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Optimal Portfolio Choice under Regime Switching, Skew and Kurtosis Preferences

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Abstract

This paper proposes a new tractable approach to solving multi-period asset allocation problems. We assume that investor preferences are defined over moments of the terminal wealth distribution such as its skew and kurtosis. Time-variations in investment opportunities are driven by a regime switching process that can capture bull and bear states. We develop analytical methods that only require solving a small set of difference equations and thus are very convenient to use. These methods are applied to a simple portfolio selection problem involving choosing between a stock index and a risk-free asset in the presence of bull and bear states in the return distribution. If the market is in a bear state, investors increase allocations to stocks the longer their time horizon. Conversely, in bull markets it is optimal for investors to decrease allocations to stocks the longer their investment horizon.

Key words: Optimal Asset Allocation, Regime Switching, Skew and Kurtosis Preference.

1. Introduction

Optimal asset allocation has generated considerable interest in finance since the seminal papers by Merton (1969) and Samuelson (1969). Examples of recent studies include Ang and Bekaert (2001), Barberis (2000), Brandt (1999), Brennan, Schwarz and Lagnado (1997), Campbell and Viceira (1999, 2001), Kandel and Stambaugh (1996) and Lynch (2001). Only in very special cases such as under mean-variance or power utility with constant investment opportunities or under logarithmic utility can exact solutions to an investor's multi-period portfolio choice be derived in closed form. Unfortunately, the assumption of constant investment opportunities is at odds with considerable empirical evidence which indicates that asset returns are partially predictable.[1]

Faced with these limitations, recent papers have used numerical techniques such as quadrature methods (Ang and Bekaert (2001), Lynch (2001)) or Monte Carlo simulations (Barberis (2000)) to characterize optimal portfolio holdings. Unfortunately, these methods have their own limitations. Quadrature methods may not be very precise when the underlying asset return distributions are not

[1]See, e.g., Campbell (1987), Keim and Stambaugh (1986), Fama and French (1988) and Pesaran and Timmermann (1995).

Gaussian, as is strongly suggested by empirical research, c.f. Bollerslev et al. (1992) and Gallant and Tauchen (1989). While Monte Carlo methods do not suffer from this problem, they can be computationally expensive to use as they rely on discretization of the state space and use grid methods. This imposes severe constraints on multi-asset problems.[2]

This paper proposes a new tractable approach to optimal multi-period asset allocation which is both convenient to use and offers new insights into an investor's asset allocation problem in the presence of regime switching. We assume that investor preferences are defined over a finite number of moments of terminal wealth and thus incorporate the skew, kurtosis and possibly even higher order moments of the wealth distribution. Our approach follows recent papers in the asset pricing literature such as Harvey and Siddique (2000) and Dittmar (2002) that emphasize the need to consider moments of returns other than just the mean and variance.

Our model of investor preferences is combined with an assumption that the distribution of asset returns is driven by a regime switching process. There is now a large body of empirical evidence suggesting that returns on stocks and other financial assets can be captured by this class of models.[3] While a single Gaussian distribution generally does not provide an accurate description of stock returns, the regime switching models that we consider have far better ability to approximate the return distribution and can capture outliers, fat tails and skew.

Using this setup, we develop analytical methods for deriving the moments of the wealth distribution that only require solving a small set of difference equations corresponding to the number of regimes in the return distribution. When coupled with a utility specification that incorporates skew and kurtosis preferences, the otherwise complicated numerical problem of optimal asset allocation is reduced to that of solving for the roots of a low-order polynomial. Our solution is closed-form in the sense that it is computable with a finite number of elementary operations.

We apply our methods to a simple portfolio selection problem involving a US stock portfolio and a risk-free asset. We find evidence of two regimes in US stock returns, namely a bear state with high volatility and low mean returns and a bull state with high mean returns and low volatility. Both states are persistent and their presence generates predictability in the stock return distribution. Unsurprisingly it is optimal for investors to hold more stocks when the perceived probability of the bull state is high. Since the probability of switching to a bear state grows with the investor's horizon, a buy-and-hold investor will hold less in stocks the longer the investment horizon provided that the market starts from a bull state. In contrast, if the market starts from a bear state, stocks are unattractive in the short-run but become more attractive in the longer run since a bull state will almost certainly emerge. This creates an upward-sloping demand schedule for stocks as a function

[2]In continuous time, closed-form solutions obtain under less severe restrictions. For instance Kim and Omberg (1996) work with preferences in the HARA class defined over final wealth and assume that the single risky asset return is mean-reverting. Under identical assumptions on preferences and risk premia, Wachter (2002) shows that a closed-form solution obtains even when interim consumption is possible if markets are complete (predictors are perfectly negatively correlated with risky returns).

[3]See, e.g., Ang and Bekaert (2002), David and Veronesi (2001), Gray (1996), Perez-Quiros and Timmermann (2000), Turner, Startz and Nelson (1989), Whitelaw (2001).

of the investment horizon.

The plan of the paper is as follows. Section 2 describes investor preferences as defined over moments of the terminal wealth distribution. Section 3 introduces the regime-switching model for asset returns and documents the presence of regimes in US stock returns. Section 4 solves the asset allocation problem and derives the moments of the wealth distribution up to an arbitrary order both for the popular case with two states and the general case with multiple assets and any number of states. Section 5 provides an empirical application of our methods to US stock returns while Section 6 studies the effect of portfolio rebalancing and Section 7 concludes. An Appendix provides details of the main technical results in the paper.

2. Investor Preferences

We are interested in studying the optimal asset allocation problem at time t for an investor with a T-period investment horizon. Suppose that the investor's utility function $U(W_{t+T};\boldsymbol{\theta})$ only depends on wealth at time $t+T$, W_{t+T}, and a set of parameters, $\boldsymbol{\theta}$. The investor maximizes expected utility by choosing among h risky assets which pay continuously compounded returns $\mathbf{r}_t^s \equiv (r_{1t}\ r_{2t}\ ...\ r_{ht})'$. We collect these portfolio weights in an $h \times 1$ vector $\boldsymbol{\omega}_t \equiv (\omega_{1t}\ \omega_{2t}\ ...\ \omega_{ht})'$ and complete the asset menu by an $h+1$-th risk-free asset such that $1-\boldsymbol{\omega}_t'\boldsymbol{\iota}_h$ is invested in the risk-free security which has a continuously compounded return of r^f. The portfolio selection problem solved by a buy-and-hold investor with unit wealth is

$$\begin{aligned} &\max_{\boldsymbol{\omega}_t} \quad E_t\left[U(W_{t+T};\boldsymbol{\theta})\right] \\ &s.t.\ W_{t+T} = \left\{(1-\boldsymbol{\omega}_t'\boldsymbol{\iota}_h)\exp\left(Tr^f\right) + \boldsymbol{\omega}_t'\exp\left(\mathbf{R}_{t+T}^s\right)\right\} \end{aligned} \tag{1}$$

where $\mathbf{R}_{t+T}^s \equiv \mathbf{r}_{t+1}^s + \mathbf{r}_{t+2}^s + ... + \mathbf{r}_{t+T}^s$ is the vector of continuously compounded risky returns over the $T-$period investment horizon. Accordingly, $\exp\left(\mathbf{R}_{t+T}^s\right)$ is a vector of cumulated returns. Short-selling can be imposed through the constraint $\omega_{it} \in [0,1]$ for $i = 1,2,...,h$. Rebalancing is introduced in Section 6, but we exclude this for the moment to keep the problem simple.

For general preferences there is no closed-form solution to (1). Given the economic importance of problems such as (1), it is not surprising that numerous approaches have been suggested for its solution. Campbell and Viceira (1999, 2001) develop analytical approximations to the investor's Euler equation and intertemporal budget constraint and solve for asset holdings and optimal consumption when time-variations in investment opportunities are driven by a state variable that follows a first-order autoregressive process. Assuming power utility and return predictability from the dividend yield, Barberis (2000) resorts to Monte Carlo simulation methods to solve (1). Under a similar specification for stochastic investment opportunities, Lynch (2001) uses Gaussian quadrature methods to approximate the objective function. In the presence of regime switching in asset returns, Ang and Bekaert (2001) also apply Gaussian quadrature techniques. These methods have yielded important insights into the solution to (1), but are often computationally expensive or impose specific conditions on the stochastic process driving asset returns.

2.1. *Preferences over Moments of the Wealth Distribution*

Building on the work of Scott and Horvath (1980), Harvey and Siddique (2000) and Dittmar (2002) we follow a different approach and study preference functionals that improve over classical preferences such as mean-variance by taking into account a generic number of moments (m) of the wealth process.

For this purpose, we consider an m-th order Taylor expansion of a generic utility function $U(W_{t+T};\boldsymbol{\theta})$ around a wealth level v_T:

$$U(W_{t+T};\boldsymbol{\theta}) = \sum_{n=0}^{m} \frac{1}{n!} U^{(n)}(v_T;\boldsymbol{\theta})\,(W_{t+T} - v_T)^n + R_m, \tag{2}$$

where $R_m = o((W_{t+T} - v_T)^m)$ and $U^{(0)}(v_T;\boldsymbol{\theta}) = U(v_T;\boldsymbol{\theta})$. $U^{(n)}(.)$ denotes the $n-$th derivative of the utility function with respect to terminal wealth. Suppose the utility function $U(W_{t+T};\boldsymbol{\theta})$ is continuously differentiable with $U'(W_{t+T};\boldsymbol{\theta}) > 0$, $U''(W_{t+T};\boldsymbol{\theta}) < 0$, for all W_{t+T}, and that, for all $n \geq 3$, the following conditions hold:

$$\begin{aligned} U^{(n)}(W_{t+T};\boldsymbol{\theta}) &> 0, \\ U^{(n)}(W_{t+T};\boldsymbol{\theta}) &= 0, \text{ or} \\ U^{(n)}(W_{t+T};\boldsymbol{\theta}) &< 0, \end{aligned} \tag{3}$$

Assumption (3) is what Scott and Horvath (1980) call strict consistency for moment preference. It simply states that the n-th order derivative is either always negative, always positive, or everywhere zero for all possible wealth levels. Under these assumptions, Scott and Horvath show that the following restrictions follow:

$$\begin{aligned} U^{(3)}(W_{t+T};\boldsymbol{\theta}) &> 0 \qquad U^{(4)}(W_{t+T};\boldsymbol{\theta}) < 0 \\ U^{(n\ odd)}(W_{t+T};\boldsymbol{\theta}) &> 0 \qquad U^{(n\ even)}(W_{t+T};\boldsymbol{\theta}) < 0 \end{aligned} \tag{4}$$

In particular, $U^{(3)}(W_{t+T};\boldsymbol{\theta}) < 0$ can be proven to violate the assumption of positive marginal utility, so we must have $U^{(3)}(W_{t+T};\boldsymbol{\theta}) > 0$. Likewise, $U^{(4)}(W_{t+T};\boldsymbol{\theta}) > 0$ would violate the assumption of strict risk-aversion. More generally, the strict consistency requirement in (3) therefore implies that all the odd derivatives of $U(W_{t+T};\boldsymbol{\theta})$ are positive while all the even derivatives are negative.

Provided that the Taylor series (2) converges, that the distribution of wealth is uniquely determined by its moments, and that the order of sums and integrals can be exchanged, (2) extends to the expected utility functional:

$$E_t[U(W_{t+T};\boldsymbol{\theta})] = \sum_{n=0}^{m} \frac{1}{n!} U^{(n)}(v_T;\boldsymbol{\theta}) E_t[(W_{t+T} - v_T)^n] + \hat{R}_m,$$

where $\hat{R}_m$ is another remainder term. We thus have

$$E_t[U(W_{t+T};\boldsymbol{\theta})] \approx \hat{E}_t[U^m(W_{t+T};\boldsymbol{\theta})] = \sum_{n=0}^{m} \frac{1}{n!} U^{(n)}(v_T;\boldsymbol{\theta}) E_t[(W_{t+T} - v_T)^n], \tag{5}$$

where the approximation improves as $m \to +\infty$. Many classes of Von-Neumann Morgenstern expected utility functions can thus be well approximated by a function of the form:

$$\hat{E}_t[U^m(W_{t+T};\boldsymbol{\theta})] = \sum_{n=0}^{m} \kappa_n E_t[(W_{t+T} - v_T)^n], \tag{6}$$

with $\kappa_0 > 0$, and κ_n positive (negative) if n is odd (even). We call these objectives $m-$moment preference functionals.

3. The Return Process

A large empirical literature has documented the presence of persistent 'regimes' in a variety of financial time series. Ang and Bekaert (2002), Driffill and Sola (1994), Gray (1996), Hamilton (1988)) find evidence of multiple states in the dynamics of interest rates, while Ang and Bekaert (2001), David and Veronesi (2001), Perez-Quiros and Timmermann (2000), Turner, Starz and Nelson (1989) and Whitelaw (2001) provide evidence for stock market returns. Typically these states capture periods of high and low volatility in returns.

Following this literature, suppose that the vector of h continuously compounded returns, $\mathbf{r}_t = (r_{1t}, r_{2t}, ..., r_{ht})'$, follows a Markov switching vector autoregressive process driven by a common state variable, S_t, that takes integer values between 1 and k:

$$\mathbf{r}_t = \boldsymbol{\mu}_{s_t} + \sum_{j=1}^{p} \mathbf{A}_{j,s_t} \mathbf{r}_{t-j} + \boldsymbol{\varepsilon}_t. \tag{7}$$

Here $\boldsymbol{\mu}_{s_t} = (\mu_{1st}, ..., \mu_{hs_t})'$ is a vector of intercepts in state s_t, $\mathbf{A}_{j,s_t}$ is an $h \times h$ matrix of autoregressive coefficients associated with the j-th lag in state s_t, and $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, ..., \varepsilon_{ht})' \sim N(\mathbf{0}, \Omega_{s_t})$ is a vector of Gaussian return innovations with zero mean vector and state-dependent covariance matrix Ω_{s_t}:

$$\Omega_{s_t} = E\left[\left(\mathbf{r}_t - \boldsymbol{\mu}_{s_t} - \sum_{j=1}^{p} \mathbf{A}_{j,s_t} \mathbf{r}_{t-j} \right) \left(\mathbf{r}_t - \boldsymbol{\mu}_{s_t} - \sum_{j=1}^{p} \mathbf{A}_{j,s_t} \mathbf{r}_{t-j} \right)' | s_t \right].$$

The state-dependence of the covariance matrix captures the possibility of heteroskedastic shocks to asset returns, which is supported by strong empirical evidence, c.f. Bollerslev et al. (1992). Each state is assumed to be the realization of a first-order, homogeneous Markov chain and the transition probability matrix, $\mathbf{P}$, governing the evolution in the common state variable, S_t, is given by

$$\Pr(s_t = j | s_{t-1} = i) = p_{ij}, \quad i, j = 1, .., k. \tag{8}$$

Conditional on knowing the state next period, the return distribution is Gaussian. However, since future states are never known in advance, the return distribution is a mixture of normals with the mixture weights reflecting the current state probabilities and the transition probabilities.

There are many advantages to modelling returns as mixtures of Gaussian distributions. As pointed out by Marron and Wand (1992), mixtures of normal distributions provide a very flexible

family that can be used to approximate numerous other distributions.[4] They can capture skew and kurtosis in a way that is easily characterized as a function of the mean, variance and persistence parameters of the underlying states. They can also accommodate predictability and serial correlation in returns and volatility clustering since they allow the first and second moments to follow a step function driven by shifts in the underlying regime process, c.f. Timmermann (2000).

Even in the absence of autoregressive terms, (7) implies time-varying investment opportunities. For example, the conditional mean of asset returns is an average of the vector of mean returns, $\boldsymbol{\mu}_{s_t}$, weighted by the current state probabilities $(\Pr(s_t = 1|\Im_t), .., \Pr(s_t = k|\Im_t))'$, conditional on information available at time t, which we denote by $\Im_t$. Since these state probabilities vary over time, the expected return will also change. In addition, our approach is very flexible and can readily be extended to incorporate a range of predictor variables such as the dividend yield. This is done simply by expanding the vector $\mathbf{r}_t$ with additional predictor variables, $\mathbf{z}_t$ and modeling their joint process $\mathbf{y}_t = (\mathbf{r}_t' \ \mathbf{z}_t')'$.

3.1. *Regimes in US Stock Returns*

Our empirical application considers one of the most commonly studied portfolio problems in finance, namely the allocation to a broad portfolio of US stocks and a risk-free asset. Before proceeding further, we thus consider whether the regime switching model (7) applies to US stock returns. We examine returns on the value-weighted portfolio of NYSE stocks provided by the Center for Research in Security Prices (CRSP). The risk-free rate is measured by the 30-day T-bill rate. We model excess returns defined as the difference between the stock return and the T-bill rate. Our data are monthly and cover the sample period 1952:6 - 1999:12. Returns are continuously compounded.

The first question that arises is of course whether multiple regimes are required to model US stock returns. To answer this we considered the single-state specification tests suggested by Davies (1977) and Garcia (1998).[5] These rejected the linear specification very strongly.[6]

The next issue is to determine the number of regimes. For this purpose we adopted two methods. We first considered statistical information criteria that trade off fit against parsimony. The Schwarz information criterion which consistently selects the true model in large samples chose a two-state specification without any lags.[7]

We also adopted an approach based on specification tests for the entire return distribution. Calculation of expected utility in (1) requires integrating over the entire probability distribution of returns. It is important to use a model for stock returns whose predictive density is not misspec-

[4]Mixtures of normals can also be viewed as a nonparametric approach if the number of states, k, is allowed to grow with the sample size.

[5]These tests account for the problem that arises because the regime switching models have parameters that are unidentified under the null hypothesis of a single regime. This means that standard critical values cannot be used in the hypothesis testing.

[6]A likelihood ratio test of the null of $k = 1$ vs. the alternative of $k = 2$ for a model with state-dependent means and variances yields a test statistic of 53.2 which carries a p-value of 0.000.

[7]See Bossaerts and Hillion (1999) for a discussion and application of information criteria in models of financial returns.

ified, so we conducted a set of specification tests that consider the entire conditional probability distribution of excess returns. These tests are based on the so-called probability integral transform examined by Diebold, Gunther and Tay (1998). We follow Berkowitz (2001) in considering four separate tests for misspecification related to the first four moments of stock returns in addition to any evidence of serial correlation in the normalized residuals. Table 1 shows that the single-state model was strongly rejected, while a two-state model with state-dependent mean and variance passed all tests at the 10% significance level. This is the model we use in our subsequent analysis.

To interpret the two states from an economic perspective, we present parameter estimates in Table 2 and plot the smoothed state probabilities in Figure 1. First consider the parameter estimates. In the linear benchmark model, the mean excess return is 0.67% per month while the volatility is 4.2% per month. This, however, conceals two very different states. In the first state the mean return is -0.93% and the volatility is 6.3% per month. In the second state the mean return is 1.11% and, at 3.3%, the volatility is around half its level in the first state. The first state is thus a high-volatility bear state while the second state is a low-volatility bull state. Interestingly, mean returns in both states are significantly different from zero at the 5% critical level. The persistence of the bear state (0.81) is considerably lower than that of the bull state (0.95). As a consequence, the average duration of a bear state is 5 months, while it is 20 months for the bull state.

Figure 1 shows that the bear state probability is high around most official recession periods, but also rises on many other occasions characterized by high volatility in returns. There does not appear to be a stable lead-lag pattern between the bear state probabilities and official recession periods. Most of the time it is clear what state the market is in and the state probabilities are far away from 0.5.

4. **The Portfolio Allocation Problem**

This section characterizes the solution to the investor's optimal asset allocation problem when preferences are defined over moments of terminal wealth (6) while returns follow the regime switching process (7). We first study the problem under the simplifying assumption of a single risky asset ($n = 1$), a regime switching process with two states ($k = 2$) and no autoregressive terms ($p = 0$). For this case, the return process is simply

$$\begin{aligned} r_t &= \mu_{s_t} + \sigma_{s_t}\varepsilon_t, \quad s_t = 1, 2, \\ \Pr(s_t &= i | s_{t-1} = i) = p_{ii}, \; i = 1, 2 \end{aligned} \tag{9}$$

Concentrating on this case allows us to convey intuition for the more general results. It also provides an accurate model in many empirical applications, c.f. Section 3.1. With a single risky asset, the wealth process is simply

$$W_{t+T} = \left\{ (1 - \omega_t) \exp\left(Tr^f\right) + \omega_t \exp\left(R_{t+T}\right) \right\} \tag{10}$$

where $R_{t+T} \equiv r_{t+1} + r_{t+2} + ... + r_{t+T}$ is the continuously compounded return on the risky asset over the T periods and ω_t is the stock holding. Without loss of generality, initial wealth is normalized

at one.

For a given value of ω_t, the only unknown component in (10) is the cumulated return, $\exp(R_{t+T})$. To be able to use the results from Section 2, our first task is to characterize the moments of this term.

4.1. *Moments of the Cumulated Return Distribution with two States*

We are interested in deriving the n-th central moment of the cumulated return distribution:

$$\tilde{M}_{t+T}^{(n)} = E\left[(\exp(r_{t+1} + ... + r_{t+T}) - E[\exp(r_{t+1} + ... + r_{t+T})])^n\right].$$

It turns out that it is easier to derive recursive expressions for the non-central moments.[8] Under the assumption of two states, $k = 2$, the nth non-central moment of the cumulated returns is given by

$$\begin{aligned} M_{t+T}^{(n)} &= E\left[(\exp(r_{t+1} + ... + r_{t+T}))^n \,|\Im_t\right] \\ &= \sum_{s_{t+T}=1}^{2} E\left[(\exp(r_{t+1} + ... + r_{t+T}))^n \,|s_{t+T}, \Im_t\right] \Pr(s_{t+T}|\Im_t) \\ &\equiv M_{1t+T}^{(n)} + M_{2t+T}^{(n)}, \end{aligned} \tag{11}$$

where we used the total probability theorem. The nth conditional moment $M_{it+1}^{(n)}$ satisfies the recursions

$$\begin{aligned} M_{it+T}^{(n)} &= E\left[\exp(n(r_{t+1} + ... + r_{t+T-1}))|s_{t+T}\right] E\left[\exp(nr_{t+T})|s_{t+T}, \Im_t\right] \Pr(s_{t+T}|\Im_t) \\ &= \left(M_{it+T-1}^{(n)} p_{ii} + M_{-i,t+T-1}^{(n)}(1 - p_{-i-i})\right) \exp\left(n\mu_i + \frac{n^2}{2}\sigma_i^2\right), \quad (i = 1, 2) \end{aligned}$$

where we used the notation $-i$ for the converse of state i, i.e. $-i = 2$ when $i = 1$ and vice versa. In more compact notation we have

$$\begin{aligned} M_{1t+1}^{(n)} &= \alpha_1^{(n)} M_{1t}^{(n)} + \beta_1^{(n)} M_{2t}^{(n)} \\ M_{2t+1}^{(n)} &= \alpha_2^{(n)} M_{1t}^{(n)} + \beta_2^{(n)} M_{2t}^{(n)}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \alpha_1^{(n)} &= p_{11} \exp\left(n\mu_1 + \frac{n^2}{2}\sigma_1^2\right) \\ \beta_1^{(n)} &= (1 - p_{22}) \exp\left(n\mu_1 + \frac{n^2}{2}\sigma_1^2\right) \\ \alpha_2^{(n)} &= (1 - p_{11}) \exp\left(n\mu_2 + \frac{n^2}{2}\sigma_2^2\right) \\ \beta_2^{(n)} &= p_{22} \exp\left(n\mu_2 + \frac{n^2}{2}\sigma_2^2\right). \end{aligned}$$

[8] The centered moments, $\tilde{M}_{t+T}^{(n)}$ can be derived from the first n non-central moments simply by expanding $E\left[(\exp(r_{t+1} + ... + r_{t+T}) - E[\exp(r_{t+1} + ... + r_{t+T})|\Im_t])^n \,|\Im_t\right]$.

Equation (12) can be reduced to a set of second order difference equations:

$$M_{it+2}^{(n)} = (\alpha_1^{(n)} + \beta_2^{(n)})M_{it+1}^{(n)} + (\alpha_2^{(n)}\beta_1^{(n)} - \beta_2^{(n)}\alpha_1^{(n)})M_{it}^{(n)}, \quad (i = 1, 2) \tag{13}$$

Collecting the two regime-dependent moments into a 2×1 vector $\boldsymbol{\vartheta}_{it+T}^{(n)} \equiv (M_{it+T}^{(n)} \; M_{it+T-1}^{(n)})'$, equation (13) can be written in companion form

$$\boldsymbol{\vartheta}_{it+T}^{(n)} = \begin{bmatrix} \alpha_1^{(n)} + \beta_2^{(n)} & \alpha_2^{(n)}\beta_1^{(n)} - \beta_2^{(n)}\alpha_1^{(n)} \\ 1 & 0 \end{bmatrix} \boldsymbol{\vartheta}_{it+T-1}^{(n)} \equiv \mathbf{A}^{(n)}\boldsymbol{\vartheta}_{it+T-1}^{(n)}.$$

Substituting backwards we get the following equation for the ith conditional moment:

$$\boldsymbol{\vartheta}_{it+T}^{(n)} = \left(\mathbf{A}^{(n)}\right)^T \boldsymbol{\vartheta}_{it}^{(n)}.$$

The elements of $\mathbf{A}^{(n)}$ only depend on the mean and variance parameters of the two states $(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2)$ and the state transition parameters, (p_{11}, p_{22}).

Applying similar principles at $T = 1, 2$ and letting $\pi_{1t} = \Pr(s_t = 1|\Im_t)$, the initial conditions used in determining the nth moment are as follows:

$$\begin{aligned}
M_{1t+1}^{(n)} &= (\pi_{1t}p_{11} + (1 - \pi_{1t})(1 - p_{22})) \exp\left(n\mu_1 + \frac{n^2}{2}\sigma_1^2\right), \\
M_{1t+2}^{(n)} &= p_{11}\left(\pi_{1t}p_{11} + (1 - \pi_{1t})(1 - p_{22})\right) \exp\left(2n\mu_1 + n^2\sigma_1^2\right) + \\
&\quad + (1 - p_{22})\left(\pi_{1t}(1 - p_{11}) + (1 - \pi_1)p_{22}\right) \exp\left(n(\mu_1 + \mu_2) + \frac{n^2}{2}(\sigma_1^2 + \sigma_2^2)\right), \\
M_{2t+1}^{(n)} &= (\pi_{1t}(1 - p_{11}) + (1 - \pi_1)p_{22}) \exp\left(n\mu_2 + \frac{n^2}{2}\sigma_2^2\right), \\
M_{2t+2}^{(n)} &= p_{22}\left(\pi_{1t}(1 - p_{11}) + (1 - \pi_1)p_{22}\right) \exp\left(2n\mu_2 + n^2\sigma_2^2\right) + \\
&\quad + (1 - p_{11})\left(\pi_{1t}p_{11} + (1 - \pi_{1t})(1 - p_{22})\right) \exp\left(n(\mu_1 + \mu_2) + \frac{n^2}{2}(\sigma_1^2 + \sigma_2^2)\right).
\end{aligned} \tag{14}$$

Finally, using (11) we get an equation for the nth moment of the cumulated return.

$$M_{t+T}^{(n)} = M_{1t+T}^{(n)} + M_{2t+T}^{(n)} = \mathbf{e}_1'\boldsymbol{\vartheta}_{1t+T}^{(n)} + \mathbf{e}_2'\boldsymbol{\vartheta}_{2t+T}^{(n)} = \mathbf{e}_1'\left(\mathbf{A}^{(n)}\right)^T \boldsymbol{\vartheta}_{1t}^{(n)} + \mathbf{e}_2'\left(\mathbf{A}^{(n)}\right)^T \boldsymbol{\vartheta}_{2t}^{(n)}, \tag{15}$$

where $\mathbf{e}_i$ is a 2×1 vector of zeros except for unity in the ith place.

Having obtained the moments of the cumulated return process, it is simple to compute the expected utility by using (6) and (10):

$$\begin{aligned}
\hat{E}_t[U^m(W_{t+T}; \boldsymbol{\theta})] &= \sum_{n=0}^{m} \kappa_n \sum_{j=0}^{n} (-1)^{n-j} v_T^{n-j} \binom{n}{j} E_t[W_{t+T}^j] \\
&= \sum_{n=0}^{m} \kappa_n \sum_{j=0}^{n} (-1)^{n-j} v_T^{n-j} \binom{n}{j} \sum_{i=0}^{j} \binom{j}{i} \omega_t^i M_{t+T}^i \left((1 - \omega_t)\exp\left(Tr^f\right)\right)^{j-i}.
\end{aligned} \tag{16}$$

The first order condition is obtained by differentiating with respect to ω_t :

$$\frac{\partial \hat{E}_t[.]}{\partial \omega_t} = \sum_{n=0}^{m} \kappa_n \sum_{j=0}^{n} (-1)^{n-j} v_T^{n-j} \binom{n}{j} \sum_{i=1}^{j} i \binom{j}{i} \omega_t^{i-1} M_{t+T}^i \left((1-\omega_t) \exp\left(T r^f \right) \right)^{j-i} = 0.$$

Second order conditions are satisfied by our earlier assumptions about the derivatives of $U(.)$. Notice that the first order condition takes the form of the roots of an $n-1$th order polynomial in ω_t, which are easily obtained. The optimal solution for ω_t corresponds to the root for which (16) has the highest value.

4.1.1. Expected Returns

As an illustration of the moment equations, we set $n = 1$ and consider the expected value of the cumulated return on the risky asset. The characteristic equation associated with (13) reduces to[9]

$$r^2 - (\alpha_1 + \beta_2) r - (\alpha_2 \beta_1 - \beta_2 \alpha_1) = 0$$

with solution

$$\begin{aligned} r_1, r_2 &= \frac{1}{2} \left\{ \alpha_1 + \beta_2 \pm \sqrt{(\alpha_1 + \beta_2)^2 + 4(\alpha_2 \beta_1 - \beta_2 \alpha_1)} \right\} \\ &= \frac{1}{2} \left\{ \alpha_1 + \beta_2 \pm \sqrt{(\alpha_1 - \beta_2)^2 + 4 \alpha_2 \beta_1} \right\}. \end{aligned} \tag{17}$$

Both roots are real since the term inside the square root is positive. The solution thus takes the form

$$\begin{aligned} M_{1t+T} &= C_1 r_1^T + C_2 r_2^T \\ M_{2t+T} &= C_3 r_1^T + C_4 r_2^T. \end{aligned} \tag{18}$$

The constants C_1, C_2, C_3 and C_4 can be derived by evaluating (18) at $T = 1, 2$:

$$\begin{aligned} C_1 &= \frac{M_{1t+2}^{(1)} - M_{1t+1}^{(1)} r_2}{r_1^2 - r_1 r_2}, \\ C_2 &= \frac{M_{1t+1}^{(1)} - C_1 r_1}{r_2} \\ C_3 &= \frac{M_{2t+2}^{(1)} - M_{2t+1}^{(1)} r_2}{r_1^2 - r_1 r_2}, \\ C_4 &= \frac{M_{2t+1}^{(1)} - C_3 r_1}{r_2}. \end{aligned} \tag{19}$$

The expected value of the cumulated return on the risky asset is thus given by

$$M_{t+T}^{(1)} = M_{1t+T}^{(1)} + M_{2t+T}^{(1)} = (C_1 + C_3) r_1^T + (C_2 + C_4) r_2^T. \tag{20}$$

This is readily evaluated for arbitrary horizons, T. Higher order moments give rise to very similar solutions.

[9] For simplicity, we suppress the superscripts on the α, β values.

4.2. *General Results*

So far we have ignored that in many applications $\mathbf{r}_t$ is a vector of returns on a multi-asset portfolio. The number of states, k, may also exceed two. For the general case with h risky assets and k states, the wealth process is

$$W_{t+T} = \boldsymbol{\omega}_t' \exp\left(\sum_{i=1}^{T} \mathbf{r}_{t+i}\right) + (1 - \boldsymbol{\omega}_t' \boldsymbol{\iota}_h) \exp(r^f T).$$

The moments of the wealth process are complicated to derive and involve lots of cross-product terms. For example, in the case with only two risky assets, the third moment is

$$\begin{aligned}
E_t[W_{t+T}^3] &= E_t\left[\omega_{1t}^3 \exp\left(3\sum_{i=1}^{T} r_{1,t+i}\right) + 3\omega_{1t}^2\omega_{2t} \exp\left(2\sum_{i=1}^{T} r_{1,t+i} + \sum_{i=1}^{T} r_{2,t+i}\right)\right. \\
&\quad \left. +3\omega_{1t}\omega_{2t}^2 \exp\left(\sum_{i=1}^{T} r_{1,t+i} + 2\sum_{i=1}^{T} r_{2,t+i}\right) + \omega_{2t}^3 \exp\left(3\sum_{i=1}^{T} r_{2,t+i}\right)\right] \\
&\quad +3E_t\left[\omega_{1t}^2 \exp\left(2\sum_{i=1}^{T} r_{1,t+i}\right) + 2\omega_{1t}\omega_{2t} \exp\left(\sum_{i=1}^{T} r_{1,t+i} + \sum_{i=1}^{T} r_{2,t+i}\right) + \right. \\
&\quad \left. +\omega_{2t}^2 \exp\left(2\sum_{i=1}^{T} r_{2,t+i}\right)\right](1 - \omega_{1t} - \omega_{2t}) \exp(r^f T) \\
&\quad +3E_t\left[\omega_{1t} \exp\left(\sum_{i=1}^{T} r_{1,t+i}\right) + \omega_{2t} \exp\left(\sum_{i=1}^{T} r_{2,t+i}\right)\right] \\
&\quad \times (1 - \omega_{1t} - \omega_{2t})^2 \exp(2r^f T) + (1 - \omega_{1t} - \omega_{2t})^3 \exp(3r^f T).
\end{aligned}$$

The complexity of the moment expressions grows by an order of magnitude for larger values of k and h. It is therefore necessary to have a simple, recursive procedure for evaluating the moments of the cumulated returns. This is provided in Proposition 1:

Proposition 1. *Under the regime-switching process (7) and m−moment preferences (6), the expected utility is given by*

$$\begin{aligned}
\hat{E}_t[U^m(W_{t+T};\boldsymbol{\theta})] &= \sum_{n=0}^{m} \kappa_n \sum_{j=0}^{n} (\text{-}1)^{n-j} v_T^{n-j} {}_nC_j E_t[W_{t+T}^j] \\
&= \sum_{n=0}^{m} \kappa_n \sum_{j=0}^{n} (\text{-}1)^{n-j} v_T^{n-j} \binom{n}{j} \sum_{i=0}^{j} \binom{j}{i} E_t[\left(\boldsymbol{\omega}_t' \exp\left(\mathbf{R}_{t+T}^s\right)\right)^i]((1\text{-}\boldsymbol{\omega}_t'\boldsymbol{\iota}_h) \exp\left(Tr^f\right))^{j-i}
\end{aligned}$$

The nth moment of the cumulated return on the risky asset portfolio is

$$E_t[\left(\boldsymbol{\omega}_t' \exp\left(\mathbf{R}_{t+T}^s\right)\right)^n] = \sum_{n_1=0}^{n} \cdots \sum_{n_h=0}^{n} (\omega_1^{n_1} \times ... \times \omega_h^{n_h}) M_{t+T}^{(n)}(n_1, ..., n_h),$$

and $M_{t+T}^{(n)}(n_1, ..., n_h)$ can be evaluated recursively, using (A4) in the Appendix.

The appendix derives this result. Proposition 1 is very convenient to use to derive the expected utility. The solution is closed-form in the sense that it reduces the expected utility calculation to a finite number of steps each of which can be solved by elementary operations.

5. **Empirical Application to US Stock Returns**

This section considers the effect of regime switching on optimal stock holdings in the context of a simple model with a single risky asset (US stocks) and a risk-free asset. Initially we focus on the decisions of a buy-and-hold investor. Section 6 introduces portfolio rebalancing.

To apply the methods in Section 4, we need to determine how many moments, m, to include in the preference specification. We follow Dittmar (2002) and use $m = 4$. The utility function thus accounts for preferences specified over the skew and kurtosis of terminal wealth. As shown by Kimball (1993), this choice can also be justified on the basis that non-satiation, decreasing absolute risk aversion, and decreasing absolute prudence determine the signs of the first four derivatives of $U(W_{t+T};\boldsymbol{\theta})$.[10]

The weights on the first four moments of the wealth distribution are determined to ensure that our results can be compared to those in the existing literature. Most studies on optimal asset allocation use power utility so our benchmark is

$$U(W_{t+T};\theta) = \frac{W_{t+T}^{1-\theta}}{1-\theta},\ \theta > 0. \tag{21}$$

For a given coefficient of relative risk aversion, θ, the functional form (21) serves as a guide in setting values of $\{\kappa_n\}_{n=0}^m$ in (6).[11] Expanding the powers of $(W_{t+T} - v_T)$ and taking expectations, we obtain the following expression for the four-moment preference function:

$$\hat{E}_t[U^4(W_{t+T};\theta)] = \kappa_0(\theta) + \kappa_1(\theta)E_t[W_{t+T}] + \kappa_2(\theta)E_t[W_{t+T}^2] + \kappa_3(\theta)E_t[W_{t+T}^3] + \kappa_4(\theta)E_t[W_{t+T}^4] \tag{22}$$

where

$$\begin{aligned}
\kappa_0(\theta) &\equiv v_T^{1-\theta}\left[(1-\theta)^{-1} - 1 - \frac{1}{2}\theta - \frac{1}{6}\theta(\theta+1) - \frac{1}{24}\theta(\theta+1)(\theta+2)\right] \\
\kappa_1(\theta) &\equiv \frac{1}{6}v_T^{-\theta}\left[6 + 6\theta + 3\theta(\theta+1) + \theta(\theta+1)(\theta+2)\right] > 0 \\
\kappa_2(\theta) &\equiv -\frac{1}{4}\theta v_T^{-(1+\theta)}\left[2 + 2(\theta+1) + (\theta+1)(\theta+2)\right] < 0 \\
\kappa_3(\theta) &\equiv \frac{1}{6}\theta(\theta+1)(\theta+3)v_T^{-(2+\theta)} > 0 \\
\kappa_4(\theta) &\equiv -\frac{1}{24}\theta(\theta+1)(\theta+2)v_T^{-(3+\theta)} < 0.
\end{aligned}$$

[10] These assumptions seem reasonable. Since Arrow (1971) it has been common to assume positive and decreasing marginal utility of wealth or, equivalently, non-satiation and strict risk aversion.

[11] The power utility function is simply used as a device for calibrating the weights on the first four moments since Taylor series expansions of this function do not converge, c.f. Loistl (1976).

This expression is consistent with our earlier comments regarding the signs of the coefficients $\{\kappa_n\}_{n=0}^4$: the expected utility from final wealth increases in $E_t[W_{t+T}]$ and $E_t[W_{t+T}^3]$, so that higher expected returns and more right-skewed distributions lead to higher expected utility. Conversely, expected utility is a decreasing function of the second and fourth moments of the terminal wealth distribution.

A solution to the optimal asset allocation problem can now easily be found from (22) by solving a system of cubic equations in $\hat{\boldsymbol{\omega}}_t$ derived from the first and second order conditions

$$\nabla_{\boldsymbol{\omega}_t} \hat{E}_t[U^4(W_{t+T};\theta)]\Big|_{\hat{\boldsymbol{\omega}}_t} = \mathbf{0}', \quad H_{\boldsymbol{\omega}_t} \hat{E}_t[U^4(W_{t+T};\theta)]\Big|_{\hat{\boldsymbol{\omega}}_t} \text{ is negative definite.}$$

Thus $\hat{\boldsymbol{\omega}}_t$ sets the gradient, $\nabla_{\boldsymbol{\omega}_t} \hat{E}_t[U^4(W_{t+T};\theta)]$, to a vector of zeros and produces a negative definite Hessian matrix, $H_{\boldsymbol{\omega}_t} \hat{E}_t[U^4(W_{t+T};\theta)]$.[12]

5.1. *Empirical Results*

Since the return distribution is very different in the bull and bear state, the state probability perceived by investors is a key determinant of their asset holdings. Similarly, the investment horizon is important since the two regimes capture a mean reverting component in stock returns. Investors can be fairly sure that the current state will apply in the short-run, particularly in case of the more persistent bull state. Regime switching is, however, more likely to occur at longer investment horizons.

This observation is key to understanding Figure 2 which plots the optimal allocation to stocks as a function of the investment horizon and the bull state probability. This figure imposes the short-sales constraint, $\omega_t \in [0, 1]$. The figure reveals a very interesting interaction between the underlying state probabilities and the investment horizon. To interpret the figure, suppose that the initial bull state probability equals one. Starting from the bull state, investors are 95% certain that the bull state will continue next month and this makes stocks an attractive investment. At the shortest investment horizon, the allocation to stocks is therefore 100%. However, as the investment horizon grows there is a higher chance of switching to the unattractive bear state, so investors allocate less to stocks. In contrast, starting from the bear state, stocks are not very attractive to short-term investors. However, as the investment horizon grows, there is a high chance that the market will switch to the bull state and stocks become increasingly attractive.[13]

To isolate the effects of state beliefs on the optimal stock holdings, Figure 3 shows optimal investments for different values of the bull state probability. It is particularly clear from this figure that, starting from the bull state, the stock demand schedules are downward sloping as a function of the investment horizon. Conversely, starting from the bear state, the stock demand schedules are upward sloping.

[12] In practice, choosing the point around which the Taylor series expansion is computed, v_T, can be cumbersome since this depends itself on $\boldsymbol{\omega}_t$ which is unknown. To resolve this issue, we set $v_T = E_t[W_{t+T-1}]$, which is the expected value of the investor's wealth for a $T-1$ period investment horizon.

[13] The flat segments in Figure 2 reflect the short-sales constraint.

The second plot in Figure 3 shows results under power utility based on Monte Carlo simulations and grid methods. Simulation techniques in asset allocation problems are notoriously slow even in simple setups such as ours, where $h = 1$. As a matter of fact, in our example maximization of (22) based on the closed-form results derived in Section 4 lowers the computation time by a factor of 50 when compared to using simulation methods. These gains are likely to be larger by an order of magnitude in multivariate asset allocation problems. Interestingly, the optimal stock holdings under power utility are very similar to those based on our four-moment specification, suggesting that our approach can be used as an alternative to the traditional techniques based on power utility.

Using the optimal asset allocation weights, we can compute the first four moments of the wealth distribution as a function of the bull state probability and the investment horizon. Figure 4 shows the outcome of this exercise. The mean return profile is largely proportional to Figure 1 since, for given values of the bull state probability and the investment horizon, mean returns are proportional to the stock holdings. Two effects contribute to the volatility of the optimal portfolio. For a given portfolio allocation the volatility declines as a function of the bull state probability since stock returns are much more volatile in the bear state. However, the optimal stock holdings also rise as a function of the bull state probability and this effect generally dominates the first effect.[14]

Despite the fact that the underlying log-normal distributions for stock returns are individually right-skewed, the two-state model can generate negative skews. This situation arises at short investment horizons for a high bull state probability and reflects the low probability of a bad event in the form of an unexpected shift to the bear state. As the investment horizon grows, the return distribution of the optimal portfolio gets a large and positive skew. Once again, the flat segments in these curves reflect points where it is optimal not to hold stocks. Kurtosis appears not to be heavily influenced by the investment horizon or the bull state probability except, of course, around the small region with zero stock holdings where the kurtosis shifts to zero.

To study the significance of going beyond a mean-variance analysis and also considering the skew and kurtosis of the terminal wealth distribution, Figure 5 plots the optimal stock holding as a function of the bull state probability for $m = 2$, 3 and 4. The figure assumes a medium-term investment horizon of six months. Extending preferences to consider skew and kurtosis clearly leads to very different stock holdings, particularly when the bull state probability exceeds 0.5. The four-moment investor tends to hold less in stocks than the two or three-moment investor, a result of the realization that the wealth distribution has fat tails. An investor with a three-moment preference specification is, however, willing to hold more in stocks than the mean-variance investor. This is a result of the generally positive skew in the wealth distribution, which makes stocks more attractive.

[14]The only point where the second effect does not dominate is when the short sales constraint is binding, i.e. in the top corner of the volatility plot where the bull state probability exceeds 0.85 and the investment horizon is very short. For fixed stock holdings, the volatility will decrease the higher the probability of the (low volatility) bull state.

5.2. *Predictability from the Dividend Yield*

Many studies have considered optimal stock holdings in the presence of predictability from the dividend yield. We compare our results to this literature by extending our model to include the dividend yield as an additional state variable:

$$\mathbf{y}_t = \boldsymbol{\mu}_{s_t} + \sum_{j=1}^{p} \mathbf{A}_{j,s_t} \mathbf{y}_{t-j} + \boldsymbol{\varepsilon}_t \tag{23}$$

where $\mathbf{y}_t = (r_t \ \mathbf{z}_t)'$ and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \Omega_{s_t})$. Table 3 presents results from estimating a linear vector autoregression with a single lag, similar to specifications in Barberis (2000) and Campbell and Viceira (1999). In the VAR(1) model reported in Panel A the coefficient estimates on the lagged dividend yield are significant in both the return and yield equations. In contrast, the autoregressive coefficient estimates on the lagged return are insignificant.

Turning to the two-state model reported in Panel B, the coefficient estimates on the lagged excess return continue to be statistically insignificant in both states. The mean excess return is 6.24% and 6.48% per annum, while the volatility is 15.5% and 13.9% per annum in states 1 and 2, respectively. Hence the marginal return distribution does not vary much across the two states. In contrast, the marginal distribution of the dividend yield is very different in the two states. The volatility of the dividend yield is almost twice as large in state 1 (0.73% versus 0.40% per annum) and its unconditional mean is much higher in state 1 (4.54%) than in state 2 (2.50%). The different properties of the dividend yield across the two states does, of course, affect the conditional distribution of stock returns, particularly since the coefficient on the dividend yield in the excess return equation, at 0.4, is insignificant in the first state but, at 1.9, is significant in the second state.

At 0.99 and 0.97 both states are extremely persistent and their interpretation clearly very different from the earlier univariate return model. This point comes out very clearly in Figure 6 which plots the smoothed probabilities of state 2. This state now captures episodes in the early and mid-fifties, a long period from 1974 to 1982 and a short episode in 1984. Overall, the states appear to be driven by long-range fluctuations in the dividend yield.

Figure 7 plots the optimal stock holdings based on the bivariate two-state model. Since the model has been expanded to include the dividend yield as an additional state variable, we show stock holdings as a function of the probability of state 2 and the value of the dividend yield, keeping the investment horizon constant. To track the effect of the investment horizon, we present results for both a short ($T = 1$) and a long ($T = 36$) horizon.

First consider the short horizon. For low values of the dividend yield and a low probability of state 2, the investor does not hold any stocks and the short-sales constraint is binding. For higher values of the dividend yield and higher probabilities of state 2, the optimal stock holding increases and at values of the dividend yield above 4.5%, the investor puts all money in stocks. While it is clear why the optimal stock holding is an increasing function of the current dividend yield, the monotonicity in the probability of state 2 is explained by the relatively fast mean reversion characterizing the dividend yield in this state in which the dividend yield fluctuates well below its

unconditional mean. If the dividend yield starts below its unconditional mean of 4.5%, it is expected to increase. When coupled with the high sensitivity of excess returns to the dividend yield in state 2, the effect is to make the optimal stock holding an increasing function of the probability of this state.

At the long horizon ($T = 36$), the sensitivity of the stock holdings with respect to the dividend yield is much larger. This makes sense since the dividend yield captures more of the return variation, the longer the investment horizon, which makes the sensitivity of stock holdings with respect to this variable greater for large T. At lower levels of the dividend yield, the probability of state 2 continues to be important to the optimal stock holdings. However, the dividend yield clearly matters relatively more than the state probabilities at the long investment horizon.

6. **Rebalancing**

To keep the analysis simple, so far we have ignored the possibility of portfolio rebalancing. In this section we relax this assumption and allow the investor to rebalance every $\varphi = \frac{T}{B}$ months at B equally spaced points t, $t + \frac{T}{B}$, $t + 2\frac{T}{B}$, ..., $t + (B-1)\frac{T}{B}$. This requires determining the portfolio weights at the rebalancing times ω_b $(b = 0, 1, ..., B-1)$. When $B = 1$, $\varphi = T$ and the investor simply implements a buy-and-hold strategy.

Cumulated wealth can be factored out as a product of interim wealth at the rebalancing points:

$$W_{t+T} = \prod_{b=1}^{B} \frac{W_{t+\varphi b}(\omega_{b-1})}{W_{t+\varphi(b-1)}(\omega_{b-2})}, \tag{24}$$

where

$$\frac{W_{t+\varphi b}(\omega_{b-1})}{W_{t+\varphi(b-1)}(\omega_{b-2})} = \left\{(1 - \boldsymbol{\omega}'_{b-1}\boldsymbol{\iota}_h)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}'_{b-1}\exp\left(\mathbf{R}^s_{\varphi(b-1)+1\to\varphi b}\right)\right\},$$

and $\mathbf{R}^s_{\varphi(b-1)+1\to\varphi b} \equiv \mathbf{r}^s_{t+\varphi(b-1)+1} + \mathbf{r}^s_{t+\varphi(b-1)+2} + ... + \mathbf{r}^s_{t+\varphi b}$. By the law of iterated expectations, the following decomposition holds:

$$\begin{aligned}
M^{(n)}_{t+T} &= E_t[W^n_{t+T}] = E_t\left[\prod_{b=1}^{B}\left(\frac{W_{t+\varphi b}(\boldsymbol{\omega}_{b-1})}{W_{t+\varphi(b-1)}(\boldsymbol{\omega}_{b-2})}\right)^n\right] \\
&= E_t\left\{(W_{t+\varphi}(\boldsymbol{\omega}_t))^n E_{t+\varphi}\left[\left(\frac{W_{t+2\varphi}(\boldsymbol{\omega}_{t+\varphi})}{W_{t+\varphi}(\boldsymbol{\omega}_t)}\right)^n E_{t+2\varphi}\left(\left(\frac{W_{t+3\varphi}(\boldsymbol{\omega}_{t+2\varphi})}{W_{t+2\varphi}(\boldsymbol{\omega}_{t+\varphi})}\right)^n ...\right)\right]\right\} \\
&= M^{(n)}_{0\to\varphi}(\boldsymbol{\omega}_0)E_t\left\{M^{(n)}_{\varphi\to 2\varphi}(\boldsymbol{\omega}_1)E_{t+\varphi}\left[M^{(n)}_{2\varphi\to 3\varphi}(\boldsymbol{\omega}_2)E_{t+2\varphi}\left(M^{(n)}_{3\varphi\to 4\varphi}(\boldsymbol{\omega}_3)...\right)\right]\right\}
\end{aligned} \tag{25}$$

where $E_{t+\varphi(b-1)}[\cdot]$ is shorthand notation for $E\left[\cdot|\Im_{t+\varphi(b-1)}\right]$ and $M^{(n)}_{\varphi(b-1)\to\varphi b}(\boldsymbol{\omega}_{b-1})$ is the n-th (non-central) moment of the cumulated portfolio returns between $t + \varphi(b-1) + 1$ and $t + \varphi b$, calculated on the basis of time $t + \varphi(b-1)$ information:

$$\begin{aligned}
M^{(n)}_{\varphi(b-1)\to\varphi b}(\boldsymbol{\omega}_{b-1}) &\equiv E_{t+\varphi(b-1)}\left[\left(\frac{W_{t+\varphi b}(\boldsymbol{\omega}_{b-1})}{W_{t+\varphi(b-1)}(\boldsymbol{\omega}_{b-2})}\right)^n\right] \\
&= E_{t+\varphi(b-1)}\left[\left((1 - \boldsymbol{\omega}'_{b-1}\boldsymbol{\iota}_h)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}'_{b-1}\exp\left(\mathbf{R}^s_{\varphi(b-1)+1\to\varphi b}\right)\right)^n\right].
\end{aligned}$$

The decomposition in (25) shows that future moments of wealth depend on future portfolio choices, $\boldsymbol{\omega}_b$.

We use the following recursive strategy to solve the asset allocation problem under m-moment preference functionals and rebalancing:

1. Start solving the time $T-\varphi$ problem

$$\hat{\boldsymbol{\omega}}_{B-1} \equiv \arg\max_{\boldsymbol{\omega}_{B-1}} \sum_{n=0}^{m} \kappa_n(\boldsymbol{\theta}) \hat{E}_t \left[M^{(n)}_{T-\varphi \to T}(\boldsymbol{\omega}_{B-1}) \right].$$

2. Solve the time $T-2\varphi$ problem

$$\hat{\boldsymbol{\omega}}_{B-2} \equiv \arg\max_{\boldsymbol{\omega}_{B-2}} \sum_{n=0}^{m} \lambda_n^{B-1}(\boldsymbol{\theta}) \hat{E}_t \left[M^{(n)}_{T-2\varphi \to T-\varphi}(\boldsymbol{\omega}_{B-2}) \right],$$

where $\lambda_n^{B-1}(\boldsymbol{\theta}) \equiv \kappa_n(\boldsymbol{\theta}) \hat{E}_t[M^{(n)}_{T-\varphi \to T}(\hat{\boldsymbol{\omega}}_{B-1})]$ and $\hat{E}_t[M^{(n)}_{T-\varphi \to T}(\hat{\boldsymbol{\omega}}_{B-1})]$ is the n-th noncentral moment of the optimal wealth process calculated under the solution found in 1.

3. Solve the problem backward by iterating on 1. and 2. up to time $t+\varphi$, to generate a sequence of optimal portfolio choices $\{\hat{\boldsymbol{\omega}}_i\}_{i=1}^{B-1}$. The optimal time t asset allocation, $\hat{\boldsymbol{\omega}}_0 \equiv \hat{\boldsymbol{\omega}}_t$, is then found by solving

$$\hat{\boldsymbol{\omega}}_0 \equiv \arg\max_{\boldsymbol{\omega}_0} \sum_{n=0}^{m} \lambda_n^1(\boldsymbol{\theta}) M^{(n)}_{t \to t+\varphi}(\boldsymbol{\omega}_0),$$

where

$$\lambda_n^1(\boldsymbol{\theta}) \equiv \kappa_n(\boldsymbol{\theta}) \prod_{b=1}^{B-1} \hat{E}_t[M^{(n)}_{t+\varphi b \to t+\varphi(b+1)}(\hat{\boldsymbol{\omega}}_b)]. \tag{26}$$

$\hat{\boldsymbol{\omega}}_0$ is the vector of optimal portfolio weights under rebalancing every φ periods.

In practice, the algorithm 1. - 3. replaces a complex multiperiod program with a sequence of simpler, buy-and-hold portfolio choice problems (each with horizon φ) in which the original moment coefficients $\{\kappa_n(\boldsymbol{\theta})\}_{n=0}^m$ are recursively replaced with products of estimates of the conditional noncentral moments of future wealth.

6.1. *Empirical Results*

Table 4 shows empirical results for a range of rebalancing frequencies and for three scenarios concerning the initial state probability. Changes to the earlier results due to introducing rebalancing are easy to follow. In the bear state, rebalancing leads the investor to scale down stock holdings. For $\varphi \leq 3$ months, the effect is so strong that $\hat{\omega}_t = 0$ for all investment horizons and the short-sales constraint becomes binding. The intuition is that investors who can frequently adjust their portfolios prefer to delay investing in stocks in the bear state since stock returns in this state are low on average and highly volatile.

In contrast, starting from the bull state, investors aggressively buy stocks to the extent that for φ less than or equal to six months an investor only holds stocks and the short sales constraint is binding. The fact that future portfolio holdings can be adjusted in case a bear state emerges means that investors choose their current stock holdings more aggressively if they are certain that the current market is in the bull state.

Finally, we also investigated the effects of changing φ when there is substantial uncertainty about the current states which are equally likely, $\hat{\boldsymbol{\pi}}_t = (0.5\ 0,5)'$. Rebalancing leads to far less aggressive portfolio choices under this scenario. Since the initial beliefs put more weight on the bear state than the steady-state probabilities, the position in stocks is reduced as φ increases, although there is no value of φ such that either $\hat{\omega}_t = 0$ or $\hat{\omega}_t = 1$. Furthermore, optimal rebalancing can produce interesting non-monotonicities in the optimal stock holdings as a function of T. For example, when φ is six months, the stock demand schedule slopes upward for short horizons but slopes downward at longer investment horizons.

7. **Conclusion**

This paper proposed a method for optimal asset allocation under regime switching in the asset return process when investor preferences depend on a finite number of moments of the terminal wealth distribution. We show how to characterize the mean, variance, skew and kurtosis (as well as other moments of arbitrarily high order) of the wealth distribution in the form of solutions to simple difference equations. When coupled with a utility specification that incorporates skew and kurtosis preferences, our method greatly reduces the otherwise numerically complicated problem of solving for the optimal asset allocation. We apply the method to a portfolio problem considered in much of the existing literature, namely the choice between a US stock index and a risk-free asset. Our empirical findings show that the optimal portfolio weights crucially depend on the underlying state probabilities.

A number of extensions to these results would be interesting to further pursue. For instance, the portfolio choice between stocks and T-bills clearly over-simplifies portfolio decision problems solved by portfolio managers in practice. Recent work has stressed the importance of allowing for regimes in the joint return distribution of two or more assets. Ang and Bekaert (2001) show that regime switching captures comovements in international financial markets, while Guidolin and Timmermann (2002) find evidence of regimes in the joint distribution of returns on bonds and stock portfolios of small and large firms. Addressing multi-asset decision problems does not pose a particular problem to our method since we presented general results for multiple assets and multiple regimes. In fact, from a computational perspective, the main advantage of our approach is likely to be in cases where the number of assets is quite large.

Appendix

This appendix derives Proposition 1 and shows how to extend the results to include autoregressive terms in the return process.

To derive the n-th moment of the cumulated return on the risky asset holdings in the general case with multiple assets (h) and states (k), notice that

$$E_t[\left(\boldsymbol{\omega}_t' \exp\left(\mathbf{R}_{t+T}^s\right)\right)^n] = E_t\left[\sum_{n_1=1}^{n} ... \sum_{n_h=1}^{n} (\omega_1^{n_1} \times .. \times \omega_h^{n_h}) \exp(\sum_{i=1}^{T} r_{1t+i}) \times .... \times \exp(\sum_{i=1}^{T} r_{ht+i})\right]. \tag{A1}$$

where the powers $0 \leq n_i \leq n$ $(i = 1, ..., h)$ satisfy the summing-up constraint

$$n_1 + n_2 + ... + n_h = n.$$

To evaluate (A1) requires solving for moments of the form

$$\begin{aligned} M_{t+T}^{(n)}(n_1, n_2, ..., n_h) &= E_t\left[\exp\left(\sum_{i=1}^{T} r_{1t+i}\right)^{n_1} \times .... \times \exp\left(\sum_{i=1}^{T} r_{ht+i}\right)^{n_h}\right] \\ &= E_t\left[\exp\left(\sum_{l=1}^{h} n_l \sum_{i=1}^{T} r_{l,t+i}\right)\right]. \end{aligned} \tag{A2}$$

(A2) can be decomposed as follows

$$M_{t+T}^{(n)}(n_1, n_2, ..., n_h) = \sum_{i=1}^{k} M_{i,t+T}^{(n)}(n_1, n_2, ..., n_h), \tag{A3}$$

where

$$M_{i,t+T}^{(n)}(n_1, n_2, ..., n_h) = E_t\left[\exp\left(\sum_{l=1}^{h} n_l \sum_{i=1}^{T} r_{lt+i}\right) | s_{t+T} = i\right] \Pr(s_{t+T} = i).$$

Each of these terms satisfies the recursions

$$\begin{aligned} M_{i,t+T}^{(n)}(n_1, n_2, ..., n_h) &= \sum_{g=1}^{k} M_{g,t+T-1}^{(n)}(n_1, n_2, ..., n_h) E_t\left[\exp\left(\sum_{l=1}^{h} n_l r_{lt+T}\right) | s_{t+T} = i, \Im_t\right] p_{gi} \\ &= \sum_{g=1}^{k} p_{gi} M_{g,t+T-1}^{(n)}(n_1, n_2, ..., n_h) \exp\left(\sum_{l=1}^{h} n_l \mu_{il} + \sum_{l=1}^{h}\sum_{u=1}^{h} n_l n_u \frac{\sigma_{i,lu}}{2}\right) \end{aligned} \tag{A4}$$

where μ_{il} is the mean return of asset l in state i and $\sigma_{i,lu} = e_l' \Omega_i e_u$ is the covariance between r_{lt+T} and r_{ut+T} in state $i = 1, 2, ..., k$. This is an obvious generalization of our earlier result (12).

Finally, using (A1) and (A2), we get the following expression for the $n-$th moment of the cumulated return:

$$E_t[\left(\boldsymbol{\omega}_t' \exp\left(\mathbf{R}_{t+T}^s\right)\right)^n] = \sum_{n_1=0}^{n} \cdots \sum_{n_h=0}^{n} (\omega_1^{n_1} \times ... \times \omega_n^{n_h}) M_{t+T}^{(n)}(n_1, ..., n_h). \tag{A5}$$

Expected utility can now readily be evaluated in a straightforward generalization of (16):

$$\begin{aligned}\hat{E}_t[U^m(W_{t+T};\boldsymbol{\theta})] &= \sum_{n=0}^{m}\kappa_n\sum_{j=0}^{n}(-1)^{n-j}v_T^{n-j}{}_nC_jE_t[W_{t+T}^j] \\ &= \sum_{n=0}^{m}\kappa_n\sum_{j=0}^{n}(-1)^{n-j}v_T^{n-j}\binom{n}{j}\sum_{i=0}^{j}\binom{j}{i}E_t[\left(\boldsymbol{\omega}_t'\exp\left(\mathbf{R}_{t+T}^s\right)\right)^i]\left((1-\boldsymbol{\omega}_t'\boldsymbol{\iota}_h)\exp\left(Tr^f\right)\right)^{j-i}.\end{aligned}$$

Inserting (A5) into this expression gives rise to a first order condition that takes the for of an $n-1$th order polynomial in the portfolio weights.

Autoregressive Terms in the Return Process

The generalization of the results to include autoregressive terms is straightforward. To keep the notation simple, suppose $k = 2$. Using (7) the n-th noncentral moment satisfies the recursions

$$\begin{aligned}M_{i,t+T}^{(n)} &= M_{i,t+T-1}^{(n)}(n)p_{ii}\exp\left(n\mu_i + n\sum_{j=1}^{p}a_{j,i}E_t[r_{t+T-j}] + \frac{n^2}{2}\sigma_i^2\right) + \\ &\quad + M_{-i,t+T-1}^{(n)}(n)(1-p_{-i-i})\exp\left(n\mu_i + n\sum_{j=1}^{p}a_{j,i}E_t[r_{t+T-j}] + \frac{n^2}{2}\sigma_i^2\right)\end{aligned}$$

or

$$\begin{aligned}M_{1,t+1}^{(n)} &= \tilde{\alpha}_1^{(n)}M_{1,t}^{(n)} + \tilde{\beta}_1^{(n)}M_{2,t}^{(n)} \\ M_{2,t+1}^{(n)} &= \tilde{\alpha}_2^{(n)}M_{1,t} + \tilde{\beta}_2^{(n)}M_{2,t}^{(n)},\end{aligned}$$

where now

$$\begin{aligned}\tilde{\alpha}_1^{(n)} &= p_{11}\exp\left(n\mu_1 + n\sum_{j=1}^{p}a_{j,1}E_t[r_{t+T-j}] + \frac{n^2}{2}\sigma_1^2\right) \\ \tilde{\beta}_1^{(n)} &= (1-p_{22})\exp\left(n\mu_1 + n\sum_{j=1}^{p}a_{j,1}E_t[r_{t+T-j}] + \frac{n^2}{2}\sigma_1^2\right) \\ \tilde{\alpha}_2^{(n)} &= (1-p_{11})\exp\left(n\mu_2 + n\sum_{j=1}^{p}a_{j,2}E_t[r_{t+T-j}] + \frac{n^2}{2}\sigma_2^2\right) \\ \tilde{\beta}_2^{(n)} &= p_{22}\exp\left(n\mu_2 + n\sum_{j=1}^{p}a_{j,2}E_t[r_{t+T-j}] + \frac{n^2}{2}\sigma_2^2\right).\end{aligned}$$

Subject to these changes, the methods in Section 4 can be used with the only difference that terms such as $\exp\left(n\mu_i + \frac{n^2}{2}\sigma_i^2\right)$ have to be replaced by

$$\exp\left(n\mu_1 + n\sum_{j=1}^{p}a_{j,i}E_t[r_{t+T-j}] + \frac{n^2}{2}\sigma_1^2\right).$$

The term $\sum_{j=1}^{p} a_{j,i} E_t[r_{t+T-j}]$ may be decomposed in the following way:

$$\sum_{j=1}^{p} a_{j,i} E_t[r_{t+T-j}] = I_{\{j>T\}} \sum_{j=1}^{p} \left(I_{\{j \geq T\}} a_{j,i} r_{t+T-j} + I_{\{j<T\}} a_{j,i} E_t[r_{t+T-j}] \right),$$

where $E_t[r_{t+1}], ..., E_t[r_{t+T-1}]$ can be evaluated recursively, c.f. Doan et al. (1984):

$$\begin{aligned}
E_t[r_{t+1}] &= \pi_{1t} \left(\mu_1 + \sum_{j=1}^{p} a_{j,1} r_{t-j} \right) + (1 - \pi_{1t}) \left(\mu_2 + \sum_{j=1}^{p} a_{j,2} r_{t-j} \right) \\
E_t[r_{t+2}] &= \boldsymbol{\pi}_t' \mathbf{P} \mathbf{e}_1 \left(\mu_1 + \sum_{j=1}^{p} a_{j,1} E_t[r_{t+1}] \right) + (1 - \boldsymbol{\pi}_t' \mathbf{P} \mathbf{e}_1) \left(\mu_2 + \sum_{j=1}^{p} a_{j,2} E_t[r_{t+1}] \right) \\
&\ldots \\
E_t[r_{t+T-1}] &= \boldsymbol{\pi}_t' \mathbf{P}^{T-1} \mathbf{e}_1 \left(\mu_1 + \sum_{j=1}^{p} a_{j,1} E_t[r_{t+T-2}] \right) + (1 - \boldsymbol{\pi}_t' \mathbf{P}^{T-1} \mathbf{e}_1) \left(\mu_2 + \sum_{j=1}^{p} a_{j,2} E_t[r_{t+T-2}] \right).
\end{aligned}$$

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Table 1

Density Specification Tests for Regime Switching Models

This table reports tests for the transformed z-scores generated by univariate regime-switching models

$$R_t = \mu_{s_t} + \sum_{j=1}^{p} a_{js_t} y_{t-j} + \sigma_{s_t} \varepsilon_t$$

where R_t is the excess return on the value-weighted CRSP stock index, $\varepsilon_t \sim IIN(0,1)$ and s_t is governed by an unobservable, first-order Markov chain that can assume k distinct values (states). The sample period is 1952:06 – 1999:12. The tests are based on the principle that under the null of correct specification of the model, the probability integral transform of the one-step-ahead standardized forecast errors should follow an IID uniform distribution over the interval (0,1). A further Gaussian transform described in Berkowitz (2001) is applied to perform Likelihood ratio tests of the null that (under correct specification) the transformed z-scores, z^*_{t+1}, are IIN(0,1) distributed. In particular, given the transformed z-score model

$$z^*_{t+1} = \mu + \sum_{j=1}^{p} \sum_{i=1}^{l} \rho_{ij} (z^*_{t+1-i})^j + \sigma e_{t+1} ,$$

LR_2 tests the hypothesis of zero mean and unit variance under the restriction $p = l = 0$; LR_3 tests the joint hypothesis of zero mean, unit variance, and $\rho_{11} = 0$ under $p = l = 1$; LR6 tests the joint null of zero mean, unit variance, and $\rho_{11} = \rho_{12} = \rho_{21} = \rho_{22} = 0$ with $p = l = 2$.

Model	Number of parameters	Jarque-Bera test	LR_2	LR_3	LR_6
Linear	2	153.438 (0.000)	**0.000 (1.000)**	**5.005 (0.171)**	**11.985 (0.062)**
Two-state	8	**1.304 (0.521)**	**0.145 (0.930)**	**2.808 (0.422)**	**9.876 (0.130)**

Table 2

Estimates of the Two-State Switching Model

This table reports maximum likelihood estimates for a single state model and a two-state regime switching model fitted to monthly, value weighted CRSP excess returns. The regime switching model takes the form:

$$r_t = \mu_{s_t} + \sigma_{s_t}\varepsilon_t$$

where μ_{st} is the intercept in state s_t and $\varepsilon_t \sim N(0,1)$ is an unpredictable return innovation. The sample period is 1952:06 – 1999:12.

Panel A – Single State Model		
Mean excess return	0.668**	
Volatility	4.203	
Panel B – Two State Model		
Mean excess return		
Regime 1 (bear)	-0.933*	
Regime 2 (bull)	1.107**	
Volatility		
Regime 1 (bear)	6.250	
Regime 2 (bull)	3.303	
Transition probabilities	Regime 1	Regime 2
Regime 1 (bear)	0.814	0.186
Regime 2 (bull)	0.051	0.949

* = significant at 5% level; ** = significant at 1% level

Table 3

Estimates for a Bivariate Regime Switching Model: Stock Returns and Dividend Yields

This table reports maximum likelihood estimates for a bivariate VAR and a two-state regime switching model fitted to monthly excess returns and the dividend yield on the value weighted CRSP stock index. The regime switching model takes the form

$$y_t = \mu_{s_t} + \sum_{j=1}^{p} A_{js_t} y_{t-j} + \sigma_{s_t}\, \boldsymbol{\varepsilon}_t,$$

where $\mathbf{y}_t$ is a vector collecting the excess return and the dividend yield, μ_{s_t} is an intercept vector in state s_t, A_{1s_t} is a matrix of first-order autoregressive coefficients in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}]' \sim \text{I.I.D. } N(\mathbf{0}, \Omega_{s_t})$. s_t is governed by an unobservable, first-order Markov chain that can assume 2 distinct values. The data is monthly and covers the period 1952:06 – 1999:12. Panel A refers to the single state benchmark (k = 1) while panel B refers to the two-state model (k = 2). The values on the diagonals of the correlation matrices are volatilities, while off-diagonal terms are correlations.

	Panel A – Single State Model	
	Excess stock returns	Dividend yield
Mean excess return	-0.2755	0.0421
VAR(1) coefficients		
Excess stock returns	0.0734	0.2580*
Dividend yield	-0.0028	0.9862**
Correlations/Volatilities		
Excess stock returns	4.1878	
Dividend yield	-0.9335	0.1600
	Panel B – Two State Model	
	Excess stock returns	Dividend Yield
Intercepts		
Regime 1	-0.4882	0.0586*
Regime 2	-8.0818**	0.4267**
VAR(1) coefficients		
Regime 1:		
Excess stock returns	0.0855	0.3936
Dividend yield	-0.0030	0.9772**
Regime 2:		
Excess stock returns	0.0723	1.8836**
Dividend Yield	-0.0035	0.9065**
Correlations/Volatilities		
Regime 1:		
Excess stock returns	3.9960	
Dividend yield	-0.9409	0.1245
Regime 2:		
Excess stock returns	4.4658	
Dividend yield	-0.9713	0.2149
Transition probabilities	Regime 1	Regime 2
Regime 1	0.9909	0.0091
Regime 2	0.0269	0.9731

* = significant at 5% level; ** = significant at 1% level

Table 4

Optimal Asset Allocation – Effects of Rebalancing

This table reports the optimal weight to be invested in equities as a function of the rebalancing frequency φ assuming the investor has a four-moment (mean, variance, third, and fourth central moments of t+T wealth) utility function. The coefficients of the objective function are evaluated by interpreting the objective as a Taylor approximation to power utility with constant relative risk aversion equal to 5. Excess returns are assumed to be generated by the regime switching model

$$r_t = \mu_{s_t} + \sigma_{s_t}\varepsilon_t,$$

where μ_{s_t} is the intercept in state s_t and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0},1)$ is an unpredictable return innovation. The sample period is 1952:06 – 1999:12. The three panels in the table refer to alternative values of the current perception $\hat{\pi}_{s_t=2}$ of the probability of being in state 2 (bull market)

Rebalancing Frequency φ	Investment Horizon T (in months)					
	T=1	**T=3**	**T=9**	**T=12**	**T=20**	**T=30**
			Bull state ($\hat{\pi}_{s_t=2}$ = 1)			
φ = T (buy-and-hold)	1.000	1.000	0.940	0.868	0.734	0.638
φ = 12 months	1.000	1.000	0.940	0.868	1.000	1.000
φ = 6 months	1.000	1.000	1.000	1.000	1.000	1.000
φ = 3 months	1.000	1.000	1.000	1.000	1.000	1.000
φ = 2 months	1.000	1.000	1.000	1.000	1.000	1.000
φ = 1 month	1.000	1.000	1.000	1.000	1.000	1.000
			High uncertainty ($\hat{\pi}_{s_t=2}$ = 0.5)			
φ = T (buy-and-hold)	0.286	0.364	0.482	0.510	0.538	0.539
φ = 12 months	0.286	0.364	0.482	0.510	0.496	0.886
φ = 6 months	0.286	0.364	0.448	0.424	0.372	0.156
φ = 3 months	0.286	0.364	0.428	0.364	0.262	0.192
φ = 2 months	0.286	0.330	0.320	0.290	0.192	0.066
φ = 1 month	0.286	0.286	0.270	0.246	0.154	0.048
			Bear state ($\hat{\pi}_{s_t=2}$ = 0)			
φ = T (buy-and-hold)	0.000	0.000	0.204	0.284	0.398	0.438
φ = 12 months	0.000	0.000	0.204	0.284	0.234	0.182
φ = 6 months	0.000	0.000	0.084	0.084	0.052	0.028
φ = 3 months	0.000	0.000	0.000	0.000	0.000	0.000
φ = 2 months	0.000	0.000	0.000	0.000	0.000	0.000
φ = 1 month	0.000	0.000	0.000	0.000	0.000	0.000

Figure 1

Smoothed Probabilities of a Bear State in a Two-Regime Model

This figure plots smoothed probabilities for the two-state MSIH (2,0) model

$$r_t = \mu_{s_t} + \sigma_{s_t}\varepsilon_t$$

to monthly, value weighted CRSP stock index excess returns.

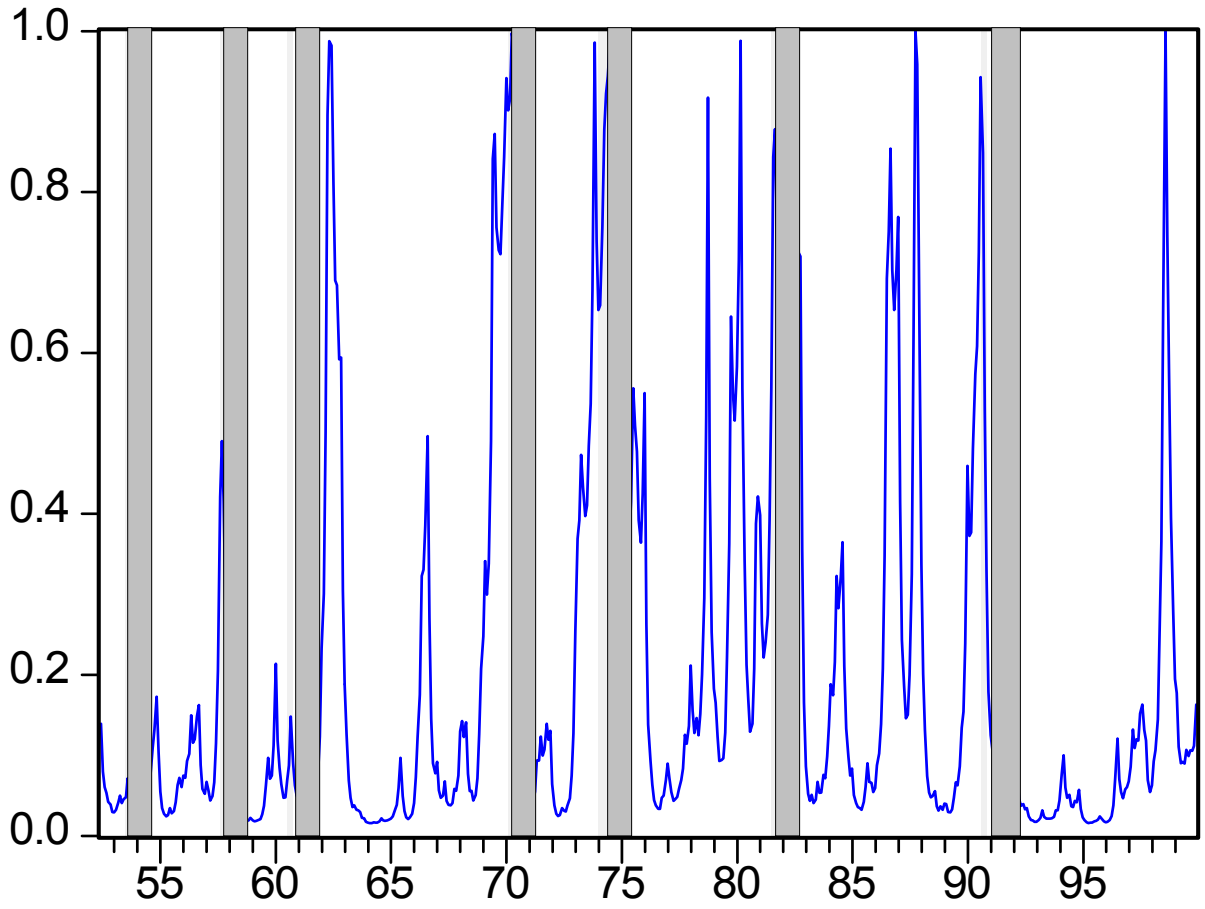


Figure 2

Optimal Portfolio Allocation under Four Moment Preferences – Effects of State Beliefs

The figure shows the optimal allocation to the value-weighted CRSP stock index as a function of the investment horizon and the current perception of the probability of being in state 2 (bull stock markets) assuming the investor has a four-moment (mean, variance third, and fourth central moments of t+T wealth) utility function. The coefficients of the objective function are evaluated by interpreting the objective as a Taylor approximation to power utility with constant relative risk aversion equal to 5. Excess stock returns are assumed to be generated by the regime switching model

$$r_t = \mu_{s_t} + \sigma_{s_t}\varepsilon_t,$$

where μ_{s_t} is the intercept in state s_t and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0},\mathbf{1})$ is an unpredictable return innovation. The sample period is 1952:06 – 1999:12. Parameters are fixed at their full-sample ML estimates.

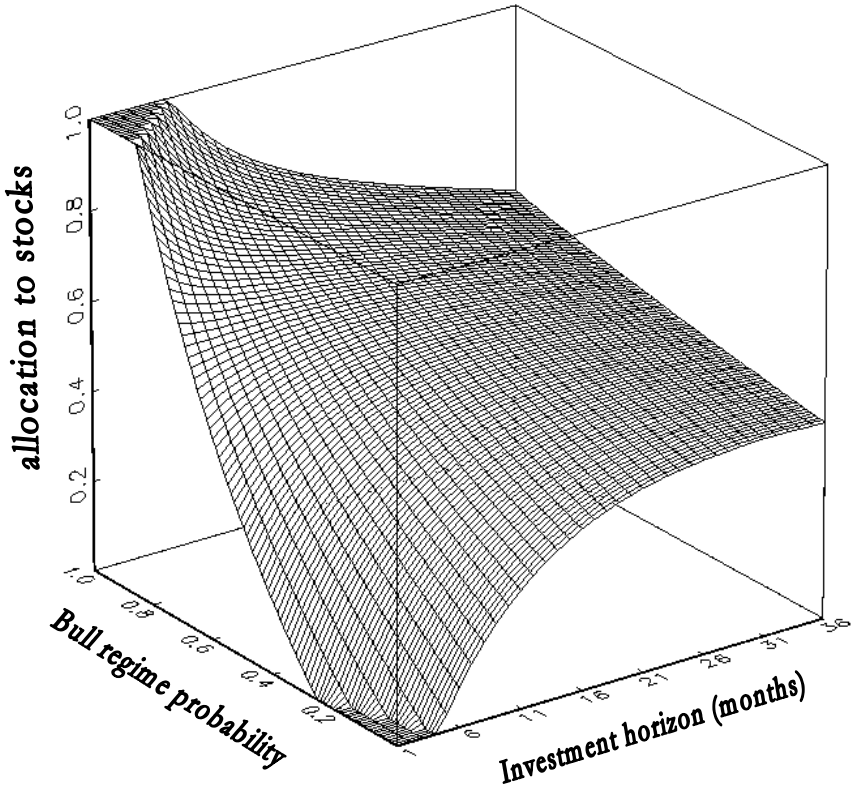


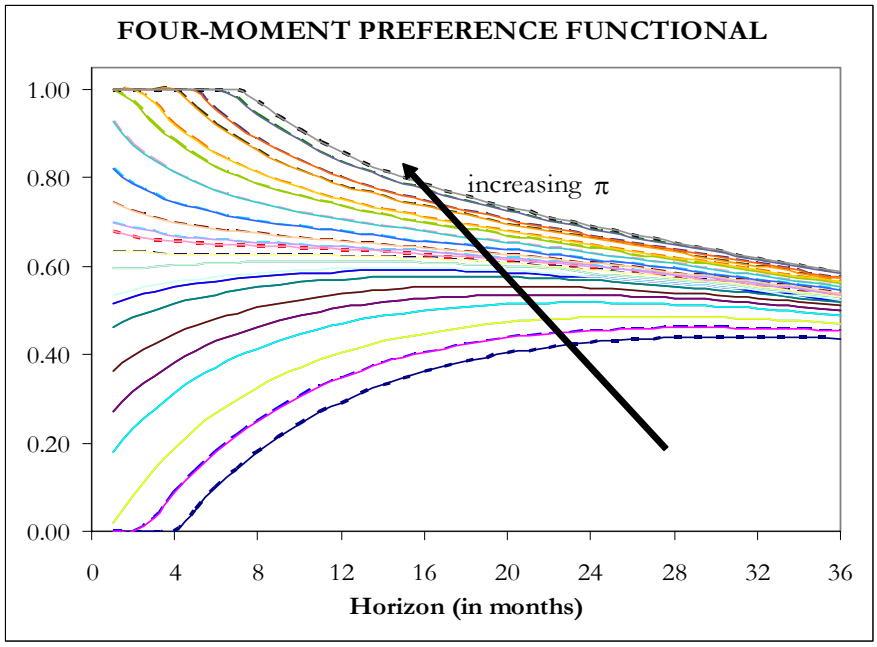
Figure 3

Optimal Portfolio Allocation under Four Moment Preferences – Effects of State Beliefs

The figures plot the change in equity investment schedules as a function of the investment horizon and the current perception of the probability of being in state 2 (bull stock markets). The upper panel assumes the investor has a four-moment (mean, variance third, and fourth central moments of t+T wealth) utility function. The coefficients of the objective function are evaluated by interpreting the objective as a Taylor approximation to power utility with constant relative risk aversion equal to 5. The bottom panel reports optimal weights calculated under power utility with coefficient of relative risk aversion equal to 5. Expectations are calculated applying Monte Carlo methods. In both cases, excess stock returns are assumed to be generated by the regime switching model

$$r_t = \mu_{s_t} + \sigma_{s_t}\varepsilon_t,$$

where μ_{s_t} is the intercept in state s_t and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0},1)$ is an unpredictable return innovation. The sample period is 1952:06 – 1999:12. Parameters are fixed at their full-sample ML estimates.



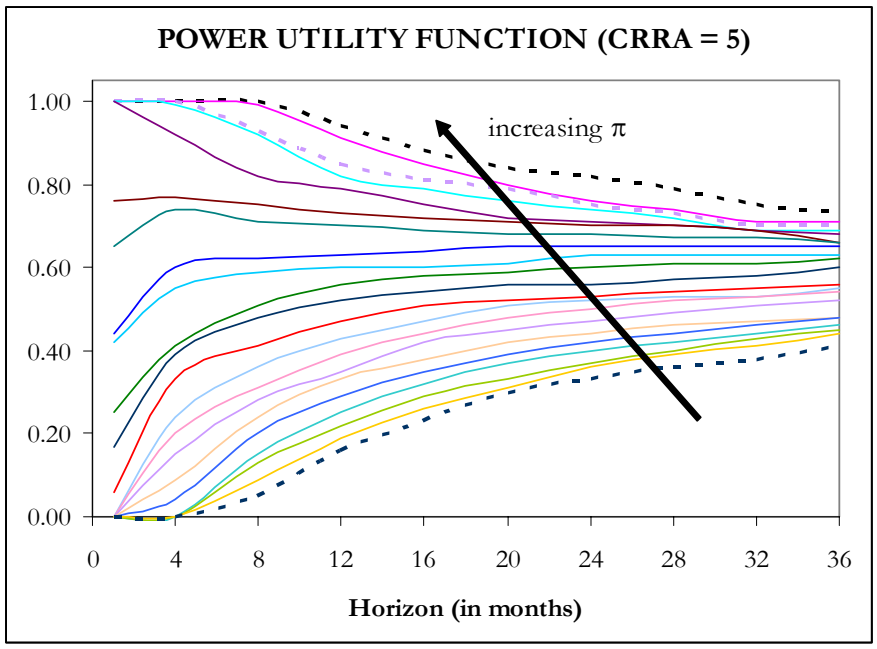


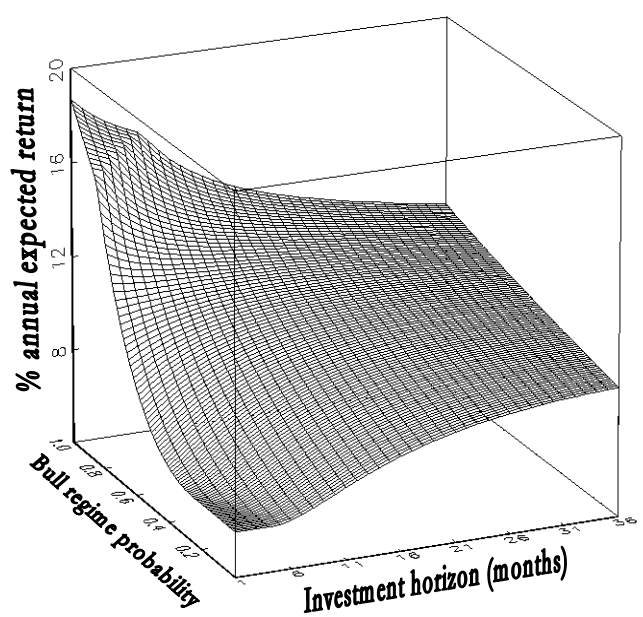
Figure 4

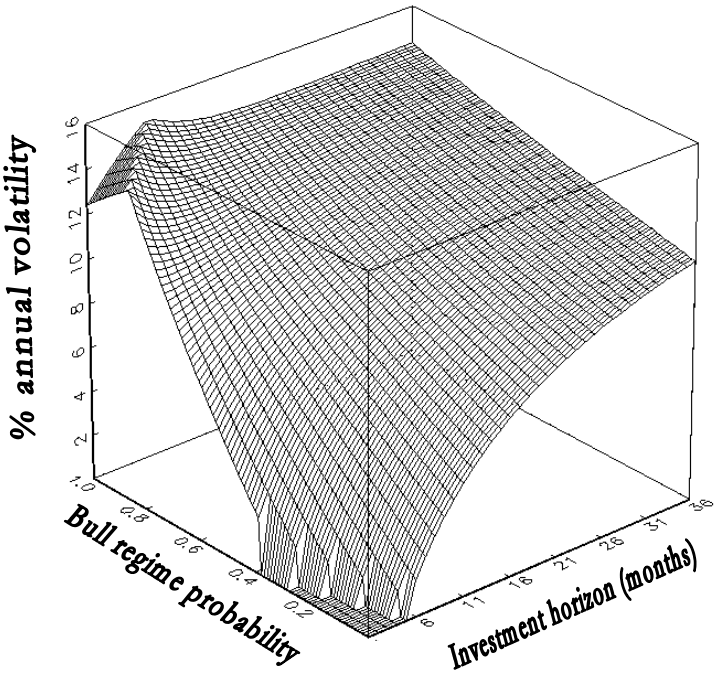
Implied Moments of T-month ahead Wealth - – Effects of State Beliefs

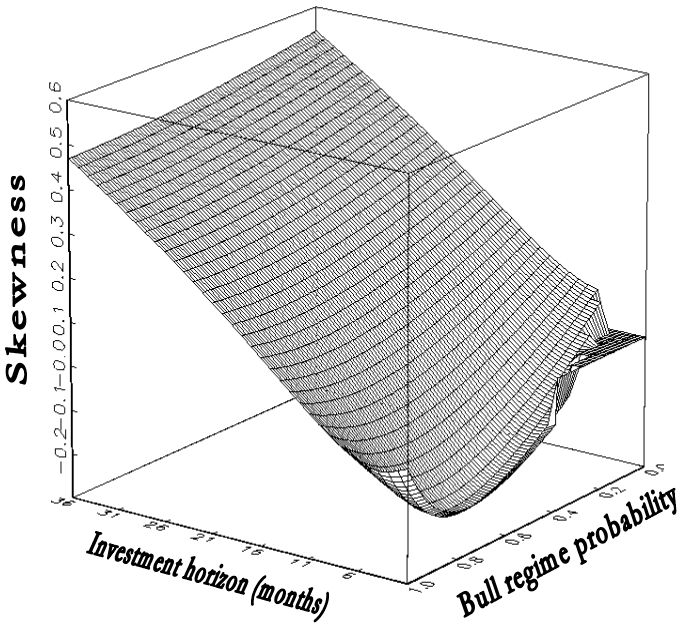
These figures plot the implied moments of wealth as a function of the investment horizon and the current perception of the probability of being in state 1 (bull stock markets) assuming the investor is optimally selecting portfolio weights of the value-weighted CRSP stock index and of one-month T-bills under a four-moment objective. Excess stock returns are assumed to be generated by the regime switching model

$$r_t = \mu_{s_t} + \sigma_{s_t}\varepsilon_t,$$

where μ_{s_t} is the intercept in state s_t and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0},\mathbf{1})$ is an unpredictable return innovation. The sample period is 1952:06 – 1999:12. Parameters are fixed at their full-sample ML estimates.







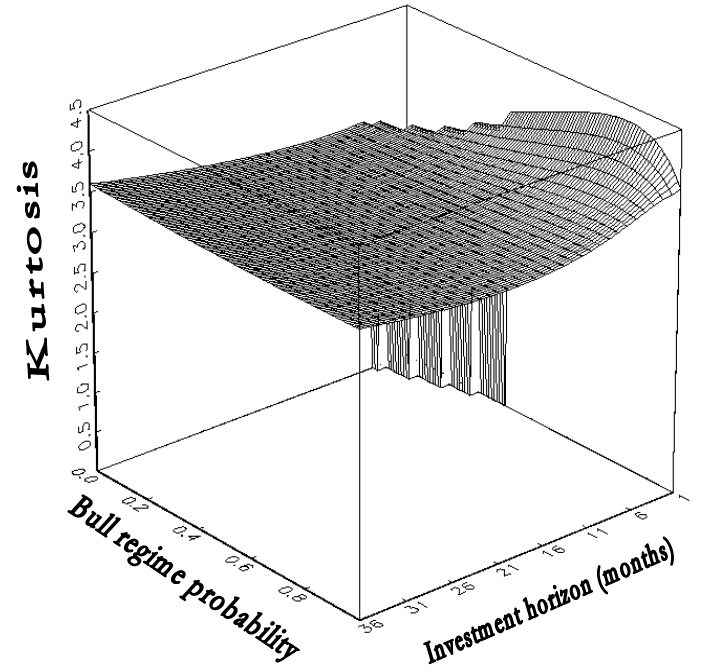


Figure 5

Effects of the order *m* on Optimal Portfolio Choices

The figures plot the optimal allocation to stocks as a function of perceived probability of a bull regime (regime 2) for three alternative choices of m: m=2 (mean-variance preferences), m=3 (a three-moment preference functional), and m=4 (four-moment functional). In the three cases, the coefficients of the objective function are evaluated by interpreting the objective as a Taylor approximation (around v_T) to power utility with constant relative risk aversion equal to θ:

$$E[U^m(W_{t+T};\theta)] = \kappa_0(v_T;\theta) + \sum_{j=1}^{m} \kappa_j(v_T;\theta) M_{t+T}^{(j)} .$$

The investment horizon (T) is six months. We assume that excess stock returns are generated by the regime switching model

$$r_t = \mu_{s_t} + \sigma_{s_t}\varepsilon_t ,$$

where μ_{s_t} is the intercept in state s_t and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0},1)$ is an unpredictable return innovation.

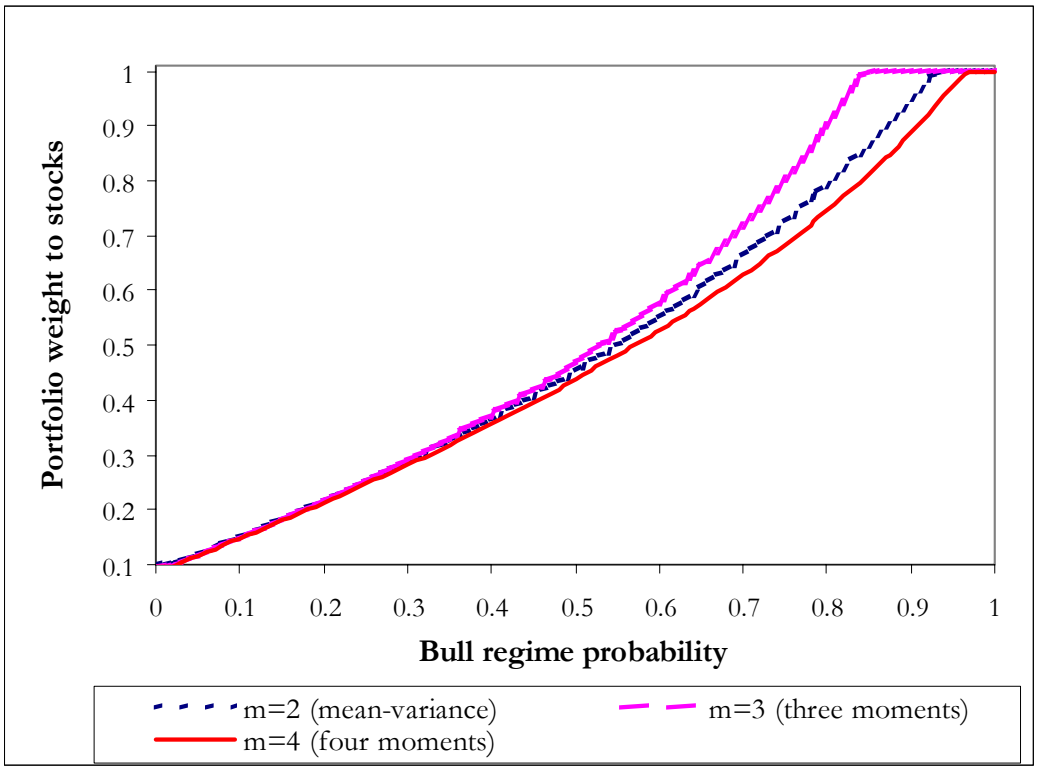


Figure 6

Smoothed Probabilities of State 2 in a Multivariate Two-Regime Model Fitted to Stock Returns and the Dividend Yield

This figure plots smoothed probabilities for the two-state MMSIAH (2,1) model $y_t = \mu_{s_t} + \sum_{j=1}^{p} a_{js_t} y_{t-j} + \sigma_{s_t}\ \boldsymbol{\varepsilon}_t$,

where $\mathbf{y}_t$ is a vector collecting the excess return and the dividend yield, μ_{s_t} is an intercept vector in state s_t, A_{1s_t} is a matrix of first-order autoregressive coefficients in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}]' \sim$ I.I.D. $N(\mathbf{0}, \Omega_{s_t})$.

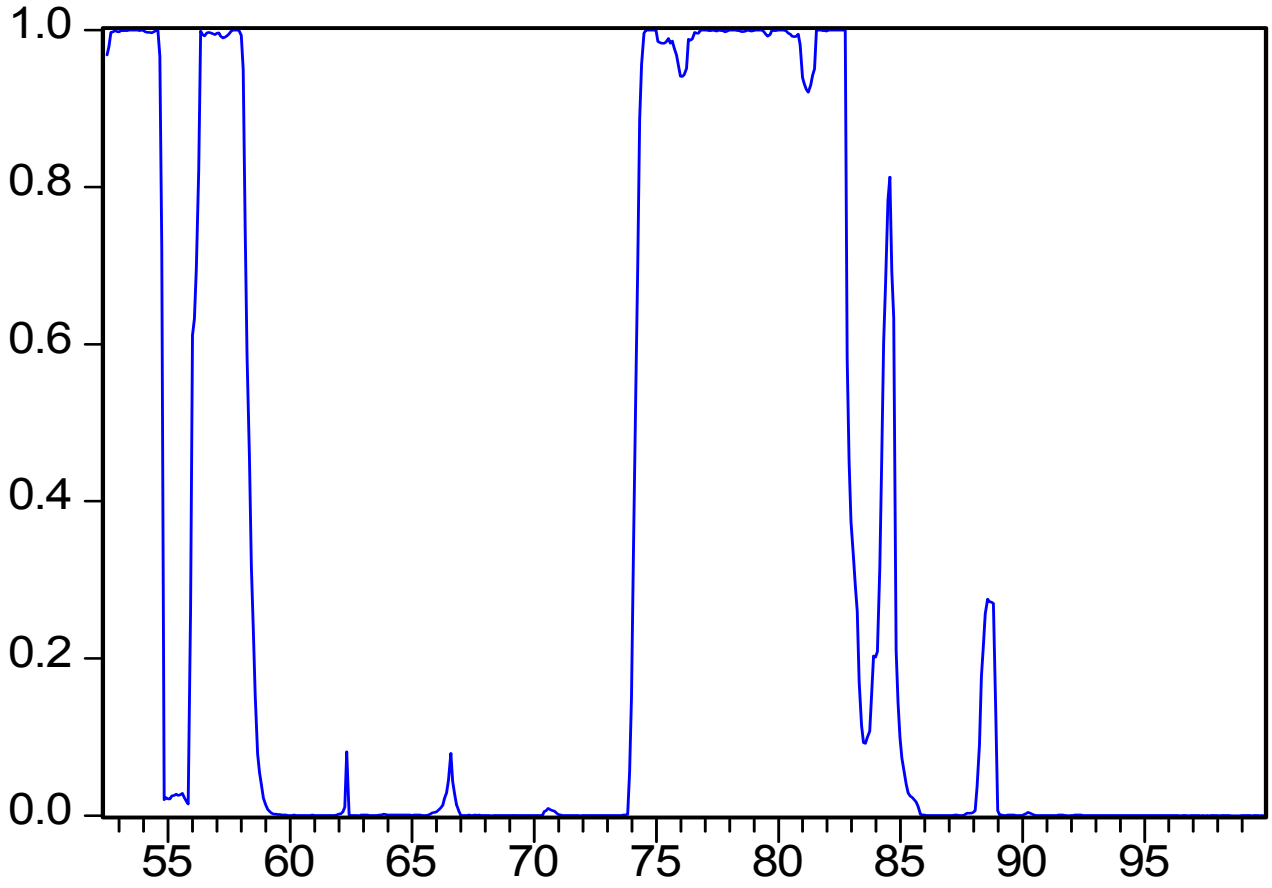


Figure 7

Effects on Optimal Asset Allocation of Changes in the Dividend Yield or State Beliefs

These figures show the optimal allocation to the value-weighted CRSP stock index as a function of the current perception of the probability of being in state 2, and of the dividend yield at the time the portfolio is chosen assuming the investor has a four-moment utility function. Excess stock returns are assumed to be generated by the regime switching model

$$y_t = \boldsymbol{\mu}_{s_t} + \sum_{j=1}^{p} A_{js_t} y_{t-j} + \sigma_{s_t} \boldsymbol{\varepsilon}_t,$$

where $\mathbf{y}_t$ is a vector collecting the excess return and the dividend yield, $\boldsymbol{\mu}_{s_t}$ is an intercept vector in state s_t, A_{1s_t} is a matrix of first-order autoregressive coefficients in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}]' \sim$ I.I.D. $N(\mathbf{0}, \Omega_{s_t})$. s_t is governed by an unobservable, first-order Markov chain that can assume 2 distinct values. The sample period is 1952:06 – 1999:12. Parameters are fixed at their full-sample ML estimates.