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BARGAINING AND MATCHING IN SMALL MARKETS

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Abstract

The present paper focuses on markets where trade is carried out through matching and bargaining and where at each date $t = 0, 1, \ldots$ a finite and exogenously given number of agents enters. Such markets are "small" in the sense that whether a match ends with trade influences matching probabilities at subsequent dates. For a small market we show that as the market becomes large, the equilibrium of the small market converges to the equilibrium of a limit market with a continuum of agents. Nonetheless, for any small market there exists a matching process such that the equilibrium of the small market significantly differs from the equilibrium of the associated large market with a continuum of agents, although equilibrium-path matching probabilities are the same in both markets. Therefore, matching models with a continuum of agents are not always a good approximation of small markets.

Keywords: matching, bargaining, market equilibrium

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1 Introduction

Recently models have been developed in which trade is conducted through sequential pairwise random matching between agents who bargain strategically over prices. These models provide a means of determining whether decentralized trading processes lead to nearly competitive outcomes when trading frictions are small (see for example, Gale [4] or Rubinstein and Wolinsky [8]). In addition, the models are useful in explaining how bargaining procedures, rates of impatience, and outside options affect market outcomes (for example Wolinsky [9]).

The present paper focuses on markets where trade is carried out through matching and bargaining and where at each date $t = 0, 1, \ldots$ a finite and exogenously given number of agents enters.[1] Such markets are "small" in the sense that at each date the number of agents in the market is finite. Thus, an agent's decision on whether to trade with his current partner influences the number of each type of agent in the market at subsequent dates. Since matching probabilities are determined by these numbers, an agent's decision whether or not to trade also influences matching probabilities at subsequent dates.

In the present paper, there are two questions of primary interest. First, does the equilibrium of a small market converge to the equilibrium of a limit market with a continuum of agents as the small market becomes large? The existing literature on markets in which an infinite measure of agents eventually enters assumes that there is a continuum of agents or that the market behaves as if there were a continuum of agents. In that case, whether or not a given match ends with trade has no effect on the measure of each type of agent in the market at the next date. We refer to markets with this property as "large." The second question of interest is whether the equilibrium of a small market is near the equilibrium of the associated large market.

As a useful analogy, the same kinds of limiting questions have been intensively investigated for exchange economies. Hildenbrand [5] gave conditions under which the core converges to the competitive equilibrium as a finite economy converges to its continuum limit. Anderson [1] gives a bound on the non-competitiveness of a fixed finite economy. In the present paper, in the context of a matching and bargaining model, we obtain a Hildenbrand-like result on the

[1] These are markets such as the housing market where market activity continues indefinitely.

convergence of equilibria of small markets to the equilibrium of a large market. We also show that there is no analog to the Anderson type bound on the difference between the equilibrium of a small market and the equilibrium of the associated large market.

To address our first question, we consider for a small market a model similar to that in Rubinstein and Wolinsky [8] (henceforth referred to as RW). We show there is a unique symmetric equilibrium in stationary strategies satisfying a natural assumption on beliefs following off-the-equilibrium-path play. In this equilibrium, if there is an excess of sellers when the market opens, then sellers receive a higher payoff than they do in the equilibrium of the associated large market. Sellers continue to receive a higher payoff in the small market than they do in the large market even as both markets become frictionless. Nonetheless, as the small market becomes large, the small market equilibrium converges to the equilibrium of a large market.[2]

When comparing the equilibria of small and large markets, we compare markets in which the matching probability of each type of agent is the same across markets. Therefore, given a small market and a matching process the associated large market is the one with matching probabilities equal to the equilibrium-path matching probabilities of the small market. Since equilibrium-path matching probabilities are the same in a small market and the associated large market, differences in the equilibria arise solely from the fact that in a small market a matched agent's decision whether or not to trade with his current partner influences matching probabilities.

To address our second question on the nearness of equilibria of small and large markets, we study a small market for a one-parameter family of matching processes. We find that the equilibrium of a small market need not be near the equilibrium of the associated large market. To be precise, consider a small market where at each date n buyers and n sellers enter and where there is an initial excess of $\Delta > 0$ sellers. For any matching process in the family under consideration, the equilibrium-path matching probabilities are the same and equal to $\alpha = \frac{n}{n+\Delta}$ for sellers and $\beta = 1$ for buyers. Moreover, there is a matching process in this family such that

[2] For the purpose of comparison the equilibrium outcome obtained in RW, where the matching probability of each type of agent is a primitive of the model, is taken as the benchmark for the large market outcome. Osborne and Rubinstein [7], p. 141, in their discussion of this model, assert that taking matching probabilities as a primitive is appropriate in a *large* market. Strictly speaking, when entry is exogenous, an agent must have measure zero if it is to have no influence on its matching probability.

sellers obtain a share of the gains to trade arbitrarily close to $\frac{1}{2}$ as frictions vanish. (This is the case even though in a small market n and Δ may be arbitrarily large.) In contrast, in the associated large market, as frictions vanish, sellers obtain a share of the surplus $\frac{\alpha}{\alpha+\beta}$ which is strictly less than $\frac{1}{2}$.

The market equilibrium of a small market depends on the matching process in an essential way: when the discount factor is less than one, the market equilibrium depends on the matching probabilities that are reached both on and off the equilibrium path. As the discount factor approaches one, however, the equilibrium depends *only* on off-the-equilibrium-path matching probabilities. This contrasts sharply with the situation in RW, where the equilibrium characterization depends only on the steady state equilibrium matching probability of each type of agent.

Of course, only equilibrium-path matching probabilities are ever observed. Thus, two small markets which have different matching processes but which to an observer appear identical (i.e. have the same number of agents and the same equilibrium-path matching probabilities) may have very different equilibria. For this reason, when the matching process is unknown, it would be difficult to determine empirically whether or not behavior in a small market is consistent with the theoretical outcome.

2 A Model of a Small Market

Price formation is modeled as a game in discrete time. To describe the game requires specifying the size of the pool of agents who can potentially be matched, how the pool evolves over time given the stream of entrants, and the characteristics of the two types of agents in the pool. Also required is a description of the matching process, the rules of the bargaining game played by matched pairs of agents, and the information of agents.

The Market: The market at date zero consists of $\hat{n}$ buyers and $\hat{n}+\Delta$ sellers. At each subsequent date a stream of n buyers and n sellers enters the market. An agent who enters at date T is said to be a generation T agent. We consider the case where $\Delta > 0$. The case where there is initially an excess of buyers is symmetric.

The Agents: Each seller is endowed with a single indivisible unit of the good and has a reservation price of zero. Each buyer is endowed with a unit of money and demands a single unit of the indivisible good with a reservation price of one. Thus, there is a one unit gain to trade in any match. Agents are von Neumann-Morgenstern expected utility maximizers and discount the future using the discount factor $\delta < 1$. A generation T agent who receives a share z of the unit gain to trade at date $t \geq T$ has utility $\delta^{t-T} z$.

Matching: At each date, each buyer remaining in the market is randomly matched to a seller. Each seller is matched to a buyer with a probability equal to the ratio of buyers to sellers in the market. At date zero, for example, each seller is matched with probability $\hat{n}/(\hat{n} + \Delta)$. If a seller is not matched, he takes no action and remains in the market at the next date.

Bargaining: Once a buyer and seller are matched, each has probability one half of being selected as the proposer. The proposer offers to his partner a share of the unit gain to trade. If his partner agrees, each member of the partnership receives his proposed share and both exit. If his partner disagrees then no trade takes place, the match is broken, and both agents remain in the market at the next date.

Information and Histories: At each date every agent in the market is informed as to whether he is matched. A matched agent is informed whether he is selected to propose a division of the surplus or to respond to his partner's proposal. If the agent is selected to respond, then he is also informed of his partner's proposal. Finally, for an agent selected to propose, following his proposal, he is informed whether his offer has been accepted or rejected. In addition, each agent has perfect recall of his own moves and the information listed above.

We distinguish among four types of histories for an agent who is in the market for the t^{th} time.[3] Let H_O^t denote the set of such histories ending with the information that the agent is matched and selected to propose. Let H_R^t denote the set of such histories where the agent has been selected to respond and has just observed his partner's offer. Let $\hat{H}^t$ denote the set of such histories where the agent was matched, but his match has just ended without trade. Finally,

[3]A generation T agent is in the market for the t^{th} time at date $T + t - 1$.

denote by H^t the set of histories ending with the information that the agent was not matched when in the market for the t^{th} time. A history in H^t or $\hat{H}^t$ contains all the information collected in t periods of market experience.

A market strategy for an agent is an offer for each history where the agent is selected to propose and a response for each history ending with an offer.

A **market strategy** is a sequence of decision rules $f = \{f^t\}_{t=1}^{\infty}$ such that $f^t : H_O^t \to [0,1]$ and $f^t : H_R^t \to \{\text{"}Y\text{"}, \text{"}N\text{"}\}$ $\forall t \geq 1$.

The market strategies of sellers and buyers are denoted by f and g, respectively. Let S denote the set of all market strategies.

A market strategy f is **stationary** if $\forall t, u \geq 1$ then (i) $h^t \in H_O^t$ and $h^u \in H_O^u$ implies $f^t(h^t) = f^u(h^u)$ and (ii) $h^t \in H_R^t$ and $h^u \in H_R^u$ implies $f^t(h^t) = f^u(h^u)$ whenever h^t and h^u both end with the same offer. If an agent's strategy is stationary, the agent makes the same offer whenever selected to propose and the agent's decision whether to accept or to reject an offer depends only on the offer. Let $\bar{S}$ denote the set of stationary strategies.

Let $U_S^T(f,g)(h)$ denote the value to a generation T seller with history h when all sellers employ the stationary strategy f and all buyers employ the stationary strategy g. Define $U_S^T(f';f,g)(h)$ to be the same value except that the seller employs the strategy f' (not necessarily stationary), while all *other* sellers employ the stationary strategy f. In a similar manner define $U_B^T(f,g)(h)$ and $U_B^T(g';f,g)(h)$ for buyers. Shortly we make an assumption which guarantees that these values are well defined for all histories.

A **market equilibrium** [Rubinstein and Wolinsky 1985]: is a pair of stationary strategies (f_*, g_*) such that for all histories h

$$U_S^T(f_*,g_*)(h) \geq U_S^T(f';f_*,g_*)(h) \quad \forall f' \in S, \quad \forall T \geq 0$$

and

$$U_B^T(f_*,g_*)(h) \geq U_B^T(g';f_*,g_*)(h) \quad \forall g' \in S, \quad \forall T \geq 0.$$

In a market equilibrium each agent's strategy is optimal following every history, including histories inconsistent with equilibrium play. Further, the strategy of each agent is optimal over all market strategies, not just stationary strategies.

The value $U_S^T(f,g)(h)$ depends not only upon the bargaining strategies of potential partners, but also depends upon a seller's expectation that he is matched at future dates. If, as in RW, matching probabilities are a primitive of the model, then that all agents of the same type play the same stationary strategy is sufficient to insure that each agent's value is well defined for every history. In a small market, however, the stock of each type of agent in the market, and hence the probability that an agent is matched, depends upon the market history. In fact, in a small market the outcome of a single match influences the matching probability of sellers at the next date: if a matched agent fails to trade, then he and his partner remain in the market at the next date increasing the stock of each type of agent by one over what it would have otherwise been. This has at least the effect of increasing the matching probability of all sellers in the market at the next date.[4]

Let $P^T(h^t)$ denote the probability that a generation T seller assigns to the event that he is matched at date $T+t$ given history $h^t \in H^t \cup \hat{H}^t$. For $f, g \in \bar{S}$, denote by $\gamma_{f,g}$ the probability a match ends with trade when the seller employs f and the buyer employs g. When all buyers employ g and all *other* sellers employ f, we assume that a seller computes $P^T(h^t)$ on the basis that each match in which he was not a member has ended with trade with probability $\gamma_{f,g}$.[5]

For a history h^t on the equilibrium path, $P^T(h^t)$ computed in this fashion coincides with the probability obtained from Bayes' rule. For histories off the equilibrium path, however, Bayes' rule no longer applies and then this assumption has force. Suppose, for example, a seller has a history $h^t \in \hat{H}^t$ (perhaps off the equilibrium path), all buyers employ $g \in \bar{S}$, all other sellers employ $f \in \bar{S}$, and that $\gamma_{f,g} = 1$. Then the seller computes $P^T(h^t)$ on the basis that each match other than his own (both at the current and previous dates) has ended with trade. Therefore, the only buyers in the market at the next date will be the buyer with whom the seller has just

[4]Whether the effect extends beyond the next date depends, in general, upon the matching technology and the probability that a matched pair of agents trades.

[5]This assumption is similar in spirit to the assumption implicit in Gale [3] that having observed off-equilibrium-path play an agent does not take this as evidence a positive measure of agents having deviated from equilibrium play. (See Osborne and Rubinstein [7] pp. 161-162 for a discussion of this assumption.) The assumption in the current paper is that the observation of off-equilibrium-path play in ones own matches is not taken as evidence that other agents in other matches have deviated from equilibrium play.

failed to trade plus the n buyers entering at the next date. Consequently, the seller expects to be matched at the next date with probability $(n+1)/(n+\Delta+1)$.

We now give the recursion equation satisfied by $U_S^T(f';f,g)(h^t)$. When a seller employs $f' \in S$, had history h^t immediately prior to becoming matched, and his partner employs $g \in \bar{S}$, then define $\gamma_{f',g}(h^t)$ to be the probability that the match ends with trade and define $S_{f',g}(h^t)$ to be the seller's expected gain conditional on the match ending with trade.[6] Given a seller has history $h^t \in H^t \cup \hat{H}^t$ and expects to be matched with probability $P^T(h^t)$ then random matching, f', and g induce a probability distribution over histories at the next date. Let $\mathcal{F}(h^t, f', g)$ denote the induced probability distribution over histories in $\hat{H}^{t+1}$ conditional on a seller being matched his $t+1^{st}$ time in the market, but the match ends without trade. The value $U_S^T(f';f,g)(h^t)$ for a seller with history $h^t \in H^t \cup \hat{H}^t$ satisfies

$$\begin{aligned} U_S^T(f';f,g)(h^t) &= \delta P^T(h^t)[\gamma_{f',g}(h^t)S_{f',g}(h^t) + (1-\gamma_{f',g}(h^t))E_{h\in\hat{H}^{t+1}}[U_S^T(f';f,g)(h)|\mathcal{F}]] \\ &\quad + \delta(1-P^T(h^t))U_S^T(f';f,g)(h^t \times (unmatched)), \end{aligned} \tag{1}$$

where $E_{h\in\hat{H}^{t+1}}[U_S^T(f';f,g)(h)|\mathcal{F}]$ is the seller's expected value conditional on his match ending without trade where the expectation is taken with respect to $\mathcal{F}$.The value $U_B^T(g';f,g)(h^t)$ for a buyer with history $h^t \in \hat{H}^t$ satisfies a similar recursion equation.

3 Characterizing the Market Equilibrium

This section characterizes the market equilibrium and establishes that a market equilibrium exists. We do not restrict attention to equilibria where matching probabilities are stationary. In the unique market equilibrium, the market converges to a steady state in one period. In period zero sellers are matched with probability $\frac{\hat{n}}{\hat{n}+\Delta}$ and, along the equilibrium path, sellers are matched with probability $\frac{n}{n+\Delta}$ in each subsequent period.[7]

[6] If $\gamma_{f',g}(h^t) = 0$ then define $S_{f',g}(h^t) = 0$.

[7] In a large market the evolution of the stock of each type of agent (and each types matching probability) is taken to be deterministic by a (usually) implicit appeal to a law of large numbers. This is despite the fact that there are well know problems with applying the law of large numbers when there are a continuum of i.i.d. random variables. (See Feldman and Gilles [2].) This is another reason to model small markets.

THEOREM 1: *A market equilibrium (f_*, g_*), if it exists, is unique and $\forall t \geq 1$ satisfies:*

$$\begin{aligned} f_*^t(h^t) &= V_B^d \quad \forall h^t \in H_O^t \\ &= \text{“}Y\text{”} \quad \forall h^t \in H_R^t \textit{ ending with an offer of } V_S^d \textit{ or greater} \\ &= \text{“}N\text{”} \quad \forall h^t \in H_R^t \textit{ ending with an offer less than } V_S^d \end{aligned}$$

and

$$\begin{aligned} g_*^t(h^t) &= V_S^d \quad \forall h^t \in H_O^t \\ &= \text{“}Y\text{”} \quad \forall h^t \in H_R^t \textit{ ending with an offer of } V_B^d \textit{ or greater} \\ &= \text{“}N\text{”} \quad \forall h^t \in H_R^t \textit{ ending with an offer less than } V_B^d. \end{aligned}$$

where

$$V_S^d = \delta \frac{\lambda + \frac{(1-\delta)(1-\lambda)}{n+\Delta+1}}{2-\delta(1-\lambda)(1+\frac{1}{n+\Delta+1})} \quad , \quad V_B^d = \delta \frac{1-\frac{\delta(1-\lambda)}{n+\Delta+1}}{2-\delta(1-\lambda)(1+\frac{1}{n+\Delta+1})} \; ,$$

and $\lambda = \frac{n}{n+\Delta}$ *is the steady state equilibrium path matching probability of sellers.*
Proof: Appendix.

Theorem 2 shows that a market equilibrium exists. The key to obtaining existence of a market equilibrium in this environment is that the probability an agent assigns to the event that he is matched at the next date only depends upon whether he is matched or unmatched at the current date. In this case, an agent's problem of choosing an optimal strategy is equivalent to a Markov decision problem.

THEOREM 2: (f_*, g_*) *as given in Theorem* 1 *is a market equilibrium.*
Proof: Appendix.

4 Comparing Small and Large Markets

In this section we compare the equilibrium outcomes of small and large markets, taking the equilibrium obtained in RW as the benchmark for the equilibrium of a large market. In the RW model type 1 and 2 agents meet new partners with probability α and β respectively (whether or not currently matched). Once matched each agent in the match has an equal probability of being selected to propose. If a proposal is rejected the match is not necessarily broken (unlike in the present paper). The match dissolves only if one or both of the agents meet a new partner.

In this case the agent(s) meeting a new partner is assumed to abandon its current partner and begin bargaining with the new partner.

The bargaining procedure in the current paper can be seen as a special case of the procedure in RW for $\alpha = \lambda$ and $\beta = 1$ when type 1 and 2 agents are interpreted as sellers and buyers, respectively, and λ is the steady state equilibrium-path matching probability of sellers in the small market. At each date a buyer begins bargaining with a new seller. A seller, however, finds a partner with probability $\lambda = \frac{n}{n+\Delta}$. In the market equilibrium of RW when $\alpha = \lambda$ and $\beta = 1$, then sellers offer $\frac{\delta}{2-\delta(1-\lambda)}$ and receive offers of $\frac{\delta\lambda}{2-\delta(1-\lambda)}$.[8]

In the equilibrium of a small market sellers make smaller offers and receive larger offers than they do in the equilibrium of the associated large market.[9] In the small market sellers offer V_B^d (see Theorem 1) which is less than $\frac{\delta}{2-\delta(1-\lambda)}$.[10] In the small market sellers receive offers of V_S^d which is greater than $\frac{\delta\lambda}{2-\delta(1-\lambda)}$. It follows immediately that unmatched sellers have a higher expected payoff in a small market than in the associated large market.

The intuition behind this result is straightforward. In a small market, following the rejection of an offer, a seller is matched at the next date with probability $\frac{n+1}{n+\Delta+1}$ which is greater than the equilibrium-path matching probability $\frac{n}{n+\Delta}$. In the associated large market a seller's matching probability upon rejecting a offer remains $\frac{n}{n+\Delta}$. Thus, a seller has a higher expected payoff when rejecting an offer in the small market and, therefore, receives larger offers since in a market equilibrium an agent receives an offer just equal to its payoff when rejecting the offer.

More surprisingly, sellers receive larger offers and make smaller offers in the small market even as it becomes frictionless. In the small market we have

$$\lim_{\delta\to 1} V_B^d = \frac{1-\frac{1-\lambda}{n+\Delta+1}}{2-(1-\lambda)(1+\frac{1}{n+\Delta+1})} \quad \textit{and} \quad \lim_{\delta\to 1} V_S^d = \frac{\lambda}{2-(1-\lambda)(1+\frac{1}{n+\Delta+1})} \ .$$

In contrast, in the associated large market sellers offer $\frac{1}{1+\lambda}$ and receive offers of $\frac{\lambda}{1+\lambda}$ as δ approaches one.

[8]In RW a type 1 agent when selected to propose receives $x_* = \frac{2(1-\delta)+\delta\alpha-\delta(1-\delta)(1-\alpha)(1-\beta)}{2(1-\delta)+\delta\alpha+\delta\beta}$ and offers to his partner $1 - x_*$. (See page 1145 of RW.) Substituting λ for α and 1 for β in $1 - x_*$ gives $\frac{\delta}{2-\delta(1-\lambda)}$.

[9]Recall that given a small market and a matching process, the associated large market is the one with matching probabilities equal to the equilibrium-path matching probabilities of the small market.

[10]Notice that $\frac{1-\delta(1-\lambda)\epsilon}{2-\delta(1-\lambda)(1+\epsilon)}$ evaluated at $\epsilon = 0$ gives the large market offer of sellers, while the same expression evaluated at $\epsilon = \frac{1}{n+\Delta+1}$ gives the small market offer. That sellers make smaller offers in the small market follows from the fact that $\frac{d}{d\epsilon}\left(\frac{1-\delta(1-\lambda)\epsilon}{2-\delta(1-\lambda)(1+\epsilon)}\right) < 0$.

We now address the question of whether the equilibrium of the small market converges to the equilibrium of a large market as the small market becomes large. Consider a sequence of small markets $\{\hat{n}_i, \Delta_i, n_i\}_{i=1}^{\infty}$. Small market i begins at date zero with $\hat{n}_i + \Delta_i$ sellers and $\hat{n}_i$ buyers. At each subsequent date an additional n_i agents of each type enter. The equilibrium-path matching probability of sellers and buyers in small market i is $\frac{n_i}{n_i+\Delta_i}$ and 1, respectively. We now state the convergence result.

COROLLARY 1: *Let* $\{\hat{n}_i, \Delta_i, n_i\}_{i=1}^{\infty}$ *be a sequence of small markets such that* $\lim_{i\to\infty} \frac{n_i}{n_i+\Delta_i} = \lambda$ *and* $\lim_{i\to\infty} n_i + \Delta_i = \infty$. *As* $i \to \infty$ *the sequence of small market equilibria converges to the equilibrium of the large market where sellers and buyers are matched with probability* λ *and* 1, respectively.

Proof: Let $V_B^d(i)$ and $V_S^d(i)$ denote the offers made and received by sellers, respectively, in the equilibrium of small market i. By Theorem 1, $V_B^d(i) = \frac{\delta(1-\frac{\delta(1-\lambda_i)}{n_i+\Delta_i+1})}{2-\delta(1-\lambda_i)(1+\frac{1}{n_i+\Delta_i+1})}$, where $\lambda_i = \frac{n_i}{n_i+\Delta_i}$. Writing the expression for $V_S^d(i)$ and taking limits gives

$$\lim_{i\to\infty} V_B^d(i) = \frac{\delta}{2-\delta(1-\lambda)} \quad and \quad \lim_{i\to\infty} V_S^d(i) = \frac{\delta\lambda}{2-\delta(1-\lambda)}. \quad \square$$

The results heretofore show that the equilibrium of a small market and the associated large market differ, but the difference becomes small when (holding the matching process fixed) the small market becomes large (i.e. n and/or Δ become large). In the next section we show that the equilibrium of any fixed small market and the associated large market are not near for some matching processes.

5 Nearness of Equilibria of Small and Large Markets

This section considers a matching model for a one-parameter family of matching processes. We show that for any matching process in this family, the equilibrium of a small market depends on both on- and off-the-equilibrium-path matching probabilities. Moreover, as the discount factor approaches one, the equilibrium depends *only* on off-the-equilibrium-path matching probabilities. Our main result is that the equilibrium of a small market need not be near the equilibrium of the associated large market as both markets become frictionless.

As before, the market at date zero consists of $\hat{n}$ buyers and $\hat{n}+\Delta$ sellers and at each subsequent date a stream of n buyers and n sellers enters. The model in the present section differs from the one studied earlier with respect to the information received by players and the matching process. Following entry at each date, every agent in the market is informed of the number of buyers in the market. Of this number, n are new entrants and k remain from previous periods. The matching process, parameterized by an integer $k^* \geq 0$, operates as follows: when $n+k$ buyers (and, therefore, $n+k+\Delta$ sellers) are in the market, the process produces n matches if $k \leq k^*$ and it produces $n+k-k^*$ matches if $k > k^*$. In all other respects the model is the same as described in section 2.

An agent's strategy is semi-stationary if the agent's offer when selected to propose depends only on the number of buyers in the market and the agent's response to an offer depends only on the number of buyers and the offer received. For any given k^* there is a unique market equilibrium in semi-stationary strategies, and in this equilibrium each match ends with trade.[11] For brevity we omit the game theoretic details of this argument as they parallel the proofs of Theorems 1 and 2 and contribute no additional insight. Instead we focus on the value equations characterizing the market equilibrium.

For matching process k^* let $V_i^k(k^*)$ denote the value to an agent of type i when the agent is unmatched, prior to random matching, and $n+k$ buyers are in the market. Suppressing the dependence of V_i^k on k^*, the value equations for matching process k^* are

$$\begin{aligned}
&for\ 0 \leq k \leq k^*: & V_S^k &= \alpha_k[\tfrac{1}{2}(1-\delta V_B^{k+1}) + \tfrac{1}{2}\delta V_S^{k+1}] + (1-\alpha_k)\delta V_S^k \\
& & V_B^k &= \beta_k[\tfrac{1}{2}(1-\delta V_S^{k+1}) + \tfrac{1}{2}\delta V_B^{k+1}] + (1-\beta_k)\delta V_B^k \\
&and & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2) \\
&for\ k > k^*: & V_S^k &= \alpha_k[\tfrac{1}{2}(1-\delta V_B^{k^*+1}) + \tfrac{1}{2}\delta V_S^{k^*+1}] + (1-\alpha_k)\delta V_S^{k^*} \\
& & V_B^k &= \beta_k[\tfrac{1}{2}(1-\delta V_S^{k^*+1}) + \tfrac{1}{2}\delta V_B^{k^*+1}] + (1-\beta_k)\delta V_B^{k^*},
\end{aligned}$$

where $\alpha_k = \frac{n}{n+\Delta+k}$ if $k \leq k^*$, $\alpha_k = \frac{n+k-k^*}{n+\Delta+k}$ if $k > k^*$, $\beta_k = \frac{n}{n+k}$ if $k \leq k^*$, and $\beta_k = \frac{n+k-k^*}{n+k}$ if $k > k^*$. These equations reflect the fact that if there are $n+k$ buyers in the market and $k \leq k^*$ then there are n matches and $n+k+1$ buyers in the market at the next period if exactly one of these matches ends in disagreement. If there are $n+k$ buyers and $k > k^*$ then there are

[11] Assuming that an off-the-equilibrium-path offer is not taken as evidence of deviations from equilibrium play in concurrent matches.

$n + k - k^*$ matches and $n + k^* + 1$ buyers in the market at the next period if exactly one of these matches ends in disagreement. Thus if there are $n + k$ buyers in the market, a matched seller's disagreement payoff is δV_S^{k+1} if $k \leq k^*$ and is $\delta V_S^{k^*+1}$ if $k > k^*$.

Values are defined by the infinite system of simultaneous linear equations given above. Nevertheless, one can characterize the solution by focusing on the four equations for $k = k^*$ and $k = k^* + 1$. Solving, one obtains

$$V_S^{k^*}(k^*) = \frac{\alpha_{k^*}(1+\delta B_{k^*})}{\delta[\alpha_{k^*}(1+\delta B_{k^*})+\beta_{k^*}(1+\delta A_{k^*})]+(1-\delta)[2+\delta A_{k^*}+\delta B_{k^*}]}$$

and (3)

$$V_B^{k^*}(k^*) = \frac{\beta_{k^*}(1+\delta A_{k^*})}{\delta[\alpha_{k^*}(1+\delta B_{k^*})+\beta_{k^*}(1+\delta A_{k^*})]+(1-\delta)[2+\delta A_{k^*}+\delta B_{k^*}]},$$

where $A_{k^*} = \alpha_{k^*} - \alpha_{k^*+1}$ and $B_{k^*} = \beta_{k^*} - \beta_{k^*+1}$.[12]

To describe the equilibrium-path evolution of the market it is convenient to take $\hat{n}$ to be equal to n. Then, on the equilibrium path, at each date zero buyers remain in the market from previous periods since each buyer is matched and each match ends with trade. Therefore, the equilibrium payoffs to sellers and buyers are $V_S^0(k^*)$ and $V_B^0(k^*)$, respectively. On the equilibrium path, sellers when selected to propose offer $\delta V_B^1(k^*)$ and buyers when selected to propose offer $\delta V_S^1(k^*)$. If there are $n + k$ buyers in the market and $k > 0$, then the market has moved off the equilibrium path. In this case, sellers offer $\delta V_B^{k+1}(k^*)$ and buyers offer $\delta V_B^{k+1}(k^*)$.

Given (3), $V_S^0(k^*)$ and $V_B^0(k^*)$ can be computed recursively using

$$V_S^k(k^*) = \frac{\frac{1}{2}\alpha_k}{1 - \delta(1 - \alpha_k)}(1 - \delta V_B^{k+1}(k^*) + \delta V_S^{k+1}(k^*)), \tag{4}$$

which follows from (2) for $k \leq k^*$. It is immediate that when the discount factor is positive but less than one, $V_S^0(k^*)$ and $V_B^0(k^*)$ depend not only on equilibrium-path matching probabilities α_0 and β_0, but also depend on the off-the-equilibrium-path matching probabilities $\alpha_1, \ldots, \alpha_{k^*+1}$ and $\beta_1, \ldots, \beta_{k^*+1}$.

Theorem 4, which follows, shows that as frictions vanish equilibrium payoffs depend only on the off-the-equilibrium-path matching probabilities α_{k^*}, α_{k^*+1}, β_{k^*}, and β_{k^*+1}. To understand this result it is useful to recall that in the RW model of a large market the equilibrium payoff to a seller as frictions vanish is

$$\frac{\alpha}{\alpha + \beta} = \frac{1}{1 + \frac{\beta}{\alpha}}, \tag{5}$$

[12]For completeness, we add that $V_S^{k^*+1}(k^*) = \frac{(\alpha_{k^*+1}+\delta A_{k^*})(1+\delta B_{k^*})}{\delta[\alpha_{k^*}(1+\delta B_{k^*})+\beta_{k^*}(1+\delta A_{k^*})]+(1-\delta)[2+\delta A_{k^*}+\delta B_{k^*}]}$ and $V_B^{k^*+1}(k^*) = \frac{(\beta_{k^*+1}+\delta B_{k^*})(1+\delta A_{k^*})}{\delta[\alpha_{k^*}(1+\delta B_{k^*})+\beta_{k^*}(1+\delta A_{k^*})]+(1-\delta)[2+\delta A_{k^*}+\delta B_{k^*}]}$.

where α and β are the matching probabilities of sellers and buyers, respectively. As frictions vanish payoffs depend upon the ratio of matching probabilities, not their magnitudes. In a small market with matching process k^*, by rejecting a sequence of k^* or more offers, a seller permanently reduces the matching probability of sellers from α_0 to α_{k^*} and of buyers from β_0 to β_{k^*}. This also reduces the ratio of the matching probabilities from $\frac{\beta_0}{\alpha_0}$ to $\frac{\beta_{k^*}}{\alpha_{k^*}}$, which from (5) increases the seller's payoff. In a market equilibrium a seller receives offers sufficiently large so that such manipulation is not carried out. Thus in a small market, as frictions vanish, it is the matching probabilities that would prevail after manipulation that determine payoffs, not equilibrium-path matching probabilities.

THEOREM 3: $\lim_{\delta\to 1} V_S^0(k^*) = \frac{\alpha_{k^*}(1+B_{k^*})}{\alpha_{k^*}(1+B_{k^*})+\beta_{k^*}(1+A_{k^*})}$ *and* $\lim_{\delta\to 1} V_B^0(k^*) = \frac{\beta_{k^*}(1+A_{k^*})}{\alpha_{k^*}(1+B_{k^*})+\beta_{k^*}(1+A_{k^*})}$.

Proof: That $\lim_{\delta\to 1} V_S^{k^*}(k^*) = \frac{\alpha_{k^*}(1+B_{k^*})}{\alpha_{k^*}(1+B_{k^*})+\beta_{k^*}(1+A_{k^*})}$ and $\lim_{\delta\to 1} V_B^{k^*}(k^*) = \frac{\beta_{k^*}(1+A_{k^*})}{\alpha_{k^*}(1+B_{k^*})+\beta_{k^*}(1+A_{k^*})}$ follows from (3). For $i \geq 0$, let $P(i)$ be the proposition "$\lim_{\delta\to 1} V_S^i(k^*) = \lim_{\delta\to 1} V_S^{k^*}(k^*)$, $\lim_{\delta\to 1} V_B^i(k^*) = \lim_{\delta\to 1} V_B^{k^*}(k^*)$, and $i \leq k^*$." $P(i)$ is obviously true for $i = k^*$. We show $P(i)$ is true and $i > 0$ implies $P(i-1)$ is true.

That $P(i)$ is true and $i > 0$ implies $0 \leq i-1 < k^*$. For $k = i-1$ we have from (4) that

$$V_S^{i-1}(k^*) = \frac{\frac{1}{2}\alpha_{i-1}}{1-\delta(1-\alpha_{i-1})}(1 - \delta V_B^i(k^*) + \delta V_S^i(k^*)).$$

Taking limits gives

$$\lim_{\delta\to 1} V_S^{i-1}(k^*) = \frac{1}{2}(1 - \lim_{\delta\to 1} V_B^{k^*}(k^*) + \lim_{\delta\to 1} V_S^{k^*}(k^*)) = \lim_{\delta\to 1} V_S^{k^*}(k^*),$$

where the first equality follows from the fact that $P(i)$ is true and the second follows from the fact that $1 - \lim_{\delta\to 1} V_B^{k^*}(k^*) = \lim_{\delta\to 1} V_S^{k^*}(k^*)$. The symmetric argument gives $\lim_{\delta\to 1} V_B^{i-1}(k^*) = \lim_{\delta\to 1} V_B^{k^*}(k^*)$. Therefore, $P(i-1)$ is true. By induction $P(i)$ is true for $0 \leq i \leq k^*$. In particular it is true for $i = 0$, which is the result. □

Finally, we show that the equilibrium of a small market with an arbitrary but finite number of agents need not be near the equilibrium its associated large market. In the small market the equilibrium-path matching probability of sellers is $\alpha_0 = \frac{n}{n+\Delta} < 1$ and of buyers is $\beta_0 = 1$ for any matching process k^*. Thus, for any matching technology in this family the associated large market is the same. Corollary 2 shows that there is matching process with k^* sufficiently large such that sellers obtain a share of the surplus arbitrarily close to $\frac{1}{2}$ as frictions vanish.

COROLLARY 2: $\lim_{k^* \to \infty} \lim_{\delta \to 1} V_S^0(k^*) = \frac{1}{2}$.

Proof:

$$\lim_{k^* \to \infty} \lim_{\delta \to 1} V_S^0(k^*) = \lim_{k^* \to \infty} \frac{1}{1 + \frac{\beta_{k^*}(1+A_{k^*})}{\alpha_{k^*}(1+B_{k^*})}} = \frac{1}{2}$$

where the first equality follows from Theorem 3 and the second equality follows from the fact that $\lim_{k^* \to \infty} \frac{\beta_{k^*}(1+A_{k^*})}{\alpha_{k^*}(1+B_{k^*})} = 1$. □

In contrast, in the associated large market, sellers obtain a share of the surplus $\frac{\alpha_0}{\alpha_0+\beta_0} < \frac{1}{2}$ as fictions vanish. We conclude that the equilibrium of a small market need not be near the equilibrium of the associated large market. The potential difference of the equilibria is increasing as $\frac{n}{n+\Delta}$ decreases.

6 Conclusion

We have shown that the equilibrium of a small market converges to the equilibrium of a large market with a continuum of agents as the small market becomes large. Nonetheless, for any fixed small market there are matching processes such that the equilibrium of the small market is not near the equilibrium of its continuum limit. Hence, a market with a continuum of agents may not be a good approximation of markets with a finite number of agents. The results of the preceding section suggest that a continuum model will only be a good approximation of a finite model if in the finite model the matching process has the property that agents can effect only small changes in matching probabilities.

7 Appendix

Before proving Theorem 1, it is convenient to prove some preliminary results for the situation where all sellers employ $f \in \bar{S}$ and all buyers employ $g \in \bar{S}$. In this case, the sum of the values of a generation zero buyer and seller whose match has just ended without trade is less than one

(Lemma 1(a)). The value to a generation zero buyer is the same for all histories where his match ended without trade his first time in the market (Lemma 1(b)). The value to a generation zero seller who is unmatched at the end of his first time in the market depends only upon whether his match ended without trade or whether he was not matched (Lemma 1 (c) and (d)).

LEMMA 1: *Let* $f, g \in \bar{S}$. *Then (a)* $\forall \hat{h}, \bar{h} \in H^1 \cup \hat{H}^1 : U_S^0(f,g)(\hat{h}) + U_B^0(f,g)(\bar{h}) < 1$, *(b)* $\forall \hat{h}, \bar{h} \in \hat{H}^1 : U_B^0(f,g)(\hat{h}) = U_B^0(f,g)(\bar{h})$, *(c)* $\forall \hat{h}, \bar{h} \in \hat{H}^1 : U_S^0(f,g)(\hat{h}) = U_S^0(f,g)(\bar{h})$, *and (d)* $\forall \hat{h}, \bar{h} \in H^1 : U_S^0(f,g)(\hat{h}) = U_S^0(f,g)(\bar{h})$.

Proof: If $\gamma_{f,g} = 0$, then $U_S^0(f,g)(h) = U_B^0(f,g)(h) = 0 \; \forall h \in H^1 \cup \hat{H}^1$ and the result is immediate. If $\gamma_{f,g} > 0$ then $S_{f,g}$ and $B_{f,g}$, the expected gain of a seller and buyer conditional on their match ending with trade are well defined and do not depend upon a history. For $h \in H^1 \cup \hat{H}^1$ we can write

$$U_S^0(f,g)(h) = \delta C_1(h) S_{f,g} + \delta^2 C_2(h) S_{f,g} + \dots \qquad (6)$$

where $C_i(h)$ represents the probability of matching and trading at date i given history $h \in H^1 \cup \hat{H}^1$. To illustrate, we compute $C_1(h)$. Given history $h \in \hat{H}^1$, for example, the seller was matched at date zero and so there were $\hat{n} - 1$ other matches. Further, a seller computes his probability of matching on the basis that each match in which he was not a member has ended with trade with probability $\gamma_{f,g}$. If x of these $\hat{n} - 1$ other matches ended with trade, then the seller is matched with probability $\frac{\hat{n}+n-x}{\hat{n}+n+\Delta-x}$ at date one. Therefore,

$$C_1(h) = \sum_{x=0}^{\hat{n}-1} \binom{\hat{n}-1}{x} \gamma_{f,g}^x (1-\gamma_{f,g})^{\hat{n}-x-1} \frac{\hat{n}+n-x}{\hat{n}+n+\Delta-x}.$$

That $C_1(h) + C_2(h) + \dots \leq 1$ implies $U_S^0(f,g)(h) \leq \delta S_{f,g}$. We can also write $U_B^0(f,g)(h)$ as

$$U_B^0(f,g)(h) = \delta\gamma_{f,g} B_{f,g} + \delta^2 (1-\gamma_{f,g})\gamma_{f,g} B_{f,g} + \dots \leq \delta B_{f,g}. \qquad (7)$$

Part (a) follows since $S_{f,g} + B_{f,g} = 1$ and $\delta < 1$.

It is immediate from (7) that $\forall \hat{h}, \bar{h} \in \hat{H}^1 : U_B^0(f,g)(\hat{h}) = U_B^0(f,g)(\bar{h})$. Since the probability a seller with history h matches and trades at date t depends only upon whether h is in $\hat{H}^1$ or in H^1, it follows immediately from (6) that $\forall \hat{h}, \bar{h} \in \hat{H}^1 : U_S^0(f,g)(\hat{h}) = U_S^0(f,g)(\bar{h})$ and $\forall \hat{h}, \bar{h} \in H^1 : U_S^0(f,g)(\hat{h}) = U_S^0(f,g)(\bar{h})$. □

A simple bargaining game is now introduced. Lemma 2 characterizes its unique perfect equilibrium. Lemma 3 establishes the relationship between a market equilibrium and the perfect equilibrium of the bargaining game with suitably chosen disagreement payoffs.

> A seller and buyer are matched. Each agent has equal probability of being selected as the proposer. The proposer offers to his partner a share of the unit gain to trade. If his partner accepts the offer then the gain is divided as proposed and each agent's payoff is the share he receives. If his partner rejects the offer then the seller and buyer receive the disagreement payoffs $V_S^d > 0$ and $V_B^d > 0$, respectively, where $V_S^d + V_B^d < 1$.

The set of histories that the seller or buyer may observe in the bargaining game is precisely $H_O^1 \cup H_R^1$. A strategy is defined as follows.

A **bargaining strategy** is a decision rule $\tilde{f}$ such that $\tilde{f} : H_O^1 \to [0,1]$ and $\tilde{f} : H_R^1 \to \{\text{"}Y\text{"}, \text{"}N\text{"}\}$.

The bargaining strategies of the seller and buyer are denoted by $\tilde{f}$ and $\tilde{g}$, respectively. Let $\tilde{S}$ denote the set of all bargaining strategies.

LEMMA 2: *There exists a unique perfect equilibrium* $(\tilde{f}, \tilde{g})$ *of the bargaining game where*

$$\begin{aligned} \tilde{f}(h^1) &= V_B^d \quad \forall h^1 \in H_O^1 \\ &= \text{"}Y\text{"} \quad \forall h^1 \in H_R^1 \textit{ ending with an offer of } V_S^d \textit{ or greater} \\ &= \text{"}N\text{"} \quad \forall h^1 \in H_R^1 \textit{ ending with an offer less than } V_S^d \end{aligned}$$

and

$$\begin{aligned} \tilde{g}(h^1) &= V_S^d \quad \forall h^1 \in H_O^1 \\ &= \text{"}Y\text{"} \quad \forall h^1 \in H_R^1 \textit{ ending with an offer of } V_B^d \textit{ or greater} \\ &= \text{"}N\text{"} \quad \forall h^1 \in H_R^1 \textit{ ending with an offer less than } V_B^d. \end{aligned}$$

Proof of Lemma 2: For $h \in H_R^1$ perfection requires the seller accept offers greater than V_S^d and reject offers less than V_S^d. In a perfect equilibrium the seller also accepts offers of V_S^d since otherwise there is no best offer for the buyer when selected to propose. The symmetric argument shows in a perfect equilibrium the buyer accepts only offers of V_B^d or greater. For $h \in H_O^1$ the seller makes the minimum acceptable offer of V_B^d and has the payoff $1 - V_B^d$. Offering less than V_B^d, the seller's payoff is $V_S^d < 1 - V_B^d$. □

LEMMA 3: *If (f_*, g_*) is a market equilibrium, then $(\tilde{f}, \tilde{g}) = (f_*^1, g_*^1)$ is a perfect equilibrium of the bargaining game where, for $h \in \hat{H}^1$, $V_S^d \equiv U_S^0(f_*, g_*)(h)$ and $V_B^d \equiv U_B^0(f_*, g_*)(h)$.*

Proof: V_B^d and V_S^d are both well defined given Lemma 1 (b) and (c). Let $\tilde{U}_i(\tilde{f}, \tilde{g})(h)$ be the value to an agent of type $i \in \{S, B\}$ in the bargaining game when the seller employs $\tilde{f}$, the buyer employs $\tilde{g}$, and the agent has history h. For a seller matched at date zero in the market game, the market strategies f_* and g_* generate the same probability distribution over outcomes ending with trade at date zero and the outcome of no trade at date zero as do the bargaining strategies $\tilde{f}$ and $\tilde{g}$ in the bargaining game. Further, the payoff to a matched seller is V_S^d in both games should the outcome of the seller's match be no trade. Therefore, $U_S^0(f_*, g_*)(h) = \tilde{U}_S(\tilde{f}, \tilde{g})(h)$ $\forall h \in H_O^1 \cup H_R^1$ and, by the symmetric argument, $U_B^0(f_*, g_*)(h) = \tilde{U}_B(\tilde{f}, \tilde{g})(h)$ $\forall h \in H_O^1 \cup H_R^1$.

Now suppose $(\tilde{f}, \tilde{g})$ is not a perfect equilibrium of the bargaining game. Then there is a history $\tilde{h} \in H_O^1 \cup H_R^1$ such that the strategy of either the seller or the buyer is not optimal. Without loss of generality suppose the seller's bargaining strategy is not optimal given $\tilde{h}$. Then there exists a bargaining strategy $\tilde{f}^1$ such that $\tilde{U}_S(\tilde{f}^1, \tilde{g})(\tilde{h}) > \tilde{U}_S(\tilde{f}, \tilde{g})(\tilde{h})$.

For the market game now consider the market strategy $\hat{f}$, where $\hat{f}^1 = \tilde{f}^1$ and $\hat{f}^t = f_*^t$ for $t > 1$. As defined, $\hat{f}$ differs from f_* only on $H_O^1 \cup H_R^1$. By the same argument as before $U_S^0(\hat{f}; f_*, g_*)(h) = \tilde{U}_S(\tilde{f}^1, \tilde{g})(h)$ $\forall h \in H_O^1 \cup H_R^1$. Thus, we have shown

$$U_S^0(\hat{f}; f_*, g_*)(\tilde{h}) = \tilde{U}_S(\tilde{f}^1, \tilde{g})(\tilde{h}) > \tilde{U}_S(\tilde{f}, \tilde{g})(\tilde{h}) = U_S^0(f_*, g_*)(\tilde{h})$$

which contradicts that (f_*, g_*) is a market equilibrium. □

We are now prepared to prove Theorem 1.

Proof of Theorem 1: Let (f_*, g_*) be a market equilibrium. By Lemma 1(a), $U_S^0(f_*, g_*)(\hat{h}) + U_B^0(f_*, g_*)(\bar{h}) < 1$ $\forall \hat{h}, \bar{h} \in \hat{H}^1$. Lemma 2 characterizes the perfect equilibrium $(\tilde{f}, \tilde{g})$ of the bargaining game with disagreement payoffs V_S^d and V_B^d when $V_S^d + V_B^d < 1$. By Lemma 3, for

$h \in \hat{H}^1$, when $V_S^d = U_S^0(f_*, g_*)(h)$ and $V_B^d = U_B^0(f_*, g_*)(h)$ then $(\tilde{f}, \tilde{g}) = (f_*^1, g_*^1)$. Since f_* and g_* are stationary, f_*^1 and g_*^1 determine f_* and g_*. Thus f_* and g_* are of the form in the theorem.

We now shown that the market equilibrium is unique. We have $\gamma_{f_*,g_*} = 1$ and $S_{f_*,g_*} = .5V_S^d + .5(1 - V_B^d)$. Then (1) reduces to

$$U_S^T(f_*, g_*)(h^t) = \delta P^T(h^t) S_{f_*,g_*} + \delta(1 - P^T(h^t)) U_S^T(f_*, g_*)(h^t \times (unmatched)) \tag{8}$$

Since $\gamma_{f_*,g_*} = 1$, then $P^T(h^t) = P^T(h^u) = \frac{n}{n+\Delta}$ $\forall h^t \in H^t, \forall h^u \in H^u$. A seller unmatched his t^{th} time in the market computes $P^T(h^t)$ on the basis that each match at date $T + t - 1$ ended with trade with probability one. Thus, he assigns probability $\frac{n}{n+\Delta}$ to the event he is matched at date $T + t$. By the same argument, the seller assigns the same probability to the event he is matched at date $T + u$ when unmatched his u^{th} time in the market. This fact and (8) implies, for any t and u, that

$$U_S^T(f_*, g_*)(h^t) = U_S^T(f_*, g_*)(h^u) \ \forall h^t \in H^t, \forall h^u \in H^u.$$

Since $\gamma_{f_*,g_*} = 1$, then $P^T(h^t) = P^T(h^u) = \frac{n+1}{n+\Delta+1}$ $\forall h^t \in \hat{H}^t, \forall h^u \in \hat{H}^u$. If a seller's current match ends without trade, then given each other match has ended with trade with probability one he assigns probability $\frac{n+1}{n+\Delta+1}$ to the event he is matched at the next date. Thus, (8) implies, for any t and u, that

$$U_S^T(f_*, g_*)(h^t) = U_S^T(f_*, g_*)(h^u) \ \forall h^t \in \hat{H}^t, \forall h^u \in \hat{H}^u.$$

Given the preceding remarks we have for any t that $V_S^d = U_S^0(f_*, g_*)(h^t)$ $\forall h^t \in \hat{H}^t$. For $h \in H^1$ define $V_S = U_S^0(f_*, g_*)(h)$. (V_S is an unmatched seller's value after random matching and is well defined by Lemma 1(d).) The preceding remarks also imply that for any t, $V_S = U_S^0(f_*, g_*)(h^t)$ $\forall h^t \in H^t$.

For $T = 0$ and $h^t \in \hat{H}^t$, (8) reduces to:

$$V_S^d = \frac{n+1}{n+\Delta+1}\delta(.5V_S^d + .5(1 - V_B^d)) + \frac{\Delta}{n+\Delta+1}\delta V_S. \tag{9}$$

For $T = 0$ and $h^t \in H^t$, (8) reduces to:

$$V_S = \frac{n}{n+\Delta}\delta(.5V_S^d + .5(1 - V_B^d)) + \frac{\Delta}{n+\Delta}\delta V_S. \tag{10}$$

Since $\gamma_{f_*,g_*} = 1$, for $h^t \in \hat{H}^t$ we have $U_B^T(f_*,g_*)(h^t) = .5\delta V_B^d + .5\delta(1 - V_S^d)$. For $T = 0$ we have

$$V_B^d = .5\delta V_B^d + .5\delta(1 - V_S^d). \tag{11}$$

In any market equilibrium (f_*, g_*) the values V_S^d, V_S, and V_B^d satisfy (9)-(11). Since these three equations have a unique solution the market equilibrium is unique. □

Proof of Theorem 2: Consider a generation T seller with history $h^t \in H_O^t$. Since all buyers employ the same stationary strategy g_*, there is no loss of generality in restricting attention to strategies where, when selected to propose, the seller either offers V_B^d or makes an unacceptable offer. Therefore, the seller's problem of choosing an optimal strategy is equivalent to a discounted dynamic programming problem with four states and with two actions $\{stop, continue\}$ where each state is defined as follows: In state 1 the seller is matched and is to make an offer. In state 2 the seller is matched and is to respond to an offer of V_S^d. In state 3 the seller is unmatched and in state 4 the seller has exited the market.

In state 1, *stop* corresponds to offering V_B^d and exiting the market (the next state is 4). *Continue* at state 1 corresponds to offering less than V_B^d in which case the seller's match ends without trade. Since $\gamma_{f_*,g_*} = 1$ the next state is 1 with probability $\frac{1}{2}\frac{n+1}{n+\Delta+1}$, is 2 with probability$\frac{1}{2}\frac{n+1}{n+\Delta+1}$, and is 3 with probability $\frac{\Delta}{n+\Delta+1}$.

In state 2, *stop* corresponds to accepting an offer of V_S^d while *continue* corresponds to rejecting such an offer. The transition probabilities are the same at state 2 as at state 1 for each of the actions. In state 3, since $\gamma_{f_*,g_*} = 1$, the transition probabilities to states 1, 2, 3 and 4 are, respectively, $\frac{1}{2}\frac{n}{n+\Delta}$, $\frac{1}{2}\frac{n}{n+\Delta}$, $\frac{\Delta}{n+\Delta}$, and 0. State 4 is an absorbing state.

The optimality equations are:

$$\begin{aligned}
V(1) &= \max\left\{1 - V_B^d + \delta V(4), \tfrac{1}{2}\tfrac{n+1}{n+\Delta+1}\delta V(1) + \tfrac{1}{2}\tfrac{n+1}{n+\Delta+1}\delta V(2) + \tfrac{\Delta}{n+\Delta+1}\delta V(3)\right\} \\
V(2) &= \max\left\{V_S^d + \delta V(4), \tfrac{1}{2}\tfrac{n+1}{n+\Delta+1}\delta V(1) + \tfrac{1}{2}\tfrac{n+1}{n+\Delta+1}\delta V(2) + \tfrac{\Delta}{n+\Delta+1}\delta V(3)\right\} \\
V(3) &= \tfrac{1}{2}\tfrac{n}{n+\Delta}\delta V(1) + \tfrac{1}{2}\tfrac{n}{n+\Delta}\delta V(2) + \tfrac{\Delta}{n+\Delta}\delta V(3) \\
V(4) &= \delta V(4)
\end{aligned}$$

A stationary policy is a function $f : \{1,2,3,4\} \to \{stop, continue\}$ which gives the action taken as a function of the state. By Theorem 2.2 of Ross [6] there is a stationary optimal policy. We now show that the policy $f(1) = f(2) = stop$ is optimal. (The actions taken at states 3 and

4 do not affect the value of a policy.) The value of this policy at state i, denoted by $V_{ss}(i)$, is given by the solution to the following system

$$\begin{aligned} V_{ss}(1) &= 1 - V_B^d + \delta V_{ss}(4) \\ V_{ss}(2) &= V_S^d + \delta V_{ss}(4) \\ V_{ss}(3) &= \tfrac{1}{2}\tfrac{n}{n+\Delta}\delta V_{ss}(1) + \tfrac{1}{2}\tfrac{n}{n+\Delta}\delta V_{ss}(2) + \tfrac{\Delta}{n+\Delta}\delta V_{ss}(3) \\ V_{ss}(4) &= \delta V_{ss}(4) \end{aligned}$$

This system has a unique solution: $V_{ss}(1) = 1 - V_B^d$, $V_{ss}(2) = V_S^d$, $V_{ss}(3) = V_S$ (which follows from (10)), and $V_{ss}(4) = 0$.

By Proposition 4.1 of Ross on policy improvement, if:

$$1 - V_B^d = \max\left\{1 - V_B^d, \frac{1}{2}\frac{n+1}{n+\Delta+1}\delta(1 - V_B^d) + \frac{1}{2}\frac{n+1}{n+\Delta+1}\delta V_S^d + \frac{\Delta}{n+\Delta+1}\delta V_S\right\} \tag{12}$$

and

$$V_S^d = \max\left\{V_S^d, \frac{1}{2}\frac{n+1}{n+\Delta+1}\delta(1 - V_B^d) + \frac{1}{2}\frac{n+1}{n+\Delta+1}\delta V_S^d + \frac{\Delta}{n+\Delta+1}\delta V_S\right\}. \tag{13}$$

then f is an optimal policy. Now, from (9) the second argument in the maximum in (12) and (13) equals V_S^d, and so (13) clearly holds. Equation (12) holds since $1 - V_B^d > V_S^d$.

Therefore, for a history $h^t \in H_O^t$ the strategy f_* is optimal. For any history $h^t \in H_R^t$, the argument above shows that the value of the optimal policy following rejection of an offer is V_S^d. Thus f_* is optimal for such histories since it calls for the seller to accept offers of V_S^d or greater and reject all other offers. The symmetric argument establishes the optimality of g_* for each buyer. □

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