http://clevelandfed.org/research/review/
Economic Review 1999 Q3

Term Structure Economics from A to B

by Joseph G. Haubrich

Joseph G. Haubrich is a consultant and economist at the Federal Reserve Bank of Cleveland. He thanks George Pennacchi for helpful comments.

Introduction

Most people understand that the term "interest rates" is plural and acknowledge the difference between the rates on a savings account, overnight federal funds rate, and 10-year Treasury bonds. Of the many differences one can point to, such as risk, issuer, or denomination, among the most basic and most important factors for determining the interest rate is the maturity, or length, of the bond. In this case, a surprisingly small amount of economics can yield some valuable insights into the relationship between interest rates on bonds of various maturities, or what is more often called the term structure of interest rates.

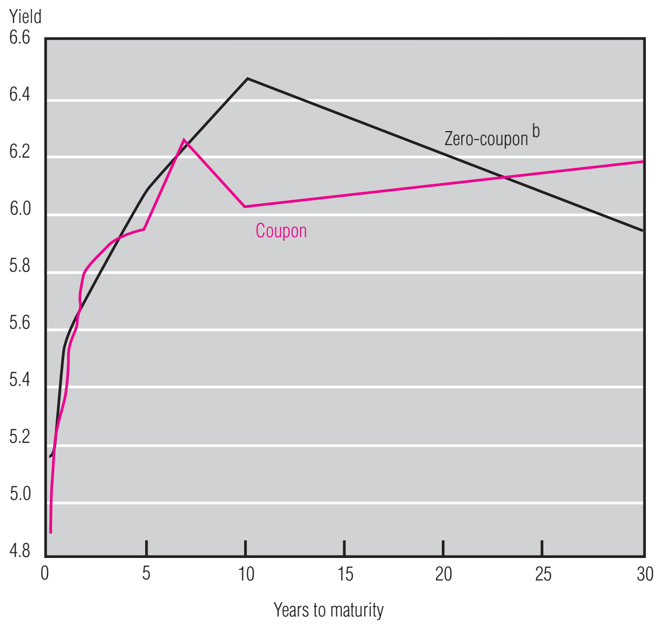
Economics tells us that at the most basic level, interest rates are a price that borrowers pay investors for moving purchasing power from the present to the future. This price obviously has both real and nominal components —the future value of the money you invest will depend on how high inflation is in the meantime. The price also reflects aspects of risk. Because you're uncertain exactly what you'll need for retirement, you're uncertain about how much consumption you should transfer into the future. Real variables, inflation, and uncertainty interact in rather complex ways, and some common perspectives ignore factors that play a key role in determining interest rates. A careful look with an economist's eye can sort out these different effects.

Term Structure versus Yield Curve

Two closely related but distinct terms are often used interchangeably. If we are interested in how interest rates vary with maturity, it is useful to look at the *yield curve*, which plots the yield to maturity of different bonds against maturity. The problem is that most Treasury bonds are coupon bonds, paying a fixed amount semi-annually. For the purist then, the yield on a five-year T-bond is really an average of the five-year interest rate on the principal and many shorter rates on the coupon payments. One solution is to look at yields on zero-coupon bonds, which have no coupons. Figure 1 shows the recent yield curves for coupon and zero-coupon bonds. Some liquidity and tax differences between coupons and zeroes lead many to prefer to estimate the pure interest rates, known as the *term structure* (of interest rates) from coupon bonds. See McCulloch, Huston, and Kwon (1993) and Dhillon and Lasser (1998) for a discussion of this. So, while the term structure is the more useful theoretical concept, the yield curve is easier to observe.

FIGURE 1

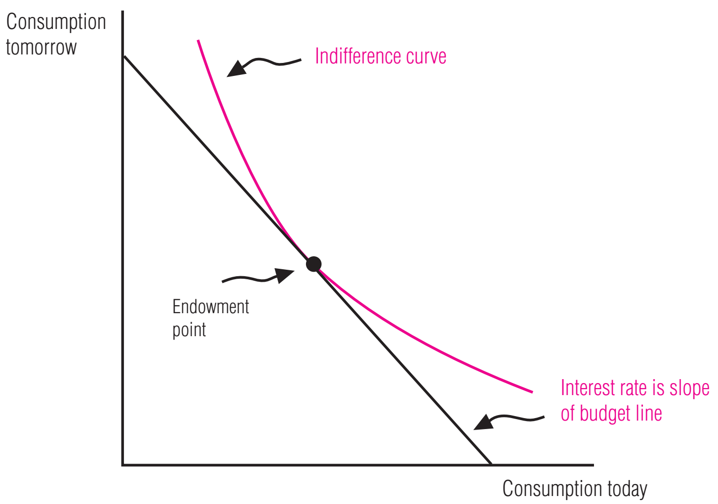
Yield Curve for October 5, 1999[a]



a. All instruments are Treasury constant-maturity series.
b. For each maturity, the yield is the average of yields on zero-coupon Treasury bonds with that maturity, as of October 5, 1999.
SOURCE: *Wall Street Journal,* October 5, 1999, p. H15.

FIGURE 2

Endowment, Preferences, and Interest Rates



SOURCE: Author's calculations.

I. Real Term Structure

To understand the interplay of factors that determine interest rates, it is easier to begin by ignoring the problem of inflation and think of real bonds. Given that a dollar tomorrow will buy just as much beef, beer, or baby-sitting time as a dollar will today, we further simplify and talk about bonds in terms of abstract consumption units (although, for the sake of concreteness, it sometimes helps to think of it as ice cream).

The economic logic behind interest rates represents an application of supply and demand. The interest rate serves as the price expressing the trade-off of consuming today versus consuming tomorrow. It adjusts to equate the supply of savings with the demand for savings. Even at this general level, we can note that an increase in the demand for savings will increase interest rates. If we specialize further, we can answer more specific questions, such as how recessions or economic growth affect interest rates.

The first step is to aggregate everyone in the economy into a single representative agent and to consider the choice problem of this agent.[1] The second step is to consider an endowment economy without production. The consumption good just drops from the trees. The last step is to assume no storage possibilities. In other words, bonds are in "zero net supply," so that when someone is borrowing, someone is lending. Any individual can save or borrow by using a "consumption loan," say, giving up one unit of consumption today for some units tomorrow, but the economy as a whole cannot.

Thus, in a very simple two-period case, in equilibrium the interest rate will adjust to make the representative agent content to hold her endowment. In figure 2, this is seen as the line tangent to the agent's indifference curve at the endowment point. The basic idea behind this simple case—where preferences and the amount of consumption today and consumption tomorrow determine the interest rate—extends to more complicated cases with uncertainty and many time periods.

1 Of course, different investors may have different preferences. Wang (1996) considers this case.

Many Periods: A More Formal Approach

Extending this analysis to many periods and to uncertainty about future consumption requires a more formal, mathematical approach. This section sets up such a model.

There is a single representative agent with preferences

$$(1) \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where E_0 denotes expectations as of period 0, β denotes the discount factor, and $u(c_t)$ denotes the utility of consumption in period t. The agent faces a budget constraint,

$$(2) \quad c_t + B_{1t} + B_{2t} \le d_t + B_{1t-1} R_{1t-1} + B_{2t-2} R_{2t-2},$$

where $B_{jt}, j=1,2$ is the amount of a bond of length j bought in period t. These bonds are perfectly safe, and at the beginning of period t investors know the gross rates of return R_{1t} and R_{2t}. The endowment, or dividend, for a period is denoted d_t.

The agent seeks to arrange consumption to maximize utility, subject to the budget constraint, so a natural way to solve the problem is to substitute (2) into (1) and obtain the first-order conditions.[2]

$$J = E_0 \sum_{t=0}^{\infty} \beta^t u(d_t + B_{1t-1} R_{1t-1} + B_{2t-2} R_{2t-2} - B_{1t} - B_{2t}),$$

$$\frac{\partial J}{\partial B_{1t}} = 0 = E_0[-\beta^t u'(d_t + B_{1t-1} R_{1t-1} + B_{2t-2} R_{2t-2} - B_{1t} - B_{2t})$$
$$+ \beta^{t+1} R_{1t} u'(d_{t+1} + B_{1t} R_{1t} + B_{2t} R_{2t} - B_{1t} - B_{2t})]$$

and

$$\frac{\partial J}{\partial B_{2t}} = 0 = E_0[-\beta^t u'(d_t + B_{1t-1} R_{1t-1} + B_{2t-2} R_{2t-2} - B_{1t} - B_{2t})$$
$$+ \beta^{t+2} R_{2t} u'(d_{t+2} + B_{1t+1} R_{1t+1} + B_{2t+1} R_{2t+1} - B_{1t+1} - B_{2t+1})].$$

We can simplify this in two ways. First, we use (2) again to get consumption back into the equations. Next, we take the perspective of time period t, where R_{1t}, R_{2t}, and c_t are known, which allows us to drop some of the expectations operators. We get

$$(3) \quad u'(c_t) = \beta E_t[R_{1t} u'(c_{t+1})]$$

and

$$(4) \quad u'(c_t) = \beta^2 E_t[R_{2t} u'(c_{t+2})].$$

These have an intuitive explanation. The left-hand side is the marginal utility of consuming one unit less in period t, that is, what you give up (in utility terms) if you invest. The right-hand side tells you what you gain: the discounted expected marginal utility of an extra R_{1t} units of consumption in a future period. The agent equates marginal cost and marginal benefits, leading to equations (3) and (4).

To focus on the interest rates, it is useful to rearrange (3) and (4) as

$$(5) \quad \frac{1}{R_{1t}} = \beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right]$$

and

$$(6) \quad \frac{1}{R_{2t}} = \beta^2 E_t\left[\frac{u'(c_{t+2})}{u'(c_t)}\right].$$

The left-hand sides of (5) and (6) are the current (date t) prices of a bond that pays one unit of consumption one period or two periods in the future. The lower the price, that is, the *less* you pay for such a bond, the *higher* the interest rate: Bond prices and rates move in opposite directions.

Even here we are not quite finished. Both R_{1t} and R_{2t} are *gross* returns, and R_{2t} in particular is a two-period gross return. For example, if the interest rate is 10 percent, R_{1t} is 1.10 and R_{2t} is 1.21. Because we want to compare the returns on bonds of different maturities, however, we need to standardize the returns—if one period is a year, we would want to annualize the returns. To transform R_{2t} into a one-period return we can take the square root.[3] The annualized return on the long (that is, two-period) bond is then

$$L_t = \sqrt{R_{2t}}.$$

This simplified model, expressed by equations (5) and (6), is the basis of an analysis that can give us a lot of insight into the term structure.

■ **2** Although the budget constraint assumes that the representative agent holds only one- and two-period bonds, the equilibrium interest rates on these bonds will be the same even if the agent can hold bonds of other maturities.

■ **3** This makes sense in the discrete time framework. In some cases, it is more convenient to take logarithms. See Campbell, Lo, and MacKinlay (1997), chapter 1.

The Expectations Hypothesis and Beyond

What do equations (5) and (6) tell us about the term structure? A good place to start is the simple case of no uncertainty, where the consumer knows everything today—all future interest rates and all future consumption endowments. Then we can rewrite (6) as

$$(7) \qquad \frac{1}{R_{2t}} = \beta^2 \left[\frac{u'(c_{t+1})}{u'(c_t)} \frac{u'(c_{t+2})}{u'(c_{t+1})}\right] = \frac{1}{R_{1t}} \bullet \frac{1}{R_{1t+1}} .$$

Put differently, $L_t = \sqrt{R_{1t} R_{1t+1}}$. The long-term rate is the average between today's short-term rate and tomorrow's short-term rate. The rational investor has two ways of moving consumption from t into $t+2$: invest in a long bond with per-period return L_t, or roll over a short-term bond, getting rate R_{1t} at the start and R_{1t+1} next period. The two ways of investing must have the same return; otherwise, the investor moves her savings from the low-return investment to the high-return investment. So, in the case of perfect certainty, the interest rates of long- and short-term bonds will adjust to keep today's long rate an average of today's and tomorrow's short rate.

Equation (7) is often seen in a slightly modified form, which, although not exactly correct, is often useful when high precision is not necessary. This approximation to (7) takes the form

$$L_t - 1 = \frac{(R_{1t} - 1) + (R_{1t+1} - 1)}{2} .$$

For example, if interest rates are 3 percent today and 7 percent tomorrow, the long-term rate should be 5 percent. This is not quite exact, as $L_t = \sqrt{(1.03)(1.07)} = 1.0498$; but, for many purposes, it is close enough.

Perfect certainty, as anyone who watches the stock market can attest, is a rather unrealistic assumption. One common way to incorporate uncertainty is to replace unknown future rates in (7) by their expectation. Thus,

$$(7a) \quad L_t = \sqrt{R_{1t} E_t R_{1t+1}} .$$

The long-term rate is an average of current and expected future short-term rates. This is often termed *the expectations hypothesis of the term structure*. A useful approach, it is not derived from (5) and (6), and it ignores the risk effect of uncertain interest rates.[4]

A more correct treatment with uncertainty comes from a closer look at equations (5) and (6). Rewrite (6) as

$$(8) \qquad \frac{1}{R_{2t}} = E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \beta \frac{u'(c_{t+2})}{u'(c_{t+1})}\right]$$

or

$$(9) \qquad \frac{1}{R_{2t}} = E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} E_{t+1} \beta \frac{u'(c_{t+2})}{u'(c_{t+1})}\right]$$

or

$$(10) \qquad \frac{1}{R_{2t}} = E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{1}{R_{1t+1}}\right].$$

To split out the risk terms, we use the standard formula

$$(11) \qquad E(XY) = E(X)\,E(Y) + \mathrm{cov}(X,Y),$$

where cov stands for the covariance of X and Y. Using (11), (10) becomes

$$(12) \qquad \frac{1}{R_{2t}} = E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)}\right] E_t \left[\frac{1}{R_{1t+1}}\right] + \mathrm{cov}\left[\beta \frac{u'(c_{t+1})}{u'(c_t)}, \beta \frac{u'(c_{t+2})}{u'(c_{t+1})}\right] .$$

A little more work yields

$$(13) \qquad \frac{1}{R_{2t}} = \frac{1}{R_{1t}} E_t \left[\frac{1}{R_{1t+1}}\right] + \mathrm{cov}\left[\beta \frac{u'(c_{t+1})}{u'(c_t)}, \beta \frac{u'(c_{t+2})}{u'(c_{t+1})}\right].$$

A lot of insight about the effect of uncertainty comes from comparing (13), the correct model with uncertainty, with (7), the correct model with perfect certainty, and (7a), the expectations hypothesis. The correct interest rate differs from the simple expectations hypothesis in two ways. The first is a Jensen's inequality term that would arise even with risk-neutral investors. The second way is a risk premium that arises precisely because investors are not risk neutral.

4 The extent to which the expectations hypothesis is a good approximation to the data is a much discussed issue. See Campbell, Lo, and MacKinlay (1997), section 10.2.

The Jensen's inequality term arises because $E_t[\frac{1}{R_{1t+1}}]$ does not equal $\frac{1}{E_t R_{1t+1}}$; indeed, $E_t\,[\frac{1}{R_{1t+1}}] \geq \frac{1}{E_t R_{1t+1}}$. The reason is that a given change in the interest rate has more effect on the price of a bond when rates are low than when rates are high.[5] The difference can be large. Hearkening back to the simple numerical example above, suppose short-term rates stand at 3 percent today and are expected to be 7 percent tomorrow, but have an even chance of being either at 3 percent or 11 percent. Using the Jensen's inequality part of (13) (ignoring the covariance term) the (annualized) yield on the long-term bond is

$$L_t = 1/\sqrt{(1.03)/[0.5\frac{1}{1.03} + 0.5\frac{1}{1.11}]} = 1.0487.$$

Thus, correctly considering uncertainty leads to an interest rate of 4.87 percent—a bit below the 4.98 percent suggested by the simple expectations hypothesis. You might not notice this on your savings account, but if you were a pension fund investing millions of dollars, it would add up.

This example highlights another key feature of the model: The interest rate can change significantly even if expected rates stay constant. If future rates become more or less uncertain, rates will change today. The numerical example showed this quite clearly: If future short-term interest rates were known with certainty to be 5 percent, then the long-term rate would be 4.98 percent. When those future rates became uncertain, the long-term rate fell to 4.87 percent.

The second way the model in (13) differs from the expectations hypothesis is that interest rates also have a risk premium. In focussing on the Jensen inequality term, we've ignored the covariance terms—in a sense, we've said that the world got riskier, but nobody cared. And we've also ignored the underlying link with consumption—when the whole point of the exercise is to stop taking interest rates as given and consider their underlying determinants. Casual inspection of (13) suggests that this general investigation might get quite complicated, as we have a covariance term involving nonlinear functions of consumption in *three* time periods. In this case, discretion is the better part of valor, and it makes sense to examine some simplified versions of the general problem.

A Specialized Example

By making a number of special assumptions, we can get to a series of explicit equations that make it easy to look at the effects of various factors on the term structure. First, specialize to log utility,[6] so that $u(c) = \log(c)$.

Recalling that in equilibrium, consumption must equal the dividend endowment for the representative agent, equations (5) and (6) reduce to

$$\frac{1}{R_{1t}} = \beta E_t(\frac{d_t}{d_{t+1}})$$

and

$$\frac{1}{R_{2t}} = \beta^2 E_t(\frac{d_t}{d_{t+2}}).$$

We further specialize by specifying a particular stochastic process for the dividends: We base it on an AR(1) process, of the form $\log d_{t+1} = g + \rho \log d_t + \Theta_{t+1}$, where Θ_{t+1} is a sequence of independent and identically distributed random variables. Adding a time trend (and normalizing the growth rate g, the process for dividends is given by:

$$(14)\quad \log d_{t+1} = gt + \sum_{k=0}^{\infty} \rho^k \Theta_{t-k}.$$

We further assume that the Θ_{t+1} terms are distributed log-normally. This lets us invoke the useful substitution that if X is distributed log-normally,

$$\log E(X) = E[\log(X)] + \tfrac{1}{2} VAR[\log(X)].$$

It also helps if we change the definition of interest rate slightly. We have been thinking about rates on a discrete time basis; if the yearly interest rate R_{1t} is 1.05, an investment of \$1 returns \$1.05 at the end of the year, and it is natural to say that the interest rate is 5 percent. When we start using logs, however, it is more convenient to consider continuously compounded rates of return, leading to the definitions

$$r_t = \log R_{1t}$$

and

$$l_t = \log L_t = \log \sqrt{R_{2t}}.$$

5 For an excellent discussion of this point, see Litterman, Scheinkman, and Weiss (1991).

6 For a more general version of this approach, see Campbell (1986), and Campbell, Lo, and MacKinlay (1997), chapter 11. See also Sargent (1987), section 3.5.

The difference between the two definitions is often small: log(1.05)=0.0488.

Taking (14) as the dividend process, these various assumptions allow equations (5) and (6) to take a relatively convenient, if not exactly simple, form:

$$(15)\quad r_{1t} = \log\frac{1}{\beta} + g + (\rho - 1)\sum_{k=0}^{\infty}\rho^k \Theta_{t-k} - \tfrac{1}{2}\sigma_\Theta^2$$

and

$$(16)\quad l_t = \log\frac{1}{\beta} + g + \tfrac{1}{2}(\rho^2 - 1)\sum_{k=0}^{\infty}\rho^k\Theta_{t-k} - \tfrac{1}{4}(1+\rho^2)\sigma_\Theta^2.$$

What Moves the Term Structure?

Equations (15) and (16) provide a reference point for illustrating term-structure economics. A variety of factors will move interest rates and the term structure. These include the value of today's shock Θ_t, the persistence of the endowment shocks ρ, the growth trend g, the variance of the endowment shocks σ_Θ^2, and the time preference parameter β. Of these, the most interesting are Θ_t, g, and σ_Θ^2.

How do interest rates react to the endowment shock Θ_t? Simple calculus shows that

$$\frac{\partial r_{1t}}{\partial \Theta_t} = (\rho - 1)$$

and

$$\frac{\partial l_t}{\partial \Theta_t} = \tfrac{1}{2}(\rho^2 - 1).$$

A positive shock today lowers interest rates as long as $\rho<1$. Income today is relatively high, so people want to save the extra income; consequently, they drive up the price of bonds and correspondingly drive down the interest rate. In addition, the term structure steepens because short rates fall more than long rates. This is because with $\rho<1$, the effect of the shock dies off, so that if income is high today, it is also expected to be higher than average next period, but not quite so high. The size of the effect, and thus the incentive to save, diminishes, leading to a smaller increase in long rates.

A somewhat different picture emerges if $\rho>1$. Then, an increase today means an even bigger increase tomorrow, depressing the incentive to save and increasing rates.[7] If $\rho=1$, then the shock has no effect on interest rates—income is expected to go up exactly as much in the next period, so there is no change in the demand for saving.

This intuition follows through to the case of changes to the growth rate of endowments, g. In that case, $\frac{\partial r_{1t}}{\partial g}=1$, and $\frac{\partial l_t}{\partial g}=1$. Growing dividends means that future dividends are expected to be greater than current dividends (similar to the case for $\rho>1$). An increase in the growth rate means that future dividends will be increasingly greater than current dividends, leading to a lessening of the desire to save today. This lower demand for savings, and thus for bonds, decreases bond prices and increases interest rates. An increase in the growth rate of dividends increases both short- and long-term interest rates one for one.

Changing the stochastic process of the dividends will also change the term structure. Consider the effects of an increase in the variance of the shocks to income, σ_Θ^2. In this case,

$$\frac{\partial r_{1t}}{\partial \sigma_\Theta^2} = -\tfrac{1}{2}$$

and

$$\frac{\partial l_t}{\partial \sigma_\Theta^2} = -\tfrac{1}{4}(1+\rho^2).$$

The increased uncertainty lowers both short- and long-term rates. The basic intuition is that as uncertainty increases, investors wish to save more "for a rainy day." The increased demand for saving drives down interest rates.[8] The yield curve steepens as long as $\rho<1$, because if shocks die out, an increased variance is less important the further out it is, and the demand for savings responds correspondingly less. Notice that though an increase in uncertainty leads to a steeper term structure, this happens not because long rates rise, but because they do not fall as far as short rates. In some sense, the increase in uncertainty is proportionally not so bad for the long term as for the short term, and thus has less of an impact on long-bond prices. This result must be interpreted carefully, however, because with a log-normal distribution, changing the variance of shocks also

■ **7** With $\rho<1$, some delicate issues arise about the existence of solutions to equation (1). For a discussion, see Campbell (1986) or Labadie (1994).

■ **8** Not every utility function displays such behavior, so the result is not completely general. See Zeldes (1989) for a good discussion.

changes the mean of the distribution. Increasing the variance here does not induce a mean-preserving spread.

II. Nominal Term Structure

In the real world, the vast majority of bonds pay off in dollars—not in gold, sides of beef, or Internet-connect time. Some bonds are indexed for inflation, but, in the United States at least, most are not. This means that bonds do not have a certain payoff in consumption terms—you don't know for sure what \$1,000 will be worth in 10 years—and bond pricing must take inflation risk into account.[9]

Fortunately, the analysis of section I can accommodate the shift to nominal interest rates relatively easily. Start by considering the *nominal* return on a bond, $R^{\$}_{1t}$, and note that to convert the dollars into consumption units and get a real return, we must consider inflation Π_{t+1}.[10] The nominal return on a bond is constant, so we can get a revised version of equation (5) as

$$(17)\quad \frac{1}{R^{\$}_{1t}} = \beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\frac{1}{\Pi_{t+1}}\right]$$

or

$$(18)\quad R^{\$}_{1t} = \frac{1}{\beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\frac{1}{\Pi_{t+1}}\right]}$$

$$= \frac{1}{\frac{1}{R_{1t}}E\frac{1}{(\Pi_{t+1})} + \beta\operatorname{cov}\left(\frac{u'(c_{t+1})}{u'(c_t)}, \frac{1}{\Pi_{t+1}}\right)}\,.$$

Equation (18) has a classic simplification due to Irving Fisher. Note that if consumption and inflation are perfectly certain (that is, there is no uncertainty), (18) reduces to $R^{\$}_{1t} = R_{1t}\bullet\Pi_{t+1}$. Shifting the perspective to rates, $R^{\$}_{1t}-1 \approx R_{1t}-1+\Pi_{t+1}$. That is, a nominal interest rate of 5 percent may be broken into a real interest rate of 3 percent and an inflation rate of 2 percent. Notice that even with perfect certainty, this approximation does not hold for high interest rates: While 5 percent is a good approximation to (1.03)(1.02)=1.0506, 50 percent is not such a good approximation to (1.30)(1.20)=1.56.

With uncertainty, the simplification becomes an even worse approximation. As illustrated earlier, uncertainty has two components. One is the Jensen's inequality term. The other is the risk premium, the covariance between the real interest rate (or consumption) and inflation. Notice that this term can be positive or negative. Uncertainty about inflation may move interest rates up or down. This may seem counterintuitive, but it makes sense. For example, if inflation covaries positively with consumption growth, a nominal bond acts as a sort of insurance. If we get lucky next period, and have a high income, we regret having saved a lot—but a high inflation rate reduces the value of our savings. If we are unlucky, and income is low next period, we wish we had saved more—but a low inflation rate increases the value of our savings. Positive covariance, though, is probably not the most important case. A variety of studies find that inflation is negatively correlated with consumption growth (or, equivalently, real interest rates; see Pennacchi [1991]), so that inflation risk in fact increases interest rates. The risk premium is positive.

Looking at longer rates merely compounds the effect of uncertainty. Thus we have

$$(19)\quad \frac{1}{R^{\$}_{2t}} = \beta^2 E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\frac{1}{\Pi_{t+1}}\frac{u'(c_{t+2})}{u'(c_{t+1})}\frac{1}{\Pi_{t+2}}\right].$$

Longer-term rates depend on how the real economy will evolve, how the price level will move, and the interactions between the two. Of course, simplifying assumptions can make (19) easier to interpret, but its more challenging form is probably more useful. What will higher consumption growth do to interest rates? Trick question—we don't really know until we have decided what will happen to inflation, and how inflation will react to the higher growth.

III. Conclusion

The Roman poet Horace once remarked that getting rid of folly was the beginning of wisdom. Something similar might be said of the term structure. Understanding the simplifications involved in averaging current and future interest rates or in subtracting off expected

■ **9** For several approaches to adding inflation to a term structure model, see Sun (1992), Campbell, Lo, and MacKinlay (1997), section 11.2.1, Labadie (1994), and den Haan (1995). Sargent (1987) provides an in-depth view of monetary economies.

■ **10** More precisely, let $P^{\$}_{1t}$ be the dollar price of a pure discount bond in time t with one period left to maturity. That is, the bond will pay \$1 in period $t+1$. Let the price level (\$/unit of consumption good) be Q_t. Then the real return is $R^{\$}_{1t} = \frac{1}{P^{\$}_{1t}} \bullet \frac{Q_t}{Q_{t+1}}$.

Returns and Compounding

This article mostly uses the *simple net return* as a measure of the interest rate, defined as $r_t = \frac{P_{t+1}}{P_t} - 1$. Exactly what this rate is depends on the length of the period, although in the financial press, returns are usually annualized and expressed as if the return were for one year. Academic work often uses continuously compounded returns, $r_t = \log(\frac{P_{t+1}}{P_t})$, because they simplify calculations. Using continuously compounded rates, equation (7) becomes

$$e^{-2l_t} = \beta^2 [\frac{u'(c_{t+1})}{u'(c_t)} \frac{u'(c_{t+2})}{u'(c_t)}] = e^{-r_{1t}} \bullet e^{-r_{1t+1}}.$$

This implies that $l_t = \frac{1}{2}(r_{1t} + r_{1t+1})$, making the long rate an exact average of the current and future short rates. Similarly, using continuously compounded rates for (18) would give an exact Fisher equation, $r^{\$}_{1t} = r_{1t} + \pi_{t+1}$.

inflation is one benefit of looking at the deeper theory of interest rates. Another benefit arises from a better understanding of how uncertainty influences interest rates.

By suggesting that current long-term interest rates are an average of current and expected short-term rates, the expectations hypothesis captures an important truth. But it is not the whole truth. We have seen how changes in the uncertainty surrounding future rates may change the term structure, even if expected rates stay the same.

The effects of uncertainty are more varied, and often more subtle, than many people realize. An increased uncertainty about future interest rates has an effect on the Jensen's inequality factor that tends to lower long-term interest rates today. An increased uncertainty about future consumption has an effect on the risk premium that tends to lower interest rates today, as people save for a rainy day, but it steepens the term structure. An increased uncertainty about inflation will increase nominal interest rates, at least if inflation and consumption covary negatively. The effects on longer rates are more complicated.

The real world is undoubtedly more complex than the model of interest rates considered here. Like a map, which can never show every detail, our model can highlight important and dangerous areas of that rather mysterious area known as the term structure. This can lead to better decisions, be they on the part of particular investors or of monetary policymakers. Examining the underlying economic theory becomes the first step in understanding the interplay between real and nominal risk factors, where they come from, and how changes in those factors matter.

References

Campbell, John Y. "Bond and Stock Returns in a Simple Exchange Model," *The Quarterly Journal of Economics*, vol. 101, no. 4 (November 1986), pp. 785–803.

Campbell, John Y., Andrew W. Lo, and A. Craig MacKinlay. *The Econometrics of Financial Markets*. Princeton, N.J.: Princeton University Press, 1997.

den Haan, Wouter J. "The Term Structure of Interest Rates in Real and Monetary Economies," *Journal of Economic Dynamics and Control,* vol. 19, no. 5/7 (1995), pp. 909–40.

Dhillon, Upinder S., and Dennis J. Lasser. "Term Premium Estimates from Zero-Coupon Bonds: New Evidence on the Expectations Hypothesis," *The Journal of Fixed Income,* vol. 8, no. 1 (June 1998), pp. 52–58.

Labadie, Pamela. "The Term Structure of Interest Rates over the Business Cycle," *Journal of Economic Dynamics and Control,* vol. 18, no. 3/4 (May–July 1994), pp. 671–97.

Litterman, Robert, José Scheinkman, and Laurence Weiss. "Volatility and the Yield Curve," *The Journal of Fixed Income,* vol. 1, no. 1 (June 1991), pp. 49–53.

McCulloch, J. Huston, and Heon-Chul Kwon. "U.S. Term Structure Data, 1947–91," Ohio State University, Working Paper no. 93–6 (March 1993).

Pennacchi, George. "Identifying the Dynamics of Real Interest Rates and Inflation: Evidence Using Survey Data," *Review of Financial Studies,* vol. 4, no. 1 (1991) pp. 53–86.

Sargent, Thomas J., *Dynamic Macroeconomic Theory*. Cambridge, Mass.: Harvard University Press, 1987.

Sun, Tong-sheng. "Real and Nominal Interest Rates: A Discrete-Time Model and Its Continuous-Time Limit," *Review of Financial Studies,* vol. 5, no. 4 (1992), pp. 581–611.

Wang, Jiang. "The Term Structure of Interest Rates in a Pure Exchange Economy with Heterogeneous Investors," *Journal of Financial Economics,* vol. 41, no. 1 (1996), pp. 75–110.

Zeldes, Stephen P. "Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence," *Quarterly Journal of Economics,* vol. 104, no. 2 (May 1989), pp. 275–98.