WORKING PAPER SERIES

On Learning and the Stability of Cycles

James B. Bullard
John Duffy

Working Paper 1995-006B
http://research.stlouisfed.org/wp/1995/95-006.pdf

PUBLISHED: Macroeconomic Dynamics, March 1998.

FEDERAL RESERVE BANK OF ST. LOUIS
Research Division
411 Locust Street
St. Louis, MO 63102

ON LEARNING AND THE STABILITY OF CYCLES

ABSTRACT

We study a general equilibrium model where the multiplicity of stationary periodic perfect foresight equilibria is pervasive. We investigate the extent of which agents can learn to coordinate on stationary perfect foresight cycles. The example economy, taken from Grandmont (1985), is an endowment overlapping generations model with fiat money, where consumption in the first and second periods of life are not necessarily gross substitutes. Depending on the value of a preference parameter, the limiting backward (direction of time reversed) perfect foresight dynamics are characterized by steady state, periodic, or chaotic trajectories for real money balances. We relax the perfect foresight assumption and examine how a population of artificial, heterogeneous adaptive agents might learn in such an environment. These artificial agents optimize given their forecasts of future prices, and they use forecast rules that are consistent with steady state or periodic trajectories for prices. The agents' forecast rules are updated by a genetic algorithm. We find that the population of artificial adaptive agents is able to eventually coordinate on steady state and low-order cycles, but not on the higher-order periodic equilibria that exist under the perfect foresight assumption.

KEYWORDS: Learning Stability, Cycles, Multiple Equilibria, Coordination, Genetic Algorithm

JEL CLASSIFICATION: D83, E32

James Bullard
Senior Economist
Federal Reserve Bank of St. Louis
411 Locust
St. Louis, MO 63102
bullard@stls.frb.org
FAX (314) 444-8731

John Duffy
Department of Economics
University of Pittsburgh
Pittsburgh, PA 15260
(412) 648-1733
jduffy+@pitt.edu
FAX (412) 648-1793

1 Introduction

1.1 Overview

The possibility of multiple stationary perfect foresight equilibria in certain classes of general equilibrium models is now well established.[1] Some of these stationary equilibria may be steady states, but others can be periodic or even chaotic. Standard general equilibrium theory makes no prediction as to which of these many stationary equilibria might be achieved in economies driven by actual human behavior. Many economists are presently trying to make sense of this situation, often by arguing that some equilibria are more likely to be eventually observed than others, and therefore that these equilibria are the more relevant ones for making predictions based on the model. One common approach involves replacing the perfect foresight assumption with an adaptive learning scheme in order to determine which of the multiple stationary equilibrium trajectories are stable under the learning dynamics.[2] However, the use of learning as an equilibrium selection device has often been limited to examples where the degree of multiplicity of stationary equilibria is not too severe. Moreover, in cases where stationary periodic equilibria have been considered, the learning analyses to date have typically been strictly local; the question of which among many equilibria would be selected under a plausible global learning analysis remains largely open.[3]

In this paper, we implement a certain type of global learning dynamic in a model that sometimes possesses coexistent steady state, periodic and chaotic perfect foresight equilibria. Our technique involves analysis of the dynamics generated by a population of artificial adaptive agents. The environment is Grandmont's (1985) endowment overlapping generations economy, in which there is a constant supply of fiat currency and where consumption in the first and second periods of life are not necessarily gross substitutes. The model supports two steady state equilibria: the Pareto inferior autarchic steady state, which is characterized by a zero demand for fiat currency, and the Pareto optimal monetary steady state, which is characterized by a positive demand for fiat currency. Under time separable

[1] For a survey, see Azariadis (1993).

[2] For a summary of this literature, see the surveys by Sargent (1993), Evans and Honkapohja (1992), and Marimon and McGrattan (1992).

[3] Of course, if any type of stationary equilibrium is locally unstable under a particular learning dynamic, it will remain globally unstable under the same learning dynamic, and can be thus be ruled out as a candidate for long-run equilibrium.

preferences, and provided that the value of the coefficient of relative risk aversion for the old agents is high enough and that of the young agents is low enough, it can be shown that stationary perfect foresight equilibria may also exist in which the equilibrium dynamics are characterized either as periodic or chaotic trajectories for real money balances, and that these complicated stationary equilibria are also Pareto optimal. However, in such cases, the forward perfect foresight dynamics are not well–defined, so that it is not clear that there is a meaningful forward dynamic that could plausibly be assumed to converge to one of these more complicated stationary equilibria. Thus while complicated perfect foresight equilibria may exist in this model it is an open question whether or how such equilibria might be achieved. Grandmont (1985) provided a complete analysis of the model under the well–defined, backward (direction of time reversed) perfect foresight dynamics, and demonstrated that the limit of these backward dynamics could be a complicated stationary equilibrium.

Our approach is to create a meaningful forward dynamic by relaxing the perfect foresight assumption and examining how a heterogeneous population of artificial adaptive agents might learn to forecast future prices in such an environment. Agents are differentiated (in addition to birth dates) by the forecast rule they employ, and each agent solves an optimization problem based on an individual-specific forecast for the future price. Agents' forecast rules are updated by a *genetic algorithm*, a population–based, stochastic, directed search algorithm that generates new rules while retaining and improving upon those rules that have performed well in the past.[4] We conduct computational experiments with economies defined in this way and report the results.

We interpret genetic algorithm learning as a powerful representation of trial-and-error learning which has important advantages over many other models in the literature. Among these are that: (1) beliefs are initially heterogeneous across agents, (2) the information requirements on agents are minimal, (3) the genetic algorithm offers a natural model for experimentation by agents with alternative forecast rules, (4) the heterogeneity of beliefs allows parallel processing to be an important feature of the economy, (5) genetic algorithm learning has been shown in other research to successfully mimic the behavior of human subjects in controlled laboratory settings, (6) the learning model can be applied even in

[4]For an introduction to genetic algorithms see Goldberg (1989) or Michalewicz (1994).

complicated problems, and finally (7) the initial heterogeneity of the population allows us to initialize the system randomly, so that we are able to obtain some sense of the "global" properties of our system under learning as opposed to the local analysis that is often employed in the learning literature.[5] These features suggest to us that genetic-algorithm-based models of learning have interesting features from an economic standpoint, and that such a model of learning will allow us to address the question of what type of behavior might reasonably be expected to arise in an actual economy or economic situation with a pervasive multiplicity of stationary equilibria.

We report the results of 1,410 computational experiments. The experiments are organized around a single preference parameter: the coefficient of relative risk aversion of the old agents (those in the second period of life). The relative risk aversion of the young agents (those in the first period of life) remains unchanged in all of the experiments. The main finding is that the population of artificial adaptive agents is able to coordinate their initially heterogeneous beliefs so as to implement steady state or low–order periodic equilibria for real money balances. For low values of the preference parameter, the artificial agents always coordinate on the monetary steady state; in this case complicated stationary equilibrium trajectories do not exist. However, as the preference parameter is increased, and the multiplicity of stationary perfect foresight equilibria becomes more pronounced, the population of artificial agents frequently fails to coordinate on the cyclic equilibria that are picked out by the limiting backward perfect foresight dynamics. Instead, coordination occurs on steady state and periodic equilibria that are of lower periodicity than the equilibria predicted by the limiting backward perfect foresight dynamics. Thus our long–run outcomes under learning tend to be simpler (lower order) than outcomes under backward perfect foresight. However, we also find that time to coordination tends to increase with the relative risk aversion of the old agents over a large portion of the parameter space, and in addition, we find that when cycles exist our systems can display qualitatively complicated dynamics for long periods of time before eventually converging to the relatively simple, low–periodicity equilibria.

We reach two main conclusions based on these results. First, in the economies we study,

[5]In this paper, we use the term "global" to describe our analysis because it is based on a random initialization scheme. We recognize that our analysis is not truly global, even computationally speaking, since we did not complete multiple experiments based on every possible initialization for a given parameterization. Such an approach is beyond the scope of this paper.

the initially heterogeneous beliefs of the agents eventually give way, leading to a situation where all agents coordinate on a perfect foresight equilibrium. Thus, our results suggest that *agents can achieve coordination even in relatively complicated situations.* Second, the stationary equilibria on which agents coordinate are always relatively simple – either a steady state or a low order cycle – and are always Pareto optimal. Our results suggest that *it is difficult for an economy comprised of optimizing agents with initially heterogeneous beliefs to coordinate on especially complicated stationary equilibria*, such as those characterized by high–order periodicities, even though such coordination is a distinct possibility in this model *a priori*. On the other hand, we are unable to rule out coordination on periodic equilibria entirely, and we find that the transient dynamics of our systems can be persistent and qualitatively complicated if (and only if) stationary periodic equilibria exist.

1.2 Recent related literature

Several recent studies have focused on certain aspects of learning in general equilibrium models closely related to those studied in this paper. Woodford (1990) analyzes the local stability of stationary sunspot equilibria in an overlapping generations economy where the sunspot variable follows a two–state Markov process. All agents in this economy use the same forecast rule, which is a version of the Robbins–Monro stochastic approximation algorithm. Woodford provides conditions under which a stationary sunspot equilibrium is (locally) an attractor under this learning algorithm. Our results amplify Woodford's conclusion in a closely related model, in that, first, a stationary perfect foresight periodic equilibrium of order two can be an attractor on the basis of our more global learning algorithm and second, these deterministic periodic equilibria can be viewed as limiting cases of the two–state stationary sunspot equilibria studied by Woodford. While Woodford's stability conditions only generally apply to two–state, stationary sunspot equilibria, our results go further, suggesting that higher order (periodicity greater than two) stationary periodic equilibria are unlikely to be attractors. That is, while equilibria characterized by high–order periodicities may in principle be locally stable in an analysis such as Woodford's, they evidently have negligible basins of attraction relative to the space of definition of the model in our analysis.

In a related study, Guesnerie and Woodford (1991) find conditions under which cycles will be stable under a learning dynamic in a one–step ahead forward looking model. In their analysis, all agents form expectations according to a simple adaptive rule in which the expected price is a convex combination of the actual and expected prices k periods in the past. Guesnerie and Woodford (1991) conclude that cycles of any order might be attractors under the learning dynamics in the systems they study, provided a condition on the map describing the equilibrium conditions is met. When this condition is not met, and such a possibility is shown always to exist depending on the nature of the equilibrium map, their systems might converge to *learning equilibria* of the type studied by Bullard (1994), limiting learning dynamics which are stationary but which are not perfect foresight equilibria and which therefore involve forecast errors in perpetuity. The Guesnerie and Woodford (1991) study suggests that *a priori*, the possibilities for the limiting learning dynamics in our system are quite open. However, our learning model is considerably more complex than one in which there is a simple adaptive learning rule used by all agents. In a similar vein, Grandmont and Laroque (1986, 1991) as well as Grandmont (1985) study general differentiable learning rules in one–step ahead systems and find conditions under which these systems may or may not converge to cycles and steady states.[6] Grandmont and Laroque (1991) argue that it is easy to construct examples where the steady state of an economic model is locally unstable under reasonably defined learning dynamics, while Grandmont and Laroque (1986) and Grandmont (1985) suggest that periodic equilibria, perhaps complicated, can be the ultimate outcome of similarly defined systems.

Evans and Honkapohja (1995) also study one step ahead forward looking systems, both stochastic and deterministic. They imagine that agents all use a certain recursive algorithm to compute the expected value of the variable of interest as a function of past observations on that variable. They assume that agents prespecify a value of k, the periodicity of the local equilibrium, so that in each case, agents are assumed to know the periodicity of the equilibrium they are attempting to learn. The Evans–Honkapohja analysis makes use of the theory of recursive stochastic systems associated with Ljung (1977) and imported to economic problems of this type by Marcet and Sargent (1989), and they also relate their necessary and sufficient conditions for stability of equilibria in the learning dynamic to the

[6]See Grandmont (1994) for a survey of this literature.

concept of *expectational stability* as formulated in Evans (1989).[7] The Evans–Honkapohja conditions suggest that local stability under learning will depend on the properties of the map describing the equilibrium conditions, and that in principle it is possible to have many equilibria which are (strongly E-) stable in the learning dynamics, so that while recursive learning might reduce the set of equilibria regarded as plausible actual outcomes, it cannot in general suggest a unique equilibrium. However, the case studied in the present paper is somewhat special in this regard in that Grandmont's (1985) example involves a unimodal map with a negative Schwartzian derivative, which in turn implies that a single periodic equilibrium among many will be (strongly E-) stable in the learning dynamics according to the Evans–Honkapohja analysis. The monetary steady state will also be (weakly E-) stable in the learning dynamics in this case, so that Evans and Honkapohja conclude that the equilibrium actually observed would depend on the exact specification of the learning rule used by the agents, in particular whether the rule is consistent with a steady state or with a stationary periodic equilibrium of order k.[8] In our model, agents' learning rules are always consistent with steady state and stationary periodic equilibria of many different orders.

We know of no research that analyzes stationary periodic equilibria such as those we are interested in under genetic algorithm learning. However, research on genetic algorithm learning by Arifovic (1994*c*) and Arifovic and Eaton (1994) does focus on coordination problems that can be severe. Arifovic (1994*c*), for example, analyzes genetic algorithm learning in the Kareken and Wallace (1981) model of exchange rates, in which the exchange rate is indeterminate in the sense that any fixed exchange rate $e \in (0, \infty)$ is a rational expectations equilibrium. Arifovic (1994*abc*) also shows that the dynamics from experiments with human subjects compare favorably to dynamics generated in the same economies with genetic algorithm learning.

While experiments with human subjects are beyond the scope of this paper, we are enthusiastic about testing the predictions of our genetic algorithm learning model against

[7]For a survey of this topic, see Evans and Honkapohja (1992).

[8]Evans, Honkapohja, and Sargent (1993) make a very different argument in the context of the same overlapping generations model analyzed in this paper. They suggest that if a significant fraction of the population uses misspecified models because they view themselves as participating in an economy driven by random processes, and these agents attempt to optimize against forecasts generated by using the unconditional distribution of past prices, then the possibility of stationary periodic equilibria will be eliminated. In the present paper, agents have heterogeneous beliefs but the existence of periodic equilibria is not threatened.

such a benchmark. Van Huyck, Cook, and Battalio (1994) have performed an experiment with human subjects in a repeated coordination game that shares some of the properties of the model studied in this paper. In particular, their coordination game has two strict, pure strategy Nash equilibria, one of which is an interior equilibrium. This interior equilibrium is shown to be stable under myopic best response learning dynamics provided that the game's payoff parameter falls within a certain range; otherwise the myopic best response dynamic leads to periodic or chaotic trajectories that cycle about the interior equilibrium. The authors consider an experimental treatment where the payoff parameter lies either within the range that assures stability of the interior equilibrium under the myopic best response dynamic or lies far enough outside this range so that the myopic best response dynamic converges to a period seven cycle. They find that, in contrast to the predictions of the myopic best response dynamic, subjects always learn over time to coordinate on the interior equilibrium, even in the parameterization where the best–response dynamics would imply that the interior equilibrium is unstable (and the period seven cycle is stable). We view these laboratory results as complementary to the findings we present in this paper, because in both papers agents fail to coordinate on relatively high order cycles. However, we differ from Van Huyck et al. (1994) in that the environment we consider sometimes displays a dense set of periodic equilibria. We also note that in the overlapping generations environment that we study, the complicated trajectories that agents are attempting to coordinate on are *equilibrium* trajectories, whereas the period seven cycle in the Van Huyck et al. (1994) experiment arises from the use of the disequilibrium, best response learning dynamic.

Experiments with human subjects that have been conducted in an overlapping generations environment similar to the one studied in this paper include Marimon and Sunder (1993, 1994, 1995), and Marimon, Spear, and Sunder (1993), with the latter paper being the one most closely related to this paper. Marimon, Spear, and Sunder (1993) study a two–period overlapping generations economy with preferences and endowments such that a period two perfect foresight equilibrium coexists with the monetary steady state.[9] Subjects make predictions of future prices, and, analogous to our set up, the optimal level of real money balances is computer generated for each subject given their price prediction. The

[9]No higher order perfect foresight periodic equilibria exist.

number of subjects in a generation ranged from 2 to 5, while the number of new generations (iterations) ranged from 27 to 67. The design of their experiment is somewhat different from ours in that the authors systematically varied the size of the incoming generation (say: 3, 2, 3, 2, ...) in their experiments as a means of inducing real variation in the equilibrium price level for the first phase of the experiment, and then subsequently kept the generation size constant to see if the subjects might learn to coordinate on the period two equilibrium once cyclicality in expectations had been established. In the experiments with changing population sizes (real shocks), they also employed a sunspot variable by alternating the color of some of the information presented on subjects' computer screens such that the sunspot was perfectly correlated with the real shock. The sunspot variable was left on at all times, even when the real shock was turned off, and agents were not informed of the presence of real shocks. The authors' main finding is that in treatments where there were real shocks in the initial phase and sunspots throughout, prices continued to display marked periodicity of order two even after the real shock was turned off. In treatments without real shocks in the initial phase (but with sunspots), subjects always learned to coordinate on the monetary steady state. The authors conclude that it is possible for human subjects to learn to coordinate both on steady state and on period two equilibria. These results match up well with our own, as our genetic algorithm based learning model predicts that both steady state and period two equilibria will be observed as long–run outcomes in a similarly specified economy. Furthermore, their time series for the price level (Figure 3, p. 89) are qualitatively similar to those generated in our computational experiments, even though our systems had neither real shocks nor a sunspot variable. Finally, these authors argue that equilibrium selection would appear to be *path dependent*, since periodic price dynamics were only observed when there were real shocks in the initial phase. We note that our model also predicts that equilibrium selection will be path dependent. By conducting a large number of computational experiments, we are able to let conditions develop spontaneously that in certain instances led our system to converge to a period two equilibrium, while Marimon, Spear, and Sunder (1993) needed inducements in the form of the real shocks in order to avoid conducting a prohibitively large number of experiments.[10] Indeed, the major advantage of

[10]In subsequent research, Marimon and Sunder (1995) observed two experimental overlapping generations economies where inflation realizations were persistently cyclical in the neighborhood of a monetary steady state of the model. These two economies were not subject to any real shocks, and there were no stationary

our computational approach is that we are able to conduct many more experiments with many more iterations across a wider variety of economies, and thus obtain a more complete picture of the long-run outcomes that might reasonably be expected in environments like this one.

Computational experiments with evolutionary algorithms (such as genetic algorithms) in economic settings have been suggested by Holland and Miller (1991) and Sargent (1993) and have been performed by Axelrod (1987), Miller (1989), Marimon, McGratten, and Sargent (1990), Binmore and Samuelson (1992), Andreoni and Miller (1993), Arthur (1994), Rust, Miller, and Palmer (1993, 1994) and Wright (1995) among others. These applications have generally not been in competitive general equilibrium environments like we have in mind. An exception is the work of Arifovic (1994*abc*).

2 The environment

The example economy is taken from Grandmont (1985). Time is discrete with integer $t \in (-\infty, \infty)$. There is a single, perishable consumption good and a constant supply of fiat money $M > 0$. There are $\frac{N}{2}$ agents per generation, where N is the total population alive at time t. Agents live for two periods, and can save between periods by holding fiat money. Agents receive strictly positive endowments of the consumption good in each period of life, $\{e_1, e_2\}$. Preferences are given by:

$$U = \frac{c_t(t)^{1-\rho_1}}{1-\rho_1} + \frac{c_t(t+1)^{1-\rho_2}}{1-\rho_2},$$

where ρ_1, $\rho_2 \in (0, \infty)$ denote the coefficient of relative risk aversion of the young and old agents, respectively. The notational convention is that subscripts denote birthdates while real time is recorded in parentheses, so that $c_i(j)$ denotes the time j consumption of the agent born at time i.[11] The agent born at time t maximizes utility subject to the pair of budget constraints

$$c_t(t) \leq e_1 - s_t(t),$$

periodic equilibria in the neighborhood of the steady state. The observed cyclic behavior for inflation was attributed to the stability of the adaptive learning system that subjects were thought to have used; the linearized learning system, evaluated at the monetary steady state, had eigenvalues that were complex, with modulus close to, or equal to one.

[11] While in principle there are many agents born at time t, we do not distinguish among them in this section.

$$c_t(t+1) \leq e_2 + \frac{P(t)}{P(t+1)} s_t(t),$$

where $s_t(t)$ denotes the amount the agent born at time t chooses to save at time t, and the price of the consumption good in terms of fiat money is denoted by $P(t)$. Combining the first order conditions with the two budget constraints, one obtains:

$$c_t(t) + c_t(t)^{(\rho_1/\rho_2)} \beta(t)^{((\rho_2-1)/\rho_2)} = e_1 + e_2\beta(t) \tag{1}$$

where $\beta(t) = \frac{P(t+1)}{P(t)}$ denotes the gross inflation factor between dates t and $t+1$. It follows from the compactness of the budget set and the strict concavity of the utility function that the young agent's consumption decision, $c_t(t)$, and therefore his savings decision, $s_t(t)$, are *uniquely* determined. Since a closed form solution for $c_t(t)$ is unavailable, we will instead rely upon numerical methods to obtain $c_t(t)$. Once the optimal consumption and savings amounts are determined, it becomes possible to define a perfect foresight equilibrium sequence for prices (equivalently, for real money balances) using the market clearing condition:

$$S(t) = \frac{M}{P(t)}$$

where $S(t)$ is aggregate savings at time t. Using this condition, and the fact that the supply of fiat money M is constant, one can derive a first order difference equation that characterizes all perfect foresight equilibria in this economy:

$$P(t) = \frac{S(t+1)}{S(t)} P(t+1).$$

Following Grandmont (1985), let us write this difference equation more compactly as:

$$P(t) = \Phi(P(t+1)). \tag{2}$$

A *perfect foresight equilibrium* is any sequence of prices $\{P(t)\}$ that satisfies equation (2). A *steady state* equilibrium is a price level $\overline{P}$ such that $\overline{P} = \Phi(\overline{P})$. It is well known (see, e.g., Gale (1973)) that in this model, there can be at most two steady state equilibria, one in which aggregate savings is zero and agents consume endowments, and possibly another in which aggregate savings is positive and the steady state gross rate of return on savings and its reciprocal, the steady state gross inflation factor, are both equal to unity. In the analysis that follows, we choose an endowment sequence which guarantees that this latter equilibrium exists, that is, that we have the *Samuelson case* in Gale's terminology.

As Grandmont (1985) demonstrates, these two steady states are not the only stationary equilibria that this economy may possess. If the offer curve is sufficiently backward bending, or equivalently, if the coefficient of relative risk aversion of the old agents, ρ_2, is large enough while the coefficient of relative risk aversion of the young agents, ρ_1, is less than or equal to unity, then in addition to steady states, the set of equilibria may also include periodic as well as chaotic equilibria.[12] Our focus in this paper is limited to the steady state and periodic equilibria that may arise in this environment. A *periodic* equilibrium of order k consists of a sequence of k prices, $\{\overline{P}_1, \overline{P}_2, ..., \overline{P}_k\}$, such that $\overline{P}_j = \Phi^k(\overline{P}_j)$, $j = 1, 2, ...k$, where Φ^k denotes the k^{th} iterate of the map Φ. Grandmont (1985) noted that the forward perfect foresight dynamics, that is, iterates of the map (1), may not be uniquely defined depending on the properties of the map $\Phi(\cdot)$. Grandmont (1985) studied this system by limiting attention to the *backward perfect foresight dynamics*, that is, sequences of prices that solve the map (1) with the direction of time reversed. A periodic equilibrium of order k in the backward perfect foresight dynamics is a sequence of k prices $\{\overline{P}_k, \overline{P}_{k-1}, ..., \overline{P}_1\}$ of the map Φ such that $\overline{P}_j = \Phi^{-k}(\overline{P}_j)$, $j = 1, 2, ..., k$. Grandmont's (1985) well–known main result was to show that periodic equilibria of any order and chaos could exist as long-run outcomes in the backward perfect foresight dynamics without abandoning the classical assumptions of utility maximization and market clearing.[13] At least for periodic equilibria, this is enough to prove existence of complicated equilibrium dynamics in the more meaningful forward perfect foresight system, because even though the forward dynamics are not uniquely defined, it is at least in principle possible that agents would choose appropriately in the forward system so as to replicate the trajectory uniquely defined in the backward perfect foresight dynamics.[14]

In Grandmont's (1985) model, when periodic equilibria exist, they coexist with other equilibria, and in particular, they coexist with steady states. Sarkovskii's (1964) theorem implies that if a periodic equilibrium of order three exists, then equilibria of every other order q, where q is an element of the set of positive integers, also exist. This raises the question of which among these many stationary equilibria might be achieved, and Grandmont (1985)

[12]Similarly, Benhabib and Day (1982) demonstrate the possibility of periodic and chaotic equilibria in the "Classical" version of this same model where the time patterns of endowments and preferences are reversed.

[13]The periodic equilibria in Grandmont's model can be viewed as deterministic versions of sunspot equilibria first studied by Shell (1977) and Azariadis (1981).

[14]While in principle there are many agents born at time t, we do not distinguish among them in this section.

addressed this question by considering the limiting backward perfect foresight dynamics and arguing that plausible learning rules exist that would, if used by all agents, cause the system to be locally convergent to the periodic equilibria isolated under backward perfect foresight. In particular, Grandmont (1985, Proposition 3.2, p. 1012) interpreted a stationary equilibrium that was stable in the backward perfect foresight dynamics as stable in the forward learning dynamics, and Grandmont gave conditions on a differentiable learning rule under which such statements would be true.

In Figure 1, we replicate Grandmont's (1985, p. 1030) Figure 4, which is a bifurcation diagram in the backward perfect foresight dynamics. In this example, which we will use throughout the remainder of the paper, $\{e_1, e_2\} = \{2, .5\}$, and the relative risk aversion of the young is fixed at $\rho_1 = .5$. The figure shows a plot of real money balances per capita, which can range between zero and the first period endowment of 2. For each value of the parameter representing the relative risk aversion of the old agents, ρ_2, the backward perfect foresight map was iterated 1000 times, using as an initial condition the peak of the unimodal map. The last 50 of the 1000 iterations are plotted in the diagram. As the relative risk aversion parameter is increased, a standard period-doubling bifurcation pattern emerges, with the limiting backward perfect foresight dynamics being the monetary steady state for low values of the risk aversion parameter and a periodic equilibrium of order three for high values of the risk aversion parameter. Between these extremes, the limiting dynamics involve cycles of mostly higher order, or perhaps of a very high order or even chaos, in which case the plot of the last 50 iterations is inadequate to characterize the equilibrium. Qualitatively, however, the figure is clear.

We now want to consider relaxing the perfect foresight assumption and introducing genetic algorithm learning into this system.

3 Learning

3.1 Representation of a heterogeneous population

At every date t, there is a finite population of N agents participating in the economy. The population is the same at every date and is equally divided between agents born at time t and those born at time $t-1$, so that the size of each generation is $\frac{N}{2}$. Each agent in the population is differentiated by a *forecast rule* for next period's price level. This forecast rule completely characterizes the agent's behavior in both periods of life, because the agent takes the optimal action (makes the optimal savings decision) based on the forecast of next period's price produced by his own forecast rule. This forecast rule is represented by an integer value $k \in [0, \bar{k}]$. The rule is to use the price level that was realized $k + 1$ periods in the past as the forecast of next period's price level. For example, consider a young agent i, born at time t, who has chosen k_i. If we let $F_t^i[P(t+1)]$ denote agent i's time t forecast of the price level expected to prevail at time $t + 1$, we can write agent i's forecast rule as:

$$F_t^i[P(t+1)] = P(t - k_i - 1).$$

This specification of the forecast rule is simple, but has the important implication that each young agent i can make optimal savings decisions in any type of stationary equilibrium of order q, so long as $q \in [0, \bar{k} + 1]$. In fact, we chose this specification precisely because it allows agents to adopt behavior consistent with steady state or periodic trajectories for prices up to the limit $\bar{k} + 1$.[15] We note that this class of forecast rules is more general than it might first appear. While one could allow agents to use past data in more complicated ways, perhaps even econometrically sophisticated ways, for our exercise to be interesting the forecast rule must ultimately allow the agent to behave in a way consistent with any of the equilibria of the model (up to some limit). In particular, if the economy is in an equilibrium with periodicity $q \leq \bar{k} + 1$, the forecast rule chosen must deliver a forecast consistent with that equilibrium.[16] If it does not, then the agents cannot coordinate on that equilibrium. For this reason, we think that it is reasonable to abstract from the question of *how* agents manipulate the data and concentrate on specifying a class of rules such that agents can in principle coordinate on any of the equilibria of the model up to some limit.[17]

In order to apply the genetic algorithm, we must first encode each agent's forecast rule as a string of length ℓ, with elements of the string chosen from the binary $\{0, 1\}$ alphabet. For example, if $\ell = 8$, the string might be 00100101. This string value can be decoded to

[15]For the same reason other researchers who have studied the stability of cycles under learning, e.g. Guesnerie and Woodford (1991) and Evans and Honkapohja (1995), have also assumed that agents form expectations based on prices that prevailed k periods in the past.

[16]This is the requirement that the forecast rule be able to *detect the cycle* discussed by Grandmont (1985) and Grandmont and Laroque (1986).

[17]For a discussion of the issue of *how* agents manipulate the data to derive forecasts see, e.g. Bray and Savin (1986) or Duffy (1994).

Bit string	k	Consistent with:
00000000	0	steady state
00000001	1	steady state, 2-cycle
00000010	2	steady state, 3-cycle
00000011	3	steady state, 2-cycle, 4-cycle
00000100	4	steady state, 5-cycle
00000101	5	steady state, 3-cycle, 6-cycle
..	.	.
..	.	.
..	.	.
11111111	255	steady state, 2-cycle, ..., 256-cycle

Table 1: Most strings are consistent with more than one stationary equilibrium.

the base 10 integer 38, which is the k-value that this agent uses in forecasting future prices. Thus, this agent would use the *actual* price level that was realized $k+1=39$ periods ago, (i.e. $P(t-39)$) as the forecast of next period's price level. Of course, if the economy had been in a steady state for each of the past 39 periods, then this forecast rule would give the same predicted price as the string 00000000. In fact, all strings representing integer values $k \in [0, \bar{k}]$ are consistent with a steady state equilibrium, and many are also consistent with periodic equilibria of other orders. The nature of this situation is described in more detail in Table 1.

3.2 How forecasts are used

Given the young agent i's time t forecast of the price level expected to prevail at time $t+1$, $F_t^i[P(t+1)]$, we can denote this same agent's forecast of the *gross inflation factor* between dates t and $t+1$ by:

$$\beta_t^i(t) = \frac{F_t^i[P(t+1)]}{P(t)}.$$

Using this forecast for gross inflation, the computer algorithm then numerically solves equation (1) for agent i's consumption amount $c_t^i(t)$. The young agent i's savings decision is given by $s_t^i(t) = e_1 - c_t^i(t)$. Aggregate savings at time t is therefore given by:

$$S(t) = \sum_{i=1}^{N/2} s_t^i(t).$$

The law of motion for the price level $P(t)$ in the *forward* dynamics of our learning model is given by:

$$P(t) = \frac{S(t-1)}{S(t)} P(t-1),$$

and thus $P(t)$ is determined once $S(t)$ is known. However, the quality of the time t young agents' forecasts cannot be evaluated until period $t+1$, when $S(t+1)$ and thus $P(t+1)$ is known. We now turn to a discussion of how we evaluate and update forecast rules.

3.3 Updating forecast rules using a genetic algorithm

While both the genetic algorithm and the overlapping generations model have some rather obvious biological overtones, we do not interpret our model of learning as one of a strict passing of genetic information from one generation to the next. Instead, we view the genetic algorithm as a convenient device with which to model communication among individual agents alive at date t. We imagine that agents who will enter the productive portion of their life at time $t+1$ (that is, agents who will be young next period) have had conversations with some of the older agents alive at time t and have devised forecast strategies based on these conversations.[18] The result is a new generation of agents—the *newborns* . The newborns become the next generation of young agents once the agents in the model are aged appropriately. Because we create the newborn generation at time t and allow these newborn agents to enter their productive lives at time $t+1$, we can make use of the entire set of genetic information available at time t to create the new generation. That is, we can allow the newborns to converse with any of the agents participating in the economy, young or old, at time t. Accordingly, we subject the entire population of N strings available at time t to the four genetic operators that make up the genetic algorithm, and create the newborn generation from this information. We now describe our implementation of these four operators: reproduction, crossover, mutation and election.

3.3.1 Reproduction

To create the newborn generation at time t, we begin by calculating the *fitness* of each string in the entire population of N agents alive at time t. First, each string is decoded to determine its forecast rule. Next, the forecast rule is applied using the history of prices

[18] See Arifovic (1994*b*) and Bullard and Duffy (1994) for a similar interpretation.

available through time $t-1$, and a price forecast is determined. The reason for restricting the forecast rule to the history of prices available through time $t-1$ is that this restriction enables us to assess how well the forecast rule *would have performed* had it been in use in the previous period. Performance is measured in terms of the two period, lifetime utility that the agent would have attained had the agent been young in the previous period and possessed the forecast rule in question. In assessing the lifetime utility value, we use the most recent realization of the gross inflation rate $\beta(t-1) = P(t)/P(t-1)$. Thus, the lifetime fitness value is a measure of the accuracy of a particular forecast rule; the higher is the lifetime fitness value, the more accurate is the forecast rule that was used.

Once the fitness values have been determined, we conduct one *selection tournament* for each of the $\frac{N}{2}$ newborns we wish to create. Each tournament consists of a random selection of two strings, with replacement, from the finite population of all N strings alive at time t. The string with the higher fitness is copied and put into the group of newborn strings. This process means that the higher fitness strings of the existing population tend to get copied into the newborn generation.[19] Although we now have a set of newborn strings, we are not finished, since we want to allow the newborn agents to experiment with new possible forecast rules which may not be part of the genetic information set available at time t. The next three operators accomplish the task of introducing new information into the system.

3.3.2 Crossover, mutation and election

The main idea of the crossover and mutation operators is to create new forecast rules by mixing portions of existing strings together and by changing individual bits with small probability. Accordingly, we randomly pair the newborn strings. Then for each pair we perform the crossover operation: with some probability $p^c > 0$, we divide the pair of strings at some randomly chosen point, $s \in [1, \ell - 1]$, where ℓ is the length of each bit string, and swap the bits to the right of the crossover point. With probability $1 - p^c$ we do not apply the crossover operation to the pair of strings. As an example, suppose that we have a pair of strings of length $\ell = 8$ and that crossover is to be performed on these two strings. Suppose

[19]The tournament method of implementing the reproduction operator of the genetic algorithm has some advantages over the "biased roulette wheel" methodology found in Goldberg (1989) and many other early genetic algorithm implementations. See Fogel (1994) or Michalewicz (1994) for a discussion. For a theoretical analysis of tournament selection, see Blickle and Thiele (1995).

also that the randomly chosen point is $s = 6$. The pair of strings are divided at this point:

$$010100|11$$

$$111001|01$$

The portion of the strings to the right of the dividing point are then swapped, creating two new strings:

$$01010001$$

$$11100111$$

The two resulting strings (even in cases where the two strings were not changed by crossover) are then subjected to some mutation: we consider each bit value $b = 0, 1$ of the two strings that result from the crossover operation, and with some probability $p^m > 0$ we replace the bit, b, with the bit $1 - b$. With probability $1 - p^m$, the bit value is not mutated.

We refer to the strings that result from crossover and mutation as *alternatives*, because we think of the newborns (strings selected through reproduction) as contemplating adopting some alternative forecast rules that may not be in use in the current population. In the standard, "biological" application of the genetic algorithm, these alternatives would simply become the newborn generation, but such an approach is less than satisfactory in economic applications where agents are appropriately viewed as making intelligent choices. In particular, the standard application of the genetic algorithm would allow our alternative strings to enter the population even if they are not strictly better (in terms of fitness) than the newborn strings from which they were created. But we want to think of our economic decision makers as being somewhat less naive. Therefore, we adopt Arifovic's (1994*a*) *election* operator: each pair of newborns is compared with the corresponding pair of alternatives. Of the four strings, only the two with the highest fitness values are allowed to remain in the generation of newborns, and the other two are discarded. Fitness of the alternatives is computed in the same way as it is in the reproduction operator. The crossover, mutation and election operators combined can be interpreted as a method of allowing new forecast rules to enter the system without forcing agents to adopt rules that are unlikely to yield high utility.[20]

[20]The election operator is properly viewed as a modification of the selection/reproduction operator of the

The reproduction, crossover, mutation and election operators have a simple economic interpretation. Being 'born' in this economy means leaving one's formative years and entering the productive portion of one's life. These newborn agents just leaving their formative years initially have no plans for the future—they are 'blank slates.' They acquire the forecast rule they will need by communicating with a few other members of society, those either one or two generations ahead of them. This communication is modeled via the reproduction operator. In our implementation, each newborn agent communicates with two randomly selected members of the society. The newborns evaluate the forecast rules that belong to these two older agents by calculating how much utility the rules would have delivered had they been in use one period in the past. Each newborn then copies the forecast rule of the two that would have delivered the most utility. This completes the first step in attaching a forecast rule to each of the incoming members of the society. But the newborns communicate further when they talk with each other and contemplate alternative forecast rules that might not be in use in the society at that time—that is, the newborns conduct a mental experiment with other possible forecast rules. This additional communication is modeled via the crossover and mutation operators. In our implementation, the newborns are paired and each pair creates two alternative forecast rules by combining parts of their existing rules into two new possibilities, and also by randomly changing small parts of the recombined forecast rules. The two alternative forecast rules are evalutated to see how much utility they would have delivered had they been in use one period in the past. Alternatives which improve upon the newborns existing forecast rules are adopted, while alternatives which do not are discarded. This last step is modeled using the election operator. Thus the incoming generation learns from the experience of the agents older than themselves and also can be innovative in introducing new forecast rules into the society.

Once the election operator has been applied, and a generation of newborn agents has

genetic algorithm. In the genetic algorithm literature, a distinction is frequently made between *pure* selection procedures and *elitist* selection procedures. See, for instance, Grefenstette (1986) or Rudolph (1994). In a pure selection procedure, strings are chosen for reproduction based solely on principles of natural selection. Elitist selection procedures involve two steps. First, pure selection is performed. Second, the elitist procedure stipulates that the best discovered string, either before or (in our case) after reproduction, always remains intact into the next generation. Thus, our reproduction operator, together with our election operator, comprise an elitist selection procedure. Rudolph (1994) has shown that an elitist selection procedure is necessary, though not sufficient for the genetic algorithm to converge asymptotically to one of the perfect foresight equilibria of the model.

been chosen, time advances to the next period. The time t population of newborn agents, created via reproduction, crossover, mutation and election, becomes the young generation alive at time $t+1$. The generation of young agents alive at time t becomes the old generation alive at time $t+1$. The generation of old agents alive at time t ceases to exist. The forecasts of the time $t+1$ young generation are determined, their savings decisions are calculated, and the aggregate savings amount $S(t+1)$ then determines the price level $P(t+1)$ according to the law of motion for prices $P(t+1) = \frac{S(t)}{S(t+1)}P(t)$. The genetic algorithm is then begun anew. The algorithm ends when either a convergence criterion (described below) is satisfied or the algorithm has reached the maximum number of iterations (generations) allowed.

3.4 Some advantages of genetic algorithm learning

We interpret genetic algorithm learning as a useful model of trial-and-error learning in a heterogeneous agent economy. This learning model has many important advantages relative to other models in the literature. First, beliefs are initially heterogeneous across agents, a feature not often modeled in the learning literature to date.[21] Second, the information requirements on agents are minimal, as they only need to know their own utility and their own forecast rule in order to make a decision. Third, the genetic algorithm offers a natural model for experimentation by agents with alternative forecast rules, an important characteristic of learning also rarely modeled in competitive general equilibrium environments in the literature to date. Fourth, the heterogeneity of beliefs allows parallel processing to be an important feature of the economy. That is, some agents are trying one forecast rule while other agents are trying other forecast rules, with the better forecast rules propagating and the poorer ones dying out. We think this is closely akin to what goes on in actual economies, where communication among agents encourages successful strategies to be quickly copied and unsuccessful ones to be discarded. Fifth, the approach to learning we study can be applied even in complicated problems such as the one studied in this paper. And finally, the initial heterogeneity of the population allows us to initialize the system randomly, so that we are able to obtain some sense of the "global" properties of our system under learning as opposed to the local analysis that is often employed in the learning literature. We think that

[21]For an alternative approach to systems with heterogeneous learning rules, see Evans, Honkapohja, and Marimon (1994).

these features provide good reasons to investigate the properties of genetic-algorithm-based models of learning in systems where learning may have an important role to play. In the present model the potentially important role for learning is equilibrium selection.

We chose to use a genetic algorithm, rather than some other learning/search algorithm, because the genetic algorithm has a number of attractive features that are particularly well-suited to the particular problem that we examine. First, unlike other search/learning algorithms, the genetic algorithm is *population-based*, and involves parallel processing of a finite set of initially heterogeneous strings; each of these strings can be viewed as representing different candidate solutions to a particular optimization problem, or, as in our interpretation, each string can be viewed as representing the belief of a different agent in the population. The population–based genetic algorithm can be regarded as a global search algorithm whereas "representative–agent" type search algorithms are necessarily local. The implicit parallelism of the genetic algorithm also works to reduce computation time as compared with enumerative search strategies and other search algorithms. As we have stressed, we think parallelism is an important feature of coordination in actual economies or economic situations. Second, the genetic algorithm is readily applied to difficult optimization problems sometimes heuristically described as "rugged surfaces." Unlike gradient–based learning algorithms, (e.g. the Robbins-Monro or the least squares learning algorithms studied by Woodford and Marcet and Sargent or other hill–climbing algorithms), the genetic algorithm does not require the taking of derivatives. This feature makes the genetic algorithm an attractive model of how populations of economic agents might update their forecast rules over time in highly nonlinear environments such as the one we consider here. Third, genetic algorithms are known to behave as excellent function optimizers. Holland (1975) has shown that genetic algorithms optimize on the trade–off between searching for new rules – *exploration* – and utilizing information discovered in the past – *exploitation*.[22] Optimization of this trade–off is important: an algorithm that engages in too much exploration discards feedback information early in the search that may prove useful, while an algorithm that engages in too much exploitation is prone to local hill–climbing and may be overly sensitive to noise. Finally, modelling learning behavior using a genetic algorithm

[22]Holland's "schema theorem" is proved by analogy with the two armed bandit problem and is found in Holland (1975, pp. 75-88). See also Goldberg (1989, pp. 28-33).

has the advantage that the exact specification of the problem that agents are attempting to solve is known at the outset. Indeed, the representation of the agent's problem is a necessary prerequisite to the application of the genetic algorithm. Thus, interpretation and evaluation of the results from applying the genetic algorithm become relatively straightforward. In contrast, neural networks and genetic programming techniques evolve structures and programs that are often difficult to interpret.

For all of these reasons, we chose the genetic algorithm over several alternative methods as a way of modelling learning behavior. This is not to say that other methods are not interesting, but only to note that the genetic algorithm approach has a number of desirable features.

3.5 Design of computational experiments

We conducted a set of 1,410 computational experiments using the genetic algorithm described above in Grandmont's example economy. We used the same set of parameter values used to generate Figure 1, namely $\{e_1, e_2\} = \{2, .5\}$, and $\rho_1 = .5$. The preference parameter ρ_2 was initially set equal to 2 and was then increased to 16 by increments of .1. We set the bit string length $\ell = 8$ which allows the agents to take actions consistent with a periodic equilibrium of an order as high as 256.[23] We set $N = 100$ (so each generation consisted of N/2=50 agents), $p^c = 1$ and $p^m = 1/\ell = .125$; we chose these values based in part on the optimal values recommended by Grefenstette (1986) and Bäck (1993) and in part because the election operator that we use assures that strings that emerge from crossover and mutation with particularly low fitness values will not enter the population, and so little is lost by allowing agents to experiment extensively with alternative strings.

For each of the experiments, we created an initial population of 50 old and 50 young agents randomly (that is, each bit in each string was set equal to 1 with probability .5 and 0 with probability .5). We initialized prices by choosing 256 random numbers from a uniform distribution on the unit interval. We then repeatedly applied the genetic algorithm and computed market clearing prices until either convergence was obtained or 2,000 generations had failed to coordinate on an equilibrium (that is, after 2,000 iterations).

[23]With this choice for ℓ, we are explicitly ruling out the possibility of agents coordinating on periodic equilibria of order greater than 256. Experiments with higher values of ℓ, not reported here, did not appear to produce qualitatively different behavior in our systems.

Convergence was checked after each iteration of the algorithm using the following convergence criterion. We checked for cycles of every possible periodicity, and we required that the most recent ten prices matched the ten corresponding prices that occurred $k+1$ periods in the past ($k \in (0, 255)$) up to a tolerance of 1×10^{-10}.[24] This criterion was met before the 2,000 iteration limit in most of our experiments. This criterion is relatively easy on high-order equilibria, since we check only ten prices on those cycles, but nevertheless we did not observe high-order equilibria.

Once convergence was obtained or the maximum number of iterations was reached, the same experiment was repeated again nine more times, so that we have ten experiments for each of the 150 values of ρ_2 and a total of 1,410 experiments. For each iteration of each experiment, we recorded several pieces of information: the price, the gross inflation rate, and the mean value of real balances held by agents in the young generation. We also recorded the mean base ten value of the strings, which we called *mean position number*, as well as the standard deviation of the position number. If the standard deviation is zero, then the strings are identical. We stress that our convergence criterion was price-based, so that the fact that strings are identical does not imply that the system has converged. As we will show, we sometimes found that strings were identical before prices met our convergence criterion.

[24]Our specific convergence criterion is described by the following pseudo code:

```
convergence=yes
tolerance = 1 × 10^-10
for s = 1, s ≤ maxcycle = 256
 for j = 0, j ≤ 9
  if [P(t − j) − P(t − j − s)] ≥ tolerance,
   convergence=no
   j = 10
  endif
 next j
 if convergence=yes,
  s = maxcycle + 1
 endif
next s
```

4 Experimental results

4.1 Overview

The main finding from our set of computational experiments is that in the economies we study, the artificial agents eventually learn to coordinate on stationary perfect foresight equilibria of order $k+1$, where $k = 0, 1$ or 3, and where $k = 0$ refers to a degenerate cycle, the monetary steady state. None of the economies we studied displayed coordination on stationary equilibria of any other periodicities. We never observed the nonmonetary steady state. We conclude from these results that simplicity may be a virtue for equilibria, as simpler equilibria are more likely to be achieved in our systems.[25] As mentioned in the introduction, our finding that artificial adaptive agents tend to coordinate on relatively simple equilibria is consistent with experimental findings with human subjects in similar environments studied by Van Huyck, Cook, and Battalio (1994) and Marimon, Spear, and Sunder (1993).

A second, related finding is that our systems under learning rarely chose a single equilibrium from among the many possible equilibria as the only limit point. Instead, for most values of ρ_2, our systems sometimes converged to one stationary equilibrium, the monetary steady state, and sometimes converged to another stationary equilibrium, a period two cycle. In one case, the learning system had nearly converged to a period four cycle, and in two other cases the system had not converged to any path resembling an equilibrium after 2,000 iterations. We conclude that while the introduction of learning into our economic systems can sharply limit the set of equilibria that might be considered reasonable in the sense that they might be achievable even in systems where the decisions are made by humans, the introduction of learning does not imply a *unique* stationary equilibrium, and there is no guarantee that convergence will even occur in finite time.

A third broad finding is that while our systems almost always converged eventually, they sometimes displayed qualitatively complicated dynamics for very long periods of time. By complicated dynamics we mean that either the price dynamics do not appear to be in a small neighborhood of any discernible equilibrium path, or that they are in such a

[25] One could view this as a "strong stability result" in Evan's (1989) sense because it is robust to "overparameterization," meaning in our case that agents consider higher order cycles even though none exist for low values of ρ_2.

neighborhood, but the path involves higher order periodic motion. These complicated "transient" dynamics sometimes occurred in a region of the parameter space where cycles exist in the backward perfect foresight dynamics, but never occurred in the region where the monetary steady state was the lone efficient equilibrium. Thus, when the set of equilibria is relatively complicated, the coordination problem is more pronounced and the economy may remain out of equilibrium for long periods of time. We will provide examples of complicated transient dynamics in the next section.

4.2 Specific results

In most cases, our systems did not converge to the same stationary equilibrium chosen by the limiting backward perfect foresight dynamics studied by Grandmont (1985). For low values of ρ_2, in particular those below 4.2, we observed convergence to the monetary steady state in every experiment, and this is the same prediction made by the limiting backward perfect foresight dynamics (see Figure 1). But as ρ_2 is increased further, the limiting backward perfect foresight dynamics display a bifurcation, with the monetary steady state losing stability and never regaining it for values of $\rho_2 > 4.2$. In contrast, in our systems with learning, the monetary steady state was always a limit point in at least one of the ten experiments conducted at each value of ρ_2. In addition, for higher values of ρ_2 and in particular those past the first bifurcation point in the backward perfect foresight dynamics, our systems often converged to a period two stationary equilibrium, even in cases where that equilibrium too had lost its stability in the backward perfect foresight dynamics. Other relatively simple periodic equilibria, such as the period three cycle that is stable in the backward perfect foresight dynamics for values of $\rho_2 > 13$, appear not to be attractive in our learning dynamics, as they were never observed in our computational experiments.[26]

The raw data from all of our 1,410 experiments is summarized in a table in the Appendix. Looking at this table, one will notice that we encountered numerous cases of *nonconvergence*, in the sense that our convergence criterion had not been met within the allotted 2,000 iterations, and that these cases were especially common for higher values of ρ_2. Nevertheless, we regard most of these cases as *near convergent* situations (rather than

[26]We note that the "relatively simple equilibria" we observed all had a periodicity that is one of the first few entries in Sarkovskii's ordering. Since three is the last entry in that ordering, one might view it as a "complicated" equilibrium.

cases of nonconvergence), since in every instance save two the system was very close to converging to a period two stationary equilibrium. If the experiments had been continued beyond the upper limit of 2,000 iterations in these cases, we have little doubt that our convergence criterion would have eventually been met, and that the system would in fact have converged to a period two equilibrium. Generally, we found that when our systems converged to the monetary steady state, they spent relatively few iterations in the immediate vicinity of this steady state value, by which we mean that the last ten prices were all equal up to five significant digits. In contrast, when our systems converged to a cycle of period two, they often spent several hundred iterations in a similar immediate neighborhood of the equilibrium before finally meeting our convergence criterion. Since this process of locking on to the period two equilibrium often took a long time, it is not surprising that our 2,000 iteration limit was often encountered in these cases.[27] This situation might have been helped somewhat by a less stringent tolerance.

The specific convergence results from our experiments can be seen by comparing Figure 2, in which we plot the limiting dynamics of our adaptive learning system for each value of ρ_2, with Figure 1, which plots the limiting backward perfect foresight dynamics for each value of ρ_2. There are slight differences in the methods used to construct these two figures. Figure 1 replicates Grandmont's (1985, p. 1030) Figure 4, and plots the last 50 of 1,000 iterations of the backward perfect foresight system for each value of ρ_2, taking as the initial condition the peak of the unimodal map. Figure 2 plots the fixed points of any of the stationary equilibria to which the system under our adaptive learning algorithm converged in any of the ten computational experiments completed at a given value of ρ_2. If the learning system failed to meet the convergence criterion within the allotted 2,000 iterations, then the last 50 of the 2,000 iterations were plotted.

As an example, consider Figure 2 for the value $\rho_2 = 5$. Some of the ten experiments conducted for this value of ρ_2 resulted in convergence to the monetary steady state, while others resulted in convergence to the period two cycle that is predicted by the limiting backward perfect foresight dynamics (see Figure 1).[28] Therefore, there are three fixed

[27] The 2,000 iteration limit was chosen in order to conserve on the amount of computer time it took to complete our computational experiments.

[28] The exact number of times the algorithm converged to a particular type of equilibrium can be found in the table presented in the Appendix.

points plotted in Figure 2 for $\rho_2 = 5$. The middle point corresponds to the steady state value for real money balances while the outer two points correspond to the stationary two–cycle values for real money balances. As one can see from Figure 2, the outcome of the ten experiments performed for $\rho_2 = 5$ is the same outcome observed for virtually all other values of $\rho_2 > 4.2$; in some experiments the learning system converged to the monetary steady state, while in others the system converged to a stationary, monetary period two equilibrium. A comparison between Figures 2 and 1 conveys the relative simplicity of the set of long–run equilibria under learning as opposed to that set under backward perfect foresight. The exceptional cases in Figure 2 occur for a single computational experiment each when $\rho_2 = 4.7$, $\rho_2 = 7.5$ and $\rho_2 = 12.1$. In the first and third of these cases, the system did not approach anything resembling an equilibrium in 2,000 iterations, and the corresponding plots of the last 50 iterations show no clear pattern in Figure 2. In one experiment where $\rho_2 = 7.5$, the system finished 2,000 iterations very close to a period 4 stationary equilibrium, and the last fifty iterations in this case are also plotted in Figure 2.[29]

The fact that there was wide variation in the number of iterations to convergence can be observed in Figure 3, which depicts the mean number of iterations to convergence for each set of the ten computational experiments conducted for each different value of ρ_2. In cases where the system failed to converge within 2,000 periods, the number of iterations to convergence included in the sum used to calculate the mean is 2,000. Thus, the reported mean number of iterations to convergence underestimates the actual mean number of iterations it would have taken the system to achieve convergence, had the system been allowed to continue in those experiments where nonconvergence results were obtained. In the figure, there is a noticeable spike in the mean number of iterations to convergence near the first backward perfect foresight bifurcation point at around $\rho_2 = 4.2$ (cf. Figure 1) as well as near (less obviously) the second and third bifurcation points. Thereafter, as ρ_2 is further increased, the bifurcations occur more and more frequently and the mean number of iterations to convergence shows a marked tendency to increase up to the point where $\rho_2 = 13$. At this point, the limiting backward perfect foresight dynamics of Figure 1 indicate the presence of

[29]As in other cases of "near convergence" where the system was close to converging to a period two cycle after 2,000 iterations, we are quite confident that this single run that is very close to a period four cycle at 2,000 iterations would eventually have met our convergence criterion.

a period three cycle, and thus, via Sarkovskii's (1964) theorem, periodic equilibria of every order q exist, where q is an element of the set of positive integers. In this complicated region, there is a marked decrease in the mean number of iterations to convergence. This decrease is due to the increased tendency of the learning system in this region to coordinate on the monetary steady state outcome rather than a period two or higher order cycle. As the data in the table found in the appendix reveal, coordination on the steady state outcome tends to increase for values of $\rho_2 > 13$. When the system converged to the monetary steady state, the number of iterations to convergence was generally less than when the system converged to a period two equilibrium.

Figures 4*abcd* illustrate further the character of our results by examining time series from experiments which converged or nearly converged. Figure 4*a* is typical of all of our 1,410 experiments, as we observed outcomes similar to this one at virtually all values of ρ_2. In the figure, real balances *per capita* is plotted on the vertical axis. We recorded real balances per capita for each iteration of each experiment, as it provides a summary of the agents' savings behavior at each point in time. However, it is important to note that only in a stationary equilibrium, possibly periodic, when all agents within a generation are saving the same amount, does the amount of real balances per capita accurately represent all individual behavior in the economy. Figure 4*a* illustrates a case of fairly rapid convergence, with the monetary steady state obtaining to within our convergence tolerance in 51 iterations.

Figure 4*b* is representative of another common outcome in our computational experiments. In this case, real balances per capita moves to within a small neighborhood of a period two equilibrium within 100 iterations, but then the system takes another 250 iterations before converging at iteration 353. This latter result was typical, as it took a long time for our systems to lock on to period two equilibria. One consequence of the fact that the system remains in the neighborhood of the period two equilibrium for such a long time is that beliefs continue to evolve long past the point where the system initially enters the neighborhood of the equilibrium. This is illustrated in Figure 4*c*, which plots the time series of the mean position number (the mean base ten value of all of the strings in the system) for the same experiment depicted in Figure 4*b*. The mean position number can range between zero and 255, and in the figure the position number initially falls rapidly and converges to one, which is a string consistent with a period two equilibrium.[30] Of course, many other strings are also consistent with a period two equilibrium (in particular, any odd string), and when the system still has not converged according to our criterion at iteration 150, some agents begin to experiment with some of these alternative strings. Subsequently, the mean position number rises until it reaches a relatively high level by the time of convergence.[31]

A portion of the time series graph for an exceptional case is presented in Figure 4*d*. This time series was generated in a single replication of a computational experiment with $\rho_2 = 7.5$. In this case, the agents in the economy nearly coordinated on a stationary equilibrium of period four although again our convergence criterion was not quite met in this case at iteration 2,000. The system depicted in the figure moved to a neighborhood of a period two equilibrium by iteration 440, but all agents had coordinated on the string 00000011 which has base ten value 3 and is consistent with both a period four and a period two equilibrium. We observed dynamics qualitatively similar to those in the first portion of Figure 4*d* in a number of other experiments, but in those cases the period four equilibrium broke down during the volatile period which occurs around iteration 700 in Figure 4*d*. The fixed points of the period four equilibrium correspond precisely to those of the cycle predicted by the limiting backward perfect foresight dynamics (compare the fixed points of Figure 4*d* with the limiting backward perfect foresight dynamics in Figure 1 at $\rho_2 = 7.5$). This case illustrates that a third type of equilibrium, a period four cycle, is possible despite being rarely observed.

Figures 5*abcd* illustrate situations of more complicated dynamics. Figure 5*a* and Figure 5*b* display the full time series for the two exceptional cases where no convergence or near convergence was observed after 2,000 iterations.[32] As the time series reveals, the agents never come close to coordinating on any path for real money balances that discernibly resembles a steady state or periodic equilibrium.[33] Nevertheless, the observed nonconvergence of the

[30]The standard deviation of the position number fell to zero between iterations 80 and 150, indicating that all the strings were identical over this period. Nevertheless, the differences in price realizations must not have been small enough to satisfy the convergence criterion.

[31]We did not observe continuing evolution of beliefs in cases where the system converged to the monetary steady state. In most instances, the agents simply coordinated on the zero string, 00000000, in these cases. In some cases the system tracked a damped oscillatory path to the steady state, with all agents coordinated on the string 00000001 or occasionally on a string with a higher position number.

[32]In these figures, lines connecting adjacent observations have been omitted in order to reduce clutter.

[33]In Figure 5a, the dynamics are bounded by the fixed points of the period two equilibrium which is the attractor under backward perfect foresight for $\rho_2 = 4.7$.

system after 2,000 iterations does not imply that the system would not eventually converge. Our experience leads us to believe that the system would eventually have converged to some type of equilibrium if it had been allowed to run for more than 2,000 iterations, but of course the question remains open.

A much more common outcome was one of complicated transient dynamics within experiments that did eventually converge or that nearly converged. Figure 5c provides a portion of the time series from one experiment. In this diagram, the mean position number is plotted above the box marking the level of mean real balances for cases where the standard deviation of the position number is zero (indicating that all agents have coordinated on a single string).[34] From iteration 175 through 350, a favored string is evidently 00011010. This string is consistent only with a periodic equilibrium of order 23 and this equilibrium has not yet occurred in the Feigenbaum cascade depicted in Figure 1. The result is a near–periodic time series which continues for more than 500 iterations. Figure 5d illustrates how a sharp qualitative change in dynamics can occur. In this experiment, the level of real balances per capita has been fluctuating without approaching an equilibrium for more than 800 iterations. A favored string is 01011110. Around iteration 960, the system dynamics change abruptly and the system approaches the steady state before converging to a period 2 equilibrium. A precise assessment of 'complicated transitory dynamics' requires further quantification, but we think we can convey the nature of our results by stating that dynamics qualitatively like those in Figures 5c and 5d were commonplace in our experiments, but never occurred for values of $\rho_2 < 4.2$. We conclude that existence of periodic equilibria is necessary but not sufficient for complicated transient dynamics in our systems with learning.

5 Conclusion

We have studied learning in a model where there can be an extensive multiplicity of stationary equilibria, many of which are periodic. It is often thought that introducing learning into such a model might help with the pervasive multiplicity problem because some of the

[34]We note again that a standard deviation of zero does not imply that our convergence criterion has been satisfied; recall that we also require the differences in price realizations must be less than the prescribed tolerance.

equilibria might not be attractors under learning and hence might not be viewed as likely long–run outcomes in an actual economy or economic situation driven by human behavior. In this paper, we have provided some evidence in favor of this notion, as introducing the criterion that equilibria should be 'learnable' does lead to a sharp reduction in the set of stationary equilibria one would view as plausible in our systems. However, we are unable to rule out convergence to periodic equilibria entirely, and our systems sometimes display qualitatively complicated transitory dynamics for long periods of time.

Experiments with human subjects might provide some further clarification of which equilibria are more likely to emerge in environments like this one. However, as noted in the introduction, genetic learning like that employed here has already been shown to be quite successful in mimicking the behavior of human subjects in controlled laboratory settings, and our results are consistent with the experimental work to date on this topic. Since experiments with human subjects are not presently feasible on the scale contemplated in this paper, we think our computational approach provides a reliable and practical alternative.

References

[1] Andreoni, J., and J. Miller (1993) "Auctions with Artificial Adaptive Agents," mimeo, Carnegie-Mellon and the University of Wisconsin-Madison.

[2] Arifovic, J. (1994*a*), "Genetic Algorithm Learning and the Cobweb Model." *Journal of Economic Dynamics and Control* 18: 3-28.

[3] Arifovic, J. (1994*b*), "Genetic Algorithms and Inflationary Economies," forthcoming, *Journal of Monetary Economics.*

[4] Arifovic, J. (1994*c*), "The Behavior of the Exchange Rate in Genetic Algorithm and Experimental Economies," working paper, Simon Fraser University.

[5] Arifovic, J., and C. Eaton. (1994), "Coordination via Genetic Learning," working paper, Simon Fraser University.

[6] Arthur, W.B. (1994), "Inductive Reasoning and Bounded Rationality," *American Economic Review* 84, 406–411.

[7] Axelrod, R.M. (1987), "The Evolution of Strategies in the Iterated Prisoner's Dilemma," in L. Davis, ed., *Genetic Algorithms and Simulated Annealing*, Morgan Kaufmann Publishers, Los Altos, CA.

[8] Azariadis, C. (1981), "Self-Fulfilling Prophecies," *Journal of Economic Theory* 25, 380-396.

[9] Azariadis, C. (1993), "The Problem of Multiple Equilibria," working paper, UCLA.

[10] Bäck, T. (1993), "Optimal Mutation Rates in Genetic Search," in S. Forrest, ed., *Proceedings of the Fifth Annual Conference on Genetic Algorithms*, San Mateo: Morgan Kaufmann.

[11] Benhabib, J. and R.H. Day (1982), "A Characterization of Erratic Dynamics in the Overlapping Generations Model," *Journal of Economic Dynamics and Control*, 4: 37–55.

[12] Binmore, K.G. and L. Samuelson (1992), "Evolutionary Stability in Repeated Games Played by Finite Automata," *Journal of Economic Theory* 57, 278-305.

[13] Blickle, T. and L. Thiele (1995), "A Mathematical Analysis of Tournament Selection," in L.J. Eshelman, ed., *Proceedings of the Sixth Annual Conference on Genetic Algorithms*, San Mateo: Morgan Kaufmann.

[14] Bray, M.M. and N.E. Savin (1986), "Rational Expectations Equilibria, Learning, and Model Specification," *Econometrica* 54, 1129-1160.

[15] Boldrin, M., and M. Woodford (1990), "Equilibrium Models Displaying Endogenous Fluctuations and Chaos: A Survey," *Journal of Monetary Economics*, 25: 189–222.

[16] Bullard, J. (1994), "Learning Equilibria." *Journal of Economic Theory*, 64: 468–485.

[17] Bullard, J., and J. Duffy. (1994), "A Model of Learning and Emulation with Artificial Adaptive Agents." working paper, FRB-St.Louis and University of Pittsburgh.

[18] Duffy, J. (1994), "On Learning and the Nonuniqueness of Equilibrium in an Overlapping Generations Model with Fiat Money," *Journal of Economic Theory* 64: 541–553.

[19] Evans, G.W. (1989), "The Fragility of Sunspots and Bubbles," *Journal of Monetary Economics*, 23: 297–317.

[20] Evans, G.W., and S. Honkapohja (1992), "Adaptive Learning and Expectational Stability: An Introduction," forthcoming in A. Kirman and M. Salmon, eds., *Learning and Rationality in Economics*, Oxford: Basil Blackwell.

[21] Evans, G.W., and S. Honkapohja (1995), "Local Convergence of Recursive Learning to Steady States and Cycles in Stochastic Nonlinear Models," *Econometrica* 63: 195–206.

[22] Evans, G.W., S. Honkapohja, and R. Marimon (1994), "Global Convergence in Monetary Models with Heterogeneous Learning Rules," working paper, University of Edinburgh, University of Helsinki, and Universitat Pompeu Fabra.

[23] Evans, G.W., S. Honkapohja, and T.J. Sargent (1993), "On the Preservation of Deterministic Cycles When Some Agents Perceive Them to be Random Fluctuations," *Journal of Economic Dynamics and Control*, 17: 705–721.

[24] Fogel, D.B. (1994), "An Introduction to Simulated Evolutionary Optimization." *IEEE Transactions on Neural Networks* , 5: 3–14.

[25] Gale, D. (1973), "Pure Exchange Equilibrium of Dynamic Economic Models," *Journal of Economic Theory* , 6: 12–36.

[26] Goldberg, D.E. (1989), *Genetic Algorithms in Search, Optimization and Machine Learning*, Menlo Park: Addison–Wesley.

[27] Grandmont, J-M (1985), "On Endogenous Competitive Business Cycles," *Econometrica*, 53: 995–1045.

[28] Grandmont, J-M (1994), "Expectations Formation and Stability of Large Socioeconomic Systems," CEPREMAP working paper no. 9424.

[29] Grandmont, J-M, and G. Laroque (1986), "Stability of Cycles and Expectations," *Journal of Economic Theory*, 40: 138–151.

[30] Grandmont, J-M, and G. Laroque (1991), "Economic Dynamics with Learning: Some Instability Examples," in W.A. Barnett, et. al., eds., *Equilibrium Theory and Applications*, Cambridge: Cambridge University Press.

[31] Grefenstette, J.J. (1986), "Optimization of Control Parameters for Genetic Algorithms," *IEEE Transactions on Systems, Man, and Cybernetics*, 16: 122–28.

[32] Guesnerie, R., and M. Woodford (1991), "Stability of Cycles with Adaptive Learning Rules," in W.A. Barnett, et. al., eds., *Equilibrium Theory and Applications*, Cambridge: Cambridge University Press.

[33] Holland, J.H. (1975), *Adaptation in Natural and Artificial Systems*, Ann Arbor: University of Michigan Press. Reprinted (1992), Cambridge: MIT Press.

[34] Holland, J.H., and J.H. Miller (1991), "Artificial Adaptive Agents in Economic Theory," *American Economic Review Papers and Proceedings* 81, 365-370.

[35] Kareken, J., and N. Wallace (1981), "On the Indeterminacy of Equilibrium Exchange Rates," *Quarterly Journal of Economics*, 96: 207–222.

[36] Ljung, L. (1977), "Analysis of Recursive Stochastic Algorithms," *IEEE Transactions on Automatic Control*, AC–22, 551–575.

[37] Marcet, A., and T.J. Sargent (1989), "Convergence of Least Squares Learning Mechanisms in Self–Referential Stochastic Models," *Journal of Economic Theory*, 48: 337–368.

[38] Marimon, R., and E. McGrattan (1992), "On Adaptive Learning in Strategic Games," forthcoming in A. Kirman and M. Salmon, eds., *Learning and Rationality in Economics*, Oxford: Basil Blackwell.

[39] Marimon, R., E. McGrattan, and T.J. Sargent (1990), "Money as a Medium of Exchange in an Economy with Artificially Intelligent Agents," *Journal of Economic Dynamics and Control*, 14, 329-373.

[40] Marimon, R., S.E. Spear and S. Sunder (1993), "Expectationally Driven Market Volatility: An Experimental Study, *Journal of Economic Theory*, 61: 74–103.

[41] Marimon, R., and S. Sunder (1993), "Indeterminacy of Equilibria in a Hyperinflationary World: Experimental Evidence," *Econometrica*, 61: 1073–1107.

[42] Marimon, R., and S. Sunder (1994), "Expectations and Learning Under Alternative Monetary Regimes: An Experimental Approach," *Economic Theory*, 4: 131–162.

[43] Marimon, R., and S. Sunder (1995), "Does a Constant Money Growth Rule Help Stabilize Inflation?: Experimental Evidence," forthcoming, *Carnegie–Rochester Conference Series on Public Policy.*

[44] Miller, J.H. (1989), "The Co-evolution of Automata in the Repeated Prisoner's Dilemma," Santa Fe Institute working paper no. 89-003.

[45] Michalewicz, Z. (1994), *Genetic Algorithms + Data Structures = Evolution Programs*, 2nd Ed., New York: Springer- Verlag.

[46] Rudolph, G. (1994), "Convergence Analysis of Canonical Genetic Algorithms," *IEEE Transactions on Neural Networks* 5, 96–101.

[47] Rust, J., J.H. Miller, and R. Palmer (1993) "Behavior of Trading Automata in a Computerized Double Auction Market," in D. Friedman and J. Rust, eds., *The Double Auction Market*, Santa Fe Institute Studies in the Sciences of Complexity vol. XIV, Addison-Wesley, Reading, MA.

[48] Rust, J., J.H. Miller, and R. Palmer (1994) "Characterizing Effective Trading Strategies: Insights from a Computerized Double Auction Model," *Journal of Economic Dynamics and Control*, 18, 61-96.

[49] Sargent, T.J. (1993), *Bounded Rationality in Macroeconomics*, Oxford: Oxford University Press.

[50] Sarkovskii, A.N. (1964), "Coexistence of Cycles of a Continuous Map of a Line into Itself," *Ukrainian Mathematics Journal*, 16: 61–71.

[51] Shell, K. (1977), "Monnaie et Allocation Intertemporelle,"mimeo, CNRS, Paris (text in english).

[52] Van Huyck, J.B., J.P. Cook, and R.C. Battalio (1994), "Selection Dynamics, Asymptotic Stability, and Adaptive Behavior," *Journal of Political Economy,* 102: 975–1005.

[53] Wright, R. (1995), "Search, Evolution and Money," *Journal of Economic Dynamics and Control* 19, 181-206.

[54] Woodford, M. (1990), "Learning to Believe in Sunspots," *Econometrica* , 58: 277–307.

A Data from computational experiments

In this appendix we present a summary of the raw data from our computational experiments. The first column in the table lists the value of ρ_2, while the second column lists the number of experiments run at that value of ρ_2. The third column gives the mean number of iterations to convergence for those experiments which produced convergence at the given value of ρ_2, and the fourth column gives the standard deviation. The last three columns give the number of experiments that converged to a steady state ($k = 0$), a stationary equilibrium of period 2 ($k = 1$), and failed to converge in 2,000 iterations (nc). As discussed in the text, we regard all but three of these non–convergent (nc) cases as instances of *near convergence* to a cycle of period 2. Of the remaining three nc cases, one was nearly convergent to a cycle of period 4, and the other two cases showed no clear pattern at the end of 2,000 iterations.

ρ_2	Replications	Mean	Std. Dev.	$k = 0$	$k = 1$	nc
2.0	10	35.6	7.7	10	0	0
2.1	10	36.9	7.2	10	0	0
2.2	10	28.4	5.1	10	0	0
2.3	10	32.0	7.4	10	0	0
2.4	10	31.9	6.4	10	0	0
2.5	10	32.0	6.0	10	0	0
2.6	10	31.0	5.4	10	0	0
2.7	10	33.7	8.6	10	0	0
2.8	10	29.5	4.5	10	0	0
2.9	10	32.8	8.7	10	0	0
3.0	10	50.9	45.3	10	0	0
3.1	10	51.2	35.1	10	0	0
3.2	10	53.6	41.0	10	0	0
3.3	10	45.0	42.2	10	0	0
3.4	10	57.0	57.0	10	0	0
3.5	10	67.1	64.5	10	0	0
3.6	10	158.4	118.6	10	0	0
3.7	10	140.1	274.6	10	0	0
3.8	10	73.3	113.9	10	0	0
3.9	10	195.4	162.4	10	0	0
4.0	10	149.6	236.9	10	0	0
4.1	10	499.8	556.1	10	0	0
4.2	10	34.1	4.9	7	0	3
4.3	10	187.6	313.1	8	2	0
4.4	10	292.2	263.6	5	5	0
4.5	10	181.7	186.4	6	4	0
4.6	10	146.0	171.0	7	3	0
4.7	10	214.4	391.9	7	2	1
4.8	10	219.6	291.5	9	1	0
4.9	10	38.4	11.7	10	0	0
5.0	10	137.4	165.2	7	3	0
5.1	10	271.0	417.7	8	2	0
5.2	10	91.8	120.1	8	2	0
5.3	10	263.9	389.7	7	3	0
5.4	10	237.9	240.4	5	5	0
5.5	10	155.7	165.9	6	4	0
5.6	10	189.0	180.7	6	4	0
5.7	10	296.3	370.7	7	3	0
5.8	10	247.6	317.1	7	3	0
5.9	10	219.5	191.7	4	6	0

ρ_2	Replications	Mean	Std. Dev.	$k=0$	$k=1$	nc
6.0	10	265.2	368.5	5	5	0
6.1	10	126.0	171.5	7	3	0
6.2	10	373.4	387.4	5	5	0
6.3	10	72.5	104.0	9	1	0
6.4	10	246.4	309.4	5	5	0
6.5	10	126.3	196.8	7	3	0
6.6	10	194.4	252.5	6	4	0
6.7	10	145.7	175.9	7	3	0
6.8	10	135.5	190.1	8	2	0
6.9	10	159.0	248.1	8	2	0
7.0	10	481.4	269.2	2	8	0
7.1	10	442.8	367.2	4	6	0
7.2	10	253.3	280.4	6	4	0
7.3	10	37.5	11.1	10	0	0
7.4	10	359.1	395.4	6	4	0
7.5	10	341.6	403.2	6	3	1
7.6	10	550.6	613.7	7	3	0
7.7	10	174.9	282.6	8	2	0
7.8	10	526.1	452.9	4	6	0
7.9	10	561.5	445.3	4	6	0
8.0	10	232.7	325.6	8	2	0
8.1	10	209.3	352.8	9	1	0
8.2	10	482.0	553.6	7	3	0
8.3	10	531.2	413.1	4	6	0
8.4	10	294.8	394.1	8	2	0
8.5	10	451.2	499.5	6	4	0
8.6	10	638.8	499.8	4	6	0
8.7	10	678.3	527.1	4	6	0
8.8	10	445.4	553.5	6	4	0
8.9	10	200.4	272.2	8	2	0
9.0	10	628.5	520.1	5	5	0
9.1	10	458.2	520.6	7	3	0
9.2	10	624.1	695.8	6	4	0
9.3	10	889.8	582.8	3	6	1
9.4	10	165.1	348.0	9	1	0
9.5	10	568.1	691.7	6	4	0
9.6	10	916.5	372.6	2	8	0
9.7	10	661.5	781.0	6	4	0
9.8	10	808.3	726.0	5	4	1
9.9	10	647.3	629.4	5	5	0

ρ_2	Replications	Mean	Std. Dev.	$k=0$	$k=1$	nc
10.0	10	409.5	509.0	7	3	0
10.1	10	69.7	100.1	10	0	0
10.2	10	548.8	600.7	5	4	1
10.3	10	751.4	679.8	6	4	0
10.4	10	728.9	671.4	4	4	2
10.5	10	683.1	730.4	5	4	1
10.6	10	629.4	680.9	5	4	1
10.7	10	911.1	757.8	4	5	1
10.8	10	687.8	664.9	5	4	1
10.9	10	520.1	551.6	7	3	0
11.0	10	627.0	817.5	6	3	1
11.1	10	1097.7	727.7	3	7	0
11.2	10	305.6	429.9	7	1	2
11.3	10	1132.9	758.9	3	6	1
11.4	10	1078.7	686.2	3	6	1
11.5	10	940.6	771.6	4	5	1
11.6	10	507.7	738.8	5	2	3
11.7	10	1030.9	814.1	4	6	0
11.8	10	926.6	788.4	4	5	1
11.9	10	635.2	804.5	7	3	0
12.0	10	824.9	843.4	5	4	1
12.1	10	312.5	562.2	5	1	4
12.2	10	190.1	402.5	9	1	0
12.3	10	108.0	74.3	3	0	7
12.4	10	588.9	813.9	5	2	3
12.5	10	785.3	842.8	4	3	3
12.6	10	81.9	66.7	8	0	2
12.7	10	918.0	817.6	4	4	2
12.8	10	53.4	16.3	5	0	5
12.9	10	528.6	724.8	6	2	2
13.0	10	311.7	518.4	8	1	1
13.1	10	322.3	587.0	8	1	1
13.2	10	251.7	540.3	8	1	1
13.3	10	50.6	10.6	8	0	2
13.4	10	228.0	519.5	8	1	1
13.5	10	249.4	569.6	8	1	1
13.6	10	296.8	597.2	7	1	2
13.7	10	234.9	681.9	9	1	0
13.8	10	224.0	491.8	7	1	2
13.9	10	198.9	288.7	9	0	1

ρ_2	Replications	Mean	Std. Dev.	$k=0$	$k=1$	nc
14.0	10	43.0	11.2	9	0	1
14.1	10	78.1	68.9	9	0	1
14.2	10	85.0	105.4	8	0	2
14.3	10	93.1	97.7	9	0	1
14.4	10	119.0	194.2	8	0	2
14.5	10	45.1	23.5	8	0	2
14.6	10	131.9	238.7	9	0	1
14.7	10	44.9	13.1	8	0	2
14.8	10	47.7	26.4	9	0	1
14.9	10	147.3	287.4	9	0	1
15.0	10	59.3	38.8	9	0	1
15.1	10	73.4	57.4	9	0	1
15.2	10	189.4	307.4	9	0	1
15.3	10	46.7	12.7	9	0	1
15.4	10	100.4	132.4	9	0	1
15.5	10	77.9	71.5	8	0	2
15.6	10	316.7	654.3	6	1	3
15.7	10	63.2	17.9	5	0	5
15.8	10	112.3	173.8	9	0	1
15.9	10	42.0	11.6	6	0	4
16.0	10	178.1	332.8	7	0	3

Figure 1

Bifurcation diagram, backward perfect foresight dynamics

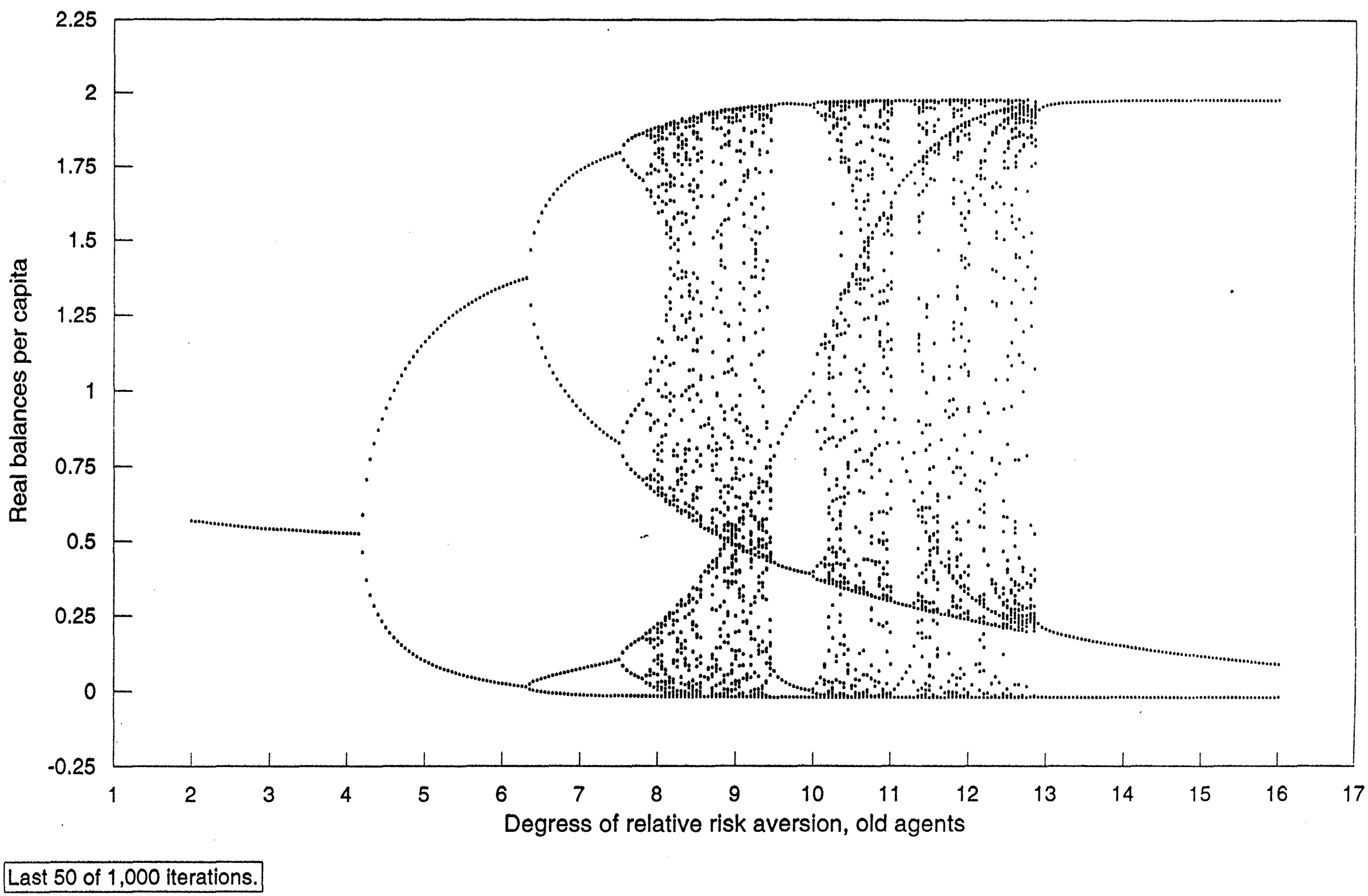
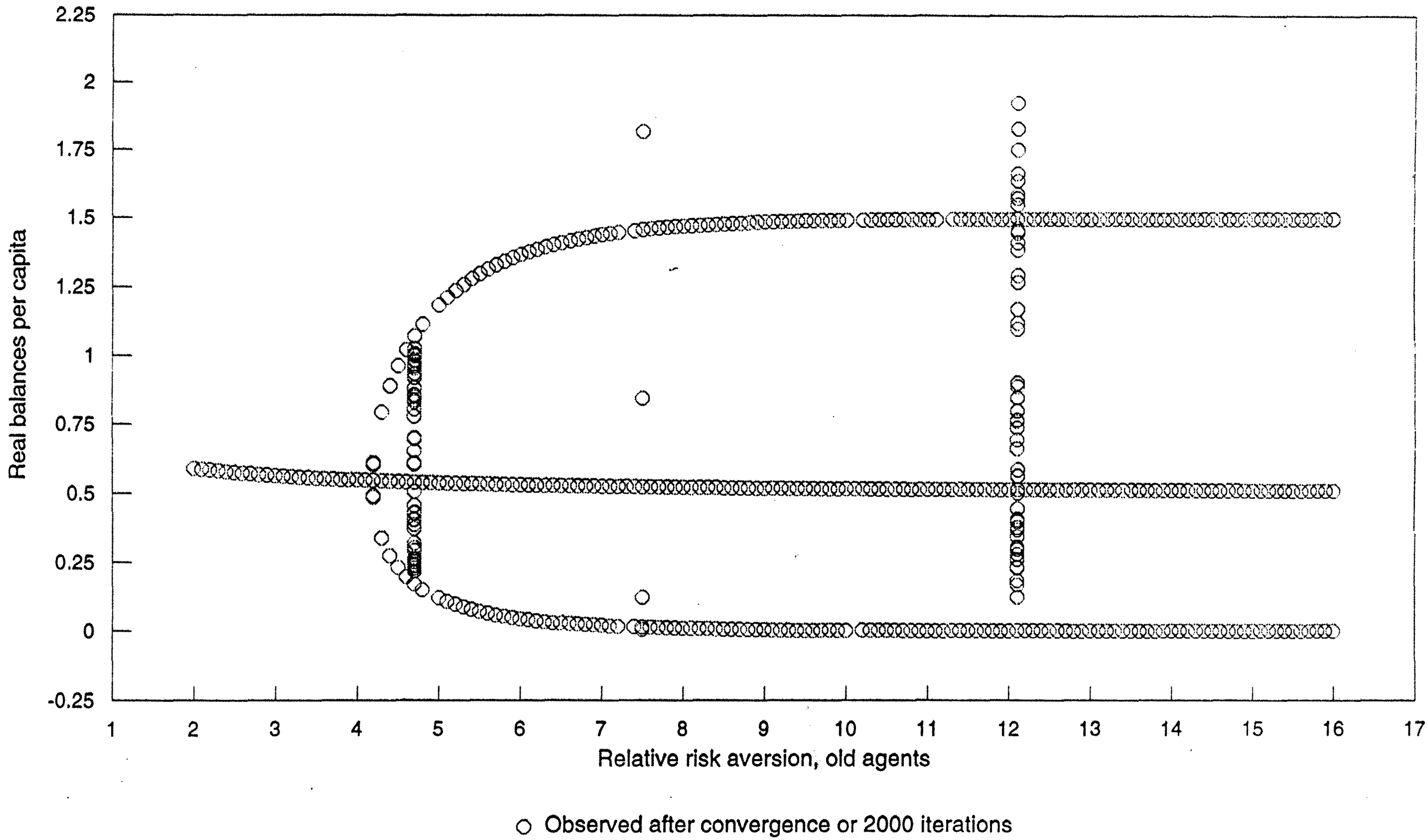


Figure 2

Limiting learning dynamics



Ten replications at each old agent relative risk aversion.
Convergence values or last 50 iterations of each replication plotted.

Figure 3

Mean Number of Iterations to Convergence

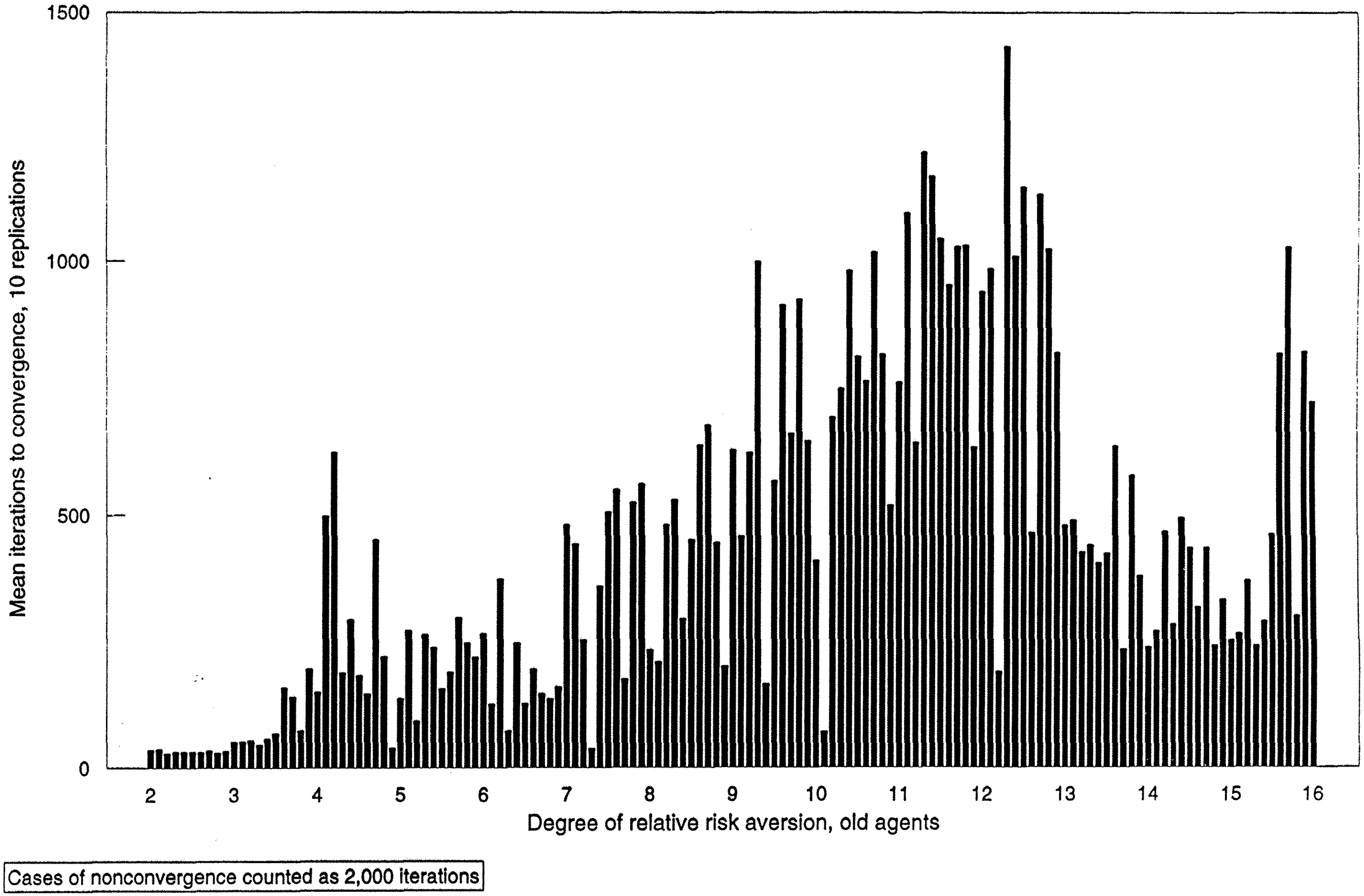
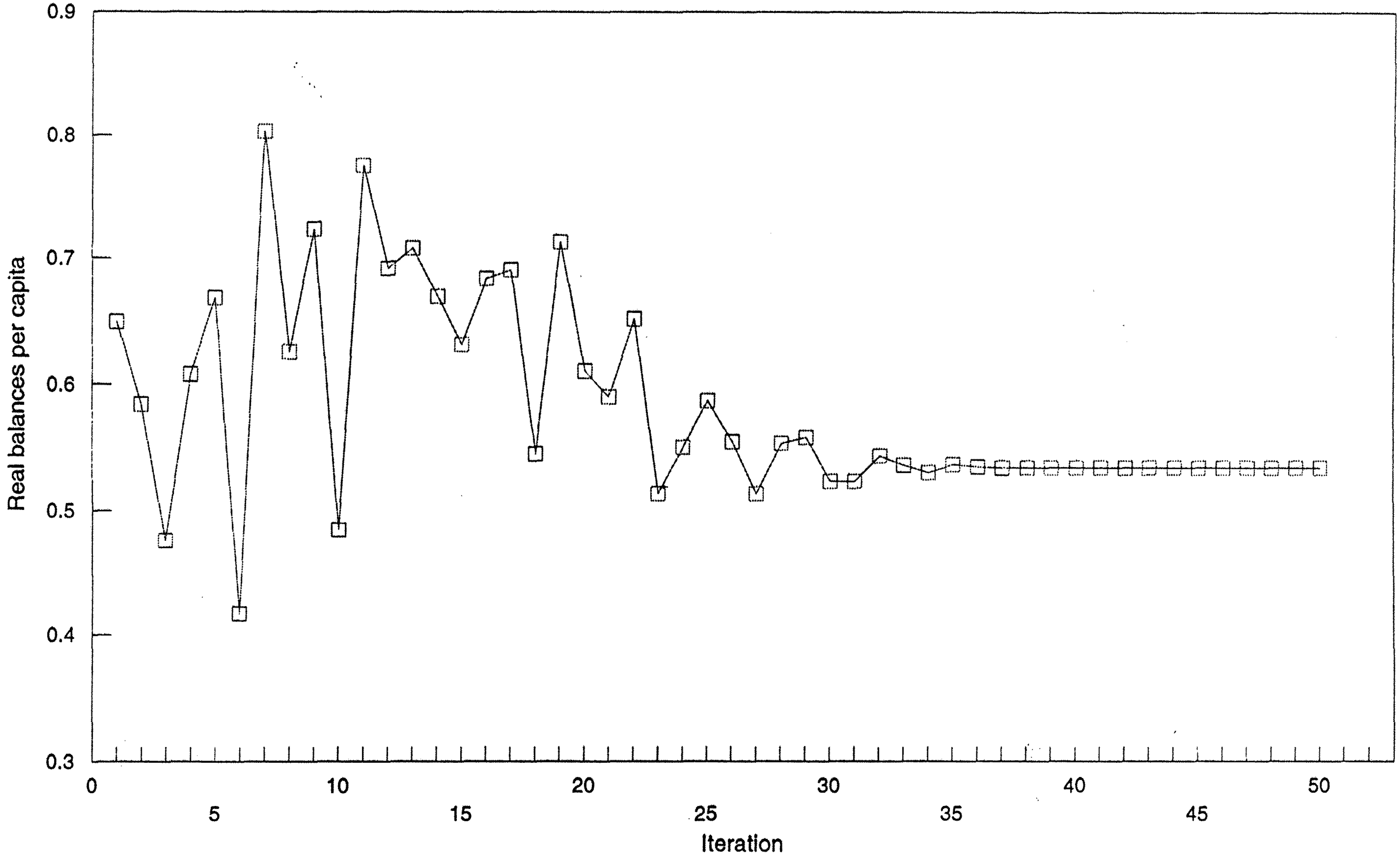


Figure 4a

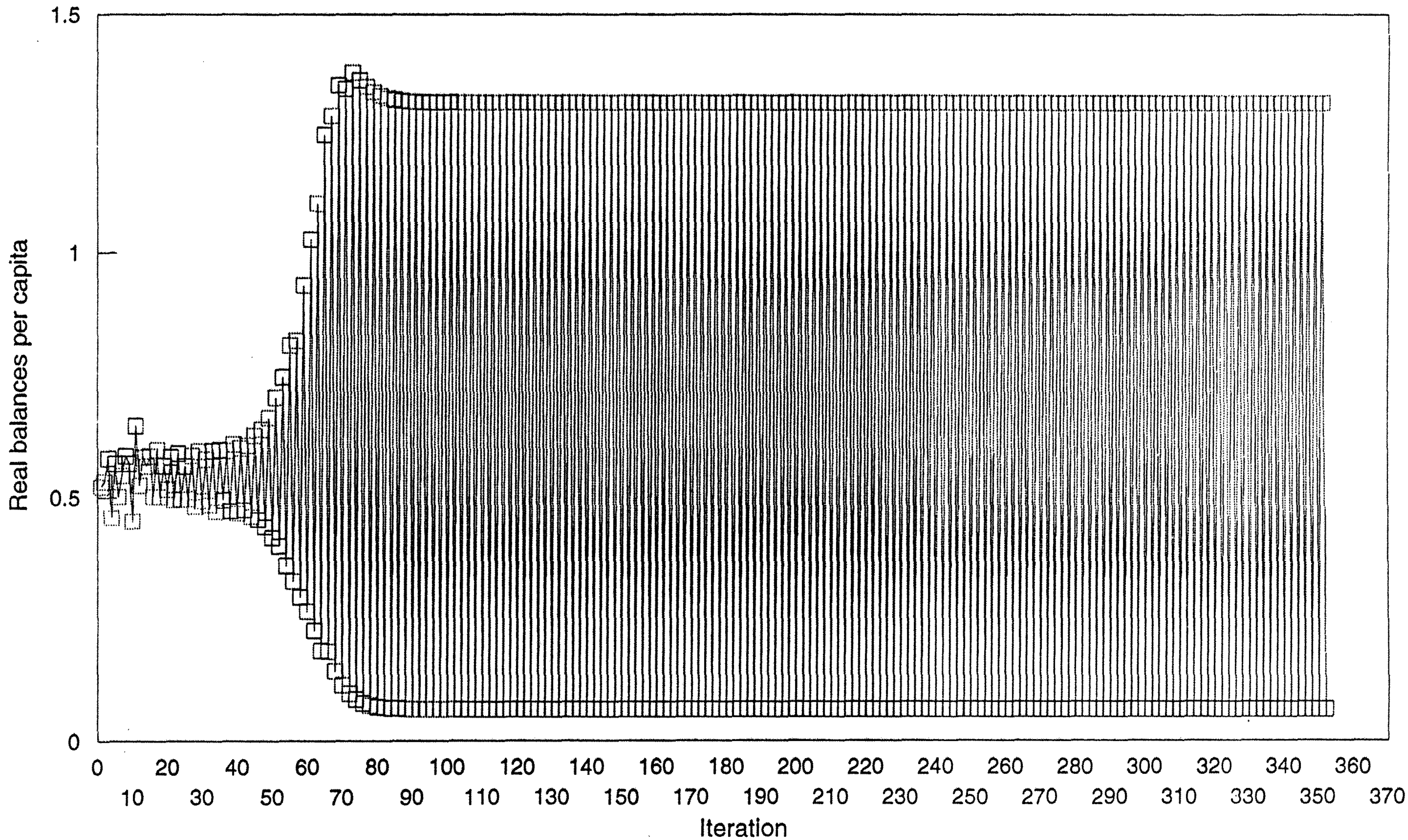
Example of convergence to the monetary steady state



Relative risk aversion of the old agents is 5.6, computational experiment 3.

Figure 4b

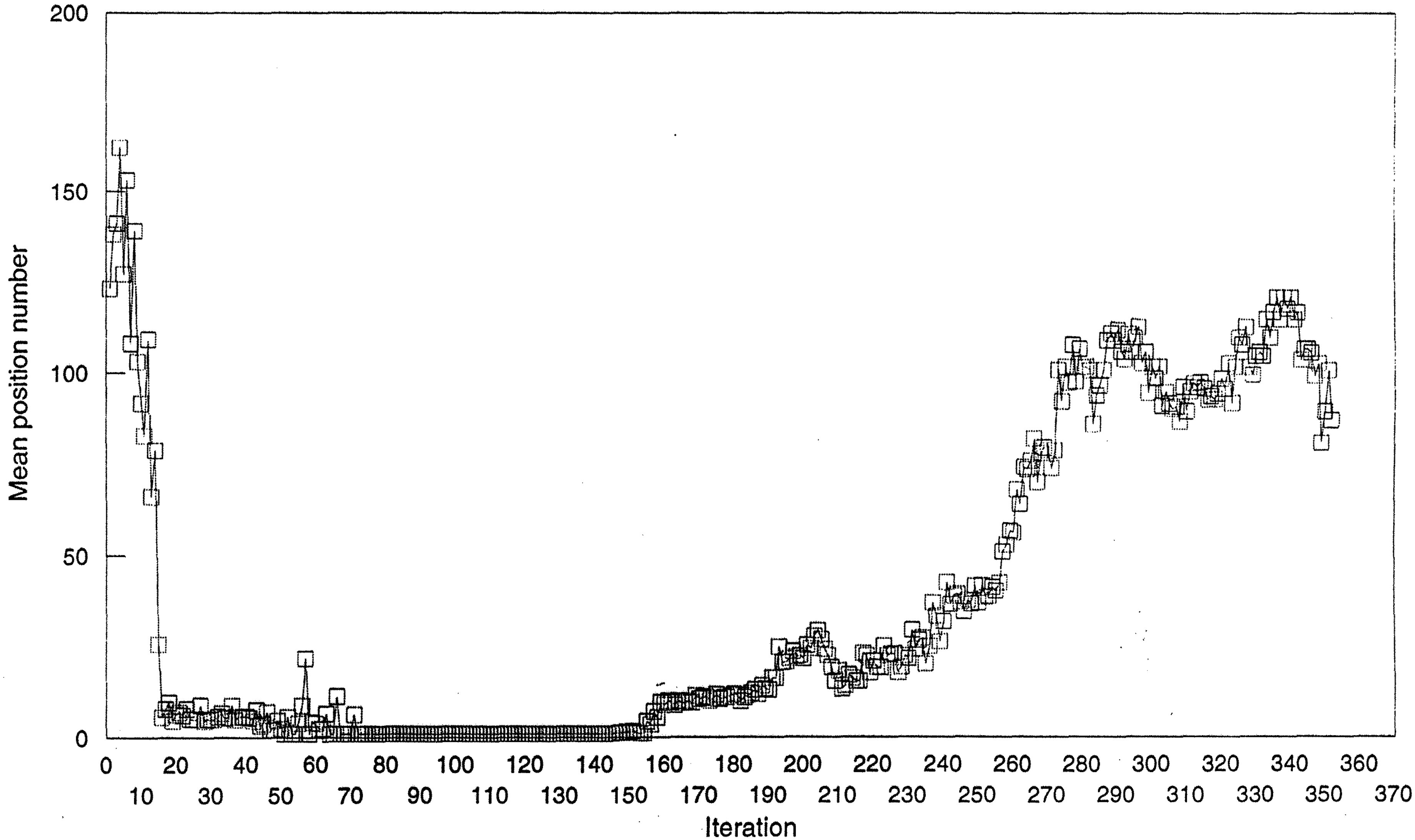
Example of convergence a periodic equilibrium of order 2



Relative risk aversion of the old agents is 5.6, computational experiment 1.

Figure 4c

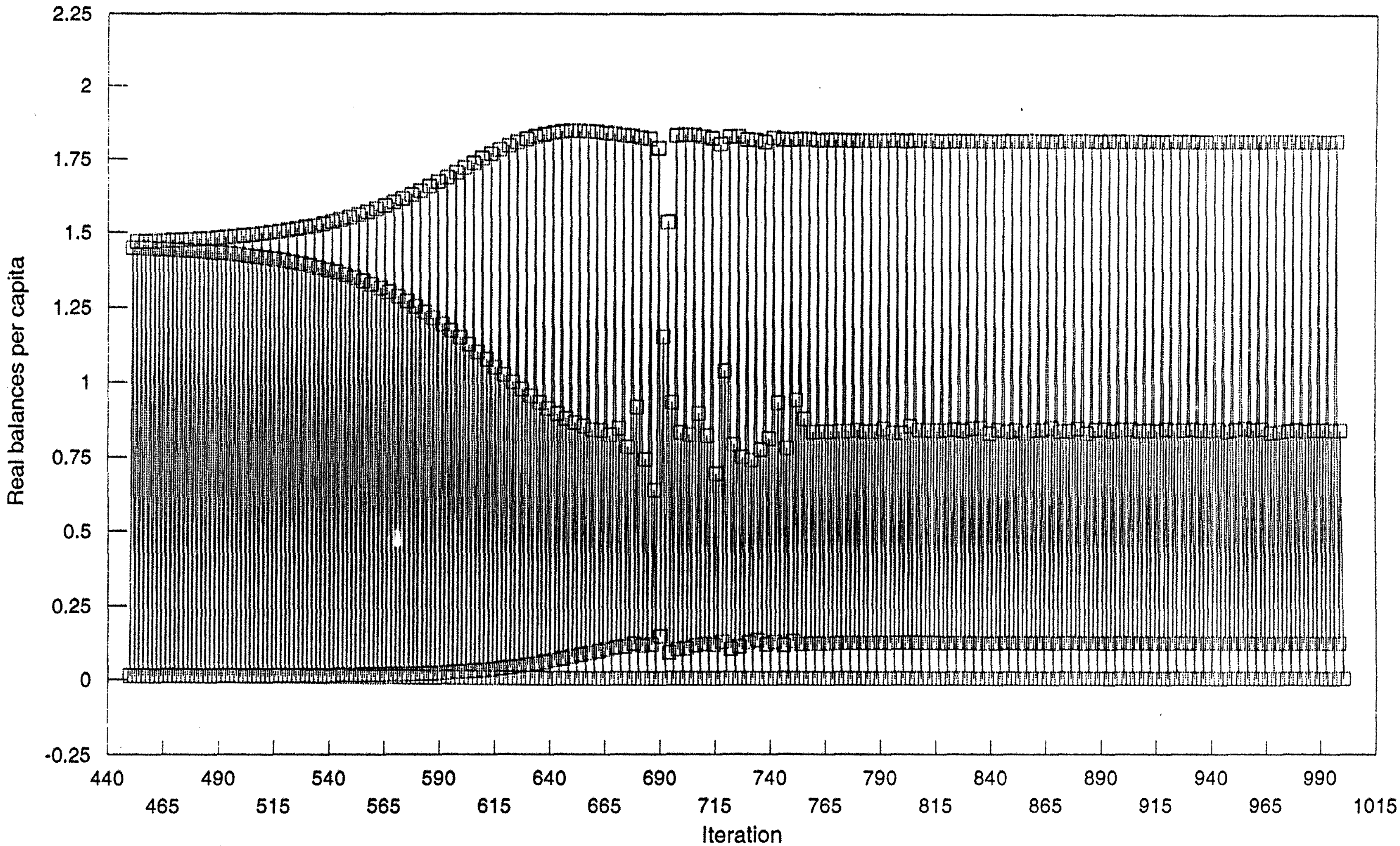
Evolution of beliefs during convergence to a periodic equilibrium of order 2



Relative risk aversion of the old agents is 5.6, computational experiment 1.

Figure 4d

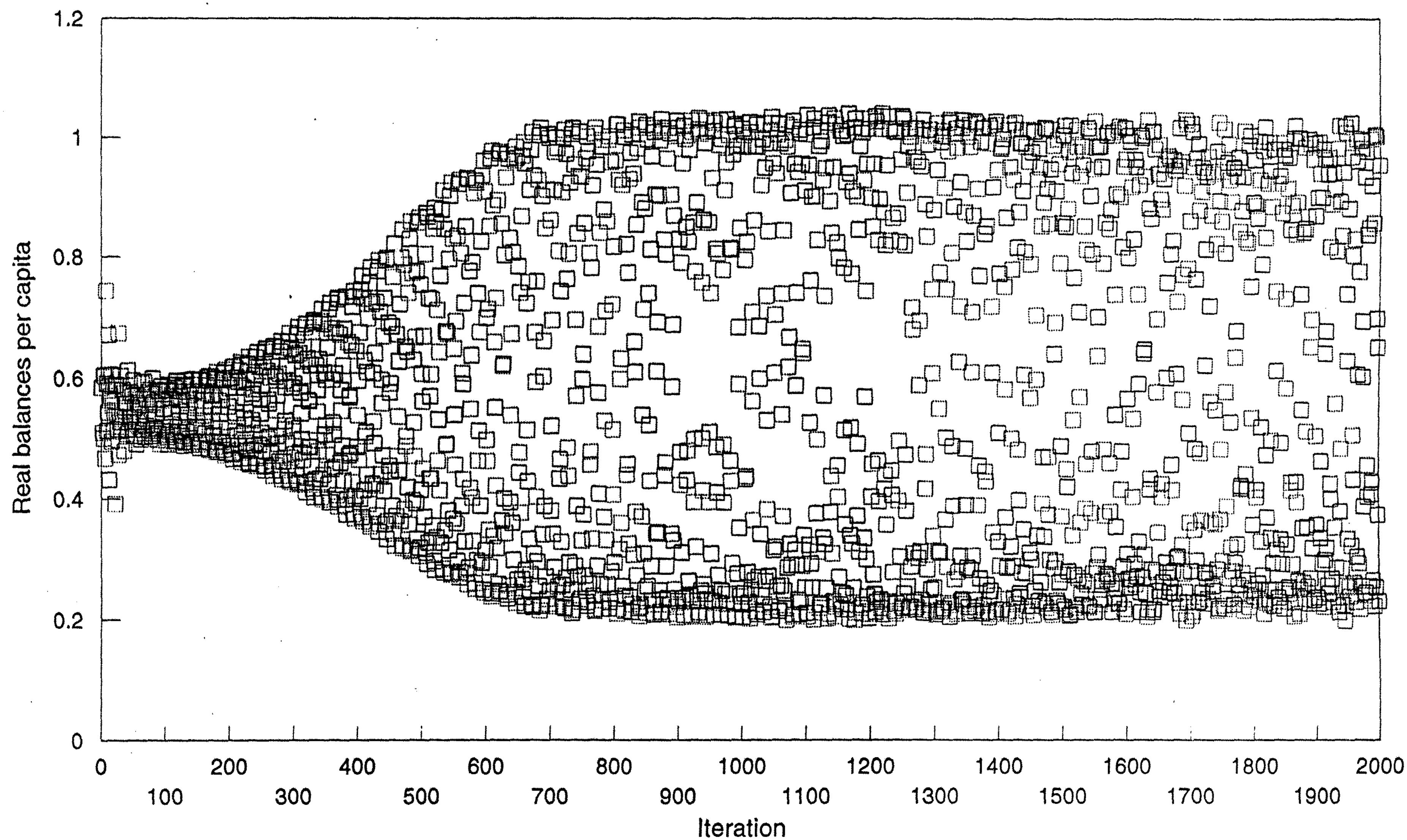
Near convergence to a periodic equilibrium of order 4



Relative risk aversion of the old agents is 7.5, computational experiment 6.

Figure 5a

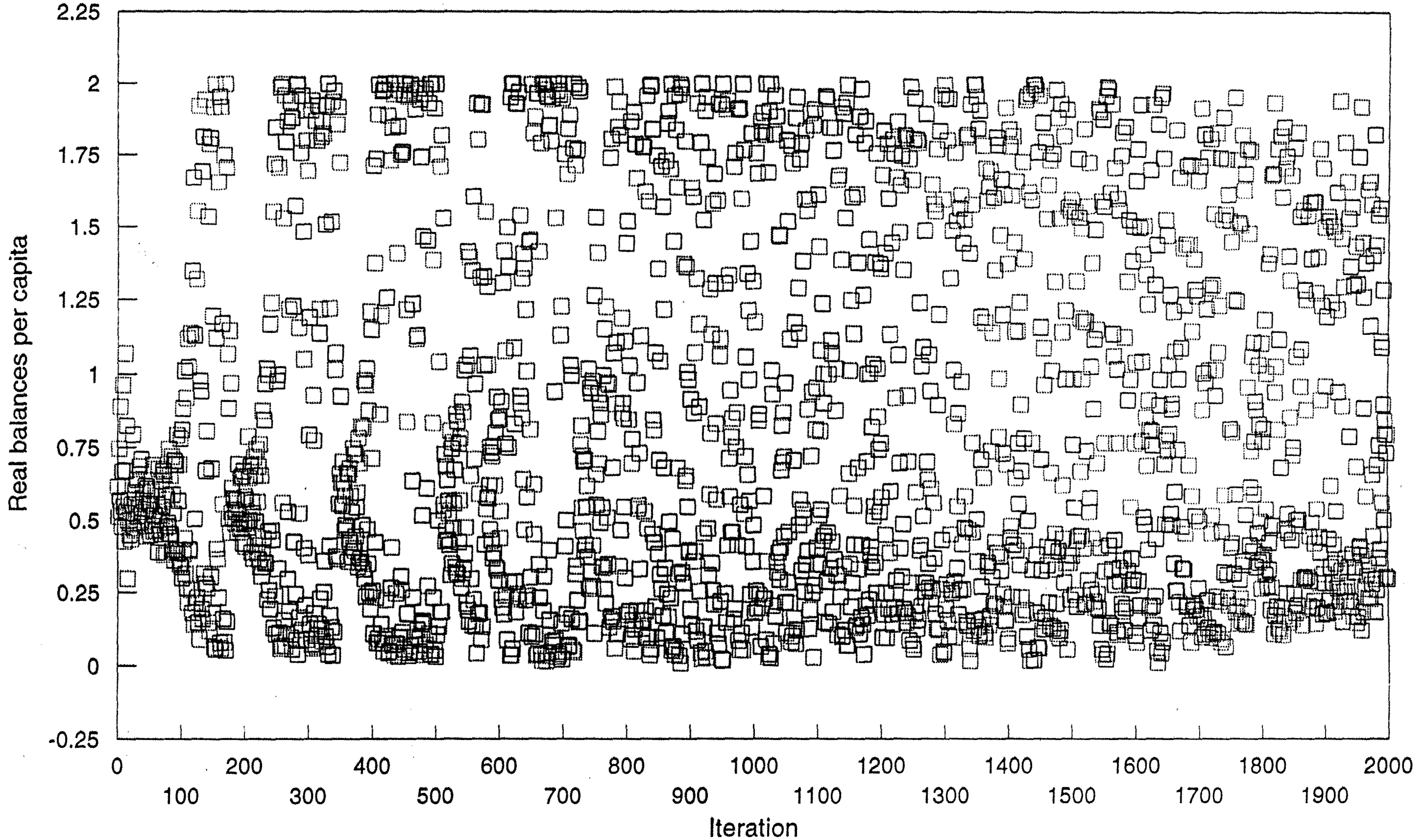
Failure to converge



Relative risk aversion of the old agents is 4.7, computational experiment 4

Figure 5b

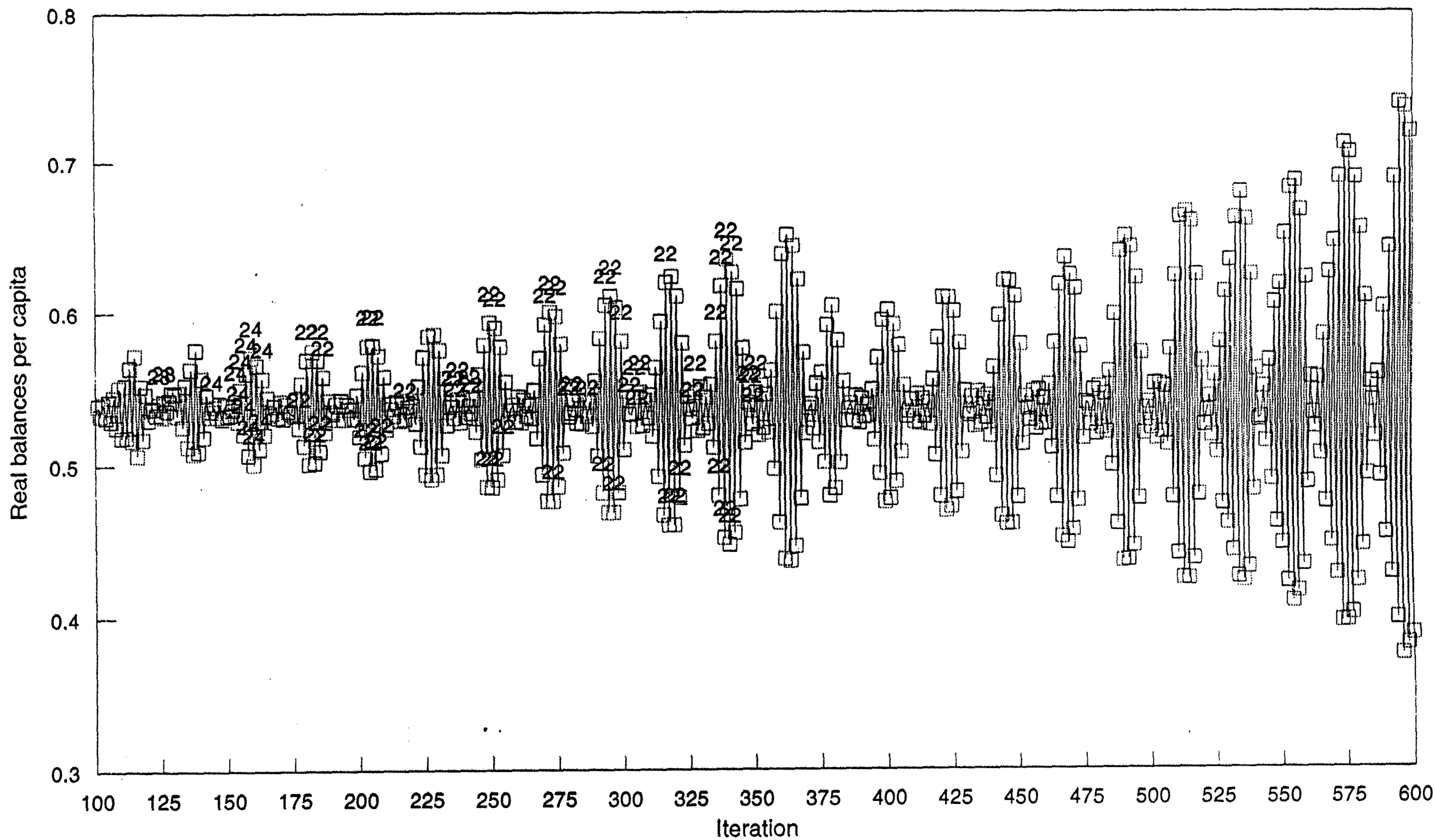
Failure to converge



Relative risk aversion of the old agents is 12.1, computational experiment 2.

Figure 5c

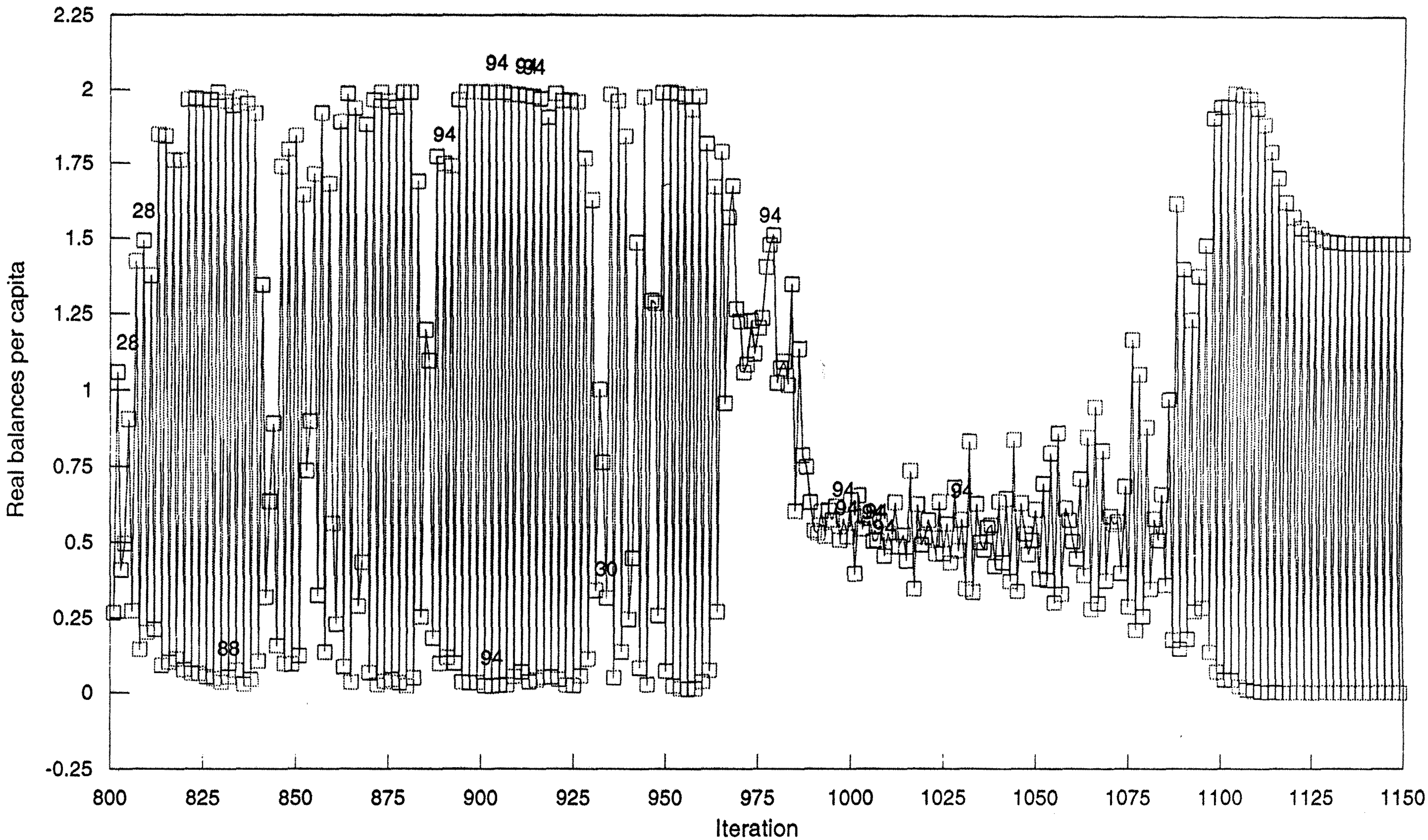
Complicated transitory dynamics



Relative risk aversion of the old agents is 5.3, computational experiment 3.
Mean position number above box when coordination obtained.

Figure 5d

Complicated transitory dynamics



Relative risk aversion of the old agents is 10.4, computational experiment 2.
Mean position number above box when coordination obtained.