `infinitesimal' variations in parameters is a non-negligible factor in turbulent markets. An economist, narrowly trained in standard mathematics will always have to resort to *ad hockeries* to handle the *infinitesimal* and the *infinite* - for example, in models capable of *relaxation oscillations*. Quite apart from aesthetics and pragmatics, it is also the case that the mathematics of nonstandard analysis is intuitively natural, pragmatically constructive and conceptually much simpler, without any of the contorted paraphernalia of the Weierstrassian `ε-δ' calisthenics.

It is to Barkley Rosser's credit that his honed analytical intuition was able to see the need for appealing to an 'accidental infinitesimal', if constrained to discourse in an inappropriate mathematical domain.

§3, Beyond the Rosserian Visions

> "I would like to make it clear that I find merit in the Catastrophe theorist's use of modern calculus and geometric techniques in models in science. In particular discontinuities can often best be understood via this kind of mathematics. For example it would be important to find a calculus oriented approach model for the computer, a machine which is intrinsically discrete. Such a calculus model would not be exact, but it would give great insight to automata theory."
>
> Smale, 1978, p.1366.

Even the gods nod – sometimes! Smale discourses as if constructive and computable analysis do not furnish a 'calculus oriented approach model for the computer, a machine which is intrinsically discrete'. What else are they, if they are not at least that? Indeed, comstructive and computable analysis are, in precise senses, much more 'exact' than the conceptual fudging that masquerades as precise analysis in standard calculus with its reliance on *Zermelo-Fraenkel* set theory and the *Axiom of Choice*. How numerical and quantitative and computational can such a calculus be- compared to the other three kinds of mathematics?

We have attempted to start from visions that Rosser has, over a period of thirty years, developed for economic theory, mathematical modelling of economic phenomena and applicable dynamic theory from the vantage point of mathematical theories of discontinuities. The inspiration from Rosser's provocative visions have helped focus our attention – and, we hope the economic profession's – on the importance of thinking about discontinuities in mathematically coherent ways. Rosser has tirelessly propagated the view that one of the best ways to understand and, perhaps, tame –formally - the unruly dynamics of the economic world would be to underpin them in a theory of dynamic discontinuities.

We have tried to clarify the mathematics of continuity, given that the pure mathematics of discontinuity seems always to have been developed as a minor corollary to characterizations of continuity in different kinds of mathematics. Our melancholy conclusion is that standard real analysis is wholly unsuited to any serious discourse about discontinuity, especially from a quantitative and computational sense – those senses that are crucial for an applied, policy oriented, science like economics.

We dismiss *ad hoc* justifications given by unreflective economists -- or would-be economists – for modelling economic phenomena in terms of standard real analysis and its version of continuity, as not worth serious consideration from the point of view of the foundations of mathematics or even from the vantage point of numerically meaningful and computationally significant mathematical formalisms. For example it is claimed – in a sense in the spirit of Smale's above criterion – by Mount and Reiter (Mount & Reiter, 2002, p.1) that:

> "Computing with real numbers .... is also relevant to applications in economic theory. Economic models typically use real variables and functions of them."

But it is not explained or even seriously discussed why `economic models typically use real variables ... .'. Similarly, discussion about continuity, discontinuity and dynamics has almost entirely been in terms of real analysis. We think this is an unproductive and, ultimately, a futile exercise. Our reasons, many of them given above, are methodological and epistemological.

The standard epistemological vision may well be the one that was propagated by Hilary Putnam and Willard Van Orman Quine (Putnam, 1979, Quine, 1948) and generally referred to as the 'indispensability argument'. The spirit of our discussion in the main body of this paper is best summarised by the deep objections raised by Solomon Feferman to the Putnam-Quine 'indispensability thesis' (Feferman, 1998). The `indispensabilists' flounder on the deep ontological issues and doubts raised about their program by Feferman (ibid, p. 284)[17]:

> "If one accepts the indispensability arguments, there still remain two critical questions:
>
> Q1. Just which mathematical entities are indispensable to current scientific theories?
>
> Q2. Just what principles concerning those entities are needed for the required mathematics?"

We believe these are the two crucial questions, even if not framed in the context of a critique of an `indispensability argument', that a mathematical economist, who relies exclusively on any one type of mathematical formalism for economic modelling, should try to answer - or, at least, keep as disciplining background criteria. Our vision in this paper is almost entirely

[17]Feferman's thoughtful closing remark and query is also relevant in the context of the mathematical economists' penchant for modelling in terms of real numbers and standard real analysis, *ibid*, p. 298:

> "[A]s long as science takes the real number system for granted, its philosophers must eventually engage the basic foundational question of modern mathematics: `What are the real numbers, really?"

The economic apologist's retort may well be that `its philosophers' are irrelevant -- or don't exist -- for the mathematical modelling enterprise of the economic theorist. This instrumentalist position is, in fact, the dominant one in mathematical economics.

disciplined by these two perceptive questions that Feferman raises against the `indispensabilists'. In other words, we take it that the serious mathematical economist is at least a closet `indispensabilist' and, therefore, the themes in this essay are grounded on: (a). casting doubt on the kind of `mathematical entities that are considered indispensable' in orthodox mathematical modelling of economic discontinuities; and, (b). questioning the kind of `principles concerning entities [such as continuity, discontinuity and dynamics]' that are claimed as `necessary for the required mathematics'. Our examples are, therefore, chosen to illustrate that the chosen `mathematical entities' and the `principles concerning these entities' are not appropriate, necessary, relevant or indispensable for mathematical economic modelling – if they are extracted from, embedded in, or belong to the framework of, standard real analysis.

Barkley Rosser raised the important questions, set the standards against which they must be considered, and provided preliminary hints towards some kind of solutions to them: to the problem of continuity, discontinuity and dynamics in economic modelling. Rosser's visions defined a part of the frontier of methodology in economic modelling. It is our duty to try to push these frontiers and develop new visions.

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