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BOOTSTRAPPING UNIT ROOT AR(1) MODELS

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Abstract

We propose a bootstrap resampling scheme for the least squares estimator of the parameter of an unstable first-order autoregressive model and we prove its asymptotic validity. This method is alternative to the invalid one studied by Basawa *et al.* (1991).

Key words:
Autoregressive processes, bootstrapping least squares estimator, unit root, bootstrap invariance principle.

AMS 1980 *subject classsifications.* Primary 62M07, 62M09; secondary 62M10, 62E20.

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1 INTRODUCTION

Let $\{X_t\}$, $t = 1, 2, \ldots$ be a first-order autoregressive process defined by

$$X_t = \beta X_{t-1} + u_t, \qquad X_0 = 0, \tag{1.1}$$

where $\{u_t\}$ is a sequence of independent and identically distributed random variables with $E(u_t) = 0$, and $V(u_t) = \sigma^2 < \infty$. Let

$$\hat{\beta}_n = \left(\sum_{t=1}^{n} X_{t-1}^2\right)^{-1} \sum_{t=1}^{n} X_t X_{t-1}$$

be the least squares estimator of β, based on a sample of n observations $(X_1, \ldots, X_n)$. In the stationary case, $|\beta| < 1$, Bose (1988) showed the asymptotic validity of the bootstrap estimators corresponding to $\hat{\beta}_n$ and in the explosive case, $|\beta| > 1$, this has been established by Basawa, Mallik, McCormick and Taylor (1989). If $|\beta| = 1$, the unstable case, $\hat{\beta}_n$ has a non-normal asymptotic distribution with a complicated density (see, e.g., Rao (1978)), so it is interesting to study the bootstrap approximation in this situation. Basawa, Mallik, McCormick, Reeves and Taylor (1991) give a bootstrap resampling scheme which leads to an asymptotic random distribution showing, in this way, that this bootstrap method is asymptotically invalid even for normally distributed errors. In this paper, we introduce a different bootstrap strategy and we prove that it correctly approaches the asymptotic distribution of $\hat{\beta}_n$.

In Section 2 we describe the bootstrap resampling and we establish its asymptotic validity. Section 3 contains the proof of this result which needs a bootstrap invariance principle given in Proposition 3.1.

2 MAIN RESULT

It is known (see Anderson (1951)) that when $\beta = 1$,

$$Z_n \to_w Z = \frac{\sigma}{2}(W^2(1) - 1)\left(\int_0^1 W^2(t)dt\right)^{-1/2} \text{ as } n \to \infty, \tag{2.2}$$

where

$$Z_n = \left(\sum_{t=1}^{n} X_{t-1}^2\right)^{1/2} (\hat{\beta}_n - \beta)$$

and $\{W(t)\}$ is a standard Wiener process.

We now describe our bootstrap resampling scheme. Let $\epsilon_t = X_t - \hat{\beta}_n X_{t-1}$, $t = 1, \ldots, n$, and define $\hat{\epsilon}_t = \epsilon_t - n^{-1}\sum_{j=1}^{n} \epsilon_j$, the centered residuals. Denote by $\hat{F}_n$ the empirical distribution function based on $\{\hat{\epsilon}_t : t = 1, \ldots, n\}$ and take a random sample $\{\epsilon_{n,t}^* : t = 1, \ldots, n\}$ from $\hat{F}_n$. So, the random variables $\{\epsilon_{n,t}^* : t = 1, \ldots, n\}$ are i.i.d. with distribution function $\hat{F}_n$, conditionally on $X_1, \ldots, X_n$. Then, the bootstrap sample $\{X_{n,t}^* : t = 1, \ldots, n\}$ is recursively obtained from the model for $\beta = 1$

$$X_{n,t}^* = X_{n,t-1}^* + \epsilon_{n,t}^*, \qquad t = 1, \ldots, n \tag{2.3}$$

with $X_{n,0}^* = 0$. The bootstrap least squares estimate is then given by

$$\hat{\beta}_n^* = \left(\sum_{t=1}^{n} X_{n,t-1}^{*2}\right)^{-1} \sum_{t=1}^{n} X_{n,t}^* X_{n,t-1}^*.$$

Let

$$Z_n^* = \left(\sum_{t=1}^{n} X_{n,t-1}^{*2}\right)^{1/2} (\hat{\beta}_n^* - 1) \tag{2.4}$$

be the bootstrap version of Z_n under $\beta = 1$. Our goal is to show that $Z_n^* \to_w Z$ almost surely and so this bootstrap resampling approaches properly the correct limiting distribution. Thus, our main result is the following.

Theorem 1 . For Z_n^* defined in (2.4), under the model (1.1) with $\beta = 1$, we have that

$$Z_n^* \to_w Z$$

for almost all sample $(X_1, X_2, \ldots)$ where Z is defined in (2.2).

Remark. Basawa *et al* (1991) take i.i.d. $\{u_t^*\}$ with distribution $N(0,1)$ and they obtain $\{\tilde{X}_t^*\}$ from

$$\tilde{X}_t^* = \tilde{\beta}_n \tilde{X}_{t-1}^* + u_t^*, \qquad \tilde{X}_0^* = 0,$$

where $\tilde{\beta}_n$ is the least squares estimate for the AR(1) model; they show that for $\tilde{\beta}_n^* = \left(\sum_{t=1}^n \tilde{X}_{t-1}^{*2}\right)^{-1} \sum_{t=1}^n \tilde{X}_t^* \tilde{X}_{t-1}^*$, the sequence

$$\tilde{Z}_n^* = \left(\sum_{t=1}^n \tilde{X}_{t-1}^{*2}\right)^{1/2} (\tilde{\beta}_n^* - \tilde{\beta}_n)$$

converges to a random distribution not approaching the asymptotic correct one. However, our resampling method works in the sense that the bootstrap distribution of Z_n^* almost surely approximates the asymptotic distribution.

3 Proof of Theorem 1

We will first establish, in Proposition 3.1, a bootstrap invariance principle. To that end consider the sequence of partial sums $S_{n,0}^* = 0$, $S_{n,k}^* = \sum_{j=1}^k \epsilon_{n,j}^*$, $k = 1, \ldots, n$, $n \in \mathcal{N}$. A sequence of continuous-time process $\{Y_n^*(t) : t \in [0,1]\}_{n=1}^\infty$ can be obtained from the sequence

$$\left\{S_{n,k}^* : k = 1, \ldots, n\right\}_{n=1}^\infty$$

by linear interpolation, i.e.,

$$Y_n^*(t) = \frac{1}{\sigma\sqrt{n}} S_{n,[nt]}^* + (nt - [nt]) \frac{1}{\sigma\sqrt{n}} \epsilon_{n,[nt]+1}^*, \; t \in [0,1], \; n \in \mathcal{N} \qquad (3.5)$$

where $[s]$ denotes the greatest integer less than or equal to s.

To prove Proposition 3.1, we need some lemmas. In Lemma 3.1 we will obtain the convergence of the finite-dimensional distributions; tightness will be proved in Lemma 3.2 and Lemma 3.3.

Hereafter, P^*, E^*, V^* will denote the respectively bootstrap probability, expectation and variance conditionally on the sample $\epsilon_1, \ldots, \epsilon_n$.

Lemma 3.1. Conditionally on $(\epsilon_1, \ldots, \epsilon_n)$ and for almost all sample paths $(\epsilon_1, \ldots, \epsilon_n)$,

$$(Y_1^*(t_1), \ldots, Y_n^*(t_d)) \to_w (W(t_1), \ldots, W(t_d)) \tag{3.6}$$

for all $(t_1, \ldots, t_d) \in [0,1]^d$.

Proof. It is enough to show that, for all $s, t \in [0,1]$,

$$(Y_n^*(s), Y_n^*(t)) \to_w (W(s), W(t)) \qquad a.s.$$

Now, conditionally on $(\epsilon_1, \ldots, \epsilon_n)$, since

$$|Y_n^*(t) - \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{[nt]} \epsilon_{n,j}^*| \le \frac{1}{\sigma\sqrt{n}} |\epsilon_{n,[nt]+1}^*|,$$

we obtain by the Čebišev inequality that

$$P^* \left\{ |Y_n^*(t) - \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{[nt]} \epsilon_{n,j}^*| > \delta \right\} \le \frac{V^*(\epsilon_{n,i}^*)}{\delta^2 \sigma^2 n}. \tag{3.7}$$

But

$$V^*(\epsilon_{n,i}^*) = \frac{\sum_{j=1}^n \hat{\epsilon}_j^2}{n} = \frac{\sum_{j=1}^n \epsilon_j^2}{n} + o(1)$$

$$= \frac{1}{n} \sum_{j=1}^n X_j^2 - (\frac{1}{n} \sum_{j=1}^n X_{j-1}^2) \hat{\beta}_n^2 + o(1)$$

$$= (\frac{1}{n} \sum_{j=1}^n X_{j-1}^2)(1 - \hat{\beta}_n^2) + \frac{1}{n} X_n^2 + o(1), \qquad 1 \le i \le n,$$

has a non degenerate limiting distribution when $\beta = 1$ and so the right hand side in (3.7) converges to zero. Therefore,

$$\|(Y_n^*(s), Y_n^*(t)) - \frac{1}{\sigma\sqrt{n}} (\sum_{j=1}^{[ns]} \epsilon_{n,j}^*, \sum_{j=1}^{[nt]} \epsilon_{n,j}^*)\| \longrightarrow_{P^*} 0 \qquad a.s.$$

and it suffices to prove that

$$\frac{1}{\sigma\sqrt{n}} (\sum_{j=1}^{[ns]} \epsilon_{n,j}^*, \sum_{j=1}^{[nt]} \epsilon_{n,j}^*) \longrightarrow_w (W(s), W(t)) \qquad a.s.$$

This is equivalent to show that

$$\frac{1}{\sigma\sqrt{n}}(\sum_{j=1}^{[ns]}\epsilon^*_{n,j}, \sum_{j=[ns]+1}^{[nt]}\epsilon^*_{n,j}) \longrightarrow_w (W(s), W(t) - W(s)) \qquad a.s.; \qquad (3.8)$$

but, since the components on the left hand side are independent, (3.8) follows from the bootstrap central limit theorem (see, e.g., Singh (1981)). □

Lemma 3.2. For any $\eta > 0$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\frac{1}{\delta}P^*\left\{\max_{1\le j\le[n\delta]+1}|S^*_{n,j}| > \eta\sigma\sqrt{n}\right\} = 0$$

conditionally on $(\epsilon_1, \ldots, \epsilon_n)$ and for almost all sample paths $(\epsilon_1, \ldots, \epsilon_n)$.

Proof. By the bootstrap central limit theorem (see, e.g., Singh (1981)), we have that $(1/\sigma\sqrt{[n\delta]+1})S^*_{n,[n\delta]+1}$ converges weakly almost surely to a standard normal random variable V. Fix $\lambda > 0$ and let $\{\varphi_k\}_{k=1}^{\infty}$ be a sequence of bounded, continuous functions on $\Re$ with $\varphi_k \downarrow \mathbf{1}_{(-\infty,\lambda]\cup[\lambda,\infty)}$. We have for each k,

$$\limsup_{n\to\infty} P^*\left\{|S^*_{n,[n,\delta]+1}| \ge \lambda\sigma\sqrt{n\delta}\right\}$$
$$\le \lim_{n\to\infty} E^*\left(\varphi_k\left(\frac{1}{\sigma\sqrt{n\delta}}S^*_{n,[n\delta]+1}\right)\right) = E^*(\varphi_k(V))$$

Then, if $k \to \infty$ we obtain

$$\limsup_{n\to\infty} P^*\left\{|S^*_{n,[n\delta]+1}| \ge \lambda\sigma\sqrt{n\delta}\right\} \le P(|V| > \lambda) \le \frac{1}{\lambda^3}E(|V|^3). \qquad (3.9)$$

We now define $\tau^*_n = \min\{j \ge 1 : |S^*_{n,j}| > \eta\sigma\sqrt{n}\}$. If $0 < \delta < \frac{\eta^2}{2}$, we have

$$P^*\left\{\max_{i\le j\le[n\delta]+1}|S^*_{n,j}| > \eta\sigma\sqrt{n}\right\} \le P^*\left\{|S^*_{n,[n\delta]+1}| \ge \sigma\sqrt{n}(\eta - \sqrt{2\delta})\right\}$$

$$+\sum_{j=1}^{[n\delta]} P^*\left\{|S^*_{n,[n\delta]+1}| < \sigma\sqrt{n}(\eta - \sqrt{2\delta})|\tau^*_n = j\right\} P^*\{\tau^*_n = j\}. \qquad (3.10)$$

But for $\tau^*_n = j$,

$$|S^*_{n,[n\delta]+1}| < \sigma\sqrt{n}(\eta - \sqrt{2\delta})$$

implies $|S^*_{n,j} - S^*_{n,[n\delta]+1}| > \sigma\sqrt{2n\delta}$, and by Čebišev inequality it follows that

$$P^*\left\{|S^*_{n,[n\delta]+1}| < \sigma\sqrt{n}(\eta - \sqrt{2\delta}) \mid \tau^*_n = j\right\}$$
$$\leq \frac{1}{2n\delta\sigma^2} V^*\left(\sum_{i=j+1}^{[n\delta]+1} \epsilon^*_{n,i}\right), \quad 1 \leq j \leq [n\delta]. \qquad (3.11)$$

Moreover, the right hand side in (3.11) is bounded above by $\frac{1}{2\sigma^2 n}\sum_{k=1}^n \hat{\epsilon}_k^2$. Therefore, going back to (3.10)

$$P^*\left\{\max_{0\leq j\leq[n\delta]+1} |S^*_{n,j}| > \eta\sigma\sqrt{n}\right\}$$
$$\leq P^*\left\{|S^*_{n,[n\delta]+1}| \geq \sigma\sqrt{n}(\eta - \sqrt{2\delta})\right\}$$
$$+\frac{1}{2\sigma^2 n}\left(\sum_{k=1}^n \hat{\epsilon}_k^2\right) P^*\{\tau^*_n \leq [n\delta]\}$$
$$\leq P^*\left\{|S^*_{n,[n\delta]+1}| \geq \sigma\sqrt{n}(\eta - \sqrt{2\delta})\right\}$$
$$+\frac{1}{2\sigma^2 n}\left(\sum_{k=1}^n \hat{\epsilon}_k^2\right) P^*\left\{\max_{0\leq j\leq[n\delta]+1} |S^*_{n,j}| > \eta\sigma\sqrt{n}\right\}.$$

It follows that

$$P^*\left\{\max_{0\leq j\leq[n\delta]+1} |S^*_{n,j}| > \eta\sigma\sqrt{n}\right\}$$
$$\leq \left(1 - \frac{1}{2\sigma^2 n}\sum_{k=1}^n \hat{\epsilon}_k^2\right)^{-1} P^*\left\{|S^*_{n,[n\delta]+1}| \geq \sigma\sqrt{n}(\eta - \sqrt{2\delta})\right\}.$$

Putting $\lambda = (\eta - \sqrt{2\delta})/\sqrt{\delta}$ in (3.9), we have

$$\limsup_{n\to\infty} \frac{1}{\delta} P^* \left\{ \max_{0\le j\le [n\delta]+1} |S^*_{n,j}| > \eta\sigma\sqrt{n} \right\} \le \frac{E(|V|^3)\sqrt{\delta}}{(\eta - \sqrt{2\delta})^3} R,$$

where R is the non degenerate limit of $\left(1 - \frac{1}{2\sigma^2 n} \sum_{k=1}^n \hat{\epsilon}_k^2\right)^{-1}$ (see the proof of Lemma 3.1). Now letting $\delta \downarrow 0$ the lemma follows. □

Lemma 3.3. For any $\eta > 0$ and $T > 0$,

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P^* \left\{ \max_{\substack{1 \le j \le [n\delta]+1 \\ 0 \le k \le [nT]+1}} |S^*_{n,k+j} - S^*_{n,k}| > \eta\sigma\sqrt{n} \right\} = 0$$

conditionally on $(\epsilon_1, \ldots, \epsilon_n)$ for almost all sample paths $(\epsilon_1, \ldots, \epsilon_n)$.

Proof. Once we have Lemma 3.2, the proof follows as in Lemma 4.19 of Karatzas and Shreve (1988), page 69, replacing S_k by $S^*_{n,k}$, $k = 1, \ldots, n, n \in \mathcal{N}$. □

Now, we establish the bootstrap invariance principle.

Proposition 3.1. Let $\{\epsilon_n\}_{n=1}^\infty$ be a sequence of independent and identically distributed random variables with mean zero and finite variance $\sigma^2 > 0$ defined on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Let $\hat{F}_n$ be the empirical distribution associated to $\hat{\epsilon}_i = \epsilon_i - \frac{\sum_{j=1}^n \epsilon_j}{n}$, $i = 1, \ldots, n$ and let $\epsilon^*_{n,i}$, $i = 1, \ldots, n$ be independent random variables with distribution $\hat{F}_n$. Define $\{Y^*_n(t) : t \in [0,1]\}_{n=1}^\infty$ by (3.5). Then $Y^*_n \longrightarrow_w W$ a.s. in $C[0,1]$, where W is the standard one-dimensional Brownian motion on $[0,1]$.

Proof. For all the sample paths in $\Omega_1 \cap \Omega_2 \cap \Omega_3$, the proof in page 71 of Karatzas and Shreve (1988) gives the tightness of $\{Y^*_n\}_{n=1}^\infty$; this and the finite dimensional convergence in (3.6) imply, by theorem 4.15 in Karatzas and Shreve (1988), the weak convergence in $C[0,1]$. □

Finally, to prove Theorem 1 we will need the following lemma.

Lemma 3.4. Let

$$R_n^* = \frac{1}{n}\sum_{i=1}^{n} Y_n^{*2}\left(\frac{i}{n}\right) - \int_0^1 Y_n^{*2}(t)dt$$

Then, conditionally on $(\epsilon_1, \ldots, \epsilon_n)$ and for almost all sample paths $(\epsilon_1, \ldots, \epsilon_n)$

$$R_n^* \longrightarrow_p 0,$$

as $n \to \infty$.

Proof. It is straightforward from Proposition 3.1. □

Proof of Theorem 1. Observe that

$$\begin{aligned} Z_n^* &= \left(\textstyle\sum_{t=1}^n X_{n,t-1}^{*2}\right)^{1/2}\left(\hat{\beta}_n^* - 1\right) \\ &= \left(\textstyle\sum_{t=1}^n X_{n,t-1}^{*2}\right)^{1/2}\left(\textstyle\sum_{t=1}^n X_{n,t-1}^{*}\epsilon_{n,t}^{*}\right) \end{aligned}$$

Now, by squaring (2.3) and by summing we obtain

$$\sum_{t=1}^{n} X_{n,t-1}^{*}\epsilon_{n,t}^{*} = \frac{1}{2}X_{n,n}^{*2} - \frac{1}{2}\sum_{t=1}^{n}\epsilon_{n,t}^{*2}. \tag{3.12}$$

Then, expressing the quantities $X_{n,t}^*$ in terms of $Y_n^*(t)$, defined in (3.5), we have

$$X_{n,n}^{*2} = n\sigma^2 Y_n^{*2}(1) \tag{3.13}$$

and

$$\sum_{t=1}^{n} X_{n,t-1}^{*2} = n\sigma^2 \sum_{k=1}^{n-1} Y_n^{*2}\left(\frac{k}{n}\right). \tag{3.14}$$

It follows from (3.12), (3.13) and (3.14) that

$$Z_n^* = \frac{\sigma}{2}\left(Y_n^{*2}(1) - \frac{1}{\sigma^2 n}\sum_{t=1}^{n}\epsilon_{n,t}^{*2}\right)\left(\frac{1}{n}\sum_{k=1}^{n-1} Y_n^{*2}\left(\frac{k}{n}\right)\right)^{-1/2}.$$

By the bootstrap weak law of large numbers and Proposition 3.1, the numerator converges to $\frac{\sigma}{2}(W^2(1) - 1)$. Moreover, from Lemma 3.4, Proposition 3.1 and the continuous mapping theorem, the denominator tends to $\left(\int_0^1 W^2(t)dt\right)^{1/2}$. Since can be easily proved that the bootstrap version of Slutsky's theorem holds, the theorem follows.□

Remark. It is straightforward to check that in the case $\beta = -1$, the result also holds if the innovations distribution is symmetric around zero.

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