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Problems in the Numerical Simulation of Models with Heterogeneous Agents and Economic Distortions

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Problems in the Numerical Simulation of Models with Heterogeneous Agents and Economic Distortions

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Abstract

Our work has been concerned with the numerical simulation of dynamic economies with heterogeneous agents and economic distortions. Recent research has drawn attention to inherent difficulties in the computation of competitive equilibria for these economies: A continuous Markovian solution may fail to exist, and some commonly used numerical algorithms may not deliver accurate approximations. We consider a reliable algorithm set forth in Feng *et al.* (2009), and discuss problems related to the existence and computation of Markovian equilibria, as well as convergence and accuracy properties. We offer new insights into numerical simulation.

KEYWORDS: Heterogeneous agents, taxes, externalities, financial frictions, competitive equilibrium, computation, simulation.

JEL Codes: C6, D5,E2.

1 Introduction

In this paper we review some fundamental issues in the computation and numerical simulation of dynamic economic models with heterogeneous agents and market distortions (e.g., externalities, distortionary taxation and financial frictions). These models are now an integral part of modern macroeconomics and finance, and play a central role in the analysis of fiscal and monetary policies, social security systems and savings over the life cycle, and the determinants of asset price volatility and interest rates.

A main problem with these models is numerical tractability. Economists use numerical simulations to get quantitative predictions. But for models with economic distortions or with an infinite number of overlapping generations their equilibrium solutions cannot be computed by associated global optimization problems as the welfare theorems do not hold. In the presence of multiple competitive equilibria, there is a problem of coordinating expectations that may result in the inability to invoke Bellman's optimality principle. Indeed, for a good discussion of these issues see the early seminal papers of Hellwig (1983) and Kydland and Prescott (1980). We then have that a Markovian representation of equilibria may only be possible when conditioning over an expanded set of state variables; moreover, this generalized Markovian law of motion may not be a continuous mapping. Therefore, both numerical dynamic programming algorithms and other well known numerical procedures have a limited application for the computation of competitive equilibria of these economies as these algorithms search for a continuous policy function.

These technical issues are mostly ignored in the applied literature, but are crucial to guarantee that a numerical solution is a sufficiently good approximation of the postulated economic model so that we can make reasonable inferences form the output of our computations. Of course, reliable methods are computationally costly and may not always be feasible, but they should be the starting point for the construction of faster algorithms with good convergence and accuracy properties.

We present our framework of analysis in Section 2. Here competitive equilibria are simply characterized by systems of Euler equations, feasibility, and market equilibrium conditions. Section 3 contains a simple example to illustrate non-existence of continuous Markov equilibria where numerical approximations over continuous policy functions produce biased results. In Section 4 we review some properties of the numerical algorithm of Feng *et al.* (2009) pertaining existence of Markov equilibria, convergence of the algorithm to a fixed-point solution, and accuracy properties. We also offer new insights into the simulation of economic models, which improve upon the rather *ad hoc* simulation method proposed in Feng *et al.* (2009). Some further comments and extensions follow in our final section,

Examples of non-existence of Markov equilibria can be found in Kubler and Schmedders (2002), Kubler and Polemarchakis (2004) and Santos (2002). Following Duffie *et al.* (1994), the existence of a Markov equilibrium in a generalized space of variables is proved in Kubler and Schmmeders (2003) for an asset pricing model with collateral constraints. Feng *et al.* (2009) extend the existence result to other economies, and redefine the Markov equilibrium solution over an expanded state of variables that includes the shadow values of investment. The addition of the shadow values of investment as state variables was originally proposed by Kydland and Prescott (1980), and later used in Phelan and Stacchetti (2001) in a competitive economy with a representative agent. The

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main insight of Feng *et al.* (2009) is to apply these techniques to competitive equilibrium models with heterogeneous agents and to show that this redefinition of the state space may be quite effective in the computation of these models. To guarantee convergence to a fixed-point solution the equilibrium operator iterates over a decreasing sequence of candidate compact equilibrium correspondences rather then over functions since no equilibrium function may be continuous. Feng *et al.* (2009) propose a discretized numerical algorithm that is shown to have good convergence and accuracy properties. This constitutes the first attempt to build numerical foundations for this type of algorithm in a context in which no equilibrium selection may be continuous.

2 The Analytical Framework

Time is discrete, $t = 0, 1, 2, \cdots$. The state of the economy includes a state vector of endogenous variables x and vector of exogenous shocks z. Vector x belongs to a compact domain X and contains all predetermined variables, such as agents' holdings of physical capital, human capital, and financial assets. The exogenous state vector follows a Markov chain $(z_t)_{t \geq 0}$ over a finite set Z. This Markovian process is described by positive transition probabilities $\pi(z'|z)$ for all $z, z' \in Z$. The initial state, $z_0 \in Z$, is known to all agents in the economy. Then $z^t = (z_1, z_2, ..., z_t) \in Z^t$ is a history of shocks, often called a date-event or node. Let y denote the vector of all other endogenous variables. These variables could be equilibrium prices or choice variables such as consumption and investment.

In various economic models the dynamics of the state vector x is conformed by a system of non-linear equations:

$$\varphi(x_{t+1}, x_t, y_t, z_t) = 0. \tag{2.1}$$

Function φ may incorporate technological constraints and individual budget constraints. Let m denote a vector of shadow values of the marginal return to investment for all assets and all agents. This vector is supposed to lie in a compact space M, and it will be a function of existing variables such as prices, rates of interest, and marginal utilities and productivities:

$$m_t = h(x_t, y_t, z_t). \tag{2.2}$$

Let us assume that a sequential competitive equilibrium exists and can be represented by a sequence $\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$ satisfying equations (2.1)-(2.2), and the additional system of equations

$$\Phi(x_t, y_t, z_t, E_t[m_{t+1}]) = 0, \tag{2.3}$$

where $E[m]$ is the expectations operator. Function Φ may describe individual optimality conditions (such as Euler equations), market-clearing conditions, and various types of restrictions such as short-sales and liquidity requirements. We consider that equations (2.1)-(2.3) fully characterize a sequential competitive equilibrium, and that φ, h, and Φ are continuous functions.

To compute the set of sequential competitive equilibria we define the Markovian equilibrium correspondence $V^*(x, z)$ containing all the equilibrium vectors m for any given state (x, z). From this correspondence V^*, we can generate recursively the set equilibria as V^* is the fixed point of an operator $B : V \longmapsto B(V)$ that links state variables to future equilibrium states. Operator B embodies all equilibrium conditions such as agents' optimization and market-clearing conditions from any initial node z to all immediate successor states z_+. More precisely, let $B(V)(x, z)$ be the set of all values $m = h(x, y, z)$ with the property: For given x, z there exist y and $m_+(z_+) \in V(x_+, z_+)$ with $z_+ \in Z$ such that

$$\Phi(x, y, z, \sum_{z_+ \in Z} \pi(z_+|z) m_+(z_+)) = 0,$$

and

$$\varphi(x_+, x, y, z) = 0.$$

Therefore, we are certain that for each $m \in B(V)(x, z)$ there are continuation values that satisfy the temporary equilibrium conditions. The following result is proved in Feng *et al*.

Theorem 2.1 *(convergence) Let V_0 be a compact-valued correspondence such that $V_0 \supset V^*$. Let $V_n = B(V_{n-1}), n \geq 1$. Then, $V_n \to V^*$ as $n \to \infty$. Moreover, V^* is the largest fixed point of the operator B, that is, if $V = B(V)$, then $V \subset V^*$*

Theorem 2.1 provides the theoretical foundations of our algorithm since we can apply operator B to any large compact set $V_0(x, z) \supset V^*(x, z)$ and then iterate until a desirable level of convergence is attained. An important advantage of our approach is that if multiple equilibria exist, we can find all of them. Note that if he equilibrium shadow value correspondence V^* is not single-valued, there could be multiple equilibrium selections – albeit none of them may be continuous. Moreover, there may not be an equilibrium function $y = g(x, z)$, and hence a simple recursive equilibrium may not exist.

Finally, after some technical considerations from correspondence V^* we can select a measurable policy function $y = g^y(x, z, m)$, and a transition function $m_+(z_+) = g^m(x, z, m; z_+)$, for all $z_+ \in Z$. These functions give a Markovian characterization of a dynamic equilibrium in the enlarged state

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space.

3 A Growth Model with Taxes

The economy is made up of a representative household and a single firm. For a given sequence of interest rates r_t and taxes τ_t the representative household solves the following optimization problem

$$\max \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

s.t.

$$c_t + k_{t+1} \leq \pi_t + (1 - \tau_t)\, r_t k_t + T_t$$

$$k_0 \text{ given, } 0 < \beta < 1,$$

$$c_t \geq 0,\ k_{t+1} \geq 0 \text{ for all } t \geq 0.$$

Here, c_t denotes consumption, k_t is the individual capital holdings. Taxes on capital income $\{\tau_t\}$ are functions of the aggregate capital stock K_t. All tax revenues are rebated back to the consumer as lump-sum transfers T_t.

The representative firm seeks to maximize one-period profits by employing the optimal amount of capital

$$\pi_t = \max_{K_t}\ f(K_t) - r_t K_t.$$

Let

$$f(K) = K^{1/3},\ \beta = 0.95.$$

We then consider the following piecewise linear tax schedule on capital rents

$$\tau(K) = \begin{cases} 0.10 & \text{if } K \leq 0.160002 \\ 0.05 - 10(K - 0.165002) & \text{if } 0.160002 \leq K \leq 0.170002 \\ 0 & \text{if } K \geq 0.170002. \end{cases}$$

This example is particularly attractive since it can be easily computed by our algorithm. Santos (2002, Prop. 3.4) shows that a continuous Markov equilibrium fails to exist for this specification of the model. There are three steady states, the middle one is unstable and has two complex eigenvalues, while the other two steady stares are saddle-path stable.

Standard algorithms for optimization problems approximating the Euler equation would solve for a continuous policy function of the form

$$k_{t+1} = g(k_t, \xi),$$

where g belongs to a finite dimensional space of continuous functions as defined by a vector of parameters ξ. We obtain an estimate for ξ by forming a discrete system of Euler equations,

$$u'(k^i, g(k^i, \xi)) = \beta u'(g(k^i, \xi), g(g(k^i, \xi), \xi)) \cdot \left[f'(g(k^i, \xi))(1 - \tau(g(k^i, \xi))) \right],$$

over as many grid points k^i as the dimensionality of the parameter space. Here, we assume that $g(k^i, \xi)$ belongs to the class of piecewise linear functions and employed a uniform grid of 5000 points over the domain $k \in [0.14..0.19]$. The resulting approximation, together with the highly accurate solution obtained under algorithm, are illustrated in Figure 3.1.
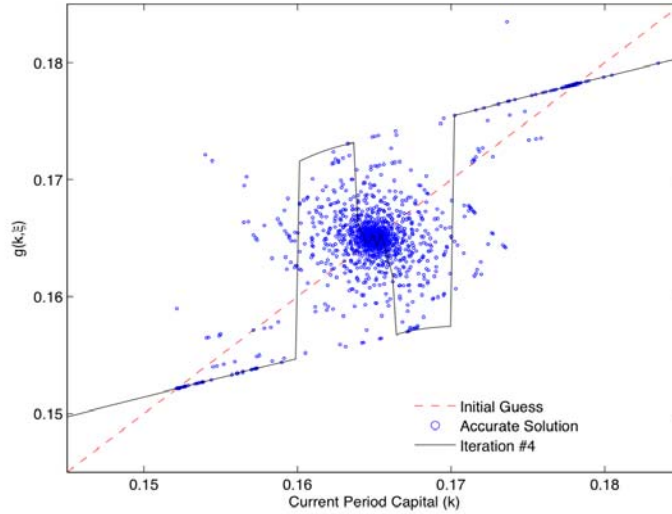

Figure 3.1: Accurate solution vs Continuous Policy Approximation.

This piecewise linear approximation of the Euler equation over piecewise continuous functions converged up to computer precision in only 3 iterations (Figure 3.2 presents the distance between consecutive function iterations). This fast convergence of the numerical method is actually deceptive because as pointed out above no continuous policy function does exist.
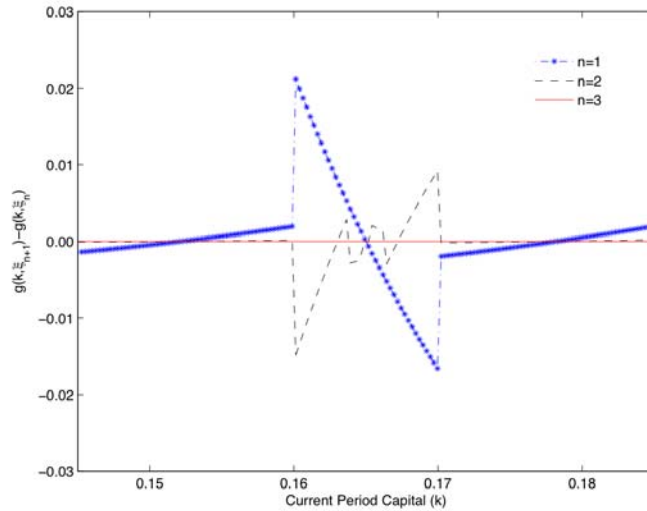
Figure 3.2: Distance Between Consecutive Continuous Function Iterations.

A further test of the fixed point solution of this algorithm, based on Euler equation residuals, produced mixed results. First, the average Euler equation residual (a standard accuracy measure) over the domain of feasible capitals is fairly small, as illustrated in Figure 3.3 below. Second, the maximum Euler equation residual is slightly more pronounced in a small area near the unstable steady state. But even in that area, the magnitude of the error is not extremely large: In three tiny intervals the Euler equation residuals are just around 0.02.
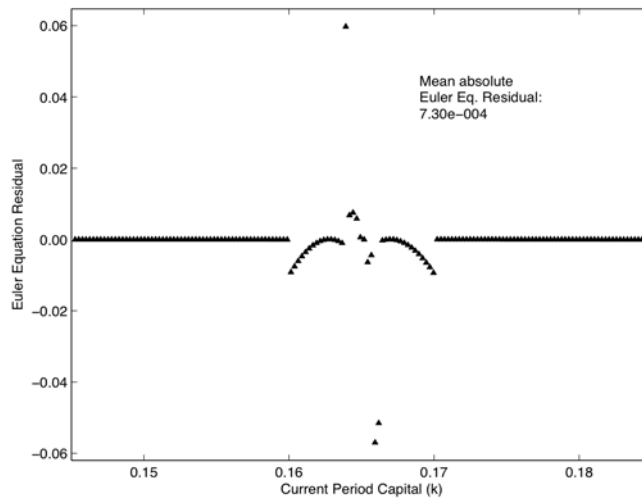

Figure 3.3: Euler Equation Residual of Continuous Approximation.

Of course, the dynamic behavior implied by the continuous function approximation is quite different

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from the true one as it displays four more steady states and substantially changes the basins of attraction of the original steady states (see Figure 3.1). Therefore, from these computational tests a researcher may be led to conclude that the purported continuous policy function should mimic well the true equilibrium dynamics.

4   Numerical Implementation

Numerical implementation of our theoretical results requires first the construction of a computable algorithm. In this section, we develop and study properties of such an algorithm. The computable algorithm uses a new operator based on discrete approximations for both the domain and the range of B. Once an equilibrium solution has been computed, we then proceed to the numerical simulation.

4.1   A Computable Algorithm

We first partition the state space into a finite set of simplices $\{X^j\}$ with non-empty interior and maximum diameter h. Over this partition we define a family of step correspondences that take constant values over each X^j. To obtain a computer representation of a step correspondence we resort to an outer approximation in which each set-value is defined by N elements. Using these two discretizations we obtain a computable approximation of operator B, which we denote by $B^{h,N}$. By a suitable selection of an initial condition V_0, the sequence $\{V_{n+1}^{h,N}\}$ defined recursively as $V_{n+1}^{h,N} = B^{h,N} V_n^{h,N}$ converges to a limit point $V^{*,h,N}$. Note that by our judicious choice of these outer approximations and of the initial condition V_0 every limit point of the sequence $\{V_{n+1}^{h,N}\}$ must contain the equilibrium correspondence V^*. The following result is proved in Feng *et al.* (2009) and it extends the convergence arguments of Beer (1980) to a dynamic setting.

Theorem 4.1 *For given h, N, and initial condition $V_0 \supseteq V^*$, consider the recursive sequence $\{V_{n+1}^{h,N}\}$ defined as $V_{n+1}^{h,N} = B^{h,N} V_n^{h,N}$. Then, (i) $V_n^{h,N} \supseteq V^*$ for all n; (ii) $V_n^{h,N} \to V^{*,h,N}$ uniformly as $n \to \infty$; and (iii) $V^{*,h,N} \to V^*$ as $h \to 0$ and $N \to \infty$.*

In conclusion, the output of our numerical algorithm is summarized by the equilibrium correspondence $V_n^{h,N}$ from operator $B^{h,N}$. By Theorem 4.1, we have that $graph[\big([B^{h,N}(V_n^{h,N})]^h\big)]$ can be made arbitrarily close to $graph[B(V^*)]$ for appropriate choices of n, h, and N. We can then choose an approximate equilibrium selection $y = g_n^{y,h,N}(x, z, m)$, and a transition function $m_+(z_+) = g_n^{m,h,N}(x, z, m; z_+)$. From these approximate equilibrium functions we can generate simulated paths $\big(x_t(z^t), y_t(z^t)\big)_{t=0}^\infty$.

4.2 Accuracy of the Simulated Moments

To assess model's predictions, analysts usually calculate moments of the simulated paths $\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$ from a numerical approximation. The idea is that the simulated moments should approach those obtained from the original model. Assuming that the optimal policy is a continuous function, Santos and Peralta-Alva (2005) establish various convergence properties of the simulated moments. They also provide examples of non-existence of stochastic steady-state solutions for non-continuous functions, and lack of convergence of empirical distributions to some invariant distribution of the model.

We now briefly outline a reliable simulation procedure that circumvents the lack of continuity of the equilibrium law of motion. We are just advancing arguments from our new paper [Santos and Peralta-Alva (2009)]. We summarize these arguments as follows:

(i) *Simulation of the computed equilibrium laws of motion* $y = g_n^{y,h,N}(x, z, m)$, *and* $m_+(z_+) = g_n^{m,h,N}(x, z, m; z_+)$. From these approximate equilibrium functions we can generate simulated paths $\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$. We prove that there are tight upper USM and lower LSM bounds such that with probability one the corresponding moments from simulated paths $\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$ stay within the prescribed bounds. More precisely, let $s = (x, m)$ and $f : S \times Z \to R_+$ be a function of interest. Let $\frac{1}{T}\sum_{t=0}^{T} f(s_t, z_t)$ represents a simulated moment or some other statistic. Then, with probability one, every limit point of $\frac{1}{T}\sum_{t=0}^{T} f(s_t, z_t)$ must be within the corresponding bounds LSM and USM.

(ii) *Existence of an invariant distribution for the original model* : An invariant distribution may actually not exist. One way to guarantee existence is to convexify the equilibrium correspondence. Thus, following Blume (1982) and Duffie *et al.* (1994) we randomize over continuation values of operator B. We construct a new operator B^{cv} that is a convex-valued correspondence in the space of probability measures. This correspondence has an invariant distribution $\mu^* \in B^{cv}(\mu^*)$.

(iii) *Accuracy of the simulated moments*: For every $\epsilon > 0$ we can consider a sufficiently good discretized operator $B^{h,N}$ and equilibrium correspondence $V_n^{h,N}$ such that for every simulated path $(s_t, z_t)_{t=0}^{\infty}$ there are invariant distributions μ^*, μ'^* of B^{cv} such that $\int f(s, z)d\mu^* - \epsilon \leq \frac{1}{T}\sum_{t=0}^{T} f(s_t, z_t) \leq \int f(s, z)d\mu'^* + \epsilon$ almost surely. Therefore, for a sufficiently fine approximation the moments from simulated paths are close to the set of moments of the invariant distributions of the model. Of course, if B^{cv} has a unique invariant distribution μ^* then $\mu'^* = \mu^*$ and the above expression reads as $\int f(s, z)d\mu^* - \epsilon \leq \frac{1}{T}\sum_{t=0}^{T} f(s_t, z_t) \leq \int f(s, z)d\mu^* + \epsilon$.

Duffie et *al.* (1994) argue that operator B^{cv} allows for some form of sunspot equilibria since the randomization proceeds over equilibrium distributions rather than over an external parameter or extraneous sunspot variable. This is not, however, the interpretation that we want to give to our simulations. For us, the primitive elements in our analysis are the Markovian equilibrium functions

which are obtained from the original equilibrium correspondences without performing arbitrary randomizations. We are therefore simulating the true model. At a later stage, to bound the range of variation of the simulated moments for both the approximate and true solutions we consider the invariant distributions of the convexified operator B^{cv}. Of course, these bounds may not be tight when this regularization changes substantially the stochastic dynamics.

5   Final Remarks

In this paper we offer a general overview of the current challenges involved in the study of dynamic models with heterogeneous agents and economic distortions. Competitive equilibria for these economies can be characterized in a recursive fashion in an enlarged state space, and can be numerically approximated by reliable methods. Competitive equilibria, however, may not be unique and present some discontinuities which may preclude the application of regular laws of large numbers in the simulation of these economies. We nevertheless outlined a simulation procedure in which we established bounds for the range of variation of the simulated moments.

What it is more important for applied work, we have provided an example in which a regular computational method may actually converge to a wrong fixed-point solution. In Feng *et al.* (2009) we consider another example of an overlapping generation economy with chaotic dynamics [Benhabib and Day (1982)] where our algorithm would work well, but more common algorithms may deliver misleading results. For more complex models with a large number of state variables [e.g., Krusell and Smith (1998)] our algorithm may not be computationally feasible, but it is our experience that some other heuristic extensions of standard algorithms may display sizable computational errors. Hence, a primary message of the present paper is a "word of caution" for the computation of economies with heterogeneous agents and economic distortions. This should lead to a rethinking of existing computational methods.

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