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Learning in a Large Square Economy

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LEARNING IN A LARGE SQUARE ECONOMY

ABSTRACT

Learning is introduced into a sequence of large square endowment economies indexed by n, in which agents live n periods. Young agents need to forecast n - 1 periods ahead in these models in order to make consumption decisions, and thus these models constitute multi-step ahead systems. Real time learning is introduced via least squares. The systems studied in this paper are sometimes locally convergent when $n = 2,3$ but are never locally convergent when $n \geq 4$. Because the economies studied are analogous, nonconvergence can be attributed solely to the multi-step ahead nature of the forecast problem faced by the agents. We interpret this result as suggesting that beliefs-outcomes interaction may be an important element in explaining actual dynamics in general equilibrium systems of this type. *Journal of Economic Literature* Classification Nos. D50, D83.

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1 Introduction

Many studies of learning in macroeconomic models focus on one-step ahead systems, where agents are viewed as forecasting a single period into the future. However, most general equilibrium macroeconomic models in use today envision that agents forecast a large but finite number of periods into the future, as in a 55-period overlapping generations model, or an infinite number of periods into the future, as in the Ramsey-Solow growth model. The main idea of this paper is to analyze learning in general equilibrium systems where agents must forecast more than one step ahead.

There is reason to believe that the dynamical systems that describe equilibrium under learning in multi-step ahead models are considerably more complicated than the analogous systems in one-step ahead models. When agents form one-step ahead forecasts, they simply apply a given learning algorithm to past data to obtain a prediction for the next period. In contrast, when agents make forecasts two or more steps ahead, they must rely on past data as well as the one-step ahead prediction. The two-step ahead agent, for instance, must apply the learning algorithm to not only the past data in order to obtain the prediction for the next period, but also to the past data and the next period prediction in order to obtain the two-step ahead forecast. This action of reapplying the algorithm is likely to make the resulting dynamical system describing equilibrium trajectories under learning complicated and nonlinear. The results presented in this paper verify this intuition.

An endowment overlapping generations model is employed in which agents live for n periods. The endowment profile over the n periods is assumed to be hump-shaped, and attention is restricted to profiles that are compatible with valued fiat currency. Bullard (1992) provides the conditions for valued fiat currency in this model, and use is made of those results in this paper. Hump-shaped endowment patterns such as those considered by Auerbach and Kotlikoff (1987) are compatible with valued fiat currency. By restricting attention to hump-shaped endowment patterns, it is possible to study adaptive learning in a sequence of monetary economies indexed by n.

One advantage of maintaining a focus on n-period overlapping generations economies is that it complements work that has already been done on learning in versions of two-period overlapping generations models. These studies serve as an important benchmark for this paper, because they provide quite a lot of information about the behavior of the system under various learning mechanisms in the two period case. Lucas (1987), for instance, studies the behavior of the model when agents forecast the next period price by computing an average of past prices; Marcet and Sargent (1989b) look at least squares learning; Woodford (1990) finds the conditions for convergence to sunspot equilibria; Arifovic (1992) studies the model when agents are artificially intelligent; and Marimon and Sunder (1992) look at the model with human subjects in an experimental setting.[1] With the exception of Woodford (1990), the general conclusion of these studies is that simple learning mechanisms exist that allow agents to eventually attain the rational expectations equilibrium where money has value–the monetary steady state. The data produced by Marimon and Sunder (1992) show that human subjects tend to settle on the monetary steady state or something close to it. Woodford (1990) provides conditions under which agents might learn to believe in

[1] See also related work by Grandmont and Laroque (1991), Guesnerie and Woodford (1991), Howitt (1992), and İmrohoroğlu (1993).

a sunspot equilibrium in a two period overlapping generations model. In this paper, such sunspot equilibria are ruled out by a gross substitutes assumption.[2]

The aim of this paper is to investigate the extent to which the qualitative results described above are robust to a simple extension in the number of periods in agents' lifetimes. In particular, the n-period model is investigated under least squares learning, and so the present paper is most comparable to the work of Marcet and Sargent (1989b). The main result is that under least squares learning, the systems studied are sometimes locally convergent (in a neighborhood of the monetary steady state) when $n = 2, 3$ but are never locally convergent when $n \geq 4$. The local convergence of the system under least squares learning in the two period model therefore constitutes a special case. Marcet and Sargent (1989b) and others have argued that the convergence results obtained in the two period model augur well for the proposition that adaptive, backward-looking agents can eventually learn to form rational expectations. The results offered in this paper cast some doubt on this proposition.[3]

We interpret our results as follows. A fundamental conceptual issue in economic theory is that if agents are viewed as learning over time, feedback effects will exist because beliefs affect outcomes and outcomes affect beliefs. Since the learning dynamics influence the actual dynamics of the economy, and vice-versa, it is an open question whether the dynamics perceived by agents will eventually coincide with the actual dynamics or not. Convergence results such as those cited above suggest that beliefs-outcomes interaction will fade asymptotically. Nonconvergence results such as those presented here suggest that beliefs-outcomes interaction may not disappear even asymptotically. Models where beliefs-outcomes interaction does not disappear can be thought of as situations where forecasting mistakes–"optimism" and "pessimism"–are important in explaining the actual dynamics of the economy. By contrast, if beliefs-outcomes interaction does eventually go away, there is no scope for such forces to drive the actual dynamics, and thus only fundamental factors play a role.

The results presented in this paper pose a challenge to economists working in the learning literature as well as to those who use the overlapping generations framework to analyze monetary economies. Our results show that a simple least squares learning rule cannot be used to support the monetary steady state of the n-period model, and we conjecture that other least-squares-based learning algorithms are also unlikely to produce convergence. If the monetary steady state of the overlapping generations environment is to be useful for theoretical or policy analysis, it is important to know what class of learning models–if any–would lead agents to believe in this outcome. We hope to encourage others to pursue this matter further by considering alternatives to least squares learning. If, on the other hand, the monetary steady state cannot be justified by a plausible learning scheme, then it is important to know what types of dynamics to expect in this model under actual expectations. Experiments with human subjects could be quite helpful in this regard.

[2] For results on large square economies under gross substitutes, see Kehoe, Levine, Mas-Colell, and Woodford (1991).

[3] The results offered in this paper have much in common, in flavor if not specifics, with Benassy and Blad (1989) and Grandmont and Laroque (1991). These authors have local nonconvergence results but model learning with bounded memory and study systems requiring one step ahead forecasts. In contrast, the focus in this paper is on the effects of an unbounded memory learning scheme in a multi-step ahead environment.

The results reported here apply for $n < 11$ in this model. As we explain at the end of the paper, we believe it is reasonable to conjecture that our results hold for any finite n. However, caution is probably warranted in extrapolating these results to infinite horizon models, which in some ways are fundamentally different. In particular, we interpret the model in this paper as one where the fundamental assumption is that agents die, and the choice of n simply divides the agents' lifetimes into a prescribed number of segments. Thus, a large n approximates a continuous time representation, not an "infinite life." Under this interpretation, the sequence of economies studied are all analogous in a manner described in Bullard (1992). The interpretation of large n in this model is important, because the restriction of $n = 2$ essentially has the effect of giving agents perfect information concerning the first portion of their lives and forcing them to make a single forecast concerning the second portion of their lives. When n is allowed to be a larger number, the agents are allowed to revise their plans more often during their lifetimes; thus, they can adjust their consumption plans as actual events unfold. In this way, the model used in this paper is one where forecast errors matter in the sense that forecast errors feed back into the plans of individual agents.

In section 2, we describe the sequence of overlapping generations economies that will be studied in this paper. A number of facts about the model are stated without making any assumptions about expectations formation. These facts are used repeatedly in the rest of the paper. In section 3, the model is closed under perfect foresight. We then pose the main question that we want to ask of the system under the assumption that agents do not have perfect foresight. In section 4, the model is closed under least squares learning. We discuss the results of our numerical analysis of the least squares learning system. Section 5 summarizes the results and provides suggestions for further research.

2 An n-period overlapping generations model

A sequence of endowment overlapping generations economies is studied. The economies are indexed by n, where n is the number of time periods in agents' lives. Time is modeled in two different ways. In the single infinity case, time t is discrete and takes on strictly positive integer values on the real line–the economy begins at time $t = 1$. In the double infinity case, the economy endures forever; in that case, time t takes on integer values on the real line. The distinction between the two cases is important for our interpretation of learning, and the differences are discussed in detail later in this section.

A government endures for all time and provides currency at a constant rate $\theta > 1$; government consumption is endogenous. Agents are identical except for birthdates, and there is one agent in each generation. The agents live for n periods and maximize utility over consumption during their lifetime. The n agents possess an endowment stream $\{w_i\}$ where $w_i > 0$, $\forall i = 1, ..., n$. Since the agents are identical except for birth dates, the endowment pattern is always the same through the life of every agent. Agents can make consumption loans for one period only. There is a single perishable good, and agents do not leave bequests to future generations. Agents can save by holding currency. Arbitrage requires that the rate of return on consumption loans is the same as the rate of return to holding currency.[4]

The model can be summarized in a few equations:

$$\begin{aligned} H(t)/P(t) &= S(t) = S[\beta(t-n+2),...,\beta(t+n-2)] & (1)\\ H(t) &= \theta H(t-1) & (2)\\ F[P(t+1)] &\equiv \beta(t)P(t) & (3)\\ F[P(t+j)] &\equiv \beta(t+j-1)F[P(t+j-1)],\ \forall\, j=2...n-1,\ n>2 & (4)\end{aligned}$$

where $H(t)$ is the currency stock at time t, $P(t)$ is the price of the good at time t, $F[P(t+j)]$ is the forecast of the time $t+j$ price at time t, $\beta(t)$ is the expected gross inflation rate–the reciprocal of the expected gross interest rate–at time t, and $S(t)$ is aggregate savings at time t, where the arguments of the function have been suppressed. Generally, $S(t)$ depends on endowments, the rate of currency creation, parameters in the utility function, and, importantly, future and past gross rates of interest. The expression for aggregate savings is generally quite complicated but can be worked out explicitly for CES preferences. The analysis in this paper centers on the fact that explicit expressions are available for aggregate savings.

The model described by equations (1)-(4) is not complete until some assumption is made about the method of forecasting used by agents at time t. For instance, one possible assumption is perfect foresight, $F[P(t+j)] = P(t+j)$. Regardless of the way the model is closed, however, equation (4) embodies the maintained assumption in this paper that the forecasting rule is applied repeatedly to obtain forecasts of prices further into the future. In the next section the model is closed under perfect foresight, and in a subsequent section the model is closed under least squares learning.

An equilibrium is a positive sequence for prices and currency such that the assumption about how expectations are formed holds. One way to think about the model is to consider the actual law of motion for prices, which is given by

$$P(t+1) = [\theta S(t)/S(t+1)]P(t),$$

as compared to the law of motion perceived by the agents, which is simply

$$P(t+1) = \beta(t)P(t).$$

When $\beta(t) = \theta$, $\forall t$, $S(t) = S(t+1)$, and thus the actual law of motion is equal to the perceived law of motion. Steady states also exist at fixed values of β such that aggregate savings is equal to zero.

3 Perfect foresight

In this section the model is closed under perfect foresight. For variables such as consumption and savings, it is important to distinguish between birthdates and real time. Therefore, for these variables a standard notational convention is adopted that subscripts denote birthdates and parentheses denote real time. Since the agents all have an identical

[4] For more detail on the n-period overlapping generations model with money, see Bullard (1992).

endowment stream through their lifetimes, it is not necessary to adopt this convention for endowments. Hence, we continue to denote endowments in the various periods of life by $w_1, w_2, ..., w_n$. Preferences are time separable logarithmic, the agent born at time t maximizing $U = \sum_{j=0}^{n-1} \ln c_t(t+j)$. This choice for preferences implies that consumption goods in the various periods of life are gross substitutes, and gross substitutes in turn implies that aggregate savings is increasing in stationary gross interest rates R (decreasing in stationary gross inflation rates β). This assumption has the effect of reducing the number of perfect foresight equilibria in the model; in particular, there will be at most two steady states.[5] Agents maximize utility subject to the budget constraint

$$c_t(t) + \sum_{i=1}^{n-1} c_t(t+i) \prod_{j=0}^{i-1} \beta(t+j) \leq w_1 + \sum_{i=1}^{n-1} w_{i+1} \prod_{j=0}^{i-1} \beta(t+j).$$

Define

$$W_k = \frac{1}{n} \left[w_1 + \sum_{i=1}^{n-1} w_{i+1} \prod_{j=k-1}^{k+i-2} \beta(t+j) \right],$$

which is equal to $c_t(t)$ when $k = 1$. Then aggregate savings at time t can be written as (for $n \geq 3$):[6]

$$S(t) = \sum_{i=1}^{n-1} w_i + \sum_{i=0}^{n-3} \sum_{j=1}^{n-2-i} w_{i+1} \prod_{k=1}^{j} \beta(t-k)^{-1} - W_1 - \sum_{i=1}^{n-2} \sum_{j=0}^{i} W_{1-i} \prod_{k=1}^{i} \beta(t-k)^{-1}.$$

As an example, consider the case where $n = 3$. The first order conditions combined with the budget constraint imply that

$$c_t(t) = W_1 \equiv \frac{1}{3} \left[w_1 + \beta(t) w_2 + \beta(t)\beta(t+1) w_3 \right].$$

The time t savings of the agent born at time t is

$$s_t(t) = w_1 - c_t(t)$$

or

$$s_t(t) = w_1 - W_1.$$

[5]When preferences are CES and the coefficient of relative risk aversion, ρ, is greater than one, the gross substitutes assumption may not hold, and in particular, it does not hold when $\rho \to \infty$. Kehoe and Levine (1990) consider a three period overlapping generations model with a hump-shaped endowment pattern and $\rho = 4$ and find that the gross substitutes assumption fails to hold. However, the endowment profile in their example is substantially hump-shaped; the ratio of the second to the first period endowment in their example is 4 whereas in the condensed Auerbach–Kotlikoff endowment profile used in this paper, the same ratio is 1.2. See the discussion of numerical endowment patterns below. When the endowment profile is flatter, one needs a large degree of risk aversion in order to obtain an aggregate savings function that is decreasing in the stationary interest rate. Nevertheless, we think it would be interesting to extend the analysis of this paper to cases where the gross substitutes assumption fails to hold. Our restriction to the case of logarithmic preferences in this paper is made to maintain comparability to previous work on learning in two period overlapping generations models, especially Marcet and Sargent (1989b).

[6]See Bullard(1992).

The time $t+1$ savings of the agent born at time t is given by

$$s_t(t+1) = w_2 + \beta(t)^{-1}s_t(t) - c_t(t+1)$$

or, from the first order conditions,

$$s_t(t+1) = w_2 + \beta(t-1)^{-1}s_{t-1}(t-1) - \beta(t-1)^{-1}W_0 \tag{5}$$

and that

$$s_{t-1}(t-1) = w_1 - W_0.$$

Thus, aggregate savings at time t can be written as

$$S(t) = s_t(t) + s_{t-1}(t)$$

or

$$S(t) = \left[1 + \beta(t-1)^{-1}\right] w_1 + w_2 - W_1 - 2\beta(t-1)^{-1}W_0.$$

The three period case will be analyzed further in later sections.

The treatment of time must be addressed in order to discuss the stationary equilibria of the model. We start with the double infinity case, $t \in (-\infty, +\infty)$, and then turn to the single infinity case, $t \in [1, +\infty)$. The latter situation is more complicated because of initial generations. An example with $n = 3$ will be used to argue that the double infinity conceptualization is appropriate for present purposes, and to motivate the questions we wish to ask of these systems under least squares learning.

3.1 The double infinity case

In this case, there are at most two stationary equilibria. One is the nonmonetary steady state, where fiat currency is not held, and the other is the monetary steady state, where fiat currency has positive value. The condition for the existence of a valued fiat currency equilibrium when $\theta = 1$ is[7]

$$\sum_{i=1}^{n} w_i \left[(n-i) - \frac{(n-1)}{2}\right] > 0.$$

The same condition with $\theta > 1$ is

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_i\theta^{i+1-j} - \sum_{i=1}^{n}\sum_{j=0}^{n-2}(j+1)w_i\theta^{i-1-j} > 0.$$

The limit of the left hand side of this latter inequality as $\theta \to \infty$ is $-\infty$. Thus it is clear that there is always a rate of currency creation high enough to force nonexistence of a valued fiat currency stationary equilibrium. Throughout the paper we restrict $\theta > 1$ but less than the value that drives currency from the system altogether. We also restrict the endowment pattern to be such that when $\theta = 1$, a valued fiat currency equilibrium exists.

Later in the analysis, the endowment pattern is further specialized in order to study numerically a sequence of analogous economies. For that purpose, the endowment pattern

[7] See Bullard (1992).

used by Auerbach and Kotlikoff (1987) is employed. They suggest that for a 55-period model, an appropriate pattern is given by

$$w_{j-20} = e^{[4.47+.033(j-20)-.00067(j-20)^2]},$$

where $j = 21, 22, ..., 76$. In this paper, the pattern is standardized so that the first endowment is one. Endowment patterns for $n \neq 55$ are then constructed by integrating appropriately the Auerbach-Kotlikoff formula. For the case where $n = 2$, for instance, the integral would be evaluated from 21 to 48.5, and then from 48.5 to 76. Standardizing on the first period endowment yields $\{w_1, w_2\} = \{1, .87\}$. Endowment patterns constructed in this way always satisfy the assumption that the condition for valued fiat currency is met when $\theta = 1$.

The stationary equilibria of the n-period perfect foresight model can be summarized fairly easily in the double infinity case. From equations (1)-(4), the model under perfect foresight can be written as

$$S(t) = \theta\beta(t-1)^{-1}S(t-1). \tag{6}$$

From equation(6) it is clear that the equilibria of the system occur at $\beta = \theta$ provided $S(\theta) > 0$, or at values of β such that $S(\beta) = 0$. The qualitative graph for $S(\beta)$ in the n-period case with log preferences is given in figure one. For all stationary $\beta > 0$ the aggregate savings function is strictly decreasing. There is always a unique nonmonetary steady state where $S(\beta) = 0$. This point will be denoted $\bar{\beta}$ in the remainder of the analysis. If $S(\theta) > 0$, a monetary steady state exists. Existence of a monetary steady state has been assured by restricting $\theta \in (1, \bar{\beta})$ and by restricting the endowment pattern to be such that the condition for valued fiat currency is satisfied when $\theta = 1$. Endowment patterns based on Auerbach-Kotlikoff schemes always satisfy this condition. Because time extends into the infinite past, the steady states at $\beta = \theta$ and $\beta = \bar{\beta}$ are stationary equilibria of the double infinity model. This model is an n-period analog of the models mentioned in the introduction.

3.2 The single infinity case

Now consider the case where $t \in [1, \bar{\beta})$. When the economy begins at time $t = 1$, there are initial generations of agents with decision-making horizons less than n. The endowment profiles of these agents are, in order from those with the shortest horizon to those with the longest horizon, $\{w_n\}$, $\{w_{n-1}, w_n\}$, ..., $\{w_2, w_3, ..., w_{n-1}, w_n\}$. The initial old will be dead in period $t = 2$ and thus make no savings decision; they simply consume their endowment w_n. All other initial generations must make savings decisions based on their remaining horizons–we call this the problem of the initial generations. Thus, when $n = 3$, the problem of the initial generations generates an aggregate savings function that is special relative to the one that would prevail at time $t = 1$ in the double infinity model. However, at time $t = n$ all agents (except the youngest) who were part of the initial generations will be dead and thus the aggregate savings function at that time will be equivalent to the one which prevails for all t in the double infinity case. In the single infinity model, equilibrium sequences must solve the problem of the initial generations. In general, a gross inflation rate sequence such as $\beta = \theta$ or $\beta = \bar{\beta}$ is not an equilibrium sequence for the single infinity model under perfect foresight. This point can be illustrated with an example by setting n=3. We want to use this example to motivate the questions we will ask of our systems under learning.

3.3 An example with n=3

First consider the double infinity conceptualization. Denoting $\lambda_1 = w_1/w_3$ and $\lambda_2 = w_2/w_3$, the system can be written as a difference equation in β. Let $a = \theta - \lambda_2 - 2$, $b = 2\lambda_1 + \lambda_2 + \theta\lambda_2 + 2\theta$, $c = \lambda_1 - \theta\lambda_2 - 2\lambda_1\theta$, and $d = -\theta\lambda_1$. Then

$$\beta(t+1) = a + \left[b + \left[c + d\beta(t-2)^{-1}\right]\beta(t-1)^{-1}\right]\beta(t)^{-1} \tag{7}$$

It is easy to check that this system has a root at $\beta = \theta$; this is the monetary steady state of the double infinity model under our assumptions that $\theta \in (1, \bar{\beta})$ and $w_1 > w_3$ (the condition for valued fiat currency when $\theta = 1$). The polynomial associated with (7) is

$$\beta^4 - a\beta^3 - b\beta^2 - c\beta - d = 0.$$

The root $(\beta - \theta)$ can be factored out leaving the cubic

$$\beta^3 + (\lambda_2 + 2)\beta^2 - (2\lambda_1 + \lambda_2)\beta - \lambda_1 = 0.$$

The roots of this system are the other potential stationary equilibria of the system. They all involve values of β, say $\bar{\beta}$, such that $S(\bar{\beta}) = 0$. Exactly one of these values will always be real and greater than zero. This value constitutes the nonmonetary steady state of the double infinity model. The other two roots will either be complex or real and negative, and thus cannot constitute stationary equilibria of the model. The roots of the cubic can be found by an application of Cardano's formulas.

Equation (7) can be written as a three dimensional first order system. Define new variables $x(t) \equiv \beta(t+1)$, $y(t) \equiv \beta(t)$, and $z(t) \equiv \beta(t-1)$. Then the system can be written as

$$\begin{aligned} x(t) &= a + y(t-1)x(t-1)^{-1} \\ y(t) &= b + z(t-1)x(t-1)^{-1} \\ z(t) &= c + dx(t-1)^{-1}. \end{aligned}$$

The steady states are now points in (x, y, z)-space. In particular, the monetary steady state occurs at $x(t) = \bar{x} = \theta$, $z(t) = \bar{z} = c + \frac{d}{\theta}$, and $y(t) = \bar{y} = b + \frac{c}{\theta} + \frac{d}{\theta^2}$, where variables with bars are simply intended to denote equilibrium values.

The dimension of the local stable manifolds of the two steady states can be found as follows. Denote $\zeta(t) = [x(t), y(t), z(t)]'$, and let $\zeta(t) = G(\zeta(t-1))$, where $\bar{\zeta}(t) = [\bar{x}, \bar{y}, \bar{z}]'$ is such that $\bar{\zeta} = G(\bar{\zeta})$. The Jacobian matrix evaluated at a steady state is

$$DG(\bar{\zeta}) = \begin{bmatrix} -\bar{y}/\bar{x}^2 & 1/\bar{x} & 0 \\ -\bar{z}/\bar{x}^2 & 0 & 1/\bar{x} \\ -d/\bar{x}^2 & 0 & 0 \end{bmatrix}$$

and the associated characteristic polynomial, denoting eigenvalues by ρ, is

$$\rho^3 + (\bar{y}/\bar{x}^2)\rho^2 + (\bar{z}/\bar{x}^3)\rho + d/\bar{x}^4 = 0.$$

Thus, in principle it is possible to find the roots of the system linearized at each of the steady states and establish the dimension of the local stable manifold for general endowment patterns. We consider hump-shaped endowment patterns characterized by $\lambda_1 \in (1,\infty)$, $\lambda_2 \in (\lambda_1,\infty)$. It is a relatively simple numerical exercise to show that for all endowment patterns in this class, there is one root inside the unit circle for the monetary steady state, and two roots inside the unit circle for the nonmonetary steady state.

Next, consider the case where $t \in [1,\infty)$. The government holds the initial stock of currency, $H(1)$. When the economy begins at time $t=1$, there are initial old agents and initial middle-aged agents. The endowment profiles of these agents are $\{w_3\}$ and $\{w_2, w_3\}$ respectively. The initial old will be dead in period $t=2$ and thus make no savings decision; they simply consume their third period endowment w_3. The initial middle-aged, "born at time zero," maximize $\ln c_o(1) + \ln c_0(2)$ subject to

$$c_0(1) + \beta(1)c_0(2) \leq w_2 + \beta(1)w_3$$

so that they save

$$s_0(1) = w_2 - \frac{1}{2}\left[w_2 + \beta(1)w_3\right].$$

The initial young save

$$s_1(1) = w_1 - \frac{1}{3}\left[w_1 + \beta(1)w_2 + \beta(1)\beta(2)w_3\right].$$

Thus,

$$S(1) = s_0(1) + s_1(1) = w_2 - \frac{1}{2}\left[w_2 + \beta(1)w_3\right] + w_1 - \frac{1}{3}\left[w_1 + \beta(1)w_2 + \beta(1)\beta(2)w_3\right].$$

Aggregate savings at time $t = 2, 3, ..., \infty$, is given by equation (5). Thus the system which starts at time $t=1$ is described by

$$S(2) = \beta(1)^{-1}\theta S(1), \tag{8}$$

where S(1) is a special version of aggregate savings, and

$$S(t) = \beta(t-1)^{-1}\theta S(t-1)$$

for $t = 3, 4, ..., \infty$. Define $e = -(\lambda_2 + 2 - \theta)$, $f = 2\lambda_1 - \lambda_2 + \theta\lambda_2 + \frac{3}{2}\theta$, and $g = \lambda_1 - 2\theta\lambda_1 - \frac{3}{2}\theta\lambda_2$. Equation (8) can be rewritten as

$$\beta(3) = e + \left[f + g\beta(1)^{-1}\right]\beta(2)^{-1}. \tag{9}$$

This system requires initial values $\beta(1)$ and $\beta(2)$, whereas the double infinity model requires three initial gross inflation rates. Thus the modeling of the initial generations lowers the dimension of the set of initial conditions by one. If the double infinity model is viewed as a three dimensional first order system, equation (9) would define a restriction on the relationship between $x(t-1), y(t-1)$, and $z(t-1)$, given by

$$x(t-1) = e + \left[f + gz(t-1)^{-1}\right]y(t-1)^{-1}.$$

The qualitative situation is as depicted in figure two where the problem has been collapsed to three dimensions. The problem of the initial generations determines a surface in (x, y, z)-space. The local stable manifolds are one and two dimensional for the monetary and nonmonetary steady states respectively. Equilibria in the single infinity model are indexed by the points in (x, y, z)-space that are both on the surface defining permissible initial conditions and on the stable manifold associated with one of the steady states. It should be stressed that figure two is merely qualitative, and that it has not been shown either what the surface defined by the problem of the initial generations looks like nor what the stable manifolds look like outside the immediate vicinity of the two steady states. In particular, it has not been shown that the surface defining permissible initial conditions actually intersects with either of the stable manifolds in the positive orthant; that is, we have not shown that equilibria exist in the single infinity model. Nevertheless, in the figure as drawn, there is a one dimensional indeterminacy associated with the nonmonetary steady state (the line of intersection of the two planes) and the monetary steady state is locally unique (its stable manifold is a line and intersects the plane of permissible initial conditions at only one point). We do not pursue our heuristic argument any further at this point because we now want to use this example to assert that the double infinity conceptualization is appropriate for the questions we want to ask of our systems under learning.

3.4 The system under learning

Two key questions can now be posed concerning the introduction of real time learning in the model. Under maintained assumptions, there are always two steady states, regardless of n. These steady states can be thought of as points in a high dimensional space. The first question concerns the double-infinity conceptualization: What are the local basins of attraction for the two steady states under least squares learning, taking as permissible initial conditions any point in the high dimensional space? This approach can be justified by thinking of the system as having been at, say, the monetary steady state and then having been slightly perturbed by a one-time perfectly unanticipated policy change (a change in θ). The second question involves the single-infinity conceptualization: How can the set of equilibrium sequences be characterized under least squares learning, taking as permissible initial conditions the set defined by the problem of the initial generations? This approach is more in the spirit of the existence of equilibria under least squares learning, and there is no concept of an unanticipated policy change.

If we adopted the single-infinity perspective for our study of learning in this model, we might want to put any learning assumption on equal footing with the perfect foresight assumption.[8] This could be accomplished in the following way. We could find, as we will in the next section, the dynamic system that describes the evolution of the economy under learning for the double infinity case. We could write this as a first order system. We could then solve the problem of the initial generations under the learning assumption. This would yield a restriction on the initial conditions of the first order system. We could then

[8]By this we mean that we might want to maintain the rule that all expected gross inflation rates are calculated from the beginning of time according to the formula we assume, and that we should regard an unanticipated policy change as inconsistent with the learning assumption in the same way that we view it as inconsistent with a perfect foresight assumption.

try to locate sequences that are consistent with the restrictions implied by the problem of the initial generations under learning, and that also have the property that they converge eventually to the monetary steady state. If we could prove the existence of a set of such learning equilibrium sequences, we could compare this set to the set of perfect foresight equilibrium sequences in the single infinity model. As we have it drawn in figure one, the set of initial conditions consistent with solving the problem of the initial generations and leading asymptotically to the monetary steady state in the single infinity sense is of dimension zero. If the set of learning equilibrium sequences was associated with a set of initial conditions of dimension, say, one or two, it could be argued that the learning mechanism in some sense supports the monetary steady state in this model, because the dimension of the set of learning equilibria would be larger than the dimension of the set of perfect foresight equilibria.

We have not pursued this approach, although we think it is a plausible way to proceed. The chief problem is that the existence of a small set (a set with lower dimension than the ambient space of the model) of equilibrium learning sequences raises the question of coordination. When agents are endowed with perfect foresight, they are sometimes viewed as knowing the model in which they operate, and thus it is at least consistent to think of the agents as computing an equilibrium sequence, even if it is the only equilibrium sequence in a large space. However, when the model is closed with a learning assumption, such a view is less tenable. The motivation behind the learning assumption is that agents simply look at the past to get the necessary future forecasts. To then turn the tables and say that agents use their knowledge of the model and their knowledge of how they learn to compute equilibrium sequences runs counter to the motivation of the learning assumption.

For this reason, we focus in the next section on the double infinity model. We find the dynamic system that describes the evolution of the economy under learning. We assume that the economy is initially at a monetary steady state, and that a small one time perfectly unanticipated policy change occurs at time $t = 1$.

4 Learning

In this section, and in the rest of the paper, we no longer assume that agents have perfect foresight. Instead, the model will be closed under a real-time learning assumption about expectations formation. Before introducing the real time learning algorithm, it will be necessary to consider how savings behavior is affected when agents do not have perfect foresight and must learn from past prices.

4.1 Aggregate savings under learning

Agents born at time t solve the maximization problem described in the previous section. These agents develop a consumption and savings plan for the n periods of their lives. The savings plan depends on the agents' forecasts of the price level that will prevail at dates $t, t+1, t+2, ..., t+n-1$; this can be written as a sequence of expected gross inflation rates. At the beginning of period $t+1$, the actual price level that prevailed in period t is available. If agents do not have perfect foresight, their forecast of the price level at t will turn out, in general, to have been incorrect. Since agents' future savings decisions were dependent in part on the forecast price, any actual price which differs from the forecast price will cause agents to want to solve a new maximization problem. The new maximization problem at time $t+1$ is the same problem the agent faced at time t, except that the agent maximizes over one less period. The same story is repeated at future dates through the second to last period of life. At each one of these dates agents act based on new information, and they form a new savings plan for the remaining periods of their lives.

The difference between the case in which agents have perfect foresight and the case in which they do not should be clear. If agents have perfect foresight, they solve only one maximization problem in the first period of their lives. If agents do not have perfect foresight, they will want to reoptimize in every period. In the case of perfect foresight, agents do not make forecast errors. When agents do not have perfect foresight, they do make forecast errors and these errors affect future savings decisions. We argue below that the interplay between forecast errors and agents' future savings decisions is responsible for our nonconvergence results under least squares learning.[9]

It is now possible to understand why the two period model should be viewed as a special case. In the two period model, agents only make one price forecast. If agents do not have perfect foresight, this price forecast will turn out, in general, to have been incorrect. However, the incorrect forecast has no consequences for agents' future savings decisions; when agents discover that their price forecast was incorrect, they are in the last period of their lives and will not be making any further savings decisions. Hence the case where forecast errors affect individual savings decisions arises in versions of the model where agents live $n \geq 3$ periods.

In deriving the aggregate savings function under learning for the case where $n \geq 3$, it will be necessary to distinguish between *expected* and *realized* values of price ratios. The following notational convention is adopted. Let $R(t)$ be the *realized* rate of interest on consumption loans, and thus currency, at time t. In this model, $R(t-1) = P(t-1)/P(t)$. Consider the model at time t, when $P(t+1)$ is yet to be realized but where $P(t)$ is known. For all dates $t, t+1, t+2, \ldots$, denote *expected* gross inflation rates by $\beta(t)$, $\beta(t+1)$, $\beta(t+2)$, and so on. For dates in the past, $t-1, t-2, t-3, \ldots$, denote past expected gross inflation rates by $\beta(t-1)$, $\beta(t-2)$, $\beta(t-3)$ and so on. Denote past realized price ratios by $R(t-1)$, $R(t-2)$, $R(t-3)$, etc. Under the assumption of perfect foresight, from equation (3), $R(t-1) = \beta(t-1)^{-1}$. But under a learning assumption, realized price ratios are not in general equal to past expected price ratios, and thus notation is required to keep track of each.

Consider first the case in which the agents live just three periods. In this case, the problem of the individual agent born at time t must be solved in two steps. In the first step, the first order conditions and the budget constraint imply

$$c_t(t) = \frac{1}{3}\left[w_1 + w_2\beta(t) + w_3\beta(t)\beta(t+1)\right]$$

so that

$$s_t(t) = \frac{2}{3}w_1 - \frac{1}{3}\left[w_2\beta(t) + w_3\beta(t)\beta(t+1)\right].$$

[9] We stress that nothing has been said so far about the particular learning algorithm that agents use to form forecasts of future prices. In fact, the discussion in this section of aggregate savings behavior under the assumption that agents do not have perfect foresight is independent of the way in which agents learn.

The expected gross inflation rates that appear are based on forecasts of $P(t+1)$ and $P(t+2)$. But once the agent becomes middle-aged, the first period interest rate is realized. Under anything other than perfect foresight, this will cause the agent to reassess planned savings. The new decision will be based on the results of a new maximization problem: maximize $\ln c_t(t+1) + \ln c_t(t+2)$ subject to

$$c_t(t+1) + \beta(t+1)c_t(t+2) \leq w_2 + \beta(t+1)w_3 + R(t)s_t(t).$$

The first order conditions and the budget constraint imply

$$c_t(t+1) = \frac{1}{2}\left[w_2 + \beta(t+1)w_3 + R(t)s_t(t)\right]$$

so that

$$s_t(t+1) = w_2 - \frac{1}{2}\left[w_2 + \beta(t+1)w_3 + R(t)s_t(t)\right].$$

This must be postdated to get aggregate savings at time t. This is

$$s_{t-1}(t) = w_2 - \frac{1}{2}\left[w_2 + \beta(t)w_3 + R(t-1)s_{t-1}(t-1)\right]$$

where

$$s_{t-1}(t-1) = w_1 - \frac{1}{3}\left[w_1 + w_2\beta(t-1) + w_3\beta(t-1)\beta(t)\right].$$

Again, the $\beta(t-1)$ terms are past forecast values as the decision about how much to save in the first period of life was based on the forecast, not the actual value. Aggregate savings in this log utility case is therefore

$$\begin{aligned} S(t) \;\; = \;\; & \frac{2}{3}w_1 - \frac{1}{3}\left[w_2\beta(t) + w_3\beta(t)\beta(t+1)\right] + \frac{1}{2}\left[w_2 - \beta(t)w_3\right] \\ & + R(t-1)\left[\frac{1}{3}w_1 - \frac{1}{6}w_2\beta(t-1) - \frac{1}{6}w_3\beta(t-1)\beta(t)\right]. \end{aligned}$$

This collapses to the perfect foresight case if $R(t-1) = \beta(t-1)^{-1}$. Now write the aggregate savings function recursively by using the actual law of motion for prices, which implies

$$R(t-1) = S(t)/\theta S(t-1).$$

Then

$$S(t) = \frac{4w_1 + 3w_2 - [2w_2 + 3w_3]\beta(t) - 2w_3\beta(t)\beta(t+1)}{6 - \frac{2w_1 - w_2\beta(t-1) - w_3\beta(t-1)\beta(t)}{\theta S(t-1)}}. \tag{10}$$

The general form of the aggregate savings function under a learning assumption in the n-period case is given by:[10]

$$S(t) = \frac{\sum_{k=0}^{n-2}\left[w_{k+1} - W_k(0)\right]}{1 - \sum_{k=1}^{n-2}\sum_{i=1}^{k}\left[\frac{n-k-1}{n-i}\right]\left[\frac{w_i - W_{i-1}(k-i+1)}{\theta^{k-i+1}S(t-k+i-1)}\right]} \tag{11}$$

[10]See Appendix A for a derivation of this expression.

where

$$W_k(s) = \frac{1}{n-k}\left[w_{k+1} + \sum_{i=k+1}^{n-1} w_{i+1} \prod_{j=0}^{i-k-1} \beta(t+j-s)\right].$$

Therefore, aggregate savings at time t depends, under any learning scheme, on $n-2$ past values of aggregate savings, and thus on the past history of price forecasts. Aggregate savings at time t also depends on the expected gross inflation rates up to $\beta(t+n-2)$ as well as on the *past expected* gross inflation rates up to $\beta(t-n+2)$. Since both past expected and realized price ratios enter the aggregate savings function, the forecast errors involved do matter in this model. It is easy to show that the aggregate savings function under learning has zeros at the same values of β as the perfect foresight savings function.

4.2 Least squares learning

Under a learning assumption, agents must choose a way to calculate a value of from past data. In this paper, we assume agents use

$$\beta(t) = \left[\sum_{s=1}^{t-1} P(s-1)^2\right]^{-1}\left[\sum_{s=1}^{t-1} P(s-1)P(s)\right],$$

that is, they calculate $\beta(t)$ as the coefficient in a first order autoregression of $P(t)$ on $P(t-1)$ using information available through time $t-1$. This formula can be written recursively.[11] The law of motion for prices implied by equations (1)-(2) combined with a recursive rewrite of the least squares formula yields

$$\beta(t+1) = \beta(t) + g(t)\left[\frac{\theta S(t-1)}{S(t)} - \beta(t)\right]$$

where $g(t) = P(t-1)^2 \sum_{s=1}^{t} P(s-1)^{-2}$. This gain term may also be written in recursive form:

$$g(t+1) = \left[g(t)^{-1}\left[\frac{\theta S(t-1)}{S(t)}\right]^{-2} + 1\right]^{-1}. \tag{12}$$

Since an explicit expression for $S(t)$ is available, the system can be written in terms of β, g, and S. For any $n \geq 4$ the difference equation in β can be written as:[12]

$$\beta(t+n-2) = A - B \tag{13}$$

where

$$A = \frac{\left(\frac{n-1}{n}\right) w_1 - \sum_{i=2}^{n-1} \frac{w_i}{n} \prod_{j=0}^{i-2} \beta(t+j) + \sum_{k=1}^{n-2}\left[w_{k+1} - W_k(0)\right]}{\frac{w_n}{n} \prod_{j=0}^{n-3} \beta(t+j)}$$

and

$$B = \frac{g(t)\theta S(t-1)\left\{1 - \sum_{k=1}^{n-2}\sum_{i=1}^{k}\left[\left(\frac{n-k-1}{n-i}\right)\frac{w_i - W_{i-1}(k-i+1)}{\theta^{k-i+1} S(t-k+i-1)}\right]\right\}}{\frac{w_n}{n}\prod_{j=0}^{n-3}\beta(t+j)\left\{\beta(t+1) - \left[1-g(t)\right]\beta(t)\right\}}.$$

[11]See Ljung and Söderström (1983) for derivations of recursive algorithms.

[12]See Appendix B.

The case where $n = 3$ is special because this process yields a quadratic in $\beta(t+1)$. This occurs because when $n = 3$, $\beta(t+1)$ enters the aggregate savings function at time t, and $\beta(t+1)$ is the expected gross inflation rate furthest in the future. Consequently, when $n = 3$ there will be two values of $\beta(t+1)$ as shown below.[13] However, for $n \geq 4$, the above equation always yields a unique value for $\beta(t+n-2)$.

The model under learning can be written as a first-order dynamical system:[14]

$$\begin{aligned}
\beta(t+n-1) &= \ell_1\left[\beta(t-n+1), ..., \beta(t+n-3), g(t), S(t-n+1), ..., S(t-2)\right] \quad (14)\\
\beta(t+n-3) &= \beta(t+n-3)\\
&\cdot\\
&\cdot\\
&\cdot\\
\beta(t-n+2) &= \beta(t-n+2)\\
g(t+1) &= \ell_2\left[\beta(t-n+1), ..., \beta(t+n-3), g(t), S(t-n+1), ..., S(t-2)\right]\\
S(t-1) &= \ell_3\left[\beta(t-n+1), ..., \beta(t+n-3), S(t-n+1), ..., S(t-2)\right]\\
S(t-2) &= S(t-2)\\
&\cdot\\
&\cdot\\
&\cdot\\
S(t-n+2) &= S(t-n+2)
\end{aligned}$$

where $\ell_1(\cdot)$, $\ell_2(\cdot)$, and $\ell_3(\cdot)$ are complicated functions defined in Appendix B. Briefly, $\ell_3(\cdot)$ is the n-period aggregate savings function under learning lagged one period. The function $\ell_1(\cdot)$ is then equation (13) for $\beta(t+n-2)$ with $\ell_3(\cdot)$ substituted in for $S(t-1)$, which is a left hand side variable in the system. The function $\ell_2(\cdot)$ can then be obtained by using the recursive formula for $g(t+1)$ given by equation (12) and substituting appropriately until only right hand side variables appear as arguments.

If we set $\gamma(t)$ equal to the left hand side of (14), we can write $\gamma(t) = G(\gamma(t-1))$, where $G(\cdot)$ is defined by the right hand side of (14). The monetary steady state occurs at $\gamma^\star = G(\gamma^\star)$, where

$$\gamma^\star = \left[\theta, \; ..., \; \theta, \; 1-\theta^{-2}, \; S(\theta), \; ..., \; S(\theta)\right]'.$$

The term $S(\theta)$ can be found by evaluating the aggregate savings function before the substitution for R is made, setting $R = \theta^{-1}$.

The dimension of the dynamical system under least squares learning is $3n-4$; hence the dimension of the system increases linearly, not exponentially. The system has the virtue that the monetary steady state is available analytically (that is, $\gamma^\star$). Unfortunately, the system is otherwise quite complicated and nonlinear, even for small values of n. The steady state at $\bar{\beta}$, for instance, is not available analytically for $n = 4$ (it would be a root of an eighth order polynomial). In order to provide a local analysis of the system in a neighborhood of the monetary steady state, we turn to numerical methods.

[13] Thus, the three period case is somewhat special, but this does not pose any problems in the subsequent analysis.

[14] See Appendix B.

Our general method is to analyze $\ell_1(\cdot)$, $\ell_2(\cdot)$, and $\ell_3(\cdot)$ through use of a symbolic processor. For each n, the endowment pattern is set numerically according to the Auerbach-Kotlikoff scheme. The steady state at β is found numerically (it is a zero of $\ell_3(\cdot)$ near $\beta = 1$). The system (14) is then linearized to obtain the resulting analytical Jacobian. The analytical Jacobian is evaluated at the monetary steady state γ^*, yielding a $(3n-4)\times(3n-4)$ matrix in the single parameter θ. For small values of n, we then compute the associated eigenvalues numerically for, say, 500 values of $\theta \in (1, \bar{\beta})$. For larger values of n, we compute numerical eigenvalues at only two values of θ, one near unity, such as the value that solves $(\theta-1)/(\bar{\beta}-1) = .1$, and one near $\bar{\beta}$, such as the value that solves $(\theta-1)/(\bar{\beta}-1) = .9$.[15]

4.3 Numerical analysis

First consider briefly the results obtained in other research for the two period model. When $n = 2$, the system converges for values of near unity, and eventually diverges as θ gets larger and approaches $\bar{\beta}$.[16] Bullard (1992) employs a bifurcation analysis in the same case and finds that a subcritical Hopf bifurcation occurs for some value $\theta_0 \in (1, \bar{\beta})$. In that case a "learning equilibrium" exists which is an invariant closed curve in the neighborhood of θ_0. Figure three summarizes qualitatively the roots of the system in the two period case as θ increases from unity to $\bar{\beta}$. Here the modulus of the roots of the Jacobian $DG(\gamma^*)$ are plotted against $(\theta-1)/(\bar{\beta}-1)$, which gives the distance from unity to $\bar{\beta}$ as one and can thus be interpreted as the percent of the distance that θ is from one to $\bar{\beta}$. There is a straight horizontal line at one in the figure which is not a root but is intended to demark the edge of the unit circle (this also applies to subsequent figures). In the figure, there are initially (for θ near unity) three real roots inside the unit circle. As θ increases, two roots combine into a complex conjugate and cross the unit circle at about $(\theta-1)/(\bar{\beta}-1) = .34$.

One way to proceed with higher values of n is simply to simulate the systems directly. Based on a simulation of this system when $n = 3$, results similar to those obtained in the two period case are available. The solutions to the quadratic are:

$$\beta(t+1)^+,\ \beta(t+1)^- = \frac{-B \pm [B^2 - 4C]^{\frac{1}{2}}}{2} \tag{15}$$

where

$$\begin{aligned} B &= a - b\beta(t)^{-1} - [1 - g(t)]\beta(t) \\ C &= b[1-g(t)] - a\beta(t)[1-g(t)] + \frac{g(t)\beta(t-1)}{2} + \frac{g(t)}{\beta(t)}\left[\frac{1}{2}\beta(t-1)\lambda_2 - \lambda_1 + \frac{3\theta S(t-1)}{w_3}\right] \end{aligned}$$

and

$$\begin{aligned} a &= \lambda_2 + \frac{3}{2} \\ b &= 2\lambda_1 + \frac{3}{2}\lambda_2 \end{aligned}$$

[15]Since we are able to compute eigenvalues directly, we do not employ the associated ordinary differential equation approach for analyzing the stability of recursive systems. On this and related topics, see Ljung and Söderström (1983), Evans and Honkapohja (1992), and Marcet and Sargent (1989a).

[16]This is a version of the result due to Marcet and Sargent (1989b).

We consider the question of stability under a small one time perfectly unanticipated change in θ. To obtain initial conditions, we assume that the system is initially in equilibrium at a monetary steady state, so that $R^{-1} = \beta = \theta$. We then simulate the system with a new value of θ, where the new value holds for $t \in (1, \infty)$. Although the system with $n = 3$ is a quadratic in $\beta(t+1)$, it is easy to establish that of the two solutions, the economically meaningful one is $\beta(t+1)^-$. This is because this solution has a steady state at $\beta = \theta$.

A series of three figures illustrates the findings. In these simulations, the endowment pattern is set according to the Auerbach-Kotlikoff scheme at {1, 1.13, .833}. The system converges when θ is near unity, and diverges when gets larger, approaching $\bar{\beta}$. There is a learning equilibrium in the neighborhood of certain values of θ. Thus, the simulation results for the case with $n = 3$ appear to conform to the analytical results for the case with $n = 2$. The three figures illustrate the dynamics beginning from a point very near the initial monetary steady state. In figure four, $\theta = 1.03$, and the dynamics are convergent to the new equilibrium at $\beta = \theta = 1.03$. In figure five, $\theta = 1.03112$, which is an approximate bifurcation point for this system. In this case, an equilibrium under learning exists which is a closed curve, and the system cycles indefinitely about the steady state at $\beta = \theta = 1.03112$. This closed curve is repelling. Finally, in figure six, the value of θ is set to 1.033, still much less than $\bar{\beta}$. In this case, the system is divergent, and the steady state has no neighborhood which constitutes a basin of attraction.

These findings are confirmed in a more compact form in figure seven. Initially, when is close to unity, all five roots of the system are inside the unit circle and the system is stable under the one time unanticipated change in policy. As θ is increased, two roots subsequently combine into complex conjugates and cross the unit circle at about $(\theta - 1)/(\bar{\beta} - 1) = .34$. This creates a subcritical Hopf bifurcation. After this point, the system is unstable. Figure seven compares favorably with figure three, and we conclude that the three period case is analogous to the two period case.

In the case where $n = 4$ the savings function can be written as:

$$S(t) = \frac{a - b\beta(t) - c\beta(t)\beta(t+1) - d\beta(t)\beta(t+1)\beta(t+2)}{\begin{array}{l} 12 - \frac{1}{\theta S(t-1)}\left[e - f\beta(t-1) - g\beta(t-1)\beta(t) - e\beta(t-1)\beta(t)\beta(t+1)\right] \\ -\frac{1}{\theta^2 S(t-2)}\left[3w_1 - w_2\beta(t-2) - w_3\beta(t-2)\beta(t-1) - w_4\beta(t-2)\beta(t-1)\beta(t)\right] \end{array}} \tag{16}$$

where $a = 9w_1 + 8w_2 + 6w_3$, $b = 3w_2 + 4w_3 + 6w_4$, $c = 3w_3 + 4w_4$, $d = 3w_4$, $e = 6w_1 + 4w_2$, $f = 2w_2 + 2w_3$, $g = 2w_3 + 2w_4$, and $h = 2w_4$. The implied dynamic system is

$$\beta(t+2) = \frac{a - b\beta(t) - c\beta(t)\beta(t+1)}{d\beta(t)\beta(t+1)} - \frac{g(t)\theta S(t-1)X}{d\beta(t)\beta(t+1)^2 - d[1 - g(t)]\beta(t)^2\beta(t+1)} \tag{17}$$

where X is the denominator of the aggregate savings function $S(t)$.

We consider the hump-shaped endowment pattern {1, 1.2, 1.11, .811} which is calculated according to the Auerbach-Kotlikoff scheme. The roots of this system, based on the evaluation of the Jacobian at the monetary steady state, are characterized in figures eight and nine. The dimension of the system is now $3n - 4 = 8$. Figure eight shows the roots between zero and two. Two roots are near zero for all θ and are not pictured. Five additional roots are real and inside the unit circle when is θ near unity. Three of these roots remain inside the unit circle for all θ. Again, two roots combine into a complex conjugate and cross the unit circle at $(\theta - 1)/(\bar{\beta} - 1) = .34$. Thus, seven of the eight roots of this system are inside the unit circle for low values of θ. Figure nine shows the roots extending the y-axis up to 50. The remaining root is real with a very large modulus when θ is near one. The modulus shrinks as θ nears $\bar{\beta}$, but never enters inside the unit circle. Thus, in the case with $n = 4$, the monetary steady state is always locally unstable under least squares learning. We are struck, however, by the similarity of figures three, seven, and eight.

This similarity carries over to figure 10, which shows the same characterization of the roots for the case with $n = 5$. Figure 10 shows the roots between zero and two. The dimension of the system is now 11. The endowment pattern is calculated according to our Auerbach-Kotlikoff scheme to be {1, 1.2, 1.23, 1.07, .798}. There are three roots near zero for all θ which are not pictured. The line at about .4 for all represents a pair of complex conjugates. Thus when θ is near one, there are nine roots inside the unit circle. When $(\theta - 1)/(\bar{\beta} - 1) = .34$, a pair of complex conjugates crosses the unit circle. Figure 11 shows that the two remaining roots are outside the unit circle for all θ and are very large for θ near one. The modulus of these roots declines but never crosses into the unit circle. Thus, the monetary steady state is again always locally unstable under the least squares learning scheme.

The dimension of the system under learning is $3n$-4, for $n \geq 3$. Thus three roots are added for each unit increase in n. We conjecture based on the results for $n = 3, 4, 5$ that each increment in n will add one additional root outside the unit circle for θ near one, and that a pair of complex conjugates will cross the unit circle at $(\theta - 1)/(\bar{\beta} - 1) = .34$. Thus, for θ near unity in the n-period case, we expect $n - 3$ roots outside the unit circle, and for θ near $\bar{\beta}$ we expect $n - 1$ roots outside the unit circle. Table one confirms our conjecture up to $n = 11$. The last line in the table represents our expectations concerning the n-period model. Thus, we predict that the n-period model will display local nonconvergence in a neighborhood of the monetary steady state under least squares learning. Based on the results presented in the table, we believe our conjecture is reasonable for any finite $n \geq 4$.[17]

5 Summary

In this paper we have considered least squares learning in a general equilibrium system with agents who must forecast two or more steps ahead in order to make consumption decisions. Models of this type are pervasive in macroeconomics. The analysis of learning in this model is considerably more complicated than in the analogous one-step ahead systems. A key aspect is that the forecast errors implied by the learning assumption "matter" in systems of this type in the sense that they enter into the aggregate savings function for the economy. Forecast errors do not matter in this sense in one-step ahead systems.

The results developed in this paper pertain to n-period endowment overlapping generations economies under least squares learning. We find that the three-period model produces results analogous to the results for least squares learning in a two-period version studied by

[17] Similar calculations for larger models of interest, such as n=55 ("annual"), n=220 ("quarterly"), or n=660 ("monthly"), appear to be technologically infeasible at this time, and we think a different approach will be needed to study these models. For n=25, for instance, we were unable even to load the aggregate savings function into memory. It may be possible, however, to push the calculations further. Our calculations were done in Mathematica on a high speed RISC-based workstation.

Marcet and Sargent (1989b) and Bullard (1992). However, for the cases where $n \geq 4$, these systems are locally nonconvergent under least squares learning. In contrast to Grandmont and Laroque (1991), it is not necessary for agents in our model to entertain thoughts of nonconvergence for the system to become locally unstable. In fact, the agents in our model are endowed with the correct equilibrium forecast rule.

One conjecture for the nonconvergence result is that the least squares mechanism studied here is not sophisticated enough to allow agents to learn perfect foresight equilibria in the n-period case. However, caution is warranted, because more sophisticated learning schemes will only make the implied dynamic systems all that much more complicated. In fact, a theme of this paper is that introducing learning into environments which are more complex than those studied to date is likely to imply that the dynamical systems describing equilibrium under learning will be complicated and nonlinear and hence unlikely to converge. Thus, remedies such as replacing our learning mechanism with a more sophisticated version of least squares, or adding a meta-learning process on top of our learning process, are in our view unlikely to succeed in making our system converge. However, this remains an open question.

An alternative to considering more sophisticated "gradient learning" methods would be to consider an evolutionary dynamic, such as Holland's genetic algorithm (Holland, 1975; Goldberg,1989). In contrast to least squares and other gradient methods, genetic algorithms are well suited to searches of highly nonlinear surfaces, since these algorithms do not require surfaces to be smooth enough so that derivatives can always be found. In fact, genetic algorithms do not rely on derivatives or any other kind of auxiliary information in determining how beliefs should be updated. Instead, a genetic algorithm evaluates the fitness of a particular belief or decision rule according to whether it improves the agents' payoff or utility. Agents who populate the genetic algorithm have heterogeneous beliefs, and are not required to possess any kind of cognitive forecasting ability. They simply make decisions based on their current beliefs. Other genetic operators are included in the genetic algorithm help to ensure that a wide surface of the decision space is examined.[18]

The main difficulty with introducing genetic algorithm learning (or other evolutionary dynamics) is that it essentially simplifies the system by eliminating the notion of agents actively forecasting n periods ahead, as envisioned in the set-up of the n-period overlapping generations model. Thus, such methods assume away what is in our opinion an important component of a large square economy. Nevertheless, we think it would be interesting to study this model under genetic algorithm learning.

Preliminary research on genetic algorithm learning is encouraging. Marimon, McGrattan, and Sargent (1990) apply versions of such methods in a Kiyotaki-Wright model of money. Arifovic (1992) examines genetic algorithm learning in a two-period overlapping generations model and finds that the genetic algorithm converges to the unique monetary steady state or the low inflation steady state of the model, even in cases where the least squares algorithm fails to converge.

Experiments of the type conducted by Marimon and Sunder (1992) and Marimon, Spear, and Sunder (1993) could be conducted for the multi-step ahead systems studied in this paper. It is an open question whether such experiments would show again that the human subjects would tend to find the monetary steady state, as they do in the two period case.

[18]See also the discussion in Sargent (1992).

This study suggests that in most cases a simple least squares learning mechanism does not describe, even approximately, such dynamics. Thus an experimental study would be quite informative as it could be used to discern whether the least squares description of learning behavior is wrong or whether these systems really do not converge under human decision-making.

Table 1

The number of roots with modulus inside and outside the unit circle when θ is close to unity and when θ is close to the value which drives currency from the system. All entries through $n = 11$ verified through direct calculation using an Auerbach–Kotlikoff endowment scheme.

	$(\theta - 1)/(\bar{\beta} - 1)$	$= .1$	$(\theta - 1)/(\bar{\beta} - 1)$	$= .9$
n	Inside	Outside	Inside	Outside
3	5	0	3	2
4	7	1	5	3
5	9	2	7	4
6	11	3	9	5
7	13	4	11	6
8	15	5	13	7
9	17	6	15	8
10	19	7	17	9
11	21	8	19	10
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
n	$2n$-1	n-3	$2n$-3	n-1

A Derivation of n-period aggregate savings function under learning

In the n-period model, an agent born at time t solves the following problem:

$$\max \sum_{i=0}^{n-1} \ln c_t(t+i)$$

subject to:

$$c_t + \sum_{i=1}^{n-1} c_t(t+i) \prod_{j=0}^{i-1} \beta(t+j) \leq w_i + \sum_{i=1}^{n-1} w_{i+1} \prod_{j=0}^{i-1} \beta(t+j)$$

Substituting the first order conditions into the budget constraint, one obtains:

$$c_t(t) = \frac{1}{n}\left[w_1 + \sum_{i=1}^{n-1} w_{i+1} \prod_{j=0}^{i-1} \beta(t+j)\right] \quad (1)$$

$$s_t(t) = w_1 - c_t(t) \quad (2)$$

At the beginning of period $t+1$ the gross interest rate, $R(t) \equiv P(t)/P(t+1)$ is realized. Since agents do not have perfect foresight, their forecast of the price ratio $\beta(t)$ will not be equal to $R(t)^{-1}$. Thus at time $t+1$, agents will choose to solve a new maximization problem over the $n-1$ remaining periods of their lives. The same story is repeated again and again for each period $t+k$, where $k = 1, 2, ...n-3, n-2$. At every date $t+k > t$ agents remaximize over the $n-k$ remaining periods of their lives. The optimal consumption and savings amounts at any date $t+k$ for an agent born at time t are given by:

$$c_t(t+k) = \frac{1}{n-k}\left[w_{k+1} + \sum_{i=k+1}^{n-1} w_{i+1} \prod_{j=k}^{i-1} \beta(t+j) + R(t+k-1)s_t(t+k-1)\right]$$

$$s_t(t+k) = w_{k+1} + R(t+k-1)s_t(t+k-1) - c_t(t+k)$$

for $k = 1, 2, ...n-3, n-2$.

Aggregate savings at any date in time, t, is found by backdating the savings functions of all the generations born prior to time t and who are alive at time t and adding these savings functions together with the savings function for the generation born at time t. Aggregate savings $S(t)$, is therefore given by the expression:[19]

$$S(t) = s_t(t) + s_{t-1}(t) + s_{t-2}(t) + ... + s_{t-n+2}(t) \quad (3)$$

Hence, we will need to obtain expressions for $c_{t-k}(t)$ and $s_{t-k}(t)$ for $k = 1, 2, ..n-3, n-2$. These are given by:

$$c_{t-k}(t) = \frac{1}{n-k}\left[w_{k+1} + \sum_{i=k+1}^{n-1} w_{i+1} \prod_{j=0}^{i-k-1} \beta(t+j) + R(t-1)s_{t-k}(t-1)\right] \quad (4)$$

$$s_{t-k}(t) = w_{k+1} + R(t-1)s_{t-k}(t-1) - c_{t-k}(t) \quad (5)$$

[19] The notational convention is that subscripts denote birthdates and real time is indicated by the date in brackets. An exception is made for endowments, w_i where the subscript $i = 1, 2, .., n$ refers to period of the agent's life in which the endowment is received.

Substituting equation (1) into (2) we have:

$$s_t(t) = w_1 - \frac{1}{n}\left[w_1 + \sum_{i=1}^{n-1} w_{i+1} \prod_{j=0}^{i-1} \beta(t+j)\right] \tag{6}$$

Similarly, substituting equation (4) into (5) we have:

$$\begin{aligned} s_{t-k}(t) &= w_{k+1} + R(t-1)s_{t-k}(t-1) \\ &\quad - \frac{1}{n-k}\left[w_{k+1} + \sum_{i=k+1}^{n-1} w_{i+1} \prod_{j=0}^{i-k-1} \beta(t+j) + R(t-1)s_{t-k}(t-1)\right] \\ &= w_{k+1} + \frac{n-k-1}{n-k}\{R(t-1)s_{t-k}(t-1)\} \\ &\quad - \frac{1}{n-k}\left[w_{k+1} + \sum_{i=k+1}^{n-1} w_{i+1} \prod_{j=0}^{i-k-1} \beta(t+j)\right] \end{aligned} \tag{7}$$

We want to eliminate the $s_{t-k}(t-1)$ terms on the right hand side of equation (7). First define the term:

$$W_k(s) \equiv \frac{1}{n-k}\left[w_{k+1} + \sum_{i=k+1}^{n-1} w_{i+1} \prod_{j=0}^{i-k-1} \beta(t+j-s)\right]$$

Now make the necessary substitutions and obtain the expressions:

$$s_t(t) = w_1 - W_0(0) \tag{8}$$

$$\begin{aligned} s_{t-k}(t) &= w_{k+1} - W_k(0) \\ &\quad + \sum_{i=1}^{k}\left(\frac{n-k-1}{n-i}\right)[w_i - W_{i-1}(k-i+1)] \prod_{j=1}^{k-i+1} R(t-j) \end{aligned} \tag{9}$$

for $k = 1, 2, \ldots n-3, n-2$. The next step is to calculate aggregate savings $S(t)$, as given by equation (3). This involves adding together equations (8)-(9) for all k. Adding these equations up, one obtains:

$$\begin{aligned} S(t) &= w_1 - W_0(0) + \sum_{k=1}^{n-2}\{w_{k+1} - W_k(0) \\ &\quad + \sum_{i=1}^{k}\left(\frac{n-k-1}{n-i}\right)[w_i - W_{i-1}(k-i+1)] \prod_{j=1}^{k-i+1} R(t-j)\} \end{aligned} \tag{10}$$

The actual law of motion for prices implies that:

$$R(t-j) = \frac{S(t-j+1)}{\theta S(t-j)} \tag{11}$$

for $j = 1, 2, \ldots n-3, n-2$. Using (11) in (10) one obtains the expression:

$$\begin{aligned} S(t) &= w_1 - W_0(0) + \sum_{k=1}^{n-2}\{w_{k+1} - W_k(0) \\ &\quad + \sum_{i=1}^{k}\left(\frac{n-k-1}{n-i}\right)\left(\frac{S(t)}{\theta^{k-i+1}S(t-k+i-1)}\right)[w_i - W_{i-1}(k-i+1)]\} \end{aligned}$$

This expression simplifies to

$$S(t) = \frac{\sum_{k=0}^{n-2}[w_{k+1} - W_k(0)]}{1 - \sum_{k=1}^{n-2}\sum_{i=1}^{k}\left[\left(\frac{n-k-1}{n-i}\right)\left(\frac{w_i - W_{i-1}(k-i+1)}{\theta^{k-i+1}S(t-k+i-1)}\right)\right]} \tag{12}$$

which is the aggregate savings function for the general n–period case. Note that for $n \geq 3$, the function is recursive.

A.1 Case where $n = 3$

If $n = 3$, then equation (12) becomes:

$$S(t) = \frac{w_1 - W_0(0) + w_2 - W_1(0)}{1 - \frac{1}{2}\left(\frac{w_1 - W_0(1)}{\theta S(t-1)}\right)} \tag{13}$$

From the definition of the term W, we have:

$$W_0(0) = \frac{1}{3}[w_1 + w_2\beta(t) + w_3\beta(t)\beta(t+1)] \tag{14}$$

$$W_0(1) = \frac{1}{3}[w_1 + w_2\beta(t-1) + w_3\beta(t-1)\beta(t)] \tag{15}$$

$$W_1(0) = \frac{1}{2}[w_2 + w_3\beta(t)] \tag{16}$$

Substituting (14)-(16) into (13), one has:

$$S(t) = \frac{\frac{2}{3}w_1 + \frac{1}{2}w_2 - \left[\frac{1}{3}w_2 + \frac{1}{2}w_3\right]\beta(t) - \frac{1}{3}w_3\beta(t)\beta(t+1)}{1 - \frac{1}{\theta S(t-1)}\left[\frac{1}{3}w_1 - \frac{1}{6}w_2\beta(t-1) - \frac{1}{6}w_3\beta(t-1)\beta(t)\right]} \tag{17}$$

Multiplying the numerator and the denominator of (17) by the least common multiple = 6, one obtains the expression found in the text:

$$S(t) = \frac{4w_1 + 3w_2 - [2w_2 + 3w_3]\beta(t) - 2w_3\beta(t)\beta(t+1)}{6 - \frac{2w_1 - w_2\beta(t-1) - w_3\beta(t-1)\beta(t)}{\theta S(t-1)}}$$

A.2 Case where $n = 4$

If $n = 4$, then equation (12) becomes:

$$S(t) = \frac{w_1 - W_0(0) + w_2 - W_1(0) + w_3 - W_2(0)}{1 - \frac{2}{3}\left(\frac{w_1 - W_0(1)}{\theta S(t-1)}\right) - \frac{1}{3}\left(\frac{w_1 - W_0(2)}{\theta^2 S(t-2)}\right) - \frac{1}{2}\left(\frac{w_2 - W_1(1)}{\theta S(t-1)}\right)} \tag{18}$$

From the definition of the term W, we have:

$$W_0(0) = \frac{1}{4}[w_1 + w_2\beta(t) + w_3\beta(t)\beta(t+1) + w_4\beta(t)\beta(t+1)\beta(t+2)] \tag{19}$$

$$W_0(1) = \frac{1}{4}[w_1 + w_2\beta(t-1) + w_3\beta(t-1)\beta(t) + w_4\beta(t-1)\beta(t)\beta(t+1)] \tag{20}$$

$$W_0(2) = \frac{1}{4}[w_1 + w_2\beta(t-2) + w_3\beta(t-2)\beta(t-1) + w_4\beta(t-2)\beta(t-1)\beta(t)] \tag{21}$$

$$W_1(0) = \frac{1}{3}\left[w_2 + w_3\beta(t) + w_4\beta(t)\beta(t+1)\right] \tag{22}$$

$$W_1(1) = \frac{1}{3}\left[w_2 + w_3\beta(t-1) + w_4\beta(t-1)\beta(t)\right] \tag{23}$$

$$W_2(0) = \frac{1}{2}\left[w_3 + w_4\beta(t)\right] \tag{24}$$

Substituting (19)-(24) into (18), one has:

$$S(t) = \frac{N}{D} \tag{25}$$

where:

$$\begin{aligned}
N \equiv\ & \frac{3}{4}w_1 + \frac{2}{3}w_2 + \frac{1}{2}w_3 - \left[\frac{1}{4}w_2 + \frac{1}{3}w_3 + \frac{1}{2}w_4\right]\beta(t) \\
& - \left[\frac{1}{4}w_3 + \frac{1}{3}w_4\right]\beta(t)\beta(t+1) - \frac{1}{4}w_4\beta(t)\beta(t+1)\beta(t+2) \\
D \equiv\ & 1 - \frac{1}{\theta S(t-1)}\left\{\frac{1}{2}w_1 + \frac{1}{3}w_2 - \frac{1}{6}\left[w_2 + w_3\right]\beta(t-1)\right. \\
& \left. -\frac{1}{6}\left[w_3 + w_4\right]\beta(t-1)\beta(t) - \frac{1}{6}w_4\beta(t-1)\beta(t)\beta(t+1)\right\} \\
& - \frac{1}{\theta^2 S(t-2)}\left\{\frac{1}{4}w_1 - \frac{1}{12}w_2\beta(t-2) - \frac{1}{12}w_3\beta(t-2)\beta(t-1)\right. \\
& \left. -\frac{1}{12}w_4\beta(t-2)\beta(t-1)\beta(t)\right\}
\end{aligned}$$

Multiplying the numerator and the denominator of (25) by the least common multiple = 12, one obtains the expression found in the text:

$$S(t) = \frac{a - b\beta(t) - c\beta(t)\beta(t+1) - d\beta(t)\beta(t+1)\beta(t+2)}{\begin{array}{c} 12 - \frac{1}{\theta S(t-1)}\left[e - f\beta(t-1) - g\beta(t-1)\beta(t) - h\beta(t-1)\beta(t)\beta(t+1)\right] \\ -\frac{1}{\theta^2 S(t-2)}\left[3w_1 - w_2\beta(t-2) - w_3\beta(t-2)\beta(t-1) - w_4\beta(t-2)\beta(t-1)\beta(t)\right]\end{array}}$$

where $a = 9w_1 + 8w_2 + 6w_3$, $b = 3w_2 + 4w_3 + 6w_4$, $c = 3w_3 + 4w_4$, $d = 3w_4$, $e = 6w_1 + 4w_2$, $f = 2w_2 + 2w_3$, $g = 2w_3 + 2w_4$, and $h = 2w_4$.

B Derivation of n-period dynamical system under learning

The least squares estimate of β can be written recursively as follows:

$$\beta(t+1) = \beta(t) + g(t)\left[\theta S(t-1)/S(t) - \beta(t)\right] \tag{1}$$

where

$$g(t) = \frac{P(t-1)^2}{\sum_{s=1}^{t} P(s-1)^2} \tag{2}$$

is the expression for the gain. In the derivation of this expression, we have made use of the fact that the *actual* law of motion for prices is given by:

$$P(t) = \left[\frac{\theta S(t-1)}{S(t)}\right] P(t-1) \tag{3}$$

Since an explicit function for $S(t)$ is available, equation (1) can be written as a difference equation in β. In order to find this difference equation, we begin by noting that $S(t)$ contains forecasts of the price ratio ranging from $\beta(t-n+2)$ on up to $\beta(t+n-2)$. Since $\beta(t+n-2)$ represents the forecast value of the price ratio furthest in the future, we will want to solve our difference equation for $\beta(t+n-2)$. The solution can be found in several steps.

First, note that equation (1) can be written as:

$$S(t)\left\{\beta(t+1) - [1-g(t)]\beta(t)\right\} = g(t)\theta S(t-1) \tag{4}$$

Next, recall that the aggregate savings function for the n-period model under learning was found to be:

$$S(t) = \frac{\sum_{k=0}^{n-2}\left[w_{k+1} - W_k(0)\right]}{1 - \sum_{k=1}^{n-2}\sum_{i=1}^{k}\left[\left(\frac{n-k-1}{n-i}\right)\left(\frac{w_i - W_{i-1}(k-i+1)}{\theta^{k-i+1}S(t-k+i-1)}\right)\right]} \tag{5}$$

where:

$$W_k(s) \equiv \frac{1}{n-k}\left[w_{k+1} + \sum_{i=k+1}^{n-1} w_{i+1}\prod_{j=0}^{i-k-1}\beta(t+j-s)\right]$$

Let us rewrite the numerator of the savings function (5) as follows:

$$\begin{aligned}
\sum_{k=0}^{n-2}\left[w_{k+1} - W_k(0)\right] &= w_1 - W_0(0) + \sum_{k=1}^{n-2}\left[w_{k+1} - W_k(0)\right] \\
&= \left(\frac{n-1}{n}\right) w_1 - \sum_{i=2}^{n-1}\frac{w_i}{n}\prod_{j=0}^{i-2}\beta(t+j) - \beta(t+n-2)\frac{w_n}{n}\prod_{j=0}^{n-3}\beta(t+j) \\
&\quad + \sum_{k=1}^{n-2}\left[w_{k+1} - W_k(0)\right]
\end{aligned}$$

Using this rewrite of the numerator of the savings function, equation (4) can be rewritten as:

$$\begin{aligned}
&\left\{\left(\frac{n-1}{n}\right) w_1 - \sum_{i=2}^{n-1}\frac{w_i}{n}\prod_{j=0}^{i-2}\beta(t+j) - \beta(t+n-2)\frac{w_n}{n}\prod_{j=0}^{n-3}\beta(t+j)\right. \\
&\qquad\left. + \sum_{k=1}^{n-2}\left[w_{k+1} - W_k(0)\right]\right\}\left\{\beta(t+1) - [1-g(t)]\beta(t)\right\} = \\
&g(t)\theta S(t-1)\left\{1 - \sum_{k=1}^{n-2}\sum_{i=1}^{k}\left[\left(\frac{n-k-1}{n-i}\right)\left(\frac{w_i - W_{i-1}(k-i+1)}{\theta^{k-i+1}S(t-k+i-1)}\right)\right]\right\}
\end{aligned}$$

or

$$\beta(t+n-2) = \frac{\left(\frac{n-1}{n}\right)w_1 - \sum_{i=2}^{n-1}\frac{w_i}{n}\prod_{j=0}^{i-2}\beta(t+j) + \sum_{k=1}^{n-2}\left[w_{k+1} - W_k(0)\right]}{\frac{w_n}{n}\prod_{j=0}^{n-3}\beta(t+j)} - \frac{g(t)\theta S(t-1)\left\{1 - \sum_{k=1}^{n-2}\sum_{i=1}^{k}\left[\left(\frac{n-k-1}{n-i}\right)\frac{w_i - W_{i-1}(k-i+1)}{\theta^{k-i+1}S(t-k+i-1)}\right]\right\}}{\frac{w_n}{n}\prod_{j=0}^{n-3}\beta(t+j)\left\{\beta(t+1) - [1-g(t)]\beta(t)\right\}} \tag{6}$$

Equation (6) describes the dynamical system under learning as reported in the text.

B.1 Case where $n = 3$

If $n = 3$, then equation (6) becomes:

$$\beta(t+1) = \frac{\frac{2}{3}w_1 - \frac{1}{3}w_2\beta(t) + w_2 - W_1(0)}{\frac{1}{3}w_3\beta(t)} - \frac{g(t)\theta S(t-1)\left\{1 - \frac{1}{2}\left(\frac{w_1 - W_0(1)}{\theta S(t-1)}\right)\right\}}{\frac{1}{3}w_3\beta(t)\left\{\beta(t+1) - [1-g(t)]\beta(t)\right\}} \tag{7}$$

From the definition of the term W, we have:

$$W_0(1) = \frac{1}{3}\left[w_1 + w_2\beta(t-1) + w_3\beta(t-1)\beta(t)\right] \tag{8}$$

$$W_1(0) = \frac{1}{2}\left[w_2 + w_3\beta(t)\right] \tag{9}$$

Substituting (8)-(9) into (7) and multiplying the numerator and denominator of the right–hand–side expression by the least common multiple = 6, one obtains:

$$\beta(t+1) = \frac{4w_1 + 3w_2 - [2w_2 + 3w_3]\beta(t)}{2w_3\beta(t)} - \frac{g(t)\left\{6\theta S(t-1) - 2w_1 + w_2\beta(t-1) + w_3\beta(t-1)\beta(t)\right\}}{2w_3\beta(t)\beta(t+1) - 2w_3[1-g(t)]\beta(t)^2}$$

Collecting the $\beta(t+1)$ terms, one obtains the quadratic equation:

$$\begin{aligned} 2w_3\beta(t)\beta(t+1)^2 + \left\{[2w_2 + 3w_3]\beta(t) - 4w_1 - 3w_2 - 2w_3[1-g(t)]\beta(t)^2\right\}\beta(t+1) \\ +[1-g(t)]\beta(t)\left\{4w_1 + 3w_2 - [2w_2 + 3w_3]\beta(t)\right\} \\ +g(t)[6\theta S(t-1) - 2w_1 + w_2\beta(t-1) + w_3\beta(t-1)\beta(t)] &= 0 \end{aligned}$$

or

$$\begin{aligned} \beta(t+1)^2 + \left\{\lambda_2 + \frac{3}{2} - \left[2\lambda_1 + \frac{3}{2}\lambda_2\right]\beta(t)^{-1} - [1-g(t)]\beta(t)\right\}\beta(t+1) \\ +[1-g(t)]\left\{2\lambda_1 + \frac{3}{2}\lambda_2 - \left[\lambda_2 + \frac{3}{2}\right]\beta(t)\right\} + \frac{1}{2}g(t)\beta(t-1) \\ + g(t)\beta(t)^{-1}\left[\frac{1}{2}\lambda_2\beta(t-1) - \lambda_1 + \frac{3\theta S(t-1)}{w_3}\right] &= 0, \end{aligned} \tag{10}$$

where $\lambda_1 = \frac{w_1}{w_3}$ and $\lambda_2 = \frac{w_2}{w_3}$. Equation (10) has solutions:

$$\beta(t+1)^+,\ \beta(t+1)^- = \frac{-B \pm [B^2 - 4C]^{\frac{1}{2}}}{2}$$

where

$$\begin{aligned} B &= a - b\beta(t)^{-1} - [1 - g(t)]\beta(t) \\ C &= b[1-g(t)] - a\beta(t)[1-g(t)] + \frac{g(t)\beta(t-1)}{2} + \frac{g(t)}{\beta(t)}\left[\frac{1}{2}\beta(t-1)\lambda_2 - \lambda_1 + \frac{3\theta S(t-1)}{w_3}\right] \end{aligned}$$

and

$$\begin{aligned} a &= \lambda_2 + \frac{3}{2} \\ b &= 2\lambda_1 + \frac{3}{2}\lambda_2 \end{aligned}$$

as reported in the text.

B.2 Case where $n = 4$

If $n = 4$, then equation (6) becomes:

$$\begin{aligned} \beta(t+2) &= \frac{\frac{3}{4}w_1 - \frac{1}{4}w_2\beta(t) - \frac{1}{4}w_3\beta(t)\beta(t+1) + w_2 - W_1(0) + w_3 - W_2(0)}{\frac{1}{4}w_4\beta(t)\beta(t+1)} \\ &\quad - \frac{g(t)\theta S(t-1)\left\{1 - \frac{2}{3}\left(\frac{w_1 - W_0(1)}{\theta S(t-1)}\right) - \frac{1}{3}\left(\frac{w_1 - W_0(2)}{\theta^2 S(t-2)}\right) - \frac{1}{2}\left(\frac{w_2 - W_1(1)}{\theta S(t-1)}\right)\right\}}{\frac{1}{4}w_4\beta(t)\beta(t+1)\left\{\beta(t+1) - [1-g(t)]\beta(t)\right\}} \end{aligned} \quad (11)$$

From the definition of the term W, we have:

$$W_0(1) = \frac{1}{4}[w_1 + w_2\beta(t-1) + w_3\beta(t-1)\beta(t) + w_4\beta(t-1)\beta(t)\beta(t+1)] \quad (12)$$

$$W_0(2) = \frac{1}{4}[w_1 + w_2\beta(t-2) + w_3\beta(t-2)\beta(t-1) + w_4\beta(t-2)\beta(t-1)\beta(t)] \quad (13)$$

$$W_1(0) = \frac{1}{3}[w_2 + w_3\beta(t) + w_4\beta(t)\beta(t+1)] \quad (14)$$

$$W_1(1) = \frac{1}{3}[w_2 + w_3\beta(t-1) + w_4\beta(t-1)\beta(t)] \quad (15)$$

$$W_2(0) = \frac{1}{2}[w_3 + w_4\beta(t)] \quad (16)$$

Substituting (12)-(16) into (11) and multiplying the numerator and denominator of the right–hand–side expression by the least common multiple = 12, one obtains the expression reported in the text:

$$\beta(t+2) = \frac{a - b\beta(t) - c\beta(t)\beta(t+1)}{d\beta(t)\beta(t+1)} - \frac{g(t)\theta S(t-1)X}{d\beta(t)\beta(t+1)^2 - d[1-g(t)]\beta(t)^2\beta(t+1)}$$

where X denotes the denominator of the aggregate savings function $S(t)$ when $n = 4$:

$$
\begin{aligned}
X &= 12 - \frac{1}{\theta S(t-1)}[e - f\beta(t-1) - g\beta(t-1)\beta(t) - h\beta(t-1)\beta(t)\beta(t+1)] \\
&\quad - \frac{1}{\theta^2 S(t-2)}[3w_1 - w_2\beta(t-2) - w_3\beta(t-2)\beta(t-1) - w_4\beta(t-2)\beta(t-1)\beta(t)]
\end{aligned}
$$

and $a = 9w_1 + 8w_2 + 6w_3$, $b = 3w_2 + 4w_3 + 6w_4$, $c = 3w_3 + 4w_4$, $d = 3w_4$, $e = 6w_1 + 4w_2$, $f = 2w_2 + 2w_3$, $g = 2w_3 + 2w_4$, and $h = 2w_4$.

B.3 First order representation of the dynamical system under learning

The dynamical system under learning (6), can be written more compactly as a first–order system. The first–order representation is more useful than equation (6) for our numerical analysis of the stability of the learning system.

We begin by deriving a first order expression for the gain parameter, g. From equation (2) we have:

$$
\begin{aligned}
g(t+1)^{-1} &= \frac{\sum_{s=1}^{t+1} P(s-1)^2}{P(t)^2} \\
&= \left[\frac{\sum_{s=1}^{t} P(s-1)^2 + P(t)^2}{P(t-1)^2}\right]\left[\frac{P(t-1)}{P(t)}\right]^2 \\
&= g(t)^{-1}\left[\frac{P(t-1)}{P(t)}\right]^2 + 1 \qquad (17)
\end{aligned}
$$

From equation (3) we know that:

$$
\frac{P(t-1)}{P(t)} = \left[\frac{\theta S(t-1)}{S(t)}\right]^{-1} \qquad (18)
$$

Substituting (18) into (17) and rearranging, we obtain a first–order law of motion for the gain:

$$
g(t+1) = \left\{g(t)^{-1}\left[\frac{\theta S(t-1)}{S(t)}\right]^{-2} + 1\right\}^{-1} \qquad (19)
$$

Equation (19), together with equation (1) and the recursive aggregate savings function derived in Appendix A (equation (12)), completely characterize the dynamics of the n–period overlapping generations model under least squares learning. All three of these equations were used in the least squares simulations and numerical analysis reported in section 4 of the paper.

Rewrite equations (19), (1), and the recursive aggregate savings function as the general functions:

$$
\begin{aligned}
\beta(t+n-2) &= f[\beta(t+n-3), .., \beta(t-n+2), g(t), S(t-1), .., S(t-n+2)] \qquad (20) \\
g(t+1) &= h[g(t), S(t), S(t-1)] \qquad (21) \\
S(t) &= k[\beta(t+n-2), .., \beta(t-n+2), S(t-1), .., S(t-n+2)] \qquad (22)
\end{aligned}
$$

Lag equation (22) by one period:

$$
S(t-1) = k[\beta(t+n-3), .., \beta(t-n+1), S(t-2), .., S(t-n+1)] \qquad (23)
$$

We envision a first-order system consisting of equations (20),(21) and (23). However, equations (20) and (21) contain the variables $S(t)$ and $S(t-1)$ and therefore do not conform to desired first-order representation. It is possible to eliminate these terms through a series of substitutions. The first step is to substitute for $S(t)$ and $S(t-1)$ in (20) and (21) using (22) and (23). This substitution yields the system:

$$\begin{aligned}
\beta(t+n-2) &= f\{\beta(t+n-3),..,\beta(t-n+2),g(t),\\
&\quad k[\beta(t+n-3),..,\beta(t-n+1),S(t-2),..,S(t-n+1)],\\
&\quad S(t-2),..,S(t-n+2)\} && (24)\\
g(t+1) &= h\{g(t),k[\beta(t+n-2),..,\beta(t-n+2),S(t-1),..,S(t-n+2)],\\
&\quad k[\beta(t+n-3),..,\beta(t-n+1),S(t-2),..,S(t-n+1)]\} && (25)\\
S(t-1) &= k[\beta(t+n-3),..,\beta(t-n+1),S(t-2),..,S(t-n+1)] && (26)
\end{aligned}$$

Next, substitute for the remaining $S(t-1)$ term in equation (25), and obtain the system:

$$\begin{aligned}
\beta(t+n-2) &= f\{\beta(t+n-3),..,\beta(t-n+2),g(t),\\
&\quad k[\beta(t+n-3),..,\beta(t-n+1),S(t-2),..,S(t-n+1)],\\
&\quad S(t-2),..,S(t-n+2)\} && (27)\\
g(t+1) &= h\{g(t),k[\beta(t+n-2),..,\beta(t-n+2),\\
&\quad k[\beta(t+n-3),..,\beta(t-n+1),S(t-2),..,S(t-n+1)],\\
&\quad S(t-2),..,S(t-n+2)],\\
&\quad k[\beta(t+n-3),..,\beta(t-n+1),S(t-2),..,S(t-n+1)]\} && (28)\\
S(t-1) &= k[\beta(t+n-3),..,\beta(t-n+1),S(t-2),..,S(t-n+1)] && (29)
\end{aligned}$$

The first-order representation can be completed by adding a number of identity mappings to the system (27)-(29). The entire first-order system is written as:

$$\begin{aligned}
\beta(t+n-1) &= \ell_1\left[\beta(t-n+1),...,\beta(t+n-3),g(t),S(t-n+1),...,S(t-2)\right]\\
\beta(t+n-3) &= \beta(t+n-3)\\
&\vdots\\
\beta(t-n+2) &= \beta(t-n+2)\\
g(t+1) &= \ell_2\left[\beta(t-n+1),...,\beta(t+n-3),g(t),S(t-n+1),...,S(t-2)\right]\\
S(t-1) &= \ell_3\left[\beta(t-n+1),...,\beta(t+n-3),S(t-n+1),...,S(t-2)\right]\\
S(t-2) &= S(t-2)\\
&\vdots\\
S(t-n+2) &= S(t-n+2)
\end{aligned}$$

This is the same system reported in the text.

B.4 Case where $n = 3$

If $n = 3$, the first-order dynamical system under least squares learning is written as:

$$\begin{aligned}
\beta(t+1) &= \ell_1[\beta(t),\beta(t-1),\beta(t-2),g(t),S(t-2)]\\
\beta(t) &= \beta(t)\\
\beta(t-1) &= \beta(t-1)\\
g(t+1) &= \ell_2[\beta(t),\beta(t-1),\beta(t-2),g(t),S(t-2)]\\
S(t-1) &= \ell_3[\beta(t),\beta(t-1),\beta(t-2),S(t-2)]
\end{aligned}$$

B.5 Case where $n = 4$

If $n = 4$, the first–order dynamical system under least squares learning is written as:

$$\begin{aligned}
\beta(t+2) &= \ell_1[\beta(t+1), \beta(t), \beta(t-1), \beta(t-2), \beta(t-3), g(t), S(t-2), S(t-3)] \\
\beta(t+1) &= \beta(t+1) \\
\beta(t) &= \beta(t) \\
\beta(t-1) &= \beta(t-1) \\
\beta(t-2) &= \beta(t-2) \\
g(t+1) &= \ell_2[\beta(t+1), \beta(t), \beta(t-1), \beta(t-2), \beta(t-3), g(t), S(t-2), S(t-3)] \\
S(t-1) &= \ell_3[\beta(t+1), \beta(t), \beta(t-1), \beta(t-2), \beta(t-3), S(t-2), S(t-3)] \\
S(t-2) &= S(t-2)
\end{aligned}$$

The dynamical systems for the $n = 3$ and $n = 4$ cases are useful in understanding how the system behaves in higher dimensions. For each additional period added to the model, the dynamical system is augmented by an additional three equations. These three equations consist of identity mappings for $\beta(t+n-3)$, $\beta(t-n+2)$, and $S(t-n+2)$. When we evaluate the Jacobian associated with the dynamical system at the monetary steady state, the characteristic polynomial will increase by an order of three for every additional period added to the model. Hence, the three period model will have 5 roots, the four period model will have 8 roots, the five period model will have 11 roots, and so on. The dimension of the first order dynamical system under least squares learning for general n is $3n - 4$.

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Figure one.

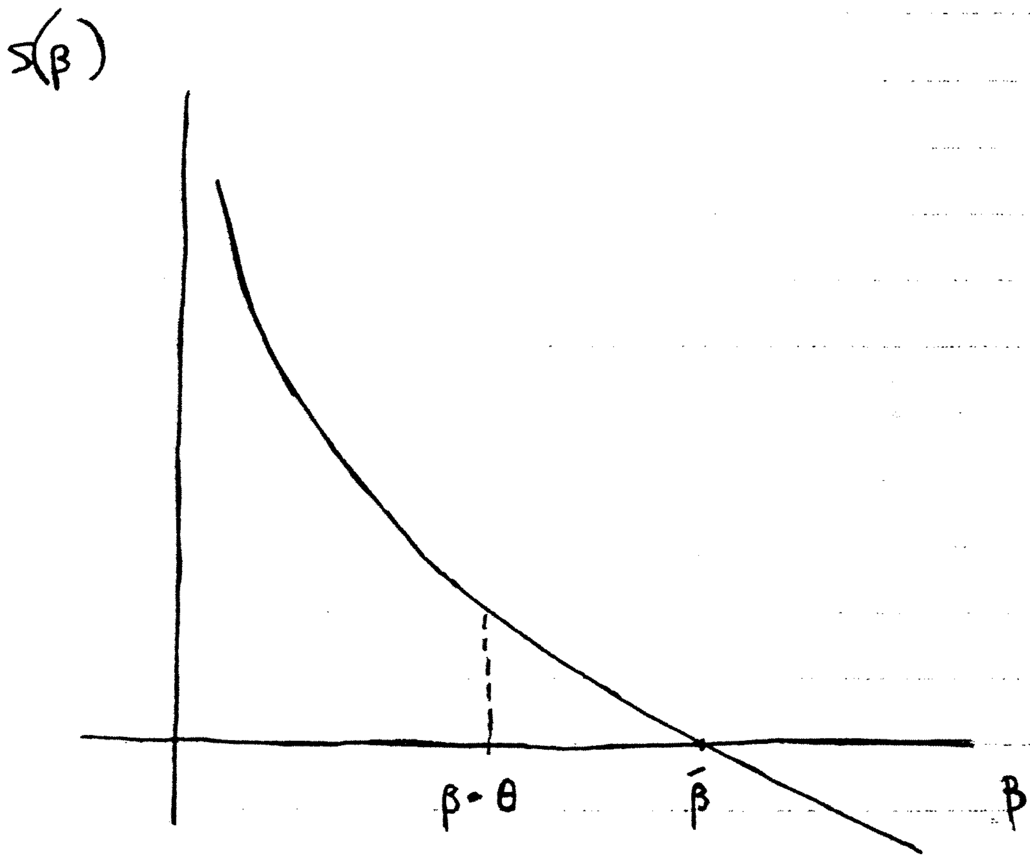


Figure two

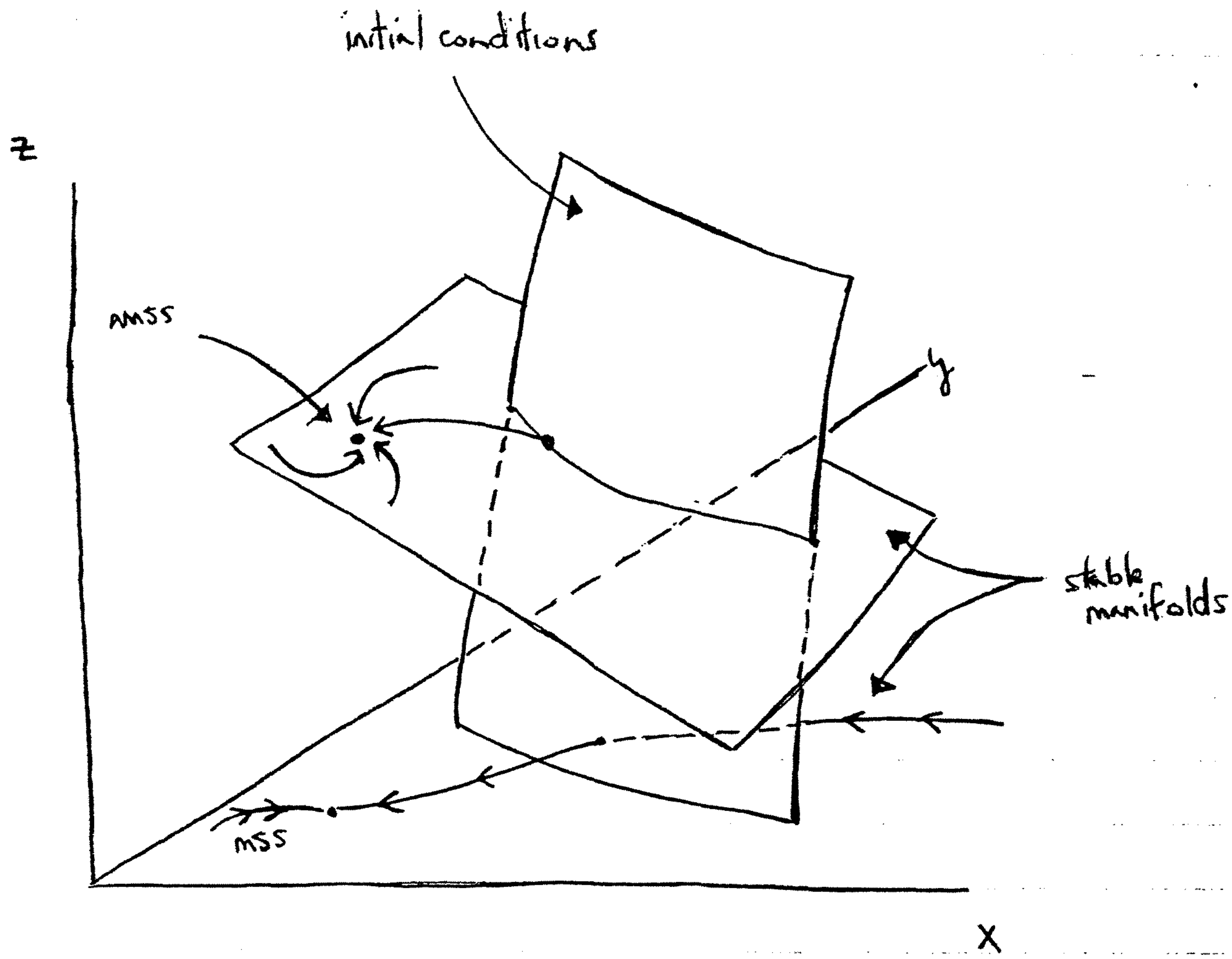


Fig 3.

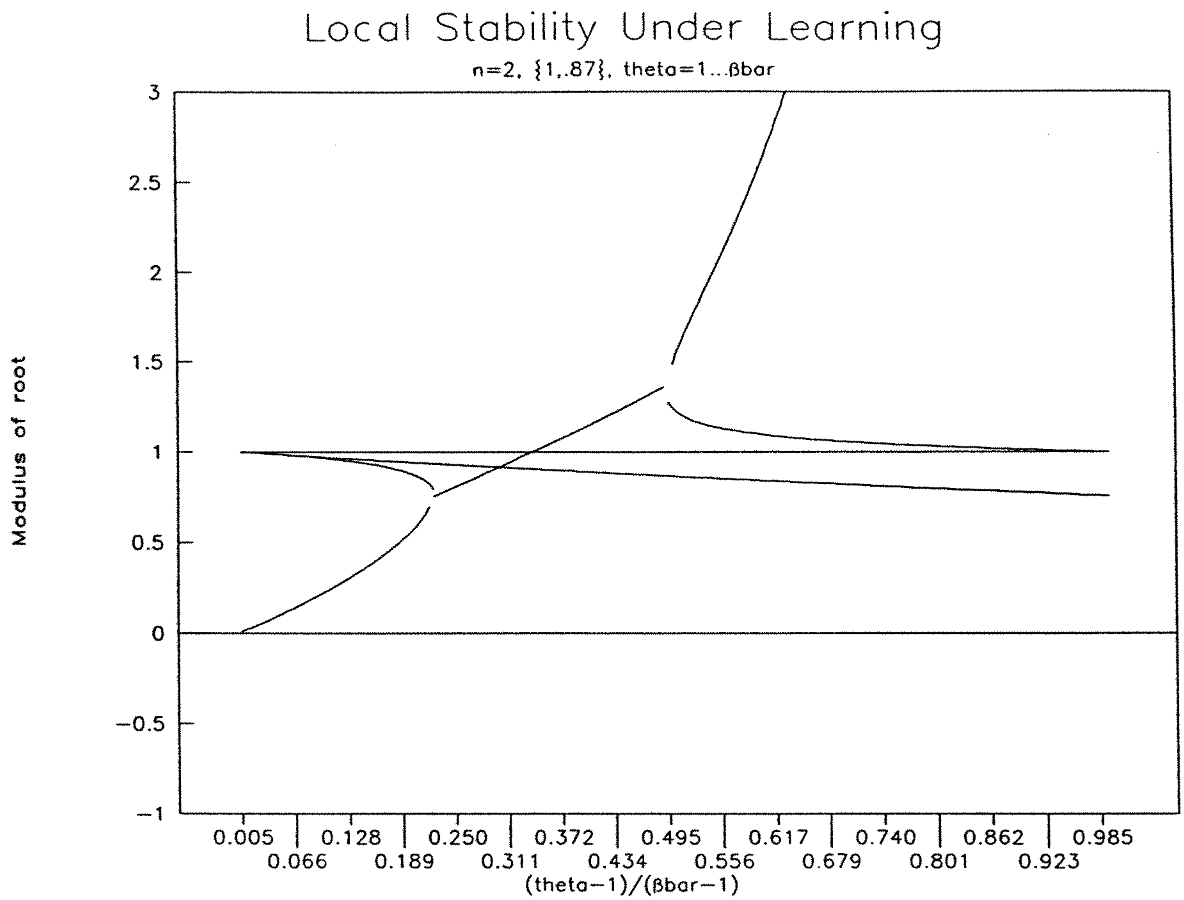
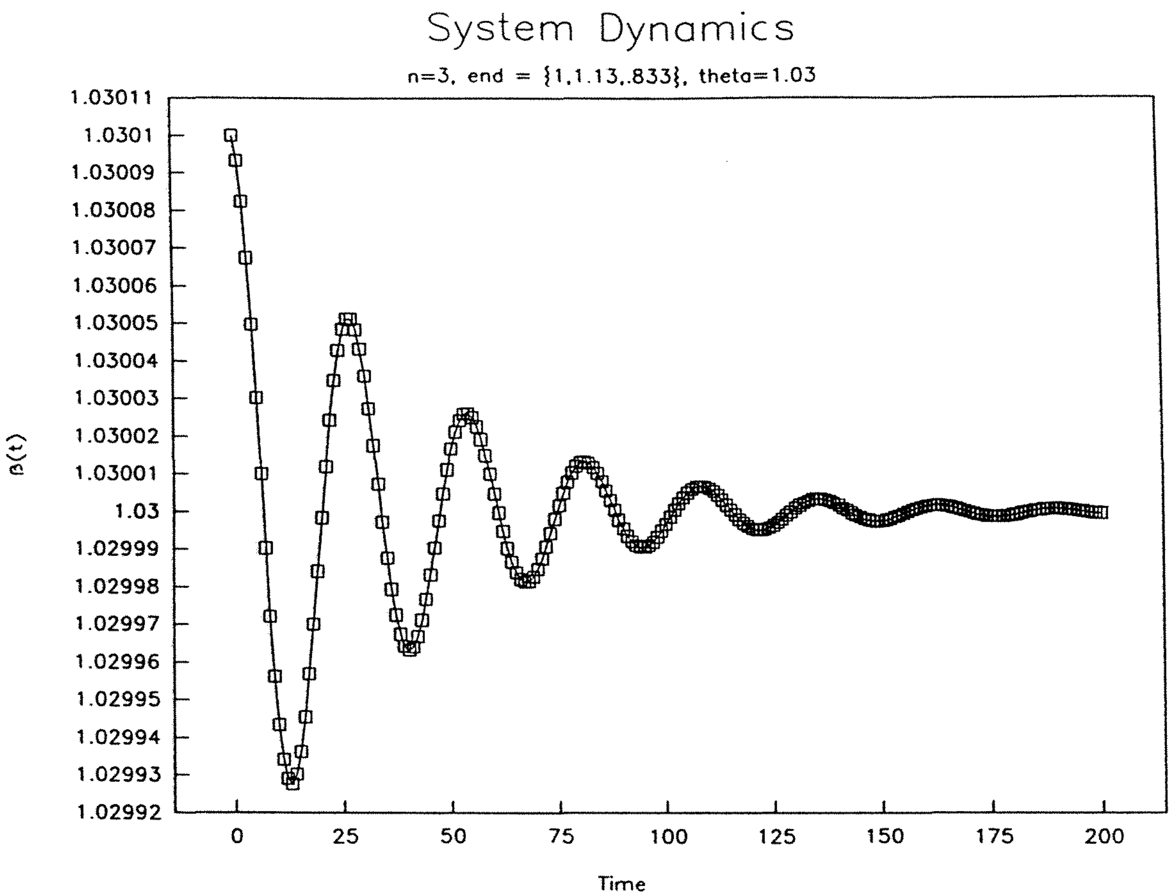

Local Stability Under Learning
n=2, {1,.87}, theta=1...βbar
Modulus of root
3
2.5
2
1.5
1
0.5
0
−0.5
−1
0.005
0.066
0.128
0.189
0.250
0.311
0.372
0.434
0.495
0.556
0.617
0.679
0.740
0.801
0.862
0.923
0.985
(theta−1)/(βbar−1)

Fig. 4



System Dynamics
n=3, end = {1,1.13,.833}, theta=1.03
1.03011
1.0301
1.03009
1.03008
1.03007
1.03006
1.03005
1.03004
1.03003
1.03002
1.03001
1.03
1.02999
1.02998
1.02997
1.02996
1.02995
1.02994
1.02993
1.02992
B(t)
0
25
50
75
100
125
150
175
200
Time

Fig. 5

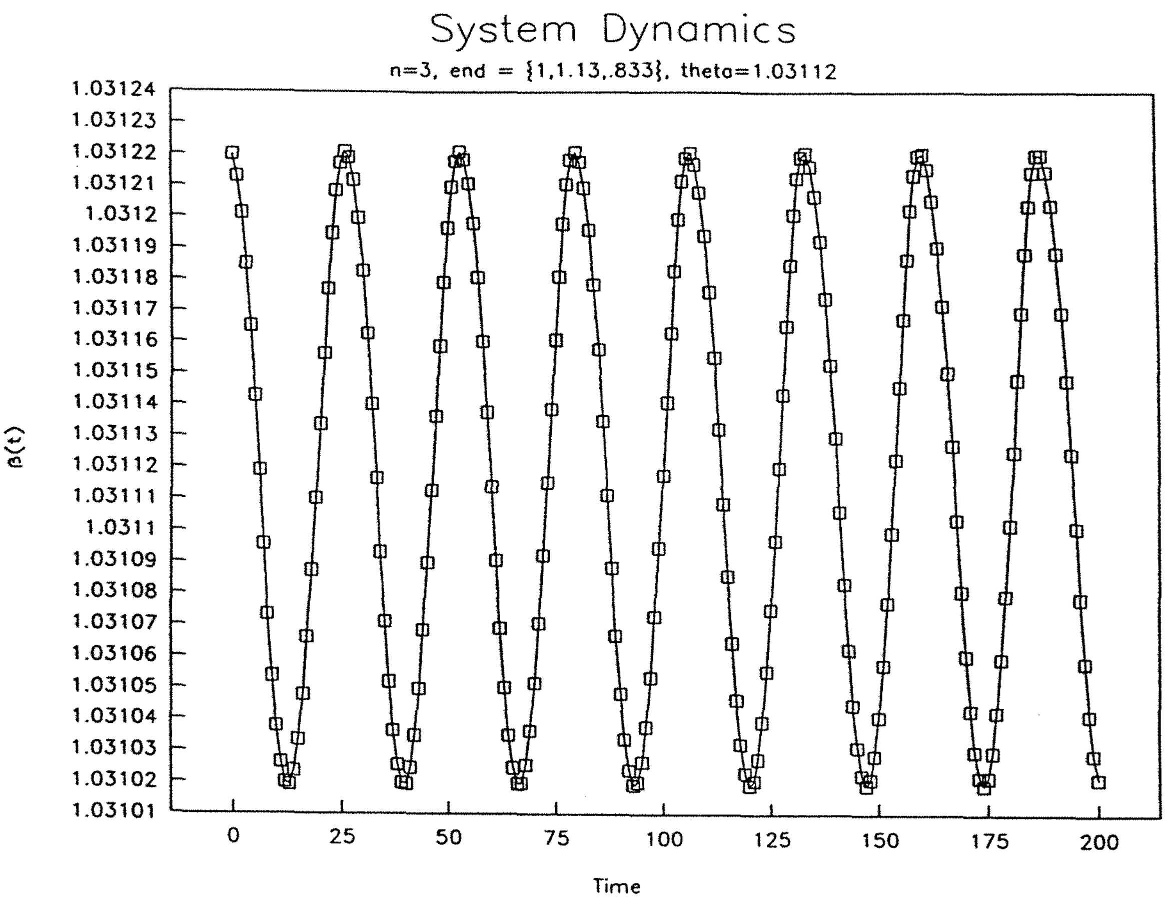
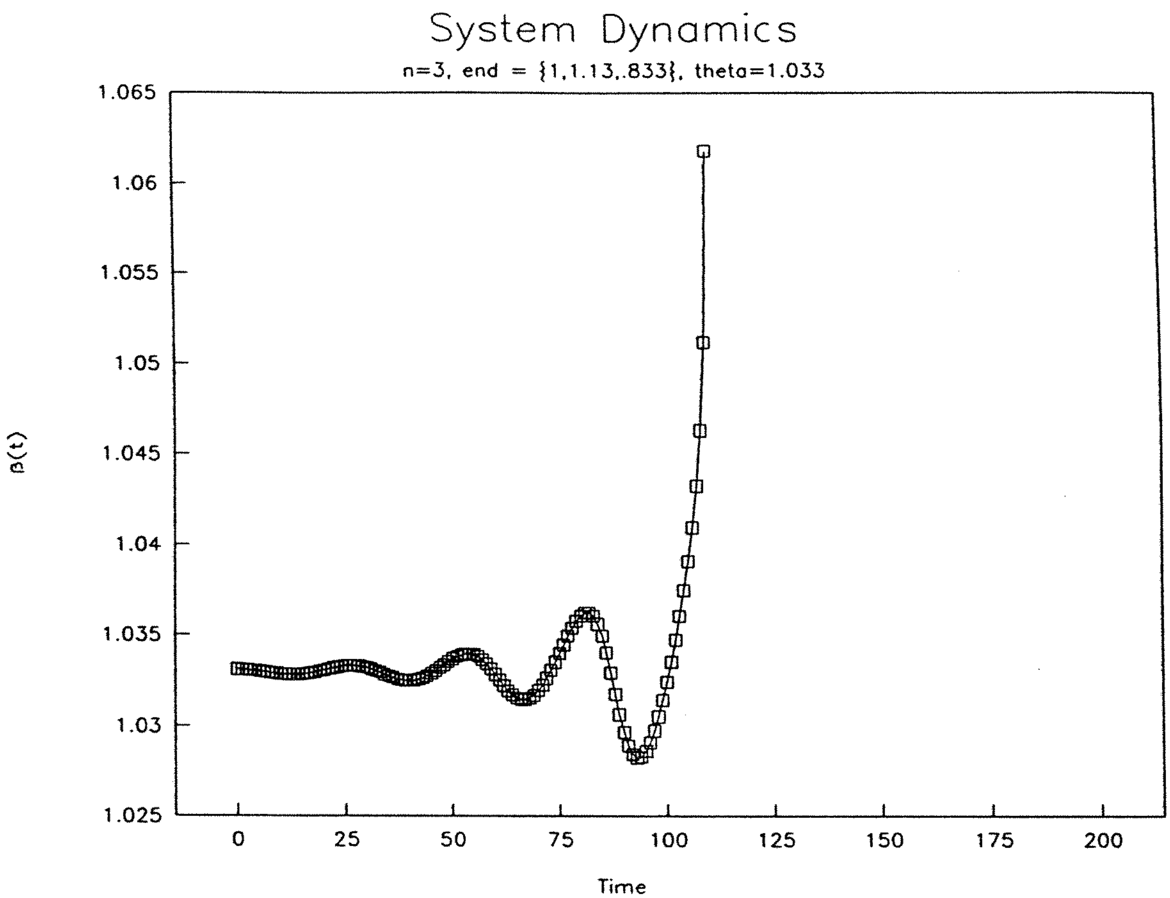
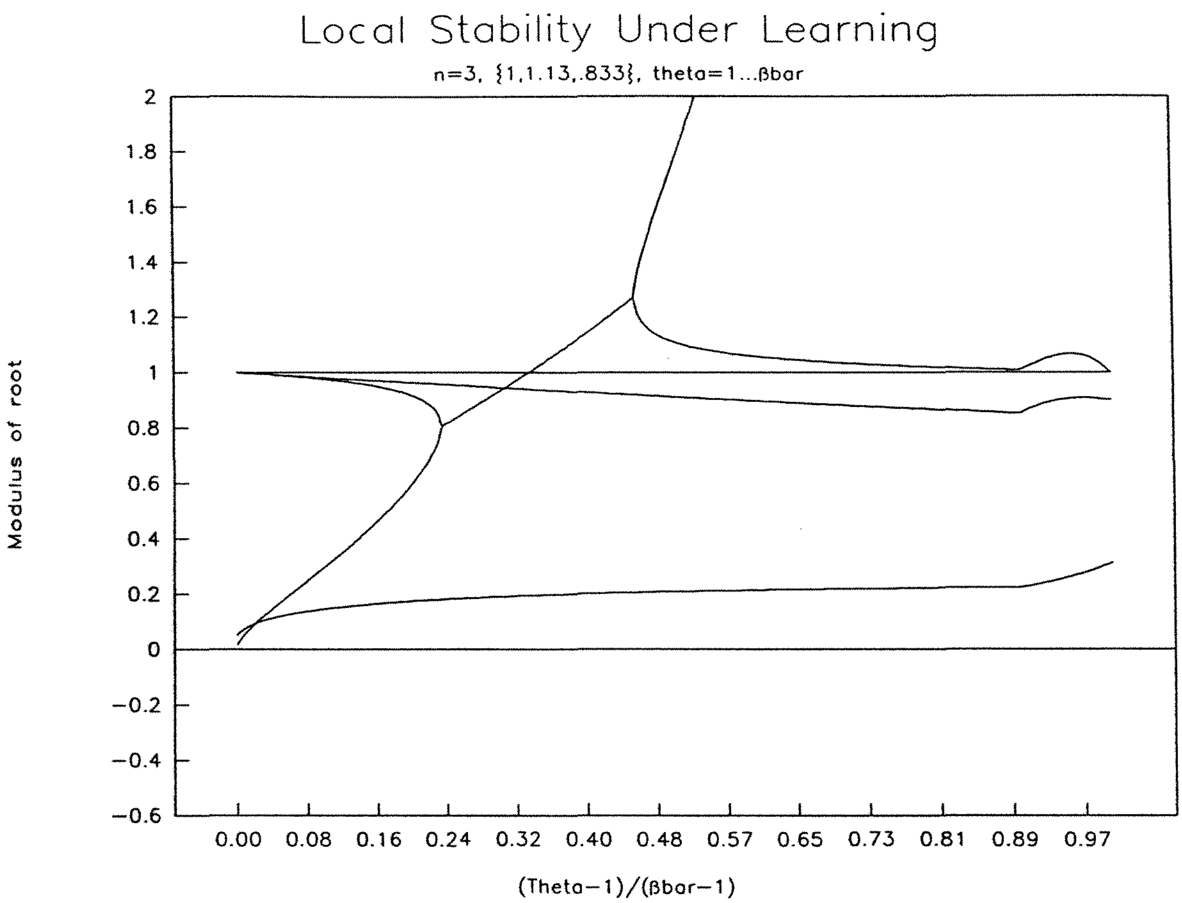

System Dynamics
n=3, end = {1,1.13,.833}, theta=1.03112
β(t)
1.03124
1.03123
1.03122
1.03121
1.0312
1.03119
1.03118
1.03117
1.03116
1.03115
1.03114
1.03113
1.03112
1.03111
1.0311
1.03109
1.03108
1.03107
1.03106
1.03105
1.03104
1.03103
1.03102
1.03101
0
25
50
75
100
125
150
175
200
Time

Fig. 6



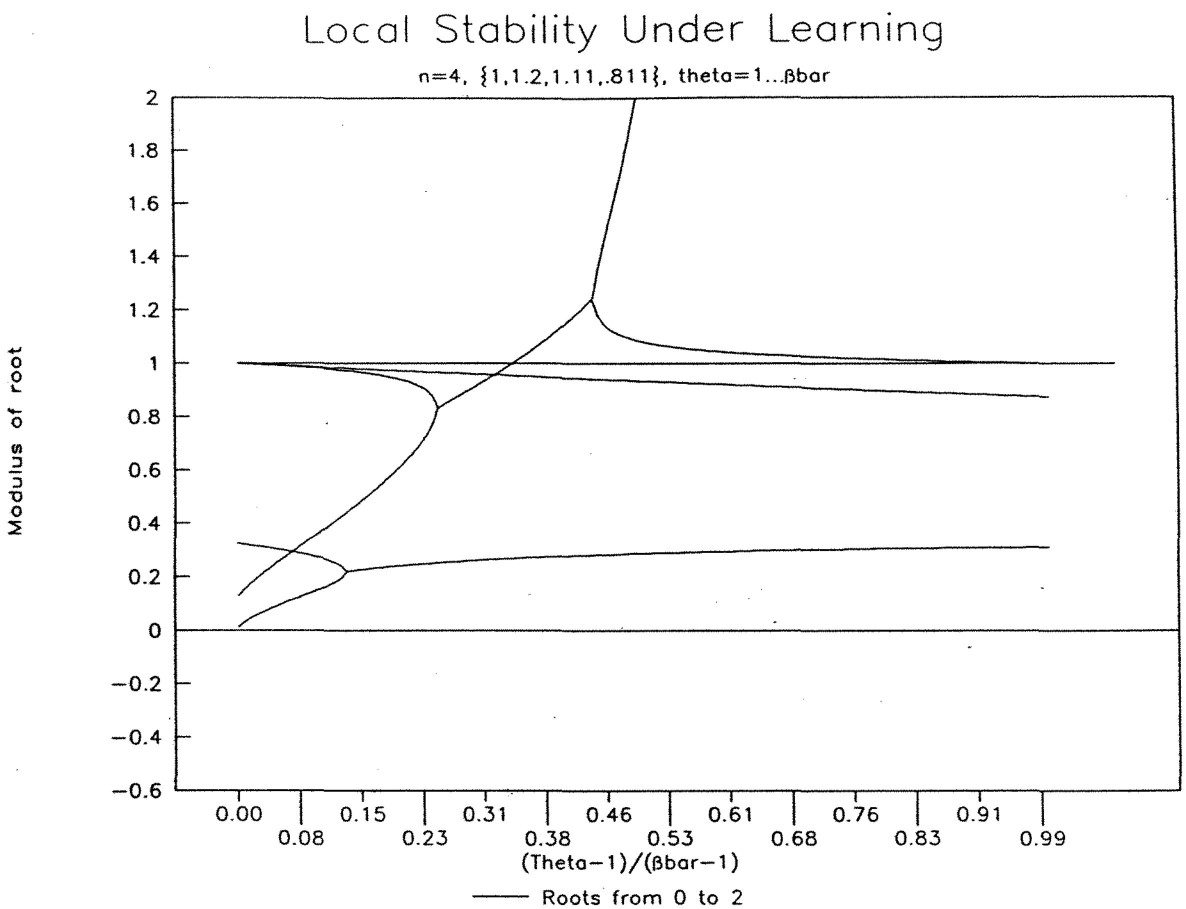
System Dynamics
n=3, end = {1,1.13,.833}, theta=1.033
1.065
1.06
1.055
1.05
1.045
1.04
1.035
1.03
1.025
B(t)
0
25
50
75
100
125
150
175
200
Time

Fig. 7



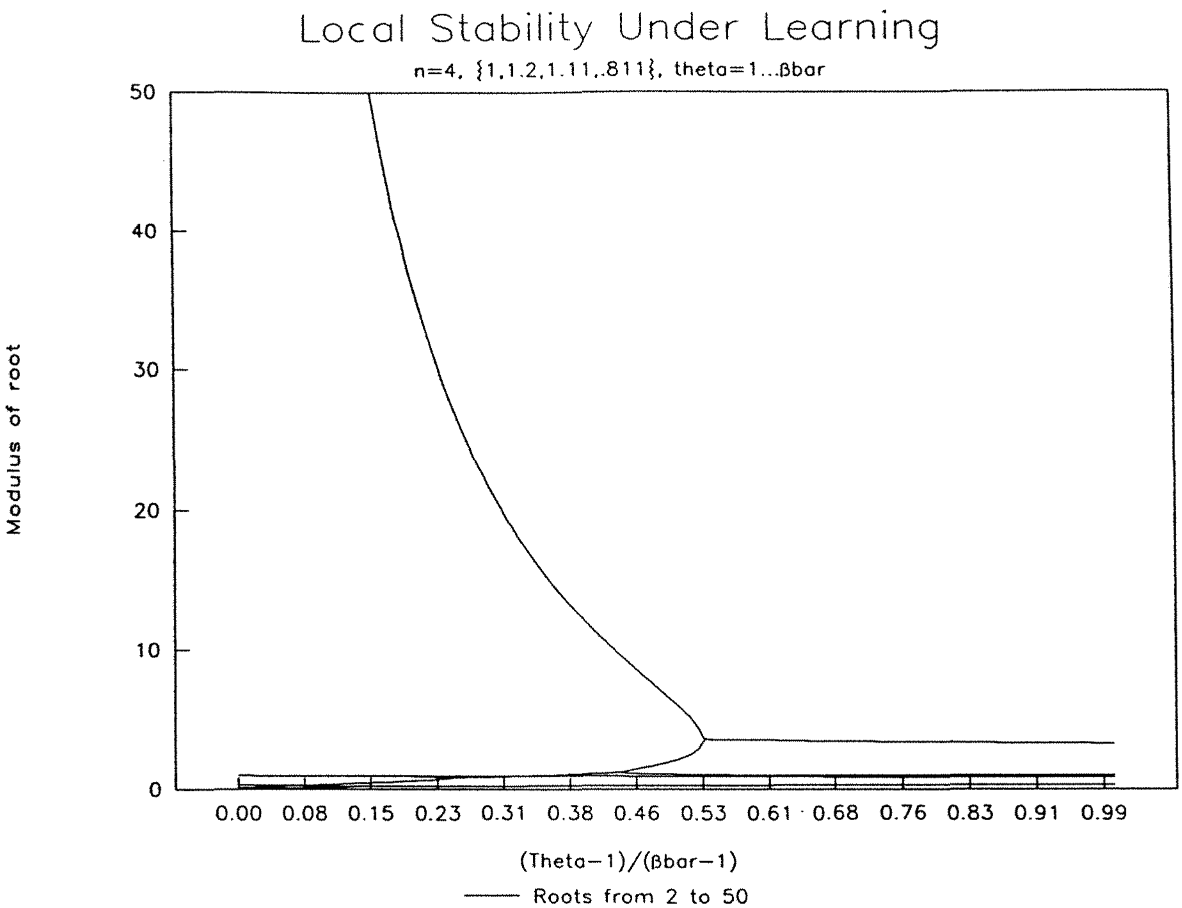
Local Stability Under Learning
n=3, {1,1.13,.833}, theta=1...βbar
Modulus of root
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
0.00 0.08 0.16 0.24 0.32 0.40 0.48 0.57 0.65 0.73 0.81 0.89 0.97
(Theta−1)/(βbar−1)

Fig. 8



Local Stability Under Learning
n=4, {1,1.2,1.11,.811}, theta=1...βbar
Modulus of root
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
0.00
0.08
0.15
0.23
0.31
0.38
0.46
0.53
0.61
0.68
0.76
0.83
0.91
0.99
(Theta−1)/(βbar−1)
Roots from 0 to 2

Fig. 9



Local Stability Under Learning
n=4, {1,1.2,1.11,.811}, theta=1...βbar
50
40
30
20
10
0
Modulus of root
0.00 0.08 0.15 0.23 0.31 0.38 0.46 0.53 0.61 0.68 0.76 0.83 0.91 0.99
(Theta−1)/(βbar−1)
Roots from 2 to 50

Fig. 10

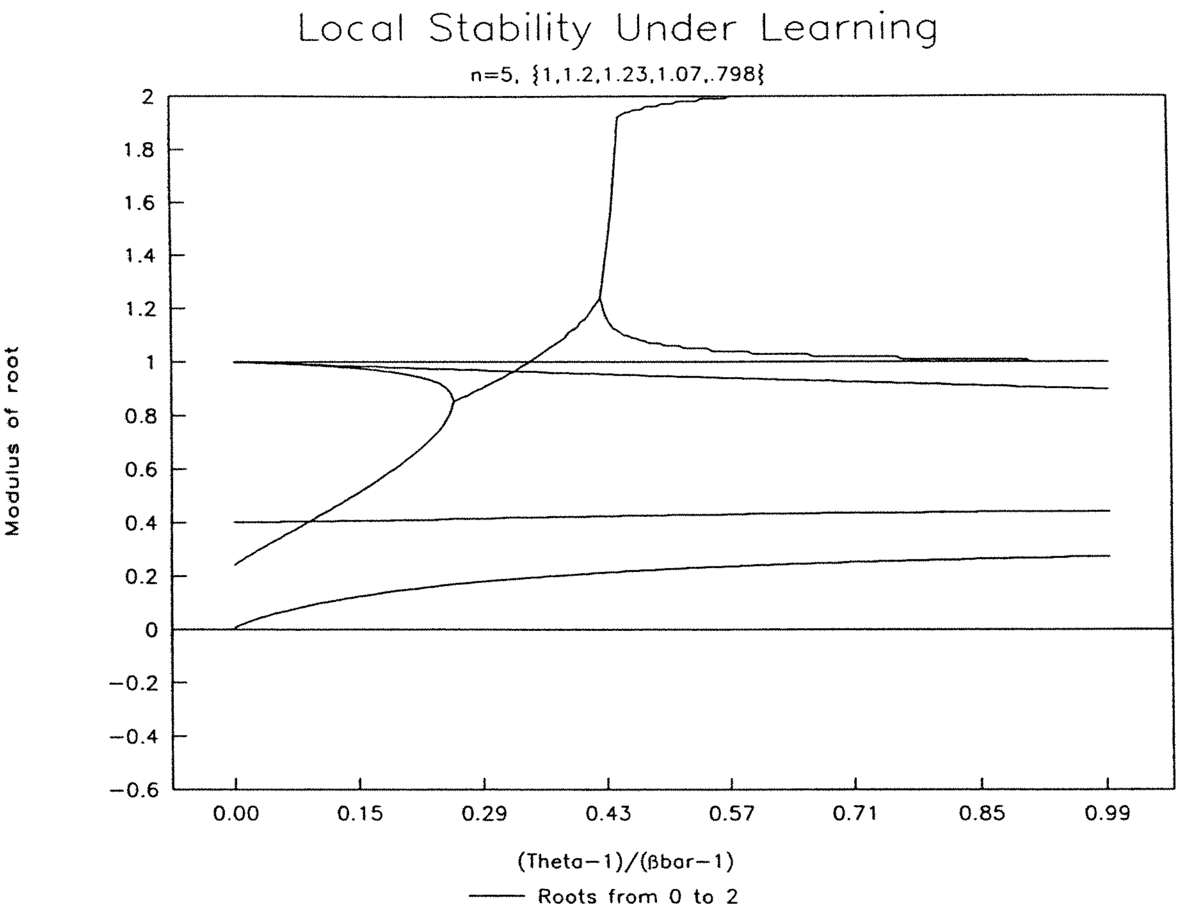
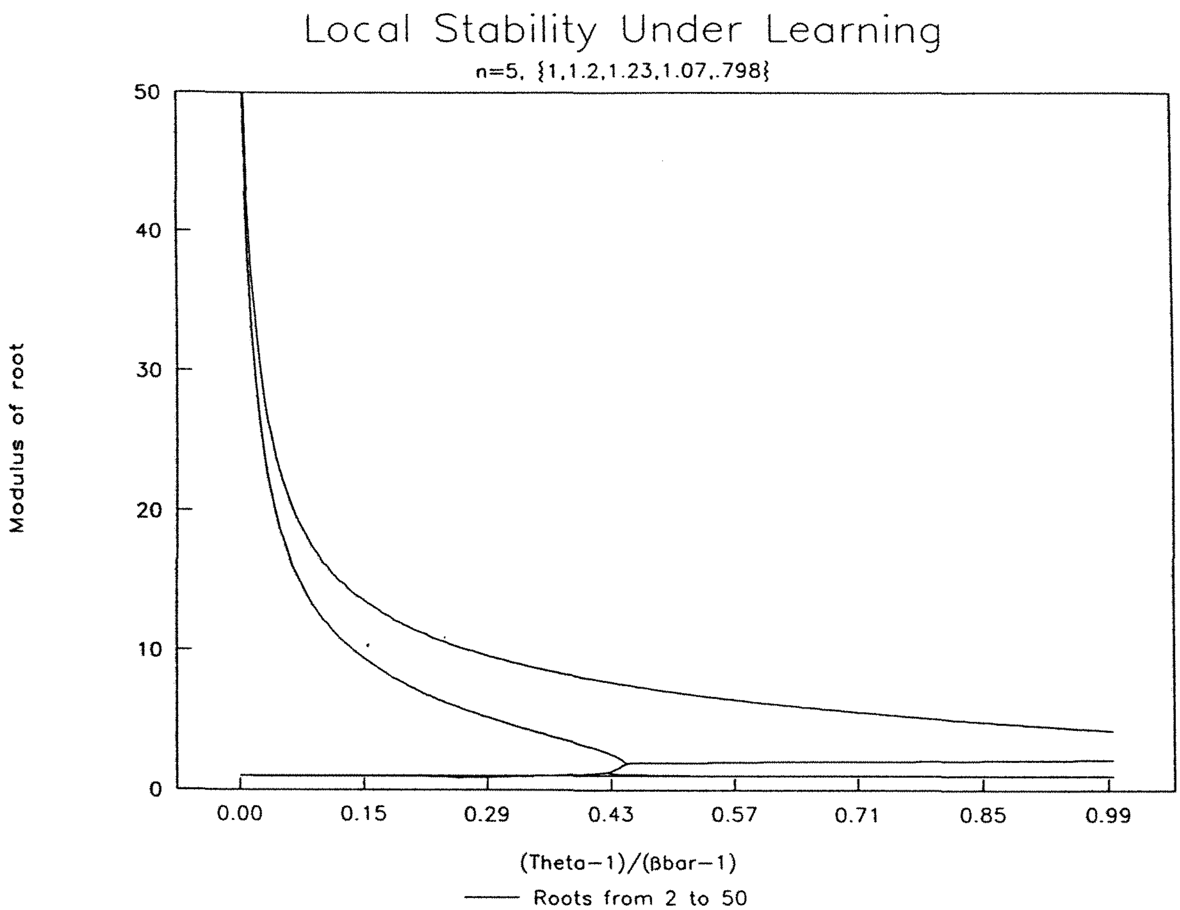

Local Stability Under Learning
n=5, {1,1.2,1.23,1.07,.798}
Modulus of root
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0.00
0.15
0.29
0.43
0.57
0.71
0.85
0.99
(Theta-1)/(βbar-1)
Roots from 0 to 2

Fig. 11


Local Stability Under Learning
n=5, {1,1.2,1.23,1.07,.798}
Modulus of root
50
40
30
20
10
0
0.00
0.15
0.29
0.43
0.57
0.71
0.85
0.99
(Theta−1)/(βbar−1)
Roots from 2 to 50