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Economy with Inventories**

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Working Paper 2005-046A
<http://research.stlouisfed.org/wp/2005/2005-046.pdf>

June 2005

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St. Louis, MO 63102

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The Multiplier:

A General Equilibrium Analysis of Multi-Stage-Fabrication
Economy with Inventories^{*}

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First Version: August, 2003

This Version: March, 2004

Abstract

This paper provides a general equilibrium multi-stage production model to explain the co-existence and co-movement of output- and input-inventories. The model offers a neoclassical perspective on the propagation mechanism of demand uncertainty. It reveals that uncertainty in demand at downstream can be transmitted and amplified towards upstream by inventory investment at all stages of production via input-output linkages, leading to a chain-multiplier effect on aggregate output and employment. The model is capable of explaining several long-standing puzzles of the business cycle associated with inventories.

JEL Classification: E22, E23, E32.

^{*}I would like to thank Karl Shell, Gaetano Antinolfi, Steve Fazzari, Patrick Pintus and seminar participants at Cornell/Penn State Macro conference, Marseilles University in France, and Washington University at St. Louis for comments.

Keywords: Inventory Investment, Input Inventory, Multi-Stage-Fabrication, Multiplier, Demand Uncertainty, Business Cycle.

1. Introduction

Given the sheer magnitude of volatility in inventory investment and its disproportionately large contribution to output fluctuations, it is fair to say that understanding inventory behavior constitutes the ultimate frontier of business cycle research.¹ There are many long-standing puzzles associated with inventory behavior. The most prominent among them are: 1) Production is more variable than sales; 2) Input order is more variable than input usage; and 3) Input inventories – defined here as raw materials and work-in-progress – are more volatile than output inventories (see Blinder, 1986, Feldstein and Auerbach, 1976, and more recently Humphreys, Maccini and Schuh 2001). While there is no shortage in supply of explanations for the first puzzle, the literature on resolving the second and third puzzle is remarkably thin, despite the fact that more than two thirds of total inventory investment in the US is contributed by input inventories (e.g., see Blinder and Maccini, 1991).

These three prominent features of production and inventory behavior are puzzling because we usually think that a firm has not only an incentive to smooth production in order to reduce production costs when facing uncertainty in output sales, but also an incentive to smooth input production/orders when factor-demand/input-usage is also uncertain. Furthermore, because output inventories are presumably more liquid and hence more effective than input inventories in meeting final demand for finished goods, it may not be profit maximizing for firms to keep most inventories in the form of raw materials and work-in-progress rather than in the form of finished output.

This paper conjectures that time lags involved in production/delivery at each stage of the input-output chain of production could be the prime explanation for these puzzles. In any economy the ultimate goal of production is always to meet final consumption demand, regardless of how many intermediate stages there are or how “round-about” the production process is. Hence, uncertainty in final consumption demand at the downstream is presumably one of the most fundamental uncertainties affecting firms and industries at all stages of production in the economy. Because of production/delivery lags, however, output must be produced in advance before demand uncertainty is resolved and input materials

¹Inventory investment accounts for only less than one percent of GNP in mean yet the drop in inventory investment has accounted for 87 percent of the drop in GNP during the average postwar recession in the United States (e.g., see Blinder and Maccini, 1991).

must be ordered in advance before production can take place. To avoid stockouts, it is conceivable that profit-seeking firms opt to hold inventories at each stage of the production process in order to fully guard against demand uncertainty. This results in a motive for inventory investment in both output and input materials. Such precautionary inventory investment may cause not only production to be more volatile than sales on the output side, but also the order of materials to be more volatile than the usage on the input side. Hence, uncertainty in final demand at the downstream is transmitted backwards to upstream both within and outside firms via the multi-stage fabrication and input-output linkage. And more importantly, during the transmission, the uncertainty in demand may be even magnified repeatedly at each link of the input-output production chain, as the input demand at downstream is also the output supply at upstream.

I set up a general-equilibrium, rational-expectations, multi-stage-fabrication model of production to put this conjecture to scrutiny. I show that with production (or delivery) lags at each stage of production, uncertainty in demand at the downstream indeed generates a motive for holding both output and input inventories by all upstream firms in order to avoid possible stockouts. Such a motive is sufficient for explaining the existence of both finished-goods inventories and intermediate-goods inventories at all stages of production observed in the actual economy. Furthermore, if shocks to final consumption demand at the downstream are serially correlated, then output production/input ordering become more variable than output sales/input usage, which causes the volatility of economic activities to increase towards upstream firms. Consequently, the variance in final consumption demand is not only propagated backwards to upstream industries via input-output linkages, but also amplified repeatedly along the production chain. This explains why in the US the variance of production at upstream industries is usually about 50 percent larger than the variance of production at downstream industries.²

It has long been well known that input inventories are far larger and more volatile than finished goods inventories (e.g., see Feldstein and Auerbach, 1976). Despite this, the vast bulk of theoretical work in the inventory literature has

²For example, according to monthly US production data for manufacturing sector (1960:01 - 1995:12), the variance of growth rate of production for raw materials is 0.00014 and that for intermediate goods is 0.00009, implying a variance ratio of 1.6. Similarly, according to monthly US data for housing sector (1968:01 - 1996:08), the variance of housing starts in total units is 130321 and the variance of housing completion in total units is 80770, implying a variance ratio of 1.6.

focused almost exclusively on finished goods inventories.³ As a consequence of neglecting input inventories, the role of inventories in amplifying and propagating the business cycle has not been well understood. This criticism of the inventory literature has also been raised recently by Humphreys, Maccini and Schuh (2001).

Following the analysis of Humphreys, Maccini and Schuh (2001), this paper takes a step further towards understanding and modeling input- and output-inventory behavior simultaneously in general equilibrium. By extending the partial equilibrium, single-stage fabrication inventory model of Reagan (1982), Able (1985), and Kahn (1987) to a general equilibrium, multi-stage fabrication model, which features separate decisions to order, use, and stock input materials and to produce, sell, and store finished goods, this paper provides a theoretical foundation for the important empirical works of Feldstein and Auerbach (1976) and Ramey (1989), and especially the more recent work of Humphreys, Maccini and Schuh (2001). In doing so, it offers not only a model of input-output inventories, but also a neoclassical perspective on the theory of aggregate demand with respect to the propagation of the business cycle. The general equilibrium approach takes into account any potential feedbacks between inventories and competitive market prices, hence it can avoid possible distortions in production and inventory behavior that may arise in partial equilibrium models. The neoclassical model reveals that small disturbances to final consumption demand can generate potentially large aggregate fluctuations in output and employment via a chain-multiplier effect induced by precautionary inventory investment. Inventory investment can thus appear to be highly destabilizing. However, since the multiplier mechanism reflects optimal responses of a perfectly functioning market economy towards demand uncertainty, it suggests that government interventions attempting to reduce the volatility of aggregate output via monetary or fiscal policies may prove counterproductive when such policies (e.g., the Taylor rule) are based solely on output volatility without looking into the causes behind.⁴

³See, e.g., Abel (1985), Amihud and Mendelson (1983), Bils and Kahn (2000), Blinder (1986), Blinder and Maccini (1991), Kahn (1987), Karlin and Carr (1962), Maccini and Zabel (1996), Reagan (1982), and Scheinkman and Schechtman (1983), among others. This is also true for much empirical work on inventories, most notably, see Blanchard (1983), Eichenbaum (1984, 1989), Ramey (1991) and West (1986).

⁴This, however, by no means implies that in the real world all government interventions are necessarily undesirable since market failures may exacerbate the neoclassical multiplier effect of inventories. But the neoclassical theory does show the potential danger of policy interventions based solely on output volatilities.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 derives equilibrium decision rules. Section 4 analyzes the empirical implications of the model for inventory fluctuations and the business cycle. Section 5 concludes the paper.

2. A General Equilibrium Model of Input and Output Inventories

Consider an economy similar to that described by Humphreys, Maccini, and Schuh (2001), in which most firms produce goods in stages. A typical firm orders input materials from an upstream supplier and uses them to produce intermediate goods, which are combined with other factor inputs to produce finished goods. Many firms sell their finished goods to downstream firms as input materials. Figure 1 presents a schematic illustration of the model (which is a near replication of the diagram in Humphreys, Maccini, and Schuh, 2001). The diagram focuses on flows through the multi-stage-production process employed by a social planner to transform raw materials (or labor) into intermediate goods inventories (or work-in-progress) and then further into output inventories (finished goods) to meet final demand of consumer (or another downstream firm). More precisely, at the beginning of a period, say $t - 2$, based on expected future demand for finished goods (c) the planner orders raw materials (n) in order to produce intermediate goods (y_2). Since production/delivery takes one period, y_2 is not available until period $t - 1$. In this next period, based on updated information about final demand (c) the planner decides how much of the intermediate goods to be kept as inventories (s_2) and how much of them (in the amount m) to be combined with other factor inputs (such as l) to produce finished goods (y_1). Again due to time lags in production/delivery, y_1 is not available until period t . In this period, the uncertainty in final demand for finished goods (c) is resolved, the planner then decides on how much of the finished goods to be kept as output inventories (s_1) for future consumption purpose and how much to be consumed in period t .

This economy can be easily decentralized or mapped to a competitive market economy in which consumers choose labor supply and final demand given market prices for output and factor payments, and downstream firms choose the order, usage, and storage of intermediated goods, as well as production and inventory investment in finished goods given expected prices of finished goods and input materials; and upstream suppliers choose production of intermediate goods and

demand for raw materials given expected output price for intermediate goods and input price for raw materials. More stages of production can be added if needed.

In this paper I adopt the social planner version of the model. By the welfare theorems, competitive equilibrium interpretations will also be applied throughout the paper to enhance understanding. In the model a social planner chooses sequentially first the order of raw materials (n) to produce intermediate goods (y_2) two periods in advance based on expected consumption ($E_{t-2}c_t$), knowing that the intermediate goods is not available for further processing until period $t - 1$. In the next period ($t - 1$) the planner then decides how much of the intermediate goods to be kept as inventories and how much to be combined with labor to produce finished goods (y_1) based on updated information regarding the expected consumption, $E_{t-1}c_t$. The resource constraint for intermediate goods is given by

$$m_t + s_{2t} = (1 - \delta_2)s_{2t-1} + y_{2t},$$

where δ_2 is the depreciation rate for intermediate goods inventories (s_2). Notice that y_{2t} is a predetermined variable when decisions for $\{m_t, s_{2t}\}$ are being made in period $t - 1$.⁵ In period t , based on realized preference shocks (θ) the planner chooses how much of the output (y_1) is to be consumed and how much is to be kept as inventories for future consumption. Again due to a one-period production/delivery lag, the finished goods output (y_1) is not available for consumption until period t . The resource constraint for finished goods is then given by

$$c_t + s_{1t} = (1 - \delta_1)s_{1t-1} + y_{1t},$$

where δ_1 is the depreciation rate for finished goods inventories and y_{1t} is predetermined in period $t - 1$. The production technologies for finished goods and intermediate goods are given by

$$y_{1t} = f(m_t, l_t)$$

and

$$y_{2t} = g(n_t)$$

⁵Denoting m_t rather than m_{t-1} as the usage of intermediate goods that is determined in period $t - 1$ is just a notation choice. The same rule applies to other variables in the paper. Thus, variables with time subscript t do not necessarily imply that they are choice variables in time period t .

respectively.⁶ The sequential optimization problem for the planner is then to solve

$$\max_{\{n_t\}} E_{-2} \left\{ \max_{\{m_t, l_t, s_{2t}\}} E_{-1} \left\{ \max_{\{c_t, s_{1t}\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [u(c_t, \theta_t) - w l_t - a n_t] \right\} \right\} \right\},$$

subject to

$$\begin{aligned} c_t + s_{1t} &= f(m_t, l_t) + (1 - \delta_1) s_{1t-1} \\ m_t + s_{2t} &= g(n_t) + (1 - \delta_2) s_{2t-1} \end{aligned}$$

for all t and two nonnegativity constraints on final-goods inventories and intermediate-goods inventories:

$$\begin{aligned} s_{1t} &\geq 0, \\ s_{2t} &\geq 0. \end{aligned}$$

The operator E_j in the objective function denotes expectations based on information available in period j , and θ in the utility function represents random shifts in consumption demand (preference shocks) and it is assumed to follow a stationary AR(1) process:

$$\theta_t = \gamma + \rho \theta_{t-1} + \varepsilon_t,$$

where ε is *i.i.d* with zero mean and variance σ_ε^2 .

In order to yield simple closed-form solutions, it is assumed that the utility function on consumption goods is quadratic,⁷

$$u(c, \theta) = \theta c - \frac{1}{2} c^2,$$

the marginal costs for labor (w) and raw materials (a) are constant, and that the two production functions satisfy constant returns to scale:

$$g(n) = n$$

$$f(m, l) = \min \{m, l\}.$$

⁶Since the crucial element in generating the results of the paper is time lag between the decision of input (e.g., n_t) and the availability of the output (e.g., y_{2t}), it does not matter whether we introduce the time lag in the resource constraint as a delivery lag or in the production function as a production lag.

⁷This assumption is not necessary for closed-form solutions but it simplifies expressions.

That is, one unit of raw materials (labor n) can be transformed into one unit of intermediate goods at the marginal cost of a , and one unit of intermediate goods (m) can be combined with one unit of labor (l) to be transformed into one unit of consumption goods at the marginal cost of $w + \lambda^m$, where λ^m denotes the shadow price (Lagrangian multiplier) of intermediate goods. The shadow price (Lagrangian multiplier) of final goods is simply the marginal utility of consumption, $\lambda^c = \theta - c$. For simplicity without loss of generality, I assume that the depreciation rates for both types of inventory goods are zero, $\delta_1 = \delta_2 = 0$. Notice that in equilibrium both forms of inventories may coexist, because without knowing precisely the optimal final demand of consumption goods the planner may opt not to transform all the intermediate goods inventories into finished goods inventories if the expected demand for finished goods next period is low. This can postpone the labor cost of producing finished goods.

The first order conditions for optimal plans for the variables $\{c, s_1, m, s_2, n\}$ are given respectively by (note: cost minimization by downstream firms implies $l = m$):

$$\theta_t - c_t = \lambda_t^c \quad (1)$$

$$\lambda_t^c = \beta E_t \lambda_{t+1}^c + \pi_t^c \quad (2)$$

$$w + \lambda_t^m = E_{t-1} \lambda_t^c \quad (3)$$

$$\lambda_t^m = \beta E_{t-1} \lambda_{t+1}^m + \pi_t^m \quad (4)$$

$$a = E_{t-2} \lambda_t^m \quad (5)$$

plus two market clearing conditions,

$$c_t + s_{1t} = m_t + s_{1t-1} \quad (6)$$

$$m_t + s_{2t} = n_t + s_{2t-1}$$

and two complementary slackness (Kuhn-Tucker) conditions associated with the nonnegativity constraints on inventories,

$$\pi_t^c s_{1t} = 0 \quad (7)$$

$$\pi_t^m s_{2t} = 0.$$

Equation (1) says that the optimal consumption level is chosen at the point where the marginal utility of consumption equals its marginal cost (the shadow price of finished goods). Equation (2) says that the optimal inventory holding of

finished goods is chosen at the point such that the cost of obtaining one extra unit of inventory (the shadow price of finished goods) equals the discounted benefit of having one extra unit of inventory available next period ($\beta E_t \lambda_{t+1}^c$) plus the value gained by relaxing the slackness constraint (π^c). Equation (3) says that the optimal usage of intermediate goods (i.e., the production of finished goods) is chosen, based on information available in period $t - 1$, such that the marginal cost of production ($w + \lambda_t^m$) – including both labor and input material costs – equals the expected value of marginal product ($E_{t-1} \lambda_t^c$). Notice that the value of finished goods (λ_t^c) is not known in period $t - 1$ when decision for finished goods production is being made. Equation (4) says that the optimal level of intermediate goods inventory is chosen such that the cost of obtaining one extra unit of intermediate-goods inventory (λ^m) equals the discounted benefit of having one extra unit of such inventory available next period ($\beta E_{t-1} \lambda_{t+1}^m$) plus the value gained by relaxing the slackness constraint (π^m). Note that the shadow value of intermediate goods (λ_t^m) is known in period $t - 1$. Equation (5) says that the optimal level of production for intermediate goods is chosen based on information in period $t - 2$ at the point so that the marginal input cost of raw materials (a) equals the expected value of marginal product ($E_{t-2} \lambda_t^m$).

A sequential equilibrium in the model is defined as sequences of a set of decision rules, $\{c_t, s_{1t}, m_t, s_{2t}, n_t, y_{1t}, y_{2t}, l_t\}_{t=0}^\infty$, such that equations (1)-(7) are satisfied. Denoting Ω_t as the information set for period t , the equilibrium decision rules of the model take the following form:

$$\begin{aligned} n_t &= n(\Omega_{t-2}), y_{2t} = y_2(\Omega_{t-2}), m_t = m(\Omega_{t-1}), l_t = l(\Omega_{t-1}), \\ y_{1t} &= y_1(\Omega_{t-1}), s_{2t} = s_2(\Omega_{t-1}), c_t = c(\Omega_t), s_{1t} = s_1(\Omega_t). \end{aligned}$$

Clearly, although all variables here have time subscript t , some of them may be functions of information sets earlier than t and hence are state variables rather than controls in period t . In particular, to avoid confusion in understanding the resource constraint (6) and the form of decision rules, notice again that the assumption of production/delivery lags for intermediate goods and finished goods are reflected by the fact that intermediate goods output y_{2t} is determined in period $t - 2$ but is available as input materials for producing finished goods output (m_t) only in period $t - 1$, hence y_{2t} is a state variable in period $t - 1$; and finished goods output y_{1t} is determined in period $t - 1$ but is available for consumption only in period t , hence $\{m_t, l_t, y_{1t}\}$ are state variables in period t . The expectation

operators and information sets in the first order conditions also reflect this fact.⁸

3. Deriving Equilibrium Decision Rules

Given output price (λ_t^c) and the availability of finished goods ($y_{1t} + s_{1t-1}$) as well as the realized demand shocks (θ) at the beginning of period t , the planner chooses consumption (c_t) and inventory holdings (s_{1t}) to maximize utility. Consider two possible cases:

Case A_1 . The demand shock is below “normal”, hence the existing supply ($y_{1t} + s_{1t-1}$) is sufficient to meet the utility maximizing level of consumption demand. In this case, we have $s_{1t} \geq 0$ and $\pi_t^c = 0$. Equation (2) implies that the shadow price of finished goods is a constant, $\lambda_t^c = \bar{\lambda}$, implying that goods price is endogenously downward sticky when inventories are allowed.⁹ The intuition for this result is straightforward. From the point of view of the representative agent, a low θ implies a low marginal utility of consumption. Hence it is optimal not to consume now but to hold the excess supply as inventories so as to be able to consume more in the future when θ maybe high. Translating this into the language of a competitive market economy, this implies that when demand is low, although firms can potentially dump all excess supply (inventory holdings) in the market to push down equilibrium price, this is not profit maximizing since inventories would be sold at equilibrium price below marginal cost. Hence it is optimal not to sell excess supply but to hold them into the future when market

⁸An alternative way of modeling the multi-stage fabrication process with production/delivery lags is to assume that the social planner solves

$$\max_{\{c_t, l_t, n_t, s_{1t}, s_{2t}\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [u(c_t, \theta_t) - w l_t - a n_t] \right\}$$

subject to

$$\begin{aligned} c_t + s_{1t} &= s_{1t-1} + y_1(m_{t-1}, l_{t-1}) \\ m_{t-1} + s_{2t-1} &= s_{2t-2} + y_2(n_{t-2}) \end{aligned}$$

where the production lags are taken into account in the production functions, $y_{1t} = y_1(m_{t-1}, l_{t-1})$, $y_{2t} = y_2(n_{t-1})$, rather than in the information sets of the objective function and the decision rules. The solutions are the same nonetheless except with some changes in notations.

⁹See Samuelson (1971) and Deaton and Laroque (1996) for a similar result in a slightly different context. Also see Reagan (1982).

price or demand becomes high. This results in equilibrium market price being not responsive to changes in demand when demand lies below certain threshold level (as inventories effectively render the supply curve perfectly elastic), offering a plausible explanation for price stickiness (see Blinder, 1982, for more discussions and references therein on this point).

Equation (1) then implies that the optimal consumption level is given by

$$c_t = \theta_t - \bar{\lambda},$$

and equation (6) implies that the optimal level of inventory holdings is given by

$$s_{1t} = y_{1t} + s_{1t-1} - \theta_t + \bar{\lambda}.$$

Since $s_{1t} \geq 0$, the threshold preference shock determining that the economy is in case A_1 is then given by $y_{1t} + s_{1t-1} - \theta_t + \bar{\lambda} \geq 0$ or equivalently, by

$$\theta_t \leq y_{1t} + s_{1t-1} + \bar{\lambda}.$$

Case B_1 . The demand shock is above “normal”, and the existing goods supply ($y_{1t} + s_{1t-1}$) falls short in meeting the consumption demand. In this case we have $s_{1t} = 0$ and $\pi_t^c > 0$. The optimal consumption policy is then to consume the entire stock given by the right-hand side of equation (6),

$$c_t = y_{1t} + s_{1t-1}.$$

Notice that the probability distribution of case A_1 and case B_1 (i.e., the probability of stocking out in period t) depends not only on demand shocks but also on the supply (production) level of finished goods (y_1) determined in period $t-1$. To choose y_{1t} optimally in period $t-1$ without knowing the demand shock in period t , equation (3) shows that y_{1t} should be chosen such that the marginal cost of production (labor cost plus the shadow price of intermediate goods, $w + \lambda_t^m$) equals the expected next period value of the output (the shadow price), $E_{t-1}\lambda_t^c$. Denoting $[\underline{\alpha}, \bar{\alpha}]$ as the support of the innovation in demand (ε) and denoting

$$z_t \equiv E_{t-1}s_{1t} = y_{1t} + s_{1t-1} - (E_{t-1}\theta_t - \bar{\lambda})$$

as the optimal target level of inventory holdings such that the firm will stock out only if $\varepsilon \geq z$, then the expected price for finished goods can be expanded into two terms,

$$E_{t-1}\lambda_t^c = \int_{\underline{\alpha}}^z \bar{\lambda} f(\varepsilon) d\varepsilon + \int_z^{\bar{\alpha}} (\theta_t - y_{1t} - s_{1t-1}) f(\varepsilon) d\varepsilon,$$

where $\lambda^c = \bar{\lambda}$ with probability $\int_{\underline{\alpha}}^z f(\varepsilon)d\varepsilon$ if there is no stockout (i.e., case A_1), and $\lambda^c = \theta - c$ with probability $\int_z^{\bar{\alpha}} f(\varepsilon)d\varepsilon$ if there is a stockout (i.e., case B_1). Note in the latter case optimal consumption is determined by $c_t = y_{1t} + s_{1t-1}$. Given the law of motion for the preference shocks, $\theta_t = E_{t-1}\theta_t + \varepsilon_t$, equation (3) then becomes

$$\begin{aligned} w + \lambda_t^m &= \bar{\lambda} \int_{\underline{\alpha}}^z f(\varepsilon)d\varepsilon + \underbrace{(E_{t-1}\theta_t - y_{1t} - s_{1t-1})}_{\bar{\lambda} - z} \int_z^{\bar{\alpha}} f(\varepsilon)d\varepsilon + \int_z^{\bar{\alpha}} \varepsilon_t f(\varepsilon)d\varepsilon \quad (8) \\ &= \bar{\lambda} - z \int_z^{\bar{\alpha}} f(\varepsilon)d\varepsilon + \int_z^{\bar{\alpha}} \varepsilon_t f(\varepsilon)d\varepsilon \\ &\equiv \Gamma(z_t), \end{aligned}$$

where $\Gamma(\cdot)$ is a nonlinear but monotonically decreasing function of z .¹⁰ Since z is proportional to y_1 by definition, the higher y_1 , the less likely there is a stockout in period t , given expected demand $E_{t-1}\theta_t$. Thus equation (8) implicitly determines the optimal level of final goods production, y_{1t} , in period $t - 1$.

Turning to the left-hand side of equation (8), the price of intermediate goods (λ^m) depends on the tightness of the intermediate goods market. Hence z_t is time dependent if λ_t^m is. There are two cases to consider for the possible values of λ^m :

Case A_2 . The demand for intermediate goods (y_1) is below “normal”. In this case we have $s_{2t} \geq 0$ and $\pi_t^m = 0$. Hence equation (4) implies that intermediate goods price is a constant,

$$\lambda_t^m = \beta a,$$

where the right side is derived based on equation (5). The rationale for this endogenously constant (sticky) price when demand is below a threshold level is similar to that discussed above regarding the finished goods price λ^c . The

¹⁰To see this, differentiating $\Gamma(z)$ with respect to z gives,

$$\frac{\partial \Gamma}{\partial z} = - \int_z^{\bar{\alpha}} f(\varepsilon)d\varepsilon + z f(z) - z f(z) < 0.$$

Also note that $\bar{\lambda} = \beta(w + \beta a)$ (see proposition 3.3 for proof) and $\lambda^m \geq \beta a$, hence

$$\Gamma(\bar{\alpha}) = \bar{\lambda} < w + \lambda^m < \bar{\lambda} - \underline{\alpha} = \Gamma(\underline{\alpha}),$$

where the lower support, $\underline{\alpha} < 0$, is an arbitrarily negative number. This implies that the solution for a cut-off value z exists and is unique.

interpretation for βa is, in case there is no stockout, the firm gets to save on the marginal cost of production next period, and the discounted value for this is precisely βa . In this case, equation (8) implies that the optimal target inventory level z_t based on period $t-1$ information is also a constant: $z_t = k$, where k solves $\Gamma(k) = w + \beta a$. This implies that the optimal demand for intermediate goods (y_1) is determined by the equation,

$$k = y_{1t} + s_{1t-1} - E_{t-1}\theta_t + \bar{\lambda},$$

or equivalently,

$$y_{1t} = (E_{t-1}\theta_t - \bar{\lambda}) + (k - s_{1t-1}). \quad (9)$$

Hence, the optimal demand of intermediate goods by a downstream (finished-goods producing) firm is characterized by a policy that specifies a constant target level for finished-goods inventory holdings ($k = E_{t-1}s_{1t}$) or a target level of inventory investment ($k - s_{1t-1}$), such that orders (purchases) of intermediate goods move one-for-one with expected consumption demand ($E_{t-1}\theta_t - \bar{\lambda}$) given the target inventory investment level ($k - s_{1t-1}$), provided that we are in case A_2 (i.e., provided there is no stockout in the intermediate goods market: $s_{2t} \geq 0$). Using equation (7), the requirement $s_{2t} \geq 0$ implies that the threshold level of expected demand that determines the probability of stockout in the intermediate goods market is given by,

$$y_{2t} + s_{2t-1} - y_{1t} \geq 0,$$

or equivalently, by

$$E_{t-1}\theta_t - \bar{\lambda} \leq y_{2t} + s_{2t-1} - (k - s_{1t-1}),$$

where we have substituted out y_{1t} using the optimal policy (9). That is, if optimal expected demand for finished goods (which equals consumption demand, $E_{t-1}c_t = E_{t-1}\theta_t - \bar{\lambda}$, plus inventory investment demand, $k - s_{1t-1}$) is less than the potential supply of finished goods (which equals the total supply of intermediate goods, $y_{2t} + s_{2t-1}$), then some intermediate goods should be kept as inventories (i.e., $s_{2t} \geq 0$) and the optimal volume of orders (usages) for intermediate goods is characterized by the policy (9). Since demand equals supply in equilibrium, policy (9) is also the policy for production of finished goods ($y_1 = f(m, l)$) given expected demand for finished goods.

Case B_2 . The demand for intermediate goods is above “normal”. In this case, the supply of intermediate goods cannot meet its demand. Hence, $s_{2t} = 0$ and

$\pi_t^m > 0$, implying that there is a stockout in intermediate goods. Hence, the optimal volume of orders for intermediate goods in equilibrium is simply

$$y_{1t} = y_{2t} + s_{2t-1}.$$

In this case, the tightness in the intermediate goods market pushes goods price upwards, so that $\lambda_t^m = \beta a + \pi_t^m > \beta a$. Clearly, whether the economy is in case A_2 or case B_2 depends not only on expected demand shock in period $t - 1$ ($E_{t-1}\theta_t$), but also on the availability (production) of intermediate goods ($y_{2t} + s_{2t-1}$), which is determined earlier in history (period $t - 2$).

To determine the optimal supply of intermediate goods (y_2) and the optimal orders for raw materials (n) in period $t - 2$ based on expected demand for intermediate goods, the planner needs to compute the total demand for intermediate goods (= input demand as factors of production by upstream firms plus inventory investment demand for intermediate goods). Equation (5) shows that y_{2t} should be chosen such that the marginal cost of production, a , equals the expected next period value of marginal product (the shadow price of intermediate goods), $E_{t-2}\lambda_t^m$. Since λ^m depends on the tightness of the intermediate goods market (as discussed above under case A_2 and case B_2), there are two possibilities to consider. In the case A_2 , $\lambda^m = \beta a$. In the case B_2 , $\lambda^m = E_{t-1}\lambda^c - w = E_{t-1}(\theta_t - c_t) - w$ (by equations 3 and 1).

Denote

$$\begin{aligned} \zeta_t &\equiv E_{t-2}s_{2t} \\ &= y_{2t} + s_{2t-1} - \underbrace{E_{t-2}(k - s_{1t-1})}_0 - (E_{t-2}\theta_t - \bar{\lambda}) \\ &= y_{2t} + s_{2t-1} - (E_{t-2}\theta_t - \bar{\lambda}) \end{aligned}$$

as the optimal target level of inventory holdings for intermediate goods such that the firm stocks out if and only if $\varepsilon_{t-1} \geq \zeta$.¹¹ In equation (5), the expected price of intermediate goods based on information available in period $t - 2$ can then be expanded into two terms,

$$a = E_{t-2}\lambda_t^m = \int_{\underline{\alpha}}^{\zeta} \beta a f(\varepsilon) d\varepsilon + \int_{\zeta}^{\bar{\alpha}} (E_{t-1}\theta_t - E_{t-1}c_t - w) f(\varepsilon) d\varepsilon, \quad (10)$$

¹¹Note that in the above expression, the equilibrium concept, $E_{t-2}s_{1t-1} = E_{t-1}s_{1t} = k$, is applied because the optimal supply of intermediate goods (y_2) in any period should always be chosen such that not only does intermediate goods inventory level meet its target but also does the finished goods inventory level (whose expected target is k in any period as shown previously).

where the first term represents the probability of case A_2 and the second represents the probability of case B_2 . Equation (10) implicitly determines ζ_t and consequently the optimal production policy for y_2 . The following proposition shows that the optimal solution for ζ_t is a constant.

Proposition 3.1. *The optimal target for intermediate goods inventory stock is a constant, $\zeta_t = \tilde{k}$.*

Proof. See Appendix 001. ■

Hence by the definition of ζ we have

$$y_{2t} = (E_{t-2}\theta_t - \bar{\lambda}) + (\tilde{k} - s_{2t-1}). \quad (11)$$

Equation (11) says that in equilibrium the optimal production plan for intermediate goods (y_2) is to meet the expected consumption demand ($E_{t-2}\theta_t - \bar{\lambda}$) plus a target level for inventory investment in intermediate goods ($\tilde{k} - s_{2t-1}$). Notice that this is very similar to the policy of choosing finished goods production (y_{1t}) in equation (9) except in this case the possibility of a stockout in the raw material market is not considered here in the model (extending the model to include such a possibility is straightforward).

Proposition 3.2. *The optimal target for intermediate goods inventories is higher than that for finished goods inventories,*

$$\tilde{k} > k.$$

Proof. See Appendix 002. ■

The intuition behind proposition 2 can be understood as follows. The probability of a stockout in period t in the final consumption goods market is affected by the supply of finished goods ($y_{1t} = f(m, l)$) determined in period $t - 1$, which in turn is affected by the demand and supply of intermediate goods as well as by the demand and supply of raw materials ($y_{2t} = g(n)$) in the primary goods market determined in period $t - 2$. Hence, in a multi-stage production economy equilibrium consumption depends ultimately on the production decisions made by the most remote upstream firms at the start of the production chain which supplies intermediate goods to downstream firms. However, because stocking out and holding inventories are both costly, in order to determine optimally how much to produce at the upstream, firms must form correct expectations on final demand

facing the most remote downstream. The longer the chain of production (i.e., the larger the number of production stages), the harder it is to forecast final demand at downstream, hence the more precautionary inventory investment it is needed in order to avoid possible stockouts.

Proposition 3.3. *The equilibrium decision rules for consumption (c), finished goods inventory holding (s_1), finished goods production (y_1) - which also equals equilibrium usage of intermediate goods (m), intermediate goods production (y_2) - which also equals equilibrium demand of raw materials (n), and intermediate goods inventory holding (s_2) are given respectively by:*

$$\begin{aligned}
s_{1t} &= \begin{cases} k - \varepsilon_t & ; \text{ if } \varepsilon_t \leq k \text{ \& } \varepsilon_{t-1} \leq \tilde{k} \\ \tilde{k} - \varepsilon_t - \rho\varepsilon_{t-1} & ; \text{ if } \varepsilon_t \leq k \text{ \& } \varepsilon_{t-1} > \tilde{k} \\ 0 & ; \text{ if } \varepsilon_t > k \end{cases} \\
c_t &= \begin{cases} E_{t-1}\theta_t - \bar{\lambda} + \varepsilon_t & ; \text{ if } \varepsilon_t \leq k \\ E_{t-1}\theta_t - \bar{\lambda} + k & ; \text{ if } \varepsilon_t > k \text{ \& } \varepsilon_{t-1} \leq \tilde{k} \\ E_{t-2}\theta_t - \bar{\lambda} + \tilde{k} & ; \text{ if } \varepsilon_t > k \text{ \& } \varepsilon_{t-1} > \tilde{k} \end{cases} \\
y_{1t} &= \begin{cases} E_{t-1}\theta_t - \bar{\lambda} + \varepsilon_{t-1} & ; \text{ if } \varepsilon_{t-1} \leq k \text{ \& } \varepsilon_{t-2} \leq \tilde{k} \\ E_{t-1}\theta_t - \bar{\lambda} + k - \tilde{k} + \varepsilon_{t-1} + \rho\varepsilon_{t-2} & ; \text{ if } \varepsilon_{t-1} \leq k \text{ \& } \varepsilon_{t-2} > \tilde{k} \\ E_{t-1}\theta_t - \bar{\lambda} + k & ; \text{ if } k < \varepsilon_{t-1} \leq \tilde{k} \\ E_{t-2}\theta_t - \bar{\lambda} + \tilde{k} & ; \text{ if } \varepsilon_{t-1} > k \end{cases} \\
s_{2t} &= \begin{cases} \tilde{k} - \varepsilon_{t-1} - \rho\varepsilon_{t-2} & ; \text{ if } \varepsilon_{t-1} \leq k \text{ \& } \varepsilon_{t-2} \leq \tilde{k} \\ 2\tilde{k} - k - (1 + \rho)\varepsilon_{t-1} - \rho\varepsilon_{t-2} & ; \text{ if } \varepsilon_{t-1} \leq k \text{ \& } \varepsilon_{t-2} > \tilde{k} \\ \tilde{k} - k - \rho\varepsilon_{t-1} & ; \text{ if } k < \varepsilon_{t-1} \leq \tilde{k} \\ 0 & ; \text{ if } \varepsilon_{t-1} > k \end{cases} \\
y_{2t} &= \begin{cases} E_{t-2}\theta_t - \bar{\lambda} + \varepsilon_{t-2} + \rho\varepsilon_{t-3} & ; \text{ if } \varepsilon_{t-2} \leq k \text{ \& } \varepsilon_{t-3} \leq \tilde{k} \\ E_{t-2}\theta_t - \bar{\lambda} + k - \tilde{k} + (1 + \rho)\varepsilon_{t-2} + \rho\varepsilon_{t-3} & ; \text{ if } \varepsilon_{t-2} \leq k \text{ \& } \varepsilon_{t-3} > \tilde{k} \\ E_{t-2}\theta_t - \bar{\lambda} + k + \rho\varepsilon_{t-2} & ; \text{ if } k < \varepsilon_{t-2} \leq \tilde{k} \\ E_{t-2}\theta_t - \bar{\lambda} + \tilde{k} & ; \text{ if } \varepsilon_{t-2} > k \end{cases}
\end{aligned}$$

where $\bar{\lambda} = \beta(w + \beta a)$.

Proof. See Appendix 003. ■

The interpretation for $\bar{\lambda} = \beta(w + \beta a)$ is reasonably straightforward. Given that the slackness constraint for finished goods does not bind in the current period (hence the slackness constraint for intermediate goods will not bind in the next period according to proposition 3.2), the expected value of carrying one extra unit of finished goods inventory into the next period is determined by two terms: first, it can save on the labor cost of production next period (w); second, it can save on the intermediate goods used in production whose expected value next period equals the discounted marginal cost of raw materials (βa) given that the slackness constraint on intermediate goods does not bind next period. The present value of this compounded marginal cost is then $\beta(w + \beta a)$. Hence $\bar{\lambda}$ is simply the competitive price of finished goods when the economy has inventories.

4. Understanding Inventory Fluctuations

The following propositions characterize some important implications of the model for inventory fluctuations. To fix concepts, note that for finished goods sector, the production is y_{1t} and the sales are c_t ; for intermediate goods sector, the usage is $m_t (= y_{1t})$, and the orders are y_{2t} (note that the one period production time for transforming n into y_2 can also be interpreted as a delivery time during which n units of goods are delivered from an upstream firm to an intermediate-goods producing firm which can produce finished goods instantaneously). Since we do not consider inventories of raw materials (n) in this model, the usage in n equals the orders of n . Extending the model to a 3-stage-fabrication economy to allow for raw material inventories is straightforward. To facilitate the proofs, we denote $P \equiv \Pr[\varepsilon_t \leq k]$ and $\tilde{P} \equiv \Pr[\varepsilon_t \leq \tilde{k}]$. Clearly, $\tilde{P} > P$ since $\tilde{k} > k$.

Proposition 4.1. *The variance of output exceeds the variance of sales for finished goods, and the variance of input ordering exceeds the variance of input usage for intermediate goods, provided that demand shocks are positively auto-correlated.*

Proof. See Appendix 004. ■

The intuition for this proposition is reasonably straightforward. If we can understand why production (y_1) is more variable than sales (c), then we can also understand why orders (y_2) are more variable than usages (y_1), because usages are like demand (sales) and orders are equivalent to supply (production).

Since optimal production under demand uncertainty is to meet expected sales plus the target inventory investment, output will respond to an innovation in sales by more than one-for-one. This is because a positive demand shock has two effects: it reduces inventories one-for-one, and it increases expected demand. The first effect alone would make production exactly as volatile as sales in order to replenish the inventory stock; the second effect further increases production if demand shocks are serially correlated ($\rho > 0$) so that the expected next period shock is strictly positive. This important insight has already been provided by Kahn (1987) using a single-stage production partial equilibrium model of output inventories. Here it is shown that it continues to hold in a general equilibrium multi-stage production model.

Since production of finished goods (y_1) requires intermediate goods (m) as input ($y_1 = f(m, l)$), the volatility in sales at the very bottom of the production chain is translated into volatility in production ($\sigma_{y_1}^2$), which in turn is translated into volatility in sales (σ_m^2) from the upstream firms which supply intermediate goods to downstream as inputs. The transmission and amplification of uncertainty continues until the most remote upstream firms are reached. Hence the variance of production/orders exceeds the variance of sales/usages at all stages of production and the variances keep increasing along the chain of production towards upstream firms.

This chain-multiplier effect thus explains why in the US production of raw materials at upstream firms is usually about 50 percent more volatile than production of finished goods at downstream firms.

Proposition 4.2. *Productions are positively correlated across all stages of fabrication.*

Proof. See Appendix 005. ■

This proposition simply explains a well known phenomenon of sectorial comovements observed in the US economy. In the existing RBC literature, such comovements are explained by aggregate technology shocks (see Hornstein, 2000, for detailed discussions on why sectorial technology shocks cannot explain such comovements). Here it is shown that such comovements may be explained also by demand shocks to the downstream production sector.

Proposition 4.3. *Input inventories are more volatile than output inventories.*

Proof. See Appendix 006.■

The puzzle that inventories of materials and good-in-progress are significantly more volatile than inventories of finished goods was first documented by Feldstein and Auerbach (1976), but it has never been explained in the literature. The multi-stage production model offers a clear-cut explanation for this puzzle. The reason for the volatility difference between the two types of inventories is that uncertainty in final demand gets transmitted and amplified along the chain of production towards upstream. Hence inventories held by upstream firms (i.e., intermediate-goods inventories) are more volatile than inventories held by downstream firms (i.e., finished-goods inventories) due to the chain-multiplier effect.

5. Conclusion

Due to lack of models that can feature both output- and input-inventories simultaneously, several well known puzzles pertaining to inventory fluctuations and the business cycle have not been well explained by dynamic optimization theory. By presenting a general equilibrium, multi-stage production model of inventories with separate decisions to order, use, and stock input materials and to produce, sell, and store finished output, this paper offers not only a model of input-output inventories but also a neoclassical perspective on the theory of aggregate demand. It shows that due to production/delivery lags, firms opt to hold both output- and input-inventories so as to guard against demand uncertainty at all stages of production. As a result, not only is production more volatile than sales but also input-ordering more volatile than input-usage, giving rise to a chain-multiplier mechanism that propagates and amplifies demand shocks at downstream towards upstream via input-output linkages. This multiplier effect induced by precautionary inventory investment at each production stage is shown to be able to explain several long-standing puzzles of the business cycle documented in the inventory literature.

Since the chain-multiplier reflects nothing more than optimal responses of a perfectly functioning market economy towards demand uncertainty, interventionary policies attempting to reduce aggregate output and employment volatilities via demand management may prove counter-productive, unless the policies can be targeted specifically at reducing demand uncertainty or shortening the time lags involved in firms' production/delivery.

The model may be extended to explain durable goods inventories and sea-

sonal cycles in production and inventories. It is well known that the durable goods sector is far more volatile than the nondurable goods sector and that seasonal movements in production and inventories are much larger than business-cycle frequency movements. These stylized facts have not been well explained by theories. Clearly, durability of consumption goods and seasonal demand at Christmas can only be modeled through preferences by making demand endogenous. The general equilibrium inventory model with preference shifts outlined in this paper provides a suitable framework for addressing these issues in the future.

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Appendix 001 (proof for proposition 3.1):

Note that

$$E_{t-1}(\theta_t - c_t) = \int_{\underline{\alpha}}^k \bar{\lambda} f(\varepsilon) d\varepsilon + \int_k^{\bar{\alpha}} (\theta_t - y_{1t} - s_{1t-1}) f(\varepsilon) d\varepsilon.$$

We can rewrite (10) as

$$\begin{aligned} a &= \int_{\underline{\alpha}}^{\zeta} \beta a f(\varepsilon) d\varepsilon + \int_{\zeta}^{\bar{\alpha}} \left(\int_{\underline{\alpha}}^k \bar{\lambda} f(\varepsilon) d\varepsilon + \int_k^{\bar{\alpha}} (\theta_t - \underbrace{y_{1t}}_{y_{2t} + s_{2t-1}} - s_{1t-1}) f(\varepsilon) d\varepsilon - w \right) f(\varepsilon) d\varepsilon \\ &= \int_{\underline{\alpha}}^{\zeta} \beta a f(\varepsilon) d\varepsilon + \int_{\zeta}^{\bar{\alpha}} \bar{\lambda} f(\varepsilon) d\varepsilon \\ &\quad + \int_{\zeta}^{\bar{\alpha}} \left(\int_k^{\bar{\alpha}} (\theta_t - y_{2t} - s_{2t-1} - s_{1t-1} - \bar{\lambda}) f(\varepsilon) d\varepsilon \right) f(\varepsilon) d\varepsilon - w \int_{\zeta}^{\bar{\alpha}} f(\varepsilon) d\varepsilon \end{aligned}$$

where $y_{1t} = y_{2t} + s_{2t-1}$ since $s_{2t} = 0$ if $\varepsilon_{t-1} > \zeta$. Assuming that $\zeta > k$ (see proposition 3.2 for a confirmation of this assumption), hence $\varepsilon_{t-1} > \zeta$ implies $\varepsilon_{t-1} > k$, which in turn implies $s_{1t-1} = 0$. Then conditioned on $\varepsilon_{t-1} > \zeta$, the third term in the above equation becomes,

$$\begin{aligned} &\int_{\zeta}^{\bar{\alpha}} \left(\int_k^{\bar{\alpha}} \left(\underbrace{\theta_t}_{E_{t-2}\theta_t + \varepsilon_t + \rho\varepsilon_{t-1}} - y_{2t} - s_{2t-1} - \underbrace{s_{1t-1}}_0 - \bar{\lambda} \right) f(\varepsilon) d\varepsilon \right) f(\varepsilon) d\varepsilon \\ &= \int_{\zeta}^{\bar{\alpha}} \left[\underbrace{(E_{t-2}\theta_t - y_{2t} - s_{2t-1} - \bar{\lambda})}_{-\zeta} \int_k^{\bar{\alpha}} f(\varepsilon) d\varepsilon \right] f(\varepsilon) d\varepsilon + \int_{\zeta}^{\bar{\alpha}} \left[\int_k^{\bar{\alpha}} (\varepsilon_t + \rho\varepsilon_{t-1}) f(\varepsilon) d\varepsilon \right] f(\varepsilon) d\varepsilon \\ &= -\zeta \int_{\zeta}^{\bar{\alpha}} \left(\int_k^{\bar{\alpha}} f(\varepsilon) d\varepsilon \right) f(\varepsilon) d\varepsilon + \int_{\zeta}^{\bar{\alpha}} \left(\int_k^{\bar{\alpha}} (\varepsilon_t + \rho\varepsilon_{t-1}) f(\varepsilon) d\varepsilon \right) f(\varepsilon) d\varepsilon \\ &\equiv G(\zeta_t, \Omega_1), \end{aligned}$$

where Ω_1 is a set of fixed parameters. Hence we have

$$\begin{aligned} a &= \beta a \int_{\underline{\alpha}}^{\zeta} f(\varepsilon) d\varepsilon + \bar{\lambda} \left(\int_{\underline{\alpha}}^k f(\varepsilon) d\varepsilon \right) \left(\int_{\zeta}^{\bar{\alpha}} f(\varepsilon) d\varepsilon \right) + G(\zeta_t, \Omega_1) - w \int_{\zeta}^{\bar{\alpha}} f(\varepsilon) d\varepsilon \\ &\equiv \tilde{\Gamma}(\zeta_t, \Omega_2), \end{aligned}$$

where Ω_2 is a set of fixed parameters. This equation implies that the optimal inventory target (ζ_t) for intermediate goods is a constant that solves the identity $\tilde{\Gamma}(\zeta_t, \Omega_2) = a$. ■

Appendix 002 (proof for proposition 3.2):

Finished goods inventory stock satisfies

$$s_{1t} = \begin{cases} y_{1t} + s_{1t-1} - (\theta_t - \bar{\lambda}) & ; \text{ if } \varepsilon_t \leq k \\ 0 & ; \text{ if } \varepsilon_t > k \end{cases} .$$

Since the optimal production for y_{1t} satisfies

$$y_{1t} = \begin{cases} (E_{t-1}\theta_t - \bar{\lambda}) + (k - s_{1t-1}) & ; \text{ if } \varepsilon_{t-1} \leq \tilde{k} \\ y_{2t} + s_{2t-1} & ; \text{ if } \varepsilon_{t-1} > \tilde{k} \end{cases}$$

and the optimal production for y_{2t} satisfies

$$y_{2t} = (E_{t-2}\theta_t - \bar{\lambda}) + (\tilde{k} - s_{2t-1}),$$

we have

$$\begin{aligned} s_{1t} &= \begin{cases} (E_{t-1}\theta_t - \bar{\lambda}) + k - (\theta_t - \bar{\lambda}) & ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} \leq \tilde{k} \\ (E_{t-2}\theta_t - \bar{\lambda}) + \tilde{k} + s_{1t-1} - (\theta_t - \bar{\lambda}) & ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} > \tilde{k} \\ 0 & ; \text{ if } \varepsilon_t > k \end{cases} \\ &= \begin{cases} k - \varepsilon_t & ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} \leq \tilde{k} \\ s_{1t-1} + \tilde{k} - \varepsilon_t - \rho\varepsilon_{t-1} & ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} > \tilde{k} \\ 0 & ; \text{ if } \varepsilon_t > k \end{cases} \end{aligned}$$

Notice that the lagged variable (s_{1t-1}) can be further iterated backwards using the above dynamic equation:

$$s_{1t-1} = \begin{cases} k - \varepsilon_{t-1} & ; \text{ if } \varepsilon_{t-1} \leq k \ \& \ \varepsilon_{t-2} \leq \tilde{k} \\ s_{1t-2} + \tilde{k} - \varepsilon_{t-1} - \rho\varepsilon_{t-2} & ; \text{ if } \varepsilon_{t-1} \leq k \ \& \ \varepsilon_{t-2} > \tilde{k} \\ 0 & ; \text{ if } \varepsilon_{t-1} > k \end{cases} ,$$

which can lead to infinite regression, resulting in the series $\{s_{1t}\}$ being nonstationary. This cannot be an equilibrium unless

$$s_{1t-1} = 0$$

always holds. But this requires that the condition, $\varepsilon_{t-1} > k$, always hold so that $s_{1t-1} = 0$ with probability one (the third line in the above equation for s_{1t-1}). However, given the second line in the equation for s_{1t} ,

$$s_{1t} = s_{1t-1} + \tilde{k} - \varepsilon_t - \rho\varepsilon_{t-1} \quad ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} > \tilde{k} \ ,$$

if the condition, $\tilde{k} > k$, holds, then $\varepsilon_{t-1} > \tilde{k}$ automatically implies $\varepsilon_{t-1} > k$, which in turn implies (by the third line in the equation for s_{1t-1})

$$s_{1t-1} = 0 \quad ; \text{ if } \varepsilon_{t-1} > \tilde{k},$$

hence we have

$$s_{1t} = \begin{cases} k - \varepsilon_t & ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} \leq \tilde{k} \\ \tilde{k} - \varepsilon_t - \rho\varepsilon_{t-1} & ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} > \tilde{k} \ , \\ 0 & ; \text{ if } \varepsilon_t > k \end{cases}$$

which is stationary and is therefore an equilibrium. Hence, $\tilde{k} > k$ must be true. ■

Appendix 003 (proof for proposition 3.3):

The decision rule for s_{1t} is proved in the proof for proposition 3.2. The rest can be obtained by following the discussions in section 3 above using straightforward substitutions. To show that $\bar{\lambda} = \beta(w + \beta a)$, note $\lambda_t^c = \bar{\lambda}$ if $\varepsilon_t \leq k$ (i.e., if $\pi_t^c = 0$). On the other hand, since the current period shadow price of intermediate goods is known based on last period information set, as all variables in the resource constraint (7) are known in period $t-1$, we then have $E_{t-1}\lambda_t^m = \lambda_t^m$ and $E_t\lambda_{t+1}^m = \lambda_{t+1}^m$. Since $\lambda_t^m = \beta a$ if $\varepsilon_{t-1} \leq \tilde{k}$ according equation (4) and equation (5),¹² we have $\lambda_{t+1}^m = \beta a$ if $\varepsilon_t \leq \tilde{k}$. According to proposition 3.2, $k < \tilde{k}$, hence $\lambda_t^c = \bar{\lambda}$ implies $\lambda_{t+1}^m = \beta a$ (since $\varepsilon_t \leq k$ implies $\varepsilon_t \leq \tilde{k}$). According to equation (3), $E_t\lambda_{t+1}^c = w + \lambda_{t+1}^m$, substituting this into equation (2) gives

$$\lambda_t^c = \beta(w + \lambda_{t+1}^m) = \beta(w + \beta a), \quad \text{if } \varepsilon_t \leq k.$$

■

Appendix 004 (proof for proposition 4.1):

We need to show $\sigma_{y_2} > \sigma_{y_1} > \sigma_c$. Using the law of motion for preference shocks, the variance of c is given by

$$\sigma_c^2 = P(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2) + (1-P)\tilde{P}\rho^2\sigma_\theta^2 + (1-P)(1-\tilde{P})\rho^4\sigma_\theta^2.$$

¹²Equation (5) implies $E_{t-1}\lambda_{t+1}^m = a$.

The variance of y_1 is given by

$$\begin{aligned}\sigma_{y_1}^2 &= P\tilde{P}(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2 + 2\rho\sigma_\varepsilon^2) \\ &\quad + P(1 - \tilde{P})(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2 + 2\rho\sigma_\varepsilon^2 + 2\rho^3\sigma_\varepsilon^2) \\ &\quad + (1 - P)\tilde{P}\rho^2\sigma_\theta^2 \\ &\quad + (1 - \tilde{P})\rho^4\sigma_\theta^2.\end{aligned}$$

And the variance of y_2 is given by

$$\begin{aligned}\sigma_{y_2}^2 &= P\tilde{P}(\rho^4\sigma_\theta^2 + \sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2 + 2\rho^2\sigma_\varepsilon^2 + 2\rho^4\sigma_\varepsilon^2) \\ &\quad + P(1 - \tilde{P})(\rho^4\sigma_\theta^2 + (1 + \rho)^2\sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2 + 2(1 + \rho)\rho^2\sigma_\varepsilon^2 + 2\rho^4\sigma_\varepsilon^2) \\ &\quad + (1 - P)\tilde{P}(\rho^4\sigma_\theta^2 + \rho^2\sigma_\varepsilon^2 + 2\rho^3\sigma_\varepsilon^2) \\ &\quad + (1 - \tilde{P})\rho^4\sigma_\theta^2.\end{aligned}$$

All terms in these expressions are strictly positive. Notice that the last two terms in σ_y^2 exceed the last two terms in σ_c^2 . Hence, $\sigma_{y_1} > \sigma_c$ if and only if the first two terms in σ_y^2 exceed the first term in σ_c^2 , namely,

$$P\tilde{P}(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2 + 2\rho\sigma_\varepsilon^2) + P(1 - \tilde{P})(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2 + 2\rho\sigma_\varepsilon^2 + 2\rho^3\sigma_\varepsilon^2) > P(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2).$$

Since the left-hand side of this inequality can be expressed as

$$P(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2) + P\tilde{P}(2\rho\sigma_\varepsilon^2) + P(1 - \tilde{P})(\rho^2\sigma_\varepsilon^2 + 2\rho\sigma_\varepsilon^2 + 2\rho^3\sigma_\varepsilon^2),$$

where the first term is the same as the right-hand side of the inequality, hence $\sigma_y^2 > \sigma_c^2$ as long as $\rho > 0$. In order to show $\sigma_{y_2} > \sigma_{y_1}$, first notice that both $\sigma_{y_2}^2$ and $\sigma_{y_1}^2$ are increasing functions of ρ . Secondly, notice that if $\rho = 0$, then $\sigma_{y_2}^2 = \sigma_{y_1}^2 = P\tilde{P}\sigma_\varepsilon^2 + P(1 - \tilde{P})\sigma_\varepsilon^2$. And thirdly, notice that if $\rho = 1$, then

$$\begin{aligned}\sigma_{y_1}^2 &= P\tilde{P}(\sigma_\theta^2 + \sigma_\varepsilon^2 + 2\sigma_\varepsilon^2) \\ &\quad + P(1 - \tilde{P})(\sigma_\theta^2 + \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + 2\sigma_\varepsilon^2 + 2\sigma_\varepsilon^2) \\ &\quad + (1 - P)\tilde{P}\sigma_\theta^2 \\ &\quad + (1 - \tilde{P})\sigma_\theta^2,\end{aligned}$$

and

$$\begin{aligned}
\sigma_{y_2}^2 &= P\tilde{P}(\sigma_\theta^2 + \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + 2\sigma_\varepsilon^2 + 2\sigma_\varepsilon^2) \\
&\quad + P(1 - \tilde{P})(\sigma_\theta^2 + 4\sigma_\varepsilon^2 + \sigma_\varepsilon^2 + 4\sigma_\varepsilon^2 + 2\sigma_\varepsilon^2) \\
&\quad + (1 - P)\tilde{P}(\sigma_\theta^2 + \sigma_\varepsilon^2 + 2\sigma_\varepsilon^2) \\
&\quad + (1 - \tilde{P})\sigma_\theta^2,
\end{aligned}$$

and clearly each term in $\sigma_{y_2}^2$ exceeds the corresponding term in $\sigma_{y_1}^2$. Hence, as long as $\rho > 0$, we have $\sigma_{y_2} > \sigma_{y_1} > \sigma_c$. ■

Appendix 005 (proof for proposition 4.2):

Eye-ball inspection on the decision rules indicates that every possible case for c in the decision rule (sales of finished goods) is strictly positively correlated with every possible case for m ($= y_1$, sales of intermediate goods), and every possible case for y_1 in the decision rule (production of finished goods) is strictly positively correlated with every possible case for y_2 (production of intermediate goods). Hence productions as well as sales are unambiguously positively correlated across all stages of fabrication. Furthermore, these correlations increase with ρ since $E_{t-j}\theta_t = \rho^j\theta_{t-j}$. ■

Appendix 006 (proof for proposition 4.3):

The variances of the two types of inventories are given respectively by

$$\sigma_{s_1}^2 = P\tilde{P}\sigma_\varepsilon^2 + P(1 - \tilde{P})(\sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2)$$

$$\sigma_{s_2}^2 = P\tilde{P}(\sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2) + P(1 - \tilde{P})((1 + \rho)^2\sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2) + (1 - P)\tilde{P}(\rho^2\sigma_\varepsilon^2).$$

Term-by-term comparison shows that $\sigma_{s_2} > \sigma_{s_1}$ as long as $\rho > 0$ (the variances are equal if $\rho = 0$). ■

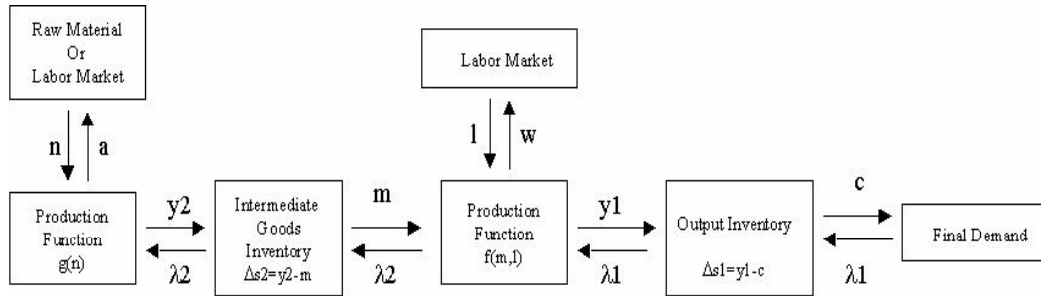


Fig. 1. Stage-of-Fabrication Model Diagram.