Productivity and the Term Structure

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Introduction

In discussions about the recent record-setting economic expansion and the accompanying record-setting bull market in stocks, two factors often receive credit: Federal Reserve interest rate policy and increased productivity. This conjunction naturally raises the question of what interest rate policy is appropriate in the face of changing productivity. Before such a policy question can be answered, however, the logically prior question of the effects of productivity changes on interest rates must be addressed. The Federal Reserve controls one particular short-term rate: the federal funds rate. Investors, consumers, and businesses generally care about long-term rates: mortgages, car loans, corporate bonds. This *Economic Review* considers how productivity changes affect the entire term structure of interest rates. It thus may serve as a prelude to thinking about how the Federal Reserve (FOMC) should move the Federal Funds rate in response to productivity changes or movement in other interest rates.

To get a handle on the economics behind productivity's effect on the term structure, this paper works out a simplified theoretical model based on the work of Cox, Ingersoll, and Ross (1985a,b) and Sun (1992). This approach abstracts from reality: it posits a very simple production structure and it ignores money and inflation.[1] This means that it ignores several vital influences on the term structure, principally how productivity affects the part of interest rates that depends on inflation and expectations about inflation.

One big advantage of writing down a theoretical model in all its detail is that we are forced to answer a host of questions that might not naturally come up. When we talk about a productivity change (or shock) is that change permanent or temporary? Do we expect it to be repeated—or offset—in the near future? To get answers we must be precise about our assumptions.

I. The Basic Economy

Whereas the Cox, Ingersoll, and Ross (CIR) model of the economy uses continuous time, this article uses discrete time to make it more comparable with other macroeconomic work. The approach builds on the explanation of dynamic portfolio theory presented in Sargent (1987).

The basic plan is to use a representative-agent framework, that is, to consider one person as a price taker and, after finding out how that person's choices depend on prices, use the results to determine what prices will clear the markets. The person decides how much wealth to invest in each of two available assets, and how much to consume now. One asset is a risky, productive investment opportunity, something like planting wheat or building a factory. The other is a one-period, risk-free, real bond, something like a government-guaranteed CD. The agent's decision depends on the assets' risk and return. The key point of the model is that the risk and return will change over time—and change in a way directly related to productivity, as a productive factory is a profitable factory. The underlying productivity changes interact with the choices made by the representative agent to yield the prices and interest rates we wish to examine.

More formally, if A_t denotes today's wealth (that is, wealth in period t), c_t denotes consumption, s_t denotes the amount put in the productive investment, and b_t denotes the amount in the bond, the basic budget constraint for this economy is: $A_t = c_t + s_t + b_t$. The transition equation, showing how wealth tomorrow depends on decisions made today, becomes: $A_{t+1} = R_t s_t + r_t b_t$, where R_t is the (gross) return on the risky investment and r_t is the return on the safe asset. Notice that R_t can be thought of as productivity: The higher R_t is, the higher the return from investing in the factory (or the

1 For papers that tackle these more difficult issues, see den Haan (1995) and Bakshi and Chen (1996).

more the factory produces for a given level of investment s_t). It is important to notice that at time t, r_t is known with certainty but R_t, being risky, is unknown.

Next, we assume that the agent has some utility function $u(c_t)$ and some discount factor β so that total utility is $U = \sum_{t=0}^{\infty} \beta^t u(c_t)$. Since the future is uncertain, expected utility is

$$U = \sum_{t=0}^{\infty} \beta^t E_0 u(c_t). \tag{1}$$

The agent's problem is to choose values for s_t and b_t to maximize (1). The appendix carries out this calculation, which results in the *Euler equation*:

$$u'(c_t) = \beta E_t \left[R_t u'(c_{t+1}) \right]. \tag{2}$$

This equation says that the agent balances the gain from consuming a little more today (left side) against the expected gain from investing that little bit and getting more tomorrow (right side).

Next, we simplify the problem in two steps by making assumptions first about the form of the utility function u and then about the stochastic process driving R_t.

We specialize utility to take the log form:

$$u(c) = \log(c). \tag{3}$$

Fortunately, as the appendix shows, log utility results in an especially simple form for investment demand, namely $s_t = kA_t$ and $b_t = \ell A_t$, where k and ℓ are just two constants (to be determined later). An even more convenient form of (2), due to Grossman and Shiller (1981), provides a compact representation for both interest rates, R_t and r_t.

$$\begin{aligned} 1 &= \beta E_t \left[R_t \frac{1}{R_t k + r_t \ell} \right] \\ 1 &= \beta E_t \left[r_t \frac{1}{R_t k + r_t \ell} \right]. \end{aligned} \tag{4}$$

General Equilibrium

Equation (4) solves the individual's portfolio selection problem. If our goal was to provide investment guidance, we could determine how much to invest in each asset by carefully specifying the stochastic processes for R_t and r_t and solving for k and ℓ. We will use (4) in a different way. Instead of taking the interest rates as given, we will use (4) to determine them, in effect using it as a demand curve. For example, if you know the demand for apples you can predict how many people will buy when the price is \$5; but the more interesting use is putting demand and supply together to find the price. In determining interest rates, similarly, the focus is on turning (4) around and using it solve for the interest rate r_t.

Equation (4) by itself does not allow a solution in purely exogenous variables, so we must bring in other aspects of the economy. The key to doing this is the representative agent: There is only one person (both a consumer and investor) in this economy, acting as a price taker. This means that in aggregate, there is no borrowing or lending: The consumer can only borrow from himself, so everything nets out to zero. The total amount invested in the safe bond is thus zero, making $\ell = 0$. Another way of saying this is that bonds are "in zero net supply." Note that the individual may still invest in the productive assets, so we have not (yet) pinned down k.

Imposing the zero-net-supply assumption, equation (4) becomes

$$\begin{aligned} 1 &= \beta E_t \left[R_t \frac{1}{R_t k} \right] = \beta E_t [1/k] \\ 1 &= \beta E_t \left[r_t \frac{1}{R_t k} \right]. \end{aligned} \tag{5}$$

Because r_t is known at time t (you know what return a safe bond will pay) and because k is a constant, the equations in (5) can be combined to solve for k and r_t, yielding $k = \beta$ and

$$\frac{1}{r_t} = E_t \left[\frac{1}{R_t} \right]. \tag{6}$$

Equation (6) provides a crucial step: it describes the return on bonds, r_t, in terms of productivity, that is, the return on the productive process, R_t. The next step is to think more carefully about how R_t moves over time and what this implies, via (6), for the interest rate.

Lognormality, Kernels, and Interest Rates

Describing how R_t moves over time, in a way both interesting and tractable, is harder than you might think. In fact, it is best done by approaching the problem indirectly. A convenient approach is to use the lognormal distribution (that is, where $\log X$ is distributed normally), because it has a particularly nice form for expectations. If X is lognormal,

$$\log E_t(X) = E_t(\log X) + 1/2 \operatorname{var}(\log X).$$

Exactly *what* should be distributed lognormally, however? For this it pays to revisit (6).

Equation (6) can be rewritten, or perhaps we should say, re-interpreted, as a way to express the *price* (not return) of the safe bond in terms of something called a pricing kernel.[2] While it is possible to give a suitable economic description of the pricing kernel (as the intertemporal marginal rate of substitution or the probability-weighted state price), at this point it's best to think of it as a step that makes the derivation easier. Consider the interest rate r_t again. Since the bond in question is a one-period, zero-coupon bond for which the owner will get one unit of consumption tomorrow for a price of P_t today, the return is $r_t = \frac{1}{P_t}$. This makes the left side of (5) $1/(r_t) = P_t$. (Recall that r_t and R_t are gross returns, that is, of the form 1.05, rather than 5 percent.)

For the right side of (6), redefine $1/(R_t)$ to be the pricing kernel, (or, as it is sometimes called by the real fun-loving types, the stochastic discount factor), M_{t+1}. These substitutions lead to

$$(7) \qquad P_t = E_t\left(M_{t+1}\right).$$

Thus, the price is just the expectation of the pricing kernel. Next, we *define* the interest rate (or yield) on the safe bond as the negative of the log of the price, so that $y_{1t} = -\log P_t$, and define m_t as $\log M_t$.[3] If m_{t+1} is distributed lognormally, then (7) becomes

$$(8) \qquad -y_{1t} = E_t\left(m_{t+1}\right) + 1/2\text{var}\left(m_{t+1}\right).$$

We arranged this detour because it is easier to put an interesting and tractable structure on m_{t+1} rather than directly on R_t. At long last, we are ready to do this.

We assume that the log pricing kernel takes the following form:

$$(9) \qquad \begin{aligned} -m_{t+1} &= x_t + x_t^{1/2}\gamma\varepsilon_{t+1}, \\ x_{t+1} &= \mu + \phi\left(x_t - \mu\right) + x_t^{1/2}\varepsilon_{t+1}. \end{aligned}$$

If we think about this equation in light of the original question we posed about the effect of productivity on interest rates, the new term, x_t, may be thought of as a factor that moves productivity around. While equation (9) may seem rather unintuitive at first, it has several nice properties that, as we show later, will carry over to interest rates. It has a long-run mean, μ, and it tends to revert to that mean with a speed that depends on ϕ. That is to say, the process has first-order serial correlation. The other term that perhaps looks a little strange is the $x^{1/2}$ factor on the shock, which makes the effect of the shock (and thus the variance of the process) depend on the level of x_t. If x_t is large, the shock will have large effects. This means that interest rates move around more when they are high than when they are low. If rates are at 10 percent, movements up to 11 or down to 9 will be common, but if rates are at 3 percent, it will be a rare move that reaches 4 or 2 percent.

This "square-root process" has another important aspect. As x drops toward zero, the variance (the effect of ε shocks) decreases, making it less likely that the process will fall below zero. For a large value of x, the odds are small that the shock would be big enough to send x_t negative. For a small value of x, the variance is very low, so the odds of going negative are also small. In the limit, with a continuous time process (as in Cox, Ingersoll, and Ross [1985b]) the probability is zero. To make life easier, we will adopt that approximation (also used in Sun [1992] and Campbell, Lo, and MacKinlay [1997]), although for discrete-time processes it is not strictly true.

If our goal was only to price bonds and other financial assets, we could simply have started with equation (7), but that would have omitted mention of any connection between productivity and interest rates, which is our main concern.

II. Term Structure

Putting the pricing equation (8) together with the assumptions on the productivity factor (9) finally puts enough structure on the problem to get some meaningful results. Substituting (9) into (8) yields

$$\begin{aligned} p_{1t} = -y_{1t} = E_t&\left[-\left(x_t + x_t^{1/2}\gamma\varepsilon_{t+1}\right)\right] \\ &+ (1/2)\,\text{var}\left[-\left(x_t + x_t^{1/2}\gamma\varepsilon_{t+1}\right)\right] \end{aligned}$$

or

$$(10) \qquad y_{1t} = x_t\left(1 - \gamma^2\sigma^2/2\right).$$

This gives the short-term yield (under the standard approximation of logs, $y_{1t} \approx r_t - 1$). This equation has a fairly intuitive explanation. The factor affecting return to capital (here, x_t) has a big influence on the interest rate, which increases with return to capital. That's not quite the end of the story, however, because investment in capital, the productive asset, is risky. The bond is safe, and therefore risk-averse investors are willing to pay a premium to put their assets in bonds. A premium price on bonds translates into a lower interest rate. So a risk factor offsets some of the direct productivity effect. Notice the importance of the square-root process

2 For more involved descriptions of using the pricing kernel to derive interest rates, the now-standard reference is Campbell, Lo, and MacKinlay (1997) chapter 11; for a more specific application along the lines of this article, see Haubrich (1999).

3 Notice that the uncertain return from t to $t+1$ is indexed as R_t, but that the associated kernel is indexed as $t+1$. This is standard usage.

here. An increase in productivity, x_t, *also* increases the variance, and thus the uncertainty, of the productivity shocks.

Long Rates

Now let's consider what the model tells us about the longer-maturity bond. With an approach analogous to that used in section I, one can obtain an expression for the price of a two-period bond, noting that the two-period yield will be the negative of one-half the log price. Thus,

$$P_{2t} = E_t\left[M_{t+1}P_{1,t+1}\right]$$

or

$$p_{2t} = E_t\left[m_{t+1} + p_{1,t+1}\right] + 1/2\text{var}\left[m_{t+1} + p_{1,t+1}\right],$$

which after substituting in (8), becomes

$$\begin{aligned}(11) \qquad p_{2t} = E_t\Big[&-\left(x_t + x_t^{1/2}\gamma\varepsilon_{t+1}\right)\\ &-x_{t+1}\left(1-\gamma^2\sigma^2\right)\Big]\\ &+1/2\text{var}\Big[-\left(x_t + x_t^{1/2}\gamma\varepsilon_{t+1}\right)\\ &-x_{t+1}\left(1-\gamma^2\sigma^2\right)\Big].\end{aligned}$$

Again using (9) to express x_{t+1} in terms of x_t and ε_{t+1}, (11) reduces to

$$\begin{aligned}p_{2t} = &-x_t\Big\{1+\left(1-\gamma^2\sigma^2/2\right)\phi\\ &-\left[\gamma+\left(1-\gamma^2\sigma^2/2\right)\right]^2\sigma^2/2\Big\}\\ &-\left(1-\gamma^2\sigma^2/2\right)(1-\phi)\mu.\end{aligned}$$

This makes the two-period yield

$$\begin{aligned}(12) \qquad y_{2t} = (1/2)\,x_t&\Big\{1+\left(1-\gamma^2\sigma^2/2\right)\phi\\ &-\left[\gamma+\left(1-\gamma^2\sigma^2/2\right)\right]^2\sigma^2/2\Big\}\\ &+(1/2)\left(1-\gamma^2\sigma^2/2\right)(1-\phi)\mu.\end{aligned}$$

A more intuitive expression for the two-period rate comes from rearranging (12) into

$$\begin{aligned}(12\text{a}) \qquad y_{2t} = (1/2)\Big\{&\left(1-\gamma^2\sigma^2/2\right)x_t\\ &+\left(1-\gamma^2\sigma^2/2\right)\left[\mu+\phi\left(x_t-\mu\right)\right]\\ &-\Big[2\gamma\left(1-\gamma^2\sigma^2/2\right)\\ &+\left(1-\gamma^2\sigma^2/2\right)^2\Big]\left(\sigma^2\right)x_t\Big\}.\end{aligned}$$

The first two terms in the brackets of (12a) describe the part of the two-period bond yield that is attributed to the expectations hypothesis of the term structure. The expectations hypothesis says that two-period interest rates ought to be the average of today's one-period interest rate and the expectation of next period's one-period interest rate. The first term,

$$\left(1-\gamma^2\sigma^2/2\right)x_t,$$

is just today's one-period interest rate. The next term,

$$\left(1-\gamma^2\sigma^2/2\right)\left[\mu+\phi\left(x_t-\mu\right)\right],$$

is the expectation of next period's interest rate. The reason x_t shows up is that productivity today has information about x_{t+1}, productivity tomorrow. The best guess for the productivity factor tomorrow is that it will revert somewhat toward the mean μ (exactly how much depends on the speed of adjustment, ϕ.) Next period's short rate ($y_{1,t+1}$) depends on what x_{t+1} is, so our best guess for next period's short-term rate is our best guess for next period's productivity factor multiplied by the factor $\left(1-\gamma^2\sigma^2/2\right)$. Notice that the second term is greater or less than the first term precisely when x_t is greater or less than μ. From the expectations perspective, if the productivity factor (and thus the short rate) is below the mean, rates are expected to increase, and so the term structure slopes upward. If rates are above the mean, they are expected to fall, and the term structure slopes downward.

The expectations hypothesis is not completely true, however, and (12a) has an additional term, accounting for risk, which tends to lower the two-period yield. For example, if $x_t = \mu$, the risk term would imply that $y_{2t} \leq y_{1t}$.

Questions about the term structure reduce to questions about the difference between equation (10), the short rate, and equation (12a), the long rate. Of course, one might ask more complicated questions involving three-period yields, four-period yields, or even seventeen-period yields. Restricting attention to one- and two-period yields eliminates questions about the shape of the yield curve, such as whether or not it is humped. Still, the key intuitions about many important questions—such as how productivity affects term structure level and slope—come through with only two yields.

Spreads

A convenient way to discuss many term-structure changes is to look at the spread between long and short yields. From (10) and (12), this becomes

$$(13)\quad y_{2t}-y_{1t}=(1/2)\left(1-\gamma^2\sigma^2/2\right)(1-\phi)\mu \\ +(1/2)\Big\{1+(\phi-2)\left(1-\gamma^2\sigma^2/2\right) \\ -\left[\gamma+\left(1-\gamma^2\sigma^2/2\right)\right]^2\sigma^2/2\Big\}x_t.$$

Now to return (and about time) to the central question of this paper: How do productivity changes affect the term structure? As may be apparent by now, getting the answer will not be easy for two very different reasons. First, the many and complicated terms in equations (10)–(13) indicate that there are fairly complicated interactions going on, and comparative statics will result in some messy algebra. A deeper reason is that the phrase "changes in productivity" now has no unambiguous meaning. Does "change" mean an increase in the average level, μ, a high (or low) value of x_t, or perhaps a higher variance, σ^2, or mean reversion parameter, ϕ? Not every change is worth looking at, but understanding a few key changes will shed light on some central aspects of the relation between productivity and the term structure.

First, consider an increase in the mean of the productivity factor μ, holding everything else constant. This indicates that the long-run average productivity of the economy has increased; we have entered a "new era" of high growth. What does this do to the term structure? A quick look at equations (10)–(13) shows that y_{1t} is unchanged, and that the effect on y_{2t} depends on the sign of

$$(14)\qquad \left(1-\gamma^2\sigma^2/2\right)(1-\phi).$$

It will also be apparent that the sign of $\left(1-\gamma^2\sigma^2/2\right)$ should be positive if a positive level of the productivity factor, x_t, implies a positive interest rate (yield). Furthermore, as long as the factor adjusts towards the mean but does not immediately jump to the mean, (that is, $0<\phi<1$), both parts of (14) will be positive, and an increase in productivity will steepen the slope of the term structure.

Intuitively, this simply says that because bonds compete with real, productive assets, when the return on those productive assets is expected to be higher in the long run, real interest rates are expected to be higher as well. If that increase doesn't show up directly in today's productivity (x_t), the part of the effect that shows up in long-term rates creates a steeper term structure.

What happens if the productivity factor itself, x_t, is higher? This corresponds to a temporary shock, an increase in productivity for a limited time. A glance at (10) shows that this increase in productivity raises short-term rates, as is to be expected. The effect on long rates and thus on the slope of the term structure is more difficult to ascertain. In fact, a direct attack along the lines of equations (13) and (14) would be unenlightening. Comparing (10) and (12a), and discussing how the productivity shocks affect expected rates and risk terms, will prove more fruitful.

An increase in x_t increases y_{1t}, as discussed above. It also may increase y_{2t}, depending on the relative sizes of the expectations effect and the risk effect. What does it mean for the slope of the term structure? Using (12a) and (10) to compare the expectations part of the two-period rate with the one-period rate shows us that

$$y_{2t}\approx(1/2)\left\{\left(1-\gamma^2\sigma^2/2\right)x_t \\ +\left(1-\gamma^2\sigma^2/2\right)\left[\mu-\phi\left(x_t-\mu\right)\right]\right\}.$$

This implies that

$$y_{2t}-y_{1t} \\ \approx(1/2)\left(1-\gamma^2\sigma^2/2\right)\left[\phi-1\left(x_t-(1-\phi)\mu\right)\right].$$

Taking the derivative with respect to x_t yields

$$(15)\qquad \partial\left(y_{2t}-y_{1t}\right)/\partial x_t \\ =(1/2)\left(1-\gamma^2\sigma^2/2\right)(\phi-1).$$

Since $\left(1-\gamma^2\sigma^2/2\right)>0$ and $\phi<1$, an increase in the productivity factor, x_t, decreases the spread between rates, meaning that the yield curve gets *flatter*. Effectively, because of reversion to the mean, a higher x_t today has less of an impact tomorrow; if x is above the mean, x_t tends to get pulled down, and if x_t is below the mean, increasing it lessens the pull upward by the mean. The net result is that the effect of the productivity shock on interest rates today is larger than the expected effect on interest rates tomorrow.

This, of course, is only one aspect of an increase in x_t. Because the economy is risky, two-period bonds are not merely the average of current and expected rates; after all, that is why equations (12) and (12a) contain variance terms. The term that remains in (12a) after accounting for the expectations hypothesis approximation is

$$-\left[2\gamma\left(1-\gamma^2\sigma^2/2\right)+\left(1-\gamma^2\sigma^2/2\right)^2\right]\left(\sigma^2/2\right)x_t.$$

Clearly this is negative, since γ, $\left(1-\gamma^2\sigma^2/2\right)$, and σ^2 are positive. An increase in x_t lowers the risk factor, decreasing two-period rates and the slope of the term structure.

Why does an increase in the productivity factor, x_t, decrease the risk factor in two-period yields? There are two parts to the answer. The first has to do with the heteroskedastic aspect of the square-root process.

An increase in x_t increases the variance of the productivity process. This means that investing in the real economy is now riskier, which leads to the second part of the answer. Because the real economy is riskier, investors will pay a premium for a safe bond that delivers them from that risk. The higher price means a lower yield. If the world really does work this way, a higher productivity shock, though good in one sense (directly higher productivity), is bad in another (higher risk).

Other Assets

Bonds are not the only financial assets around. Productivity shocks will also affect stocks, options, swaps, and other derivatives. One way to price these assets is to start with (7) and (9), specifying the return process for the asset in question. Since we've only assumed one source of uncertainty in the economy, (x_t), however, the relations between the different assets might be rather simplistic. Conceptually, at least, it is straightforward to add more shocks.

This might even be done in a way that preserves the results so far. Let the pricing kernel take the form

$$P_t = E_t\left(M_{t+1}K_{t+1}\right),$$

where K_{t+1} is independent of M_{t+1} and is a martingale (that is, $E_t\left[K_{t+1}\right] = K_t$). Then $P_t = E_t\left(M_{t+1}\right)K_t$ and $P_{2t} = E_t\left(M_{t+1}P_{1,t+1}\right)K_t$. Thus the spread between long and short rates, which depends on the ratio P_{2t}/P_t, is independent of K_t. But the extra factor, K_t, would show up in pricing other assets such as stocks.

III. Conclusion

Bond traders, stock jobbers, and risk managers all have their own reasons for understanding the course of interest rates. The Federal Reserve's Federal Open Market Committee derives its concern from its mandate for monetary policy, and that policy involves correctly setting one interest rate among many. Setting the path for the federal funds rate is itself complicated by the complex interactions of the funds rate with T-bill, mortgage, and other interest rates.

Productivity plays a crucial role in the interactions of the various interest rates, but its effect is not always simple. An increase in the long-run mean of productivity will increase long-term interest rates and cause the term structure to get steeper. An increase in today's productivity tends to increase both short- and long-term interest rates, but long-term rates move less, causing the term structure to get flatter.

Real-world productivity shifts will rarely be so cut and dried. The central, as yet unanswered, questions—such as whether recent productivity increases are permanent or temporary—matter greatly for the term structure, as they yield diametrically opposed conclusions. Thus, economic theory provides some guidance about the appropriate questions to ask. It also raises further questions. For example, in a truly “new paradigm economy” shouldn't we expect to see changes in other parameters of the productivity process, —such as the speed of adjustment—that theory tells us are important for the term structure?

So, in one sense, a more sophisticated view has complicated the matter. Just as a wine connoisseur would not hazard a recommendation until he knew whether beef or fish were being served, advice about interest rates often requires that we specify more details about the underlying economy.

Appendix

Finding the values that maximize expected lifetime utility is perhaps easiest done using dynamic programming (see Sargent [1987] for an excellent exposition). The state variables are $\left[A_t, r_t, R_{t-1}\right]$ and the control variables are $[s_t, b_t]$. Forming Bellman's equation gives

$$\text{(A.1)} \qquad V\left(A_t, r_t, R_{t-1}\right) = \max_{s_t, b_t}\Big\{u\left(A_t - s_t - b_t\right) + \beta E_t V\left(s_t R_t + b_t r_t, r_{t+1}, R_t\right)\Big\}.$$

The first-order necessary conditions for the "max" part of (A.1) are given by:

$$\text{(A.2)} \qquad \frac{\partial V}{\partial s_t} = 0 \Rightarrow -u'\left(A_t - s_t - b_t\right) + \beta E_t R_t V_1\left(s_t R_t + b_t r_t, r_{t+1}, R_t\right) = 0$$

$$\text{(A.3)} \qquad \frac{\partial V}{\partial b_t} = 0 \Rightarrow -u'\left(A_t - s_t - b_t\right) + \beta E_t r_t V_1\left(s_t R_t + b_t r_t, r_{t+1}, R_t\right) = 0.$$

Next, using the Benveniste and Scheinkman (1979) results on the differentiability of the value function V to evaluate V_1 yields $V_1\left(A_t, r_t, R_{t-1}\right) = u'\left(c_t\right)$. Substituting this into (A.2) yields the *Euler equation*, $0 = -u'\left(c_t\right) + \beta E_t\left[R_t u'\left(c_{t+1}\right)\right]$ or

$$\text{(A.4)} \qquad u'\left(c_t\right) = \beta E_t\left[R_t u'\left(c_{t+1}\right)\right].$$

(A.4) is equation (2) in the text.

Given equation (A.4), the next step is to solve for the *policy functions* $s_t\left(A_t, R_{t-1}, r_t\right)$ and $b_t\left(A_t, R_{t-1}, r_t\right)$. Substituting these into the Euler equation (A.4) gives

$$\text{(A.5)} \qquad u'\left[A_t - s_t\left(A_t, R_{t-1}, r_t\right) - b_t\left(A_t, R_{t-1}, r_t\right)\right] = \beta E_t\left\{R_t u'\left[A_t - s_t\left(A_t, R_{t-1}, r_t\right) - b_t\left(A_t, R_{t-1}, r_t\right)\right]\right\}$$

for R_t and substituting them into the corresponding Euler equation for bonds implies

$$\text{(A.6)} \qquad u'\left[A_t - s_t\left(A_t, R_{t-1}, r_t\right) - b_t\left(A_t, R_{t-1}, r_t\right)\right] = \beta E_t\left\{r_t u'\left[A_t - s_t\left(A_t, R_{t-1}, r_t\right) - b_t\left(A_t, R_{t-1}, r_t\right)\right]\right\}$$

for the bond rate, r_t.

Using log utility implies that the Euler equation (A.5) takes the form

$$\text{(A.7)} \qquad \left[A_t - s_t\left(\cdot\right) - b_t\left(\cdot\right)\right]^{-1} = \beta E_t\Big\{R_t\left[R_t s_t\left(\cdot\right) + r_t b_t\left(\cdot\right) - s_{t+1}\left(\cdot\right) + b_{t+1}\left(\cdot\right)\ \right]^{-1}\Big\}.$$

The point is now to guess a form for the policy functions s_t and b_t and to see if they work. Fortunately, log utility results in an especially simple form, namely, $s_t = kA_t$ and $b_t = \ell A_t$, where k and ℓ are just two constants (to be determined later). This transforms the Euler equation (A.7) into

$$\text{(A.8)} \qquad \left[\left(1 - k - \ell\right) A_t\right]^{-1} = \beta E_t\Big[R_t\left(R_t k A_t + r_t \ell A_t - kA_{t+1} + \ell\ A_{t+1}\right)^{-1}\Big].$$

This simplifies to

$$\left(A_t - kA_t - \ell A_t\right)^{-1} = \beta E_t\Big\{R_t\left[R_t k A_t + r_t \ell A_t\left(1 - k - \ell\right)\right]^{-1}\Big\},$$

which further reduces to equation (4) in the text.

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