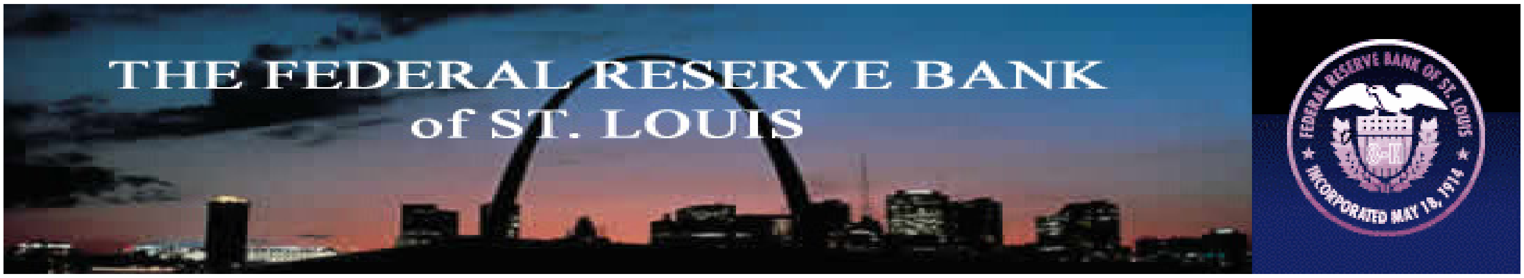

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Determinacy, Learnability, and Monetary Policy Inertia

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Determinacy, Learnability, and Monetary Policy Inertia

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Abstract. We show how monetary policy inertia can help alleviate problems of indeterminacy and non-existence of stationary equilibrium observed for some commonly-studied monetary policy rules. We also find that inertia promotes learnability of equilibrium. The context is a simple, forward-looking model of the macroeconomy widely used in the rapidly expanding literature in this area. We conclude that this might be an important reason why central banks in the industrialized economies display considerable inertia when adjusting monetary policy in response to changing economic conditions.

Keywords: Monetary policy rules, determinacy, learning, instrument instability.
JEL Classification E4, E5.

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1. Monetary policy advice

1.1. Determinacy. A fundamental issue in the evaluation of alternative monetary policy rules is the question of whether a proposed policy rule is associated with a determinate equilibrium or not. Starting with the work of Sargent and Wallace (1975), it has been shown that certain types of policy rules may be associated with large sets of rational expectations equilibria (REE) and that some of these equilibria may involve fluctuations in variables like inflation and real output due solely to self-fulfilling expectations. Such rules and the associated equilibria arguably ought to be avoided if one wishes to stabilize these variables.[1] Perhaps disconcertingly, this problem appears to be particularly acute for policy rules which may otherwise seem to be fairly realistic in terms of actual central bank behavior. For example, Clarida, Gali and Gertler (1998) have provided evidence which suggests that monetary policy for the major industrialized countries since 1979 has been *forward-looking*: Nominal interest rates are adjusted in response to *anticipated* inflation. This empirical finding is somewhat puzzling in light of the fact that such forward-looking rules are associated with equilibrium indeterminacy in many models (see, in particular, Bernanke and Woodford (1997)). Similarly, in many models policy rules which call for the monetary authority to respond aggressively to *past* values of endogenous variables (such as the previous quarter's deviations of inflation from a target level, or the output gap) can be associated with explosive instability of rational expectations equilibrium. Yet at the same time, such policy rules might also be viewed as fairly realistic in terms of actual central bank behavior in some contexts. Thus, at least two empirically relevant and seemingly ordinary-looking classes of policy rules seem to be associated with important theoretical problems.

These theoretical concerns impinge on the design of stabilization policy. Even aside from broad modeling uncertainty, there is considerable sampling variability about the estimated parameters of a given model of the macroeconomy. When a candidate class of policy rules may or may not generate indeterminacy, or explosive instability, depending on the particular parameter values of the structural model and of the policy rule, it creates something of a minefield for policy design.[2] One might, for instance, recommend a particular rule on the basis that it would generate a determinate rational expectations

[1] Some of the authors that discuss this issue most recently include Bernanke and Woodford (1997), Carlstrom and Fuerst (2000), Christiano and Gust (1999), Clarida, Gali and Gertler (2000), McCallum and Nelson (1999), Rotemberg and Woodford (1998, 1999), and Woodford (2003).

[2] See, for example, the discussion in Christiano and Gust (1999).

equilibrium, and that the targeted equilibrium would have desirable properties based on other criteria, such as utility of the representative household in the model. And yet, in reality, important parameters may lie (because of sampling variability alone) in a region associated with indeterminacy of equilibrium, or with explosive instability. Actually implementing the proposed rule could then lead to disastrous consequences. Thus, from the perspective of the design of stabilization policy, one would greatly prefer to recommend policy rules such that, even if the structural parameters actually take on values somewhat different from those that might be estimated, a determinate rational expectations equilibrium is produced.

1.2. Learnability. Even when a determinate equilibrium exists, coordination on that equilibrium cannot be assured if agents do not possess rational expectations at every point in time. It therefore seems important to analyze these systems when agents must form expectations concerning economic events using the actual data produced by the economy. In general terms, the learning approach admits the possibility that expectations might not initially be fully rational, and that, if economic agents make forecast errors and try to correct them over time, the economy may or may not reach the REE asymptotically. Thus, beyond showing that a particular policy rule reliably induces a determinate REE, one needs to show the potential for agents to learn that equilibrium.[3] In this paper, we assume the agents of the model do not initially have rational expectations, and that they instead form forecasts by using recursive learning algorithms—such as recursive least squares—based on the data produced by the economy itself. We ask whether the agents in such a world can learn the equilibria of the system induced by different classes of monetary policy feedback rules. We use the criterion of *expectational stability* (a.k.a. *E-stability*) to calculate whether rational expectations equilibria are stable under real time recursive learning dynamics or not. The research of Evans and Honkapohja (2001) and Marcet and Sargent (1989) has shown that the expectational stability of rational expectations equilibrium governs local convergence of real time recursive learning algorithms in a wide variety of macroeconomic models.

1.3. The benefits of monetary policy inertia. We conclude that it is important to recommend to central banks those policy rules which have desirable determinacy and

[3] See Bullard and Mitra (2002) and Evans and Honkapohja (2003a,b).

learnability properties, taking into consideration possible imprecision in our knowledge of structural parameters. Our main finding is that a wide variety of monetary policy rules are desirable in this sense provided the monetary authorities move cautiously in response to unfolding events. This is true both from the point of view of determinacy and of learnability of equilibrium. We model this caution, or *inertia*, on the part of the central bank by allowing the contemporaneous interest rate to respond to the lagged interest rate in the policy rule.

Inertia is a well-documented feature of central bank behavior in industrialized countries: Policymakers show a clear tendency to smooth out changes in nominal interest rates in response to changes in economic conditions. Rudebusch (1995) has provided one statistical analysis of this fact. More casually, actual policy moves are discussed among central bankers and in the business press in industrialized countries as occurring as sequences of adjustments in nominal interest rates in the same direction. This is so much the case, in fact, that policy inertia has been the source of criticism of the efforts of central bankers, as suggestions are sometimes made that policymakers have been unwilling to move far enough or fast enough to respond effectively to incoming information about the economy.

Our study provides analytical support for the idea that monetary policy inertia enhances the prospects for equilibrium determinacy and learnability in the context of a standard, small, forward-looking model which is currently the workhorse for the study of monetary policy rules. More specifically, we consider two variants of monetary policy feedback rules made famous by the seminal work of Taylor (1993, 1999*a*, 1999*b*). In one case, the central bank is viewed as adjusting a short-term nominal interest rate in response to deviations of *past values* of inflation and output from some target levels and, in order to capture interest rate smoothing, we also include a response to the deviation of the lagged interest rate from some target level. We call this the *lagged data* specification. Our second specification calls for the policymakers to react to *forecasts* of inflation deviations and the output gap, in addition to the lagged interest rate, and we call this the *forward-looking* specification.[4]

In previous studies it has been observed that there are important determinacy problems with both of these rules in the absence of inertia (see Bernanke and Woodford (1997), Bullard and Mitra (2002), Rotemberg and Woodford (1999), and Woodford (2003)). We

[4] We consider only these two classes of rules due to space constraints. We do discuss the robustness of our results to a wider class of rules when appropriate.

find that by placing a sufficiently large weight on lagged interest rate deviations in each of these classes of policy rules, the policy authorities can mitigate the threats of indeterminacy or explosive instability, and that this is one of the primary benefits of monetary policy inertia. We also argue that policy inertia actually promotes learnability of rational expectations equilibrium. Our contribution is to provide analytical results to this effect and to highlight some of the intuition behind them.

1.4. Recent related literature. Our results suggest why other, non-inertial types of policies might leave the economy vulnerable to unexpected dynamics, and hence why central banks might willingly adopt inertial behavior. Recently, several very different theories have been proposed as to why policy inertia might be observed, for instance Woodford (1999), Caplin and Leahy (1996), and Sack (1998). Our results are probably best viewed as complementary to these theories.

Bullard and Mitra (2002) study the determinacy and learnability of *simple* monetary policy rules, that is, of policy rules which only respond to inflation and output deviations, but not to lagged interest rate deviations, and so do not comment on the question of monetary policy inertia. Evans and Honkapohja (2003a) analyze learnability in a similar model, and consider different ways of implementing *optimal* monetary policy under discretion, which leads to non-inertial rules.[5]

The finding that interest-rate inertia is conducive to the existence of determinate REE has been noted by Rotemberg and Woodford (1999), Woodford (2003), Benhabib, Schmitt-Grohe, and Uribe (2003), and Carlstrom and Fuerst (2000). Our contribution with regard to determinacy is to elaborate in greater detail the reasons for the numerical findings in Rotemberg and Woodford (1999) and to show that the beneficial effects of inertia are true for a wider class of policy rules than considered in Woodford (2003). In addition, our results on determinacy are useful in understanding the effects of inertia on learning dynamics. Benhabib, Schmitt-Grohe, and Uribe (2003) find support for super-inertial interest rate policies in a somewhat different class of models where a supply side channel of monetary policy transmission is emphasized. Carlstrom and Fuerst (2000) consider models where the timing of money balances entering the utility function and the nature of sticky price assumption along off-equilibrium paths is important. They find that

[5] Evans and Honkapohja (2003b) provide a survey of the recent literature on learnability and monetary policy.

inertial forward looking policies are subject to indeterminacy problems whereas backward policies which react aggressively to past inflation can be associated with a determinate equilibrium independently of the degree of inertia.

1.5. Organization. In the next section we present the model analyzed throughout the paper. We also discuss the types of linear policy feedback rules we will use to organize our analysis, and a calibrated case which we will occasionally employ. In the subsequent sections, we present conditions for determinacy of equilibrium for the lagged and forward looking policy rules. We then turn to the question of learnability of REE under our various specifications. Section 5 considers the robustness of our results in Preston's (2005a) model. We conclude with a summary of our findings.

2. Environment

2.1. The model. We study a model developed by Woodford (2003) which we write as

$$x_t = \hat{E}_t x_{t+1} - \sigma \left(r_t - r_t^n - \hat{E}_t \pi_{t+1} \right) \tag{1}$$

$$\pi_t = \kappa x_t + \beta \hat{E}_t \pi_{t+1} \tag{2}$$

where x_t is the output gap, π_t is the period t inflation rate defined as the percentage change in the price level from $t-1$ to t, and r_t is the nominal interest rate; each of the two latter variables are expressed as a deviation from the long run level. Since we will also analyze learning we use the notation $\hat{E}_t \pi_{t+1}$ and $\hat{E}_t x_{t+1}$ to denote the possibly nonrational private sector expectations of inflation and output gap next period, respectively, whereas the same notation without the hat symbol will denote rational expectations (RE) values.[6] The parameters σ, κ, and $\beta \in (0,1)$ are structural and assumed positive on economic grounds—see Woodford (1999, 2003) for an interpretation of these constants. The "natural rate of interest" r_t^n is an exogenous stochastic term that follows the process

$$r_t^n = \rho r_{t-1}^n + \epsilon_t \tag{3}$$

where ϵ_t is *i.i.d.* noise with variance σ_ϵ^2, and $0 \leq \rho < 1$ is a serial correlation parameter.

2.2. Alternative policy rules. We close the system by supplementing equations (1), (2), and (3), which represent the behavior of the private sector, with a policy rule

[6] See Section 5 for more discussion of the role of expectations in the model under a learning assumption.

for setting the nominal interest rate representing the behavior of the monetary authority. We stress that we view identification of classes of rules that reliably produce determinacy and learnability as a prior exercise to locating an optimal rule according to some objective function assigned to the central bank. Once we isolate the characteristics of rules that reliably produce both determinacy and learnability, then one could go about finding an optimal or best-performing rule from among the ones in this set.

Taylor (1993, 1999*a*) popularized the use of interest rate feedback rules that react to information on output and inflation. Our first specification considers a case in which interest rates are adjusted in response to last quarter's observations on inflation and the output gap. This is our *lagged data* specification for our interest rate equation

$$r_t = \varphi_\pi \pi_{t-1} + \varphi_x x_{t-1} + \varphi_r r_{t-1}. \tag{4}$$

This specification is considered *operational* by McCallum (1999) since it does not call for the central bank to react to contemporaneous data on output and inflation deviations.

Our second specification assumes that the authorities set their interest rate instrument in response to their *forecasts* of output gap and inflation, so that the policy rule itself is forward-looking. Forward-looking rules have been found to describe well the actual behavior of monetary policymakers in countries like Germany, Japan, and the U.S. since 1979, as documented by Clarida, Gali, and Gertler (1998). We consider a simple version of this rule, namely[7]

$$r_t = \varphi_\pi \hat{E}_t \pi_{t+1} + \varphi_x \hat{E}_t x_{t+1} + \varphi_r r_{t-1}. \tag{5}$$

In the next section, we consider the determinacy of REE, and then we follow that with a section analyzing the learnability of equilibrium. We maintain the following assumptions throughout the paper: $\varphi_\pi \geq 0$ and $\varphi_x \geq 0$, with at least one strictly positive, $\varphi_r > 0$, $\kappa > 0$, $\sigma > 0$, and $0 < \beta < 1$. We sometimes illustrate our findings using Woodford's (1999) calibrated values, namely, $\beta = .99$, $\sigma^{-1} = .157$, $\kappa = .024$, and $\rho = .35$.

3. Inertia and determinacy

3.1. Lagged data in the policy rule. We start by considering the system when the policymaker reacts to lagged values of inflation, output, and interest rate deviations.

[7] Similar interest rate rules also arise in the context of implementing optimal discretionary monetary policies and nominal GDP targeting, see respectively Evans and Honkapohja (2003a) and Mitra (2003). One interpretation for this rule is that both policymakers and private agents have homogeneous expectations and learning algorithms. Alternately, it may be that the central bank simply targets the predictions of private sector forecasters. However, one can allow for some forms of heterogeneity in learning rules, see Honkapohja and Mitra (2005*a*, 2005*b*).

Non-inertial lagged data rules (that is, rules with $\varphi_r = 0$) can easily lead to non-existence of locally unique stationary solutions. Indeed, Bullard and Mitra (2002) note that a sufficiently aggressive response to inflation and output deviations *invariably* leads to such a situation in quantitatively important portions of the parameter space.[8] We now show that this problem need not arise if the central bank displays sufficient inertia in setting its interest rate.

In this case, our policy rule is given by equation (4), so that the complete system is given by equations (1), (2), (3), and (4). If $y_t = (x_t, \pi_t, r_t)'$, then this system can be put in the form $\hat{E}_t y_{t+1} = B_1 y_t + \varsigma r_t^n$, where

$$B_1 = \begin{bmatrix} 1+\beta^{-1}\kappa\sigma & -\beta^{-1}\sigma & \sigma \\ -\beta^{-1}\kappa & \beta^{-1} & 0 \\ \varphi_x & \varphi_\pi & \varphi_r \end{bmatrix}. \tag{6}$$

Since r_t is predetermined while x_t and π_t are free, equilibrium is determinate if and only if exactly one eigenvalue of B_1 is inside the unit circle.[9]

Woodford (2003) provides necessary and sufficient conditions for determinacy of equilibrium in such a system. The details of these calculations are given in Appendix A. The following two conditions together are shown there to be necessary for determinacy

$$\kappa(\varphi_\pi + \varphi_r - 1) + (1-\beta)\varphi_x > 0, \tag{7}$$

$$[\kappa\sigma + 2(1+\beta)]\varphi_r + 2(1+\beta) > \sigma[\kappa(\varphi_\pi - 1) + (1+\beta)\varphi_x]. \tag{8}$$

The condition (7) is precisely what Woodford (2001, 2003) calls the *Taylor principle*, whereby in the event of a permanent one percent rise in inflation, the cumulative increase in the nominal interest rate is more than one percent. However, the Taylor principle in general is not sufficient for determinacy, because another necessary condition for determinacy is condition (8). This proves the following result:

Proposition 1. *Assume that $\kappa(\varphi_\pi + \varphi_r - 1) + (1-\beta)\varphi_x > 0$ for the inertial lagged data interest rule (4). Then a necessary condition for determinacy is*

$$[\kappa\sigma + 2(1+\beta)]\varphi_r + 2(1+\beta) > \sigma[\kappa(\varphi_\pi - 1) + (1+\beta)\varphi_x]. \tag{9}$$

This proposition shows that the Taylor principle is not sufficient for determinacy, since it is also necessary that the degree of inertia φ_r be large enough. If the central bank merely

[8] The interested reader can consult Figure 2 in that paper, or similarly Figure 2.15 of Rotemberg and Woodford (1999). The analytical details for this result are in Proposition 11; see Appendix A.

[9] Our determinacy analysis follows the conventional practice of Blanchard and Kahn (1980).

responds vigorously to inflation and output without displaying enough inertia, then the condition for determinacy may be violated.

Appendix A also shows that a set of necessary and sufficient conditions required for determinacy reduce to (7), (8), and

$$\varphi_r > 2 - (1 + \kappa\sigma)\beta^{-1}. \tag{10}$$

The right hand expression in (10) is less than 1 since $\kappa > 0$, $\sigma > 0$, and $0 < \beta < 1$. These conditions show that a large enough value of φ_r will *always* result in determinacy since this contributes to satisfaction of all of the conditions (7), (8), and (10). A value of $\varphi_r \geq 1$ always satisfies (7) and (10), so that if φ_r also satisfies condition (8), the conditions for determinacy will be met. Hence, this proves:

Proposition 2. *Assume that $\varphi_r \geq 1$ for the inertial lagged data interest rule (4). Then the necessary and sufficient condition for determinacy is*

$$[\kappa\sigma + 2(1+\beta)]\varphi_r + 2(1+\beta) > \sigma[\kappa(\varphi_\pi - 1) + (1+\beta)\varphi_x]. \tag{11}$$

The analytical results given above provide intuition for a number of results obtained in more complicated forward-looking models. For instance, Rotemberg and Woodford (1999) found that large values of φ_r tend to be associated with a unique equilibrium. This is easily explained by conditions (7), (8), and (10) which are sufficient for a determinate outcome. Values of $\varphi_r \geq 1$ automatically satisfy condition (7), and condition (10) along with small values of κ, such as the one employed by Rotemberg and Woodford (1999), help to satisfy condition (8) easily. McCallum and Nelson (1999, pp. 34-35) found that policy rules with large values of φ_π or φ_x deliver *dynamically stable* (in their terminology) results, so long as there is a sufficient level of monetary policy inertia. Their explanation for this surprising finding was in terms of relatively small values of σ and κ, and can be understood from our condition (8). Relatively small values of σ and κ mean that condition (8) is likely to be easily satisfied.[10]

3.2. Forward-looking policy rule. With the forward-looking rule (5), we define $y_t = (x_t, \pi_t, r_{t-1})'$ and put the system in the form $\hat{E}_t y_{t+1} = B y_t + \varsigma r_t^n$, where

$$B = (1 - \varphi_x\sigma)^{-1} \begin{bmatrix} 1 - \beta^{-1}\kappa\sigma(\varphi_\pi - 1) & \beta^{-1}\sigma(\varphi_\pi - 1) & \sigma\varphi_r \\ -\beta^{-1}\kappa(1 - \varphi_x\sigma) & \beta^{-1}(1 - \varphi_x\sigma) & 0 \\ \varphi_x(1 + \beta^{-1}\kappa\sigma) - \beta^{-1}\kappa\varphi_\pi & \beta^{-1}(\varphi_\pi - \varphi_x\sigma) & \varphi_r \end{bmatrix}. \tag{12}$$

[10] Propositions 1 and 2 may give the impression that the Taylor principle is always necessary for determinacy. However, this is not true. See Proposition 11 in Appendix A.

Since r_{t-1} is pre-determined and x_t, π_t are free, equilibrium is determinate if and only if exactly one eigenvalue of B is inside the unit circle. As shown in Bernanke and Woodford (1997) and Bullard and Mitra (2002), a sufficiently aggressive response to inflation or output leads to indeterminacy with the rule (5) when $\varphi_r = 0$. We now turn to showing how this problem can be circumvented when policymakers adopt a sufficiently aggressive response to the lagged interest rate.[11]

The first proposition shows that if the response to the output gap φ_x is not large, then necessary conditions for determinacy are given by conditions (7) and (8). More specifically,

Proposition 3. *Assume that $\varphi_x < 2\sigma^{-1}$ for the inertial forward looking policy rule (5). Then conditions (7) and (8) are necessary for determinacy.*[12]

This again shows that the Taylor principle in general is not sufficient for determinacy, as a high degree of inertia is also necessary. In addition, we have

Proposition 4. *Assume that $\varphi_r \geq 1$ for the inertial forward looking policy rule (5). Then the necessary and sufficient condition for determinacy is (8).*

The same proposition was proved for rules responding to lagged data (Proposition 2). These results show that for given values of φ_π and φ_x, a large enough value of φ_r is invariably associated with determinacy.

3.3. Summary of the results on determinacy. We have seen the beneficial effects of a large degree of inertia in promoting determinacy for the lagged data rule and the forward looking rule. We stress that the same cannot be said for the response to inflation or output in the interest rule—a response which is too aggressive with respect to either of these variables may lead to problems of non-existence of stationary REE, or to indeterminacy of REE. The tendency of policy inertia to help generate determinacy may be an important reason why so much inertia is observed in the actual monetary policies of industrialized countries. However, too much policy inertia may cause another type of instability—that of the learning dynamics. We now turn to this topic.

[11] Woodford (2003) has considered the determinacy analysis of a variant of the forward rule where the interest rate responds to expected inflation and the *current* output gap.

[12] To economize on space, we state this and the next propostion without proof. The proofs may be obtained from the authors upon request.

4. Inertia and learnability

4.1. Lagged data in the policy rule.

The system under learning. We now consider learning, beginning with the case in which the policy authority responds to lagged data.[13] In this case, the complete system is given by equations (1), (2), (3), and (4). We analyze the expectational stability of stationary minimum state variable (MSV) solutions (see McCallum (1983)). For the analysis of learning, we need to compute the MSV solution and for this we need to obtain a relationship between the current endogenous variables (and their lags) and future expectations. This relationship is now obtained by first defining the vector of endogenous variables, $y_t = (x_t, \pi_t, r_t)'$, and by putting our system in the form $y_t = \Omega \hat{E}_t y_{t+1} + \delta y_{t-1} + \varkappa r_t^n$ where

$$\Omega = \begin{bmatrix} 1 & \sigma & 0 \\ \kappa & \beta + \kappa\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{13}$$

$$\delta = \begin{bmatrix} -\sigma\varphi_x & -\sigma\varphi_\pi & -\sigma\varphi_r \\ -\kappa\sigma\varphi_x & -\kappa\sigma\varphi_\pi & -\kappa\sigma\varphi_r \\ \varphi_x & \varphi_\pi & \varphi_r \end{bmatrix}. \tag{14}$$

The MSV solution for this model takes the form

$$y_t = \bar{a} + \bar{b} y_{t-1} + \bar{c} r_t^n \tag{15}$$

with $\bar{a} = 0$, and with $\bar{b}$ and $\bar{c}$ given by

$$\bar{b} = (I - \Omega\bar{b})^{-1}\delta, \tag{16}$$

$$\bar{c} = (I - \Omega\bar{b})^{-1}(\varkappa + \rho\Omega\bar{c}), \tag{17}$$

provided the matrix $(I - \Omega\bar{b})$ is invertible. Equation (16) potentially yields multiple solutions for $\bar{b}$ and the determinate case corresponds to the situation when there is a unique solution for $\bar{b}$ with all eigenvalues inside the unit circle. For the analysis of learning, we assume that agents have a *perceived law of motion* (PLM) of the form

$$y_t = a + b y_{t-1} + c r_t^n \tag{18}$$

corresponding to the MSV solution. We then compute the following expectation (assuming that the time t information set does not include y_t)

$$\hat{E}_t y_{t+1} = a + b\hat{E}_t y_t + c\rho r_t^n = (I + b)a + b^2 y_{t-1} + (bc + c\rho) r_t^n. \tag{19}$$

[13] Our analysis of learning is standard and follows Evans and Honkapohja (2001), Chapter 10.

Substituting these computed expectations into the model one obtains an actual law of motion (ALM)

$$y_t = (\Omega + \Omega b)a + (\Omega b^2 + \delta)y_{t-1} + (\Omega bc + \Omega c\rho + \varkappa)r_t^n. \tag{20}$$

The mapping from the PLM to the ALM takes the form

$$T(a, b, c) = ([\Omega + \Omega b]\, a,\ \Omega b^2 + \delta, \Omega bc + \Omega c\rho + \varkappa). \tag{21}$$

Expectational stability is then determined by the matrix differential equation

$$\frac{d}{d\tau}\,(a, b, c) = T\,(a, b, c) - (a, b, c)\,. \tag{22}$$

The fixed points of equation (22) give us the MSV solution $(\bar{a}, \bar{b}, \bar{c})$. We say that a particular MSV solution $(\bar{a}, \bar{b}, \bar{c})$ is expectationally stable if the MSV fixed point of the differential equation (22) is locally asymptotically stable at that point. Our system is in a form where we can apply the results of Evans and Honkapohja (2001). It can then be shown that for E-stability of any MSV solution, assuming that the time t information set is $(1, y_{t-1}', r_t^n)'$, the eigenvalues of the matrices

$$\bar{b}' \otimes \Omega + I \otimes \Omega\bar{b} - I, \tag{23}$$

$$\rho\Omega + \Omega\bar{b} - I, \tag{24}$$

$$\Omega + \Omega\bar{b} - I, \tag{25}$$

need to have negative real parts (where I denotes a conformable identity matrix). If any eigenvalue of the above matrices has a positive real part, then the MSV solution is not E-stable. Even small expectational errors would tend to drive the system away from the REE. We emphasize that the MSV solution for $\bar{b}$ directly affects the E-stability conditions and this is the key to understanding the results under learning.

A quantitative case. We illustrate regions of determinacy and E-stability for the case when the policy authorities react to lagged data in Figure 1 where we have employed the baseline parameter values. Figure 1 contains three panels, the first of which corresponds to the case where there is no policy inertia, so that $\varphi_r = 0$. The figure is drawn in (φ_π, φ_x) space, holding all other parameters at their baseline values. Vertical lines in the figure denote parameter combinations that generate determinacy, and that also generate local stability in the learning dynamics. Horizontal lines, on the other hand,

FIGURE 1. Lagged Data

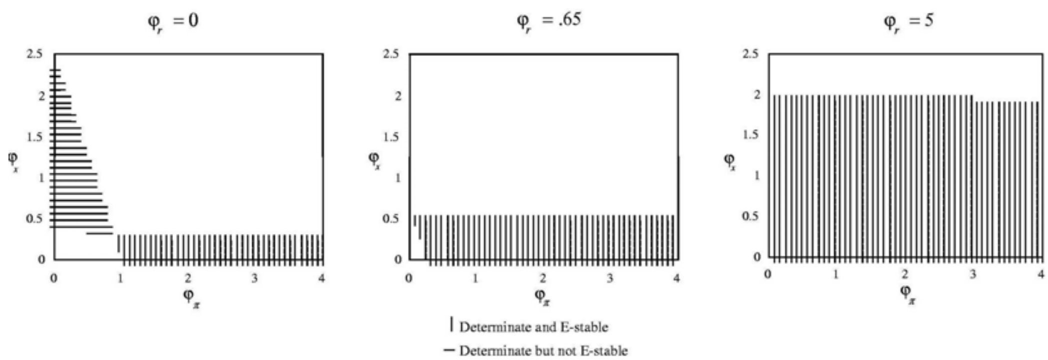

Figure 1: With $\varphi_r = 0$, the region of the parameter space associated with both determinate and learnable rational exepectations equilibria involves relatively small values for φ_x, and generally $\varphi_\pi > 1$. In the blank region, determinacy does not hold. When $\varphi_r = .65$, which is close to empirical estimates in the literature, the region of the parameter space associated with determinacy and learnability expands, relative to the no inertia case. For a large value of φ_r, such as $\varphi_r = 5$ as shown here, much of the pictured (φ_π, φ_x) space is associated with both determinacy and learnability.

indicate parameter combinations that generate determinacy, but where the unique equilibrium is unstable in the learning dynamics. In this and all figures, the blank region is not associated with determinacy.

The $\varphi_r = 0$ portion of this figure illustrates that determinacy does not always imply learnability. It also illustrates that Taylor-type rules which react aggressively to inflation, but with little or no reaction to the output gap or the lagged interest rate, tend to be associated with both determinacy and learnability. However, one concern regarding this panel might be that parameter values within an empirically relevant range are sometimes associated with equilibria which are not determinate, or which are determinate but not learnable.

The second panel of Figure 1 illustrates how the situation is improved when the degree of monetary policy inertia is increased from zero to $\varphi_r = .65$. This value is close to estimates of the degree of policy inertia based on U.S. postwar data, such as Sack (1998). In this case, the region of the (φ_π, φ_x) space associated with both determinacy and learnability of equilibrium has been enlarged. The region associated with determinate, but unlearnable, rational expectations equilibria has been eliminated. This effect becomes even more pronounced in the third panel, where a very large value of φ_r is employed, specifically, $\varphi_r = 5$. In this case, a much larger portion of the space is determinate and learnable. We conclude that larger degrees of policy inertia enhance the prospects for determinacy considerably, relative to the case where there is no policy inertia at all, in this quantitative case. In addition, learnability does not appear to be jeopardized by large degrees of policy inertia, as the determinate equilibria are also learnable, even when φ_r is large.

Intuition and analytics. We now provide some intuition and analytics for the phenomenon illustrated in Figure 1. We begin with a discussion of non-inertial, $\varphi_r = 0$, policy rules. The triangular region in the left hand panel of Figure 1 shows that there are determinate equilibria which are E-unstable in this case. We first provide intuition for this phenomenon. When $\varphi_r = 0$, the reduced form model with the interest rate rule (4) takes the form $y_t = \Omega \hat{E}_t y_{t+1} + \delta y_{t-1} + \varkappa r_t^n$ with

$$\Omega = \begin{bmatrix} 1 & \sigma \\ \kappa & \kappa\sigma + \beta \end{bmatrix}, \ \delta = \begin{bmatrix} -\varphi_x \sigma & -\varphi_\pi \sigma \\ -\kappa\varphi_x \sigma & -\kappa\varphi_\pi \sigma \end{bmatrix}, \tag{26}$$

where $y_t = [x_t, \pi_t]'$. The MSV solution continues to take the form (15) with the same solutions for $\bar{a}$ $(= 0)$, and $\bar{b}$, $\bar{c}$ given by (16) and (17). It is the feedback from lagged endogenous variables (via $\bar{b}$) in the stationary MSV solution that is the key to understanding E-instability of determinate equilibria.

In matrix form, the MSV solution for $\bar{b}$ is of the form

$$\bar{b} = \begin{bmatrix} b_{xx} & b_{x\pi} \\ b_{\pi x} & b_{\pi\pi} \end{bmatrix}, \tag{27}$$

where $b_{x\pi} = \varphi_\pi \varphi_x^{-1} b_{xx}$, $b_{\pi x} = \varphi_x \varphi_\pi^{-1} b_{\pi\pi}$ (assuming $\varphi_x, \varphi_\pi > 0$), and b_{xx} and $b_{\pi\pi}$ can be computed from equation (16); see Appendix B. Written explicitly, this MSV solution takes the form

$$x_t = b_{xx} x_{t-1} + b_{x\pi} \pi_{t-1} + ... \tag{28}$$

$$\pi_t = b_{\pi x} x_{t-1} + b_{\pi\pi} \pi_{t-1} + ... \tag{29}$$

Here the three elipses denote terms involving shocks not needed for our analysis. We conclude that $\bar{b}$ in (27) is singular, and that $|b_{xx} + b_{\pi\pi}| < 1$ is required for stationarity of the MSV solution $\bar{b}$.

We first provide an economic interpretation of E-stability. The nature of the MSV solution is crucial for E-stability since we start the system not at the REE but from within a small neighborhood of the REE of interest. If b_{xx} and $b_{\pi\pi}$ are positive (so that $b_{x\pi}$ and $b_{\pi x}$ are positive as well), then the MSV solution (28)-(29) has a perverse feature. This feature is that while an increase in either lagged output or inflation raises the nominal interest rate, the increase in the nominal interest rate is not large enough (i.e., the real interest rate falls). Consequently, current output and inflation actually *increase* which further enhances inflationary pressures *if* one starts away from the REE. If, on the other hand, agents do have rational expectations, then their beliefs will exactly match realizations and this equilibrium will be the unique one in this parameter range. It is only when agents do not have RE to start with that there will be pressure to move further away from these determinate REE owing to the perverse nature of the solution. Dynamics along off-equilibrium paths is important for E-stability but not for determinacy.

We now turn to the analytical details behind E-stability. To examine what type of MSV solutions can be E-stable, we first note that a necessary condition for E-stability is that the eigenvalues of $\Omega + \Omega\bar{b} - I$ have negative real parts and for this, the determinant of $\Omega + \Omega\bar{b} - I$, given by

$$-b_{xx}(1 - \beta + \kappa\varphi_\pi\varphi_x^{-1}) - b_{\pi\pi}(\kappa + \varphi_x\varphi_\pi^{-1})\sigma - \kappa\sigma, \tag{30}$$

must be positive. This shows that it is necessary for at least one of b_{xx} or $b_{\pi\pi}$ to be negative for E-stability since otherwise this determinant will be negative. In other words, if both b_{xx} and $b_{\pi\pi}$ are positive, the MSV solution will necessarily be E-unstable verifying the economic intuition outlined above.

This is precisely what happens in the triangular determinate but E-unstable region of Figure 1. This region corresponds to the violation of the Taylor principle and the necessary and sufficient condition for determinacy in this case is given by condition (55) in Proposition 11 (but with $\varphi_r = 0$ for this discussion). It can be easily checked that the unique stationary solution for $\bar{b}$ in this region involves $b_{xx} > 0$ and $b_{\pi\pi} > 0$ which makes this solution E-unstable. As long as φ_r is sufficiently small, the existence of a determinate equilibrium does not preclude a solution for $\bar{b}$ with both b_{xx} and $b_{\pi\pi}$ positive. As a result, a triangular determinate but E-unstable region continues to exist for small φ_r; however the size of this region shrinks as φ_r increases eventually being eliminated.

The determinate and E-stable region when $\varphi_r = 0$, on the other hand, satisfies the

Taylor principle and it can be checked numerically that these regions are characterized by MSV solutions where b_{xx} and $b_{\pi\pi}$ (and hence $b_{x\pi}$ and $b_{\pi x}$) are all *negative*. In these solutions, an increase in either lagged output or inflation increases both the nominal and the real interest rate so that contemporaneous output and inflation fall pushing the economy back towards the initial equilibrium even when agents start outside the REE and are learning using recursive least squares.

When the policy rule involves $\varphi_r > 0$, the MSV solution $\bar{b}$ takes the form

$$\bar{b} = \begin{bmatrix} b_{xx} & b_{x\pi} & b_{xr} \\ b_{\pi x} & b_{\pi\pi} & b_{\pi r} \\ \varphi_x & \varphi_\pi & \varphi_r \end{bmatrix} \tag{31}$$

with $b_{xx} = \varphi_x \varphi_r^{-1} b_{xr}$, $b_{x\pi} = \varphi_\pi \varphi_r^{-1} b_{xr}$, $b_{\pi x} = \varphi_x \varphi_r^{-1} b_{\pi r}$, and $b_{\pi\pi} = \varphi_\pi \varphi_r^{-1} b_{\pi r}$; see Appendix C. Consequently, once b_{xr} and $b_{\pi r}$ are known, the remaining unknowns can be determined from them numerically. Written explicitly the MSV solution is of the form

$$x_t = b_{xx} x_{t-1} + b_{x\pi} \pi_{t-1} + b_{xr} r_{t-1} + \ldots \tag{32}$$

$$\pi_t = b_{\pi x} x_{t-1} + b_{\pi\pi} \pi_{t-1} + b_{\pi r} r_{t-1} + \ldots \tag{33}$$

where the interest rate rule in the MSV solution is (4). A necessary condition for $\Omega + \Omega\bar{b} - I$ to have eigenvalues with negative real parts (that is, for E-stability) is that a_2, defined as

$$a_2 = -[(1-\beta)\varphi_x + \kappa\varphi_\pi]\varphi_r^{-1} b_{xr} - \sigma(\varphi_x + \kappa\varphi_\pi)\varphi_r^{-1} b_{\pi r} - \kappa\sigma, \tag{34}$$

be positive; see Appendix D for the details. This implies that at least one of b_{xr} or $b_{\pi r}$ must be negative for E-stability. This proves:

Proposition 5. *A necessary condition for an MSV solution with the lagged data interest rule (4) to be E-stable is that either* $b_{xr} < 0$ *or* $b_{\pi r} < 0$.

Proposition 5 shows that any MSV solution with both b_{xr} and $b_{\pi r}$ positive is E-unstable regardless of the degree of inertia in the policy rule. The economic intuition runs parallel to the case of the non-inertial, $\varphi_r = 0$, rule. E-stability rules out a perverse (positive) effect of the lagged interest rate on contemporaneous output and inflation in the MSV solution which is important for off-equilibrium dynamics. If agents are learning, then this MSV solution should have the right properties to enable them to converge to the REE of interest. If b_{xr} and $b_{\pi r}$ are both strictly positive, then an increase in the interest rate by the monetary authority (due to an increase in inflationary pressures in the economy) will be insufficiently large, actually *raising* future inflation and output further worsening these inflationary pressures. This in turn would raise expectations of inflation and output gap of private agents further increasing inflation and pushing the economy away from the REE, even if initially agents had started from a small neighborhood of this REE.

We emphasize that this intuition is relevant for low levels of monetary policy inertia. The situation changes markedly when the degree of inertia is large. It is easy to check that two of the eigenvalues of $\bar{b}$ in (31) at the MSV solution are zero and the third one is given by $\varphi_r + \varphi_x \varphi_r^{-1} b_{xr} + \varphi_\pi \varphi_r^{-1} b_{\pi r}$. Existence of a stationary solution for $\bar{b}$ is, therefore, equivalent to the requirement that

$$-(1+\varphi_r)\varphi_r < \varphi_x b_{xr} + \varphi_\pi b_{\pi r} < (1-\varphi_r)\varphi_r. \tag{35}$$

Without any further calculations, the right hand inequality in (35) immediately demonstrates that if $\varphi_r \geq 1$, a necessary condition for stationarity is that at least one of b_{xr} or $b_{\pi r}$ (i.e., b_{xx} or $b_{\pi\pi}$) be negative. Hence, we have the following:

Proposition 6. *Assume that $\varphi_r \geq 1$. A necessary condition for an MSV solution with the lagged data interest rule (4) to be stationary is that either $b_{xr} < 0$ or $b_{\pi r} < 0$.*

In other words, a high degree of inertia precludes a stationary MSV solution with both b_{xr} and $b_{\pi r}$ positive. Earlier, we saw that for a small degree of inertia, determinate equilibria with positive values of both b_{xr} and $b_{\pi r}$ satisfying (35) existed. These solutions were E-unstable by Proposition 5. We conclude that it is only with a high degree of inertia that the necessary conditions for both determinacy and E-stability coincide.

Imposing more conditions in the high inertia case enables us to provide a necessary and sufficient condition for E-stability below. One can check numerically that super-inertial rules (that is, rules with $\varphi_r \geq 1$) lead to determinate MSV solutions with both b_{xr} and $b_{\pi r}$ negative.[14] Appendix D shows that if the degree of inertia is large enough, the necessary and sufficient condition for E-stability simplifies to the one given in the following:

Proposition 7. *Assume that $\varphi_r \geq 1$ for the lagged interest rule (4) and consider a stationary MSV solution (i.e., one satisfying (35)) with $b_{xr} < 0$ and $b_{\pi r} < 0$. Let $\sigma\varphi_x +$*

[14] We are unable to prove this result analytically, that is, that $\varphi_r \geq 1$ implies $b_{xr} < 0$ *and* $b_{\pi r} < 0$. However, this can be easily checked numerically for plausible values of parameters (including the baseline values in Table 1) and is the basis for Proposition 7 below.

$(\beta+\kappa\sigma-1)\varphi_\pi \geq 0$ and

$$\varphi_r^+ \equiv 2^{-1}\beta^{-1}[1+\beta+\kappa\sigma+\sqrt{(1+\beta+\kappa\sigma)^2-4\beta}] > 1. \tag{36}$$

Then if $\varphi_r \geq Max\{\beta+\kappa\sigma, \varphi_r^+\}$, the necessary and sufficient condition for E-stability is[15]

$$-[(1-\beta)\varphi_x+\kappa\varphi_\pi]\varphi_r^{-1}b_{xr} - \sigma(\varphi_x+\kappa\varphi_\pi)\varphi_r^{-1}b_{\pi r} > \kappa\sigma. \tag{37}$$

We stress that $b_{xr} < 0$ and $b_{\pi r} < 0$ *per se* do not suffice for condition (37) to be satisfied. As it turns out, numerically, the determinate MSV solutions with super-inertial rules satisfy condition (37) and all such solutions are E-stable. In other words, an increase in interest rate should exert a strong dampening influence on inflation and output to reduce inflationary pressures in the economy and enable a return to the REE of interest.

4.2. Forward expectations in the policy rule.

The system under learning. With forward expectations the complete system is given by equations (1), (2), (3), and (5). We analyze E-stability of the MSV solution. After defining the vector of endogenous variables, $y_t = (x_t, \pi_t, r_t)'$, we put our system in the form $y_t = \Omega \hat{E}_t y_{t+1} + \delta y_{t-1} + \varkappa r_t^n$, where Ω and δ are given by

$$\Omega = \begin{bmatrix} \sigma(\sigma^{-1}-\varphi_x) & \sigma(1-\varphi_\pi) & 0 \\ \kappa\sigma(\sigma^{-1}-\varphi_x) & \sigma(\kappa+\beta\sigma^{-1}-\kappa\varphi_\pi) & 0 \\ \varphi_x & \varphi_\pi & 0 \end{bmatrix}, \tag{38}$$

$$\delta = \begin{bmatrix} 0 & 0 & -\sigma\varphi_r \\ 0 & 0 & -\kappa\sigma\varphi_r \\ 0 & 0 & \varphi_r \end{bmatrix}. \tag{39}$$

The MSV solutions take the same form (15) as in the case of lagged data, and the analysis of learning is also the same. Hence, assuming that the time t information set is $(1, y_{t-1}', r_t^n)'$, E-stability of any MSV solution requires that the eigenvalues of the matrices (23), (24) and (25) have negative real parts.

A quantitative case. Figure 2 illustrates how, even for this case where the policymakers are reacting to expectations of future inflation deviations and output gaps, policy inertia tends to enhance the prospects for determinacy and learnability of a REE. For low values of φ_r, such as the value $\varphi_r = 0.1$ in the first panel, we again find that

[15] We note that $\beta+\kappa\sigma > 1$, which suffices for $\sigma\varphi_x+(\beta+\kappa\sigma-1)\varphi_\pi \geq 0$, is generally satisfied for plausible values of structural parameters since the discount factor β is close to 1 (including the baseline values). In addition, for the baseline values, $\varphi_r^+ = 1.48$. We conjecture that the condition $\varphi_r \geq Max\{\beta+\kappa\sigma, \varphi_r^+\}$ can be weakened.

FIGURE 2. Forward Expectations

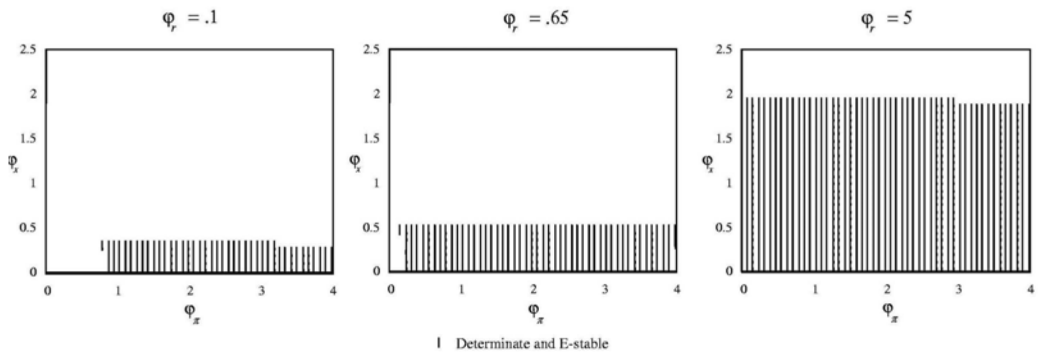

Figure 2: For small values of φ_r, forward-looking policy rules generate determinacy and learnability provided $\varphi_\pi > 1$ and φ_x is sufficiently small. For $\varphi_r = .65$, a larger region of the (φ_π, φ_x) space pictured is associated with both determinacy and learnability. Large values of φ_r generate relatively large regions of determinacy and learnability in (φ_π, φ_x) space.

active Taylor-type rules with little or no reaction to other variables are associated with both determinacy and learnability of equilibrium. However, the large region in the figure which is not associated with determinacy might be enough to limit recommendations of such rules because of uncertainty about parameter values. The second and third panels of Figure 2 show that increased policy inertia can mitigate such concerns, creating a larger region of determinacy, and in addition, that in these cases determinate equilibria are also learnable.

Intuition and analytics. We now provide some intuition and analytics for the phenomena illustrated in Figure 2. As before, it is the MSV solution for $\bar{b}$ which is crucial for E-stability. To gain further understanding, we first explore the type of stationary solutions permissible. Since it is only the lagged interest rate which appears in the model, the MSV solutions written explicitly take the form

$$x_t = b_x r_{t-1} + ..., \tag{40}$$

$$\pi_t = b_\pi r_{t-1} + ..., \tag{41}$$

$$r_t = b_r r_{t-1} + ..., \tag{42}$$

where b_x, b_π, and b_r are to be determined by solving the system of equations (16); see Appendix E for details. Assuming that $Det[I - \Omega\bar{b}] = 1 - b_x\varphi_x - b_\pi\varphi_\pi \neq 0$, the solution for b_r is given by

$$b_r = \varphi_r(1 - b_x\varphi_x - b_\pi\varphi_\pi)^{-1}. \tag{43}$$

We first consider a necessary condition for an MSV solution to be E-stable which is proved in Appendix F.

Proposition 8. *A necessary condition for E-stability of the MSV solution, (40)-(42), associated with the forward looking interest rule (5) is that* $b_x\varphi_x + b_\pi\varphi_\pi < 1$ *(which is equivalent to* $b_r > 0$ *by (43)).*

Once again, E-stability imposes restrictions on the parameters involved in the MSV solution, independently of the degree of inertia in the policy rule. In particular, it imposes the restriction that a rise in the current interest rate should necessarily lead to a rise in the future interest rate in the MSV solution. Intuitively, when $b_r > 0$, an (unexpected) rise in inflationary pressures causes the interest rate to rise today, which in turn raises the interest rate tomorrow. This rise creates downward pressure on aggregate demand and inflation reducing the inflationary pressures and pushing the economy back towards the REE. If instead $b_r < 0$, then the rise in the interest rate reduces the rate tomorrow and raises the output gap or inflation (since at least one of b_x or b_π must then be positive) which worsens the inflationary pressures pushing the economy further away from the REE. This intuition is similar to that associated with the lagged data rule.

As in the case of lagged data, we examine the type of MSV solutions that can be stationary and compare them with the E-stability conditions. The following proposition is proved in Appendix E.

Proposition 9. *Assume that* $\varphi_r \geq 1$. *The MSV solution, (40)-(42), associated with the forward looking interest rule (5), is stationary if and only if either of the following conditions hold:*

$$b_x\varphi_x + b_\pi\varphi_\pi > 1 + \varphi_r \text{ (which implies } b_r < 0), \tag{44}$$

$$b_x\varphi_x + b_\pi\varphi_\pi < 1 - \varphi_r \text{ (which implies } b_r > 0). \tag{45}$$

In other words, even with a high degree of inertia, a stationary MSV solution is *a priori* compatible with either $b_r < 0$ or $b_r > 0$. Such a stationary MSV solution could either be in the determinate or indeterminate region of the parameter space. But as in the case of lagged data, a stationary solution with $b_r < 0$ implies a perverse relation in the sense explained above. Proposition 8 immediately shows that the stationary MSV solutions possible under Proposition 9 when $b_r < 0$ are always E-unstable. Such solutions do exist in the *indeterminate* region of the parameter space as will be shown below. Hence, the only stationary MSV solutions which *can* be E-stable when $\varphi_r \geq 1$ are the ones with $b_r > 0$.

To gain further intuition, consider the case when $\varphi_x = 0$. Appendix E provides the details in this case. The MSV solution(s) for b_π are then given by a cubic polynomial. A negative solution for b_π exists (implying $b_r > 0$) which satisfies (83) when $\varphi_r + \varphi_\pi > 1$. Also, if condition (8) in Proposition 4 is violated (with $\varphi_x = 0$), then another stationary solution for b_π exists that has $b_\pi > 0$ (implying $b_r < 0$) satisfying condition (82). The latter solution is E-unstable by Proposition 8.

A determinate solution under the conditions given in Proposition 4 involves $b_\pi < 0$, $b_x < 0$, and $0 < b_r < 1$. Super-inertial rules, therefore, cause the determinate REE to have the property that a rise in the lagged interest rate of one percentage point causes a rise in the current interest rate of less than one percent, that is, $0 < b_r < 1$. In other words, a high degree of inertia rules out the stationary MSV solutions with $b_r < 0$ that would necessarily be E-unstable by Proposition 8, and instead only permits stationary solutions with $0 < b_r < 1$ which can be E-stable.

Appendix F shows that the determinate MSV solution is E-stable when φ_r is large enough. First, we recall Proposition 4 which stated that if $\varphi_r \geq 1$, then condition (8) is necessary and sufficient for determinacy. In particular, Appendix F proves the following:

Proposition 10. *Assume that $\varphi_x = 0$ and that the conditions in Proposition 4 for determinacy hold i.e., that $\varphi_r \geq 1$ and condition (8) holds for the forward rule (5). Then if $\varphi_r \geq Max\{1, \beta + \kappa\sigma\}$, the determinate equilibria are E-stable.*

The intuition behind this result follows from our discussion. A high degree of inertia forces the *determinate* MSV solution to have the property that $b_\pi < 0$, $b_x < 0$, and $0 < b_r < 1$ which results in E-stability.[16]

[16] When $\varphi_x > 0$, it is easy to check numerically that if the policy rule is super-inertial, then the determinate solutions involve $b_\pi < 0$, $b_x < 0$, and $0 < b_r < 1$ which then implies E-stability of the determinate MSV solution.

4.3. Alternative policy rules. Increasing the degree of monetary policy inertia appears to also be associated with learnability of rational expectations equilibrium in our setting. We considered only two types of (albeit plausible) interest rate rules primarily because of space constraints. However, similar results extend to other rules not reported here. In particular, this is true for rules responding to contemporaneous values of inflation, output, and the lagged interest rate as well as to contemporaneous expectations of inflation and output and the lagged interest rate—in either case, a high degree of inertia results in E-stability of the determinate REE.[17]

5. The infinite horizon model

It has been standard in the learning literature since Marcet and Sarget (1989) to replace rational expectations agents with adaptive learners. In this paper we follow this standard, taking Woodford's basic two equations for output and inflation under rational expectations (RE) and replacing RE by arbitrary subjective expectations in equations (1) and (2). Honkapohja, Mitra, and Evans (2003) call this the Euler equation approach and note that only one period ahead forecasts of output and inflation matter. They discuss how the Euler equation approach can be a plausible model of bounded rationality.

Recently, Preston (2005a,b) has argued for an alternative representation of Woodford's model under arbitrary subjective expectations. Preston derives the model under bounded rationality starting at the individual household and firm levels and obtains equations for agents' consumption and price setting that depend on forecasts into the entire infinite future; Honkapohja, Mitra, and Evans (2003) call this the infinite horizon approach. This is an interesting contribution, and we now turn to a discussion of how our results concerning inertia might be affected by taking this alternative viewpoint on bounded rationality.

Preston (2005a) shows that his model can be reduced to the following two equations for output and inflation

$$x_t = \hat{E}_t \left\{ \sum_{T=t}^{\infty} \beta^{T-t} [(1-\beta) x_{T+1} - \sigma(r_T - \pi_{T+1}) + r_T^n] \right\}, \tag{46}$$

[17] As discussed in Evans and Honkapohja (2001), E-stability conditions are in general sensitive to the information agents use in forming their forecasts. However, if we assume instead that agents use contemporaneous values of inflation and output in forming their forecasts (which McCallum (1999) would label *non-operational*, since such information is not normally available in actual economies), then a high degree of inertia continues to result in E-stability of the determinate REE.

$$\pi_t = \kappa x_t + \hat{E}_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} [\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1})] \right\}. \tag{47}$$

To these equations, one can adjoin an equation for the interest rate rule such as (4) or (5). We will conduct our analysis for the model (46) and (47) using the forward looking rule, (5), for illustrative purposes. For simplicity, we also assume that r_t^n is *i.i.d.* The MSV solution for the model continues to be of the form (15). Agents estimate a linear model

$$y_t = a_t + b_t y_{t-1} + c_t r_t^n + \epsilon_t$$

as before, where $y_t = (x_t, \pi_t, r_t)'$, ϵ_t is the error term, and a_t, b_t, and c_t are estimated from actual data. For the E-stability analysis, we may assume the estimated parameters to be time independent so that the PLM continues to be of the form (18) where

$$a = \begin{bmatrix} a_x \\ a_\pi \\ a_r \end{bmatrix}, b = \begin{bmatrix} 0 & 0 & b_x \\ 0 & 0 & b_\pi \\ 0 & 0 & b_r \end{bmatrix}, c = \begin{bmatrix} c_x \\ c_\pi \\ c_r \end{bmatrix}. \tag{48}$$

Agents compute the forecasts required in (46) and (47) by the following formula ($T \geq t+1$)

$$\hat{E}_t y_T = (I-b)^{-1}(I - b^{T-t+1})a + b^{T-t+1} y_{t-1} + b^{T-t} c r_t^n. \tag{49}$$

Using these forecasts, the ALM can be computed as

$$y_t = X_a + X_b y_{t-1} + X_c r_t^n. \tag{50}$$

where the coefficients in (50) are defined in Appendix G. The parameters a, b, c in the PLM (18) are mapped into the parameters X_a, X_b, and X_c in the ALM (50) and the fixed points of this map correspond to the MSV solution. The mapping from the PLM to the ALM can be analyzed for E-stability as before.

There are two scenarios which can be considered as discussed in Preston (2005b). The first scenario is where agents know the policy rule (5) and use this rule to form forecasts. The output gap equation is then given by (117) and the whole system by (118) in Appendix G. For the baseline Woodford parameters, the beneficial effects of inertia extend to the infinite horizon model as in the Euler equation model. Large degrees of inertia in the interest rule lead to determinacy and E-stability.

There is an alternative scenario which can be considered as discussed in Preston (2005b). This is the case where agents do not know (understand) the interest rate rule (5) and accordingly form forecasts of the interest rate from their PLM. The output gap equation in this case is given by (119) and the whole system by (120) in Appendix G. The results now change dramatically. Inertia is no longer sufficient to guarantee E-stability. One has E-instability even with large degrees of inertia. These results are similar in flavor to those in Preston (2005b). He too finds that certain forecast-based interest rules lead to E-stability when agents are endowed with knowledge of the policy rule and to E-instability when agents do not have this knowledge. As Preston discusses in some detail, this points to the virtues of transparency in policy formulation which has been emphasized in the recent literature of monetary policy.

6. Conclusion

Two key issues for the evaluation of monetary policy rules are whether they induce a determinate rational expectations equilibrium or not, and whether that equilibrium is learnable or not. We provide analytical results which indicate how an increased degree of interest rate smoothing may induce both determinacy and learnability of rational expectations equilibrium in a widely-used model of monetary policy. This is true across both of our specifications of monetary policy rules—a finding which we believe substantially alters the evaluation of these rules. Consequently, neither of these classes of policy rules—which might be considered particularly realistic in terms of actual central bank behavior—should be deemed undesirable on account of determinacy or learnability questions, once policy inertia is taken into account.

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7. Appendices

7.1. APPENDIX A (Determinacy with Lagged Data Rule) . Woodford (2003) provides necessary and sufficient conditions for determinacy of the system under a lagged data policy rule. Appendix C, Proposition C.2, in Woodford (2003) lists three possible sets of (mutually disjoint) conditions in terms of the characteristic polynomial of B_1 under which determinacy obtains. Specifically, he shows that a 3×3 matrix has exactly one eigenvalue inside the unit circle and the remaining two outside if and only if one of three cases holds. The cases are labelled I, II, and III. We will apply these conditions to B_1. First, note that the characteristic polynomial of B_1 (given in (6)), $p(\lambda)$, is given by

$$p(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 \tag{51}$$

where $A_2 = -(1+\beta^{-1}+\beta^{-1}\kappa\sigma+\varphi_r)$, $A_1 = \beta^{-1}+(1+\beta^{-1}+\beta^{-1}\kappa\sigma)\varphi_r - \sigma\varphi_x$, and $A_0 = \beta^{-1}\sigma(\kappa\varphi_\pi+\varphi_x-\sigma^{-1}\varphi_r)$. We have $p(1) = 1+A_2+A_1+A_0$ and $p(-1) = -1+A_2-A_1+A_0$, so that

$$p(1) = \beta^{-1}\sigma[\kappa(\varphi_\pi+\varphi_r-1)+(1-\beta)\varphi_x], \tag{52}$$

$$p(-1) = \beta^{-1}\sigma[\kappa(\varphi_\pi-\varphi_r-1)+(1+\beta)\varphi_x - 2\sigma^{-1}(1+\beta)(1+\varphi_r)]. \tag{53}$$

Conditions $(C.13)$ and $(C.14)$ in Woodford (2003) are necessary for both Cases II and III (and they also rule out Case I). These two conditions are that $p(1) > 0$ and $p(-1) < 0$, which reduce to (7) and (8) in the text. This proves Proposition 1.

The conditions required for Case III of Woodford are $(C.13)$, $(C.14)$, and $(C.17)$, i.e., $|A_2| > 3$. The final condition corresponds to (10) since

$$|A_2| = 1 + \beta^{-1} + \beta^{-1}\kappa\sigma + \varphi_r > 3 \tag{54}$$

iff condition (10) holds. This supplies the details behind Proposition 2.

We also state a proposition (without proof) which provides the required conditions for determinacy *when the degree of inertia is low.*

Proposition 11. *Assume that* $\kappa(\varphi_\pi + \varphi_r - 1) + (1-\beta)\varphi_x < 0$ *for the inertial lagged data interest rule (4). Then the necessary and sufficient condition for determinacy is*

$$[\kappa\sigma + 2(1+\beta)]\varphi_r + 2(1+\beta) < \sigma[\kappa(\varphi_\pi - 1) + (1+\beta)\varphi_x]. \tag{55}$$

Note that condition (55) represents violation of condition (9) in Proposition 1. In particular, it can be shown that if $\varphi_r = 0$, the necessary and sufficient condition for determinacy is given by[18]

$$(1+\beta)^{-1}[\kappa(1-\varphi_\pi) + 2(1+\beta)\sigma^{-1}] < \varphi_x < (1-\beta)^{-1}\kappa(1-\varphi_\pi). \tag{56}$$

7.2. APPENDIX B (MSV Solution for Non-Inertial Lagged Data Rule). In this case, assuming that $D \equiv b_{xx}(1-\beta b_{\pi\pi}) + (\beta+\kappa\sigma)b_{\pi\pi} + \beta b_{x\pi}b_{\pi x} + \kappa b_{x\pi} + \sigma b_{\pi x} - 1 \neq 0$, the MSV parameter values are given by the solution to the equations $b_{xx} = [(1-\beta b_{\pi\pi})\sigma\varphi_x]D^{-1}$, $b_{x\pi} = [(1-\beta b_{\pi\pi})\sigma\varphi_\pi]D^{-1}$, $b_{\pi x} = [(\kappa+\beta b_{\pi x})\sigma\varphi_x]D^{-1}$, and $b_{\pi\pi} = [(\kappa+\beta b_{\pi x})\sigma\varphi_\pi]D^{-1}$. These four equations yield $b_{x\pi} = \varphi_\pi\varphi_x^{-1}b_{xx}$ and $b_{\pi x} = \varphi_x\varphi_\pi^{-1}b_{\pi\pi}$ so that this system can be reduced to two (nonlinear) equations in two unknowns which can be solved numerically. In general, there are three solutions for $\bar{b}$ of which exactly one is stationary in the determinate region.

7.3. APPENDIX C (MSV Solution for Inertial Lagged Data Rule). We now consider the situation when $\varphi_r > 0$ in the lagged data rule. Assuming that $I - \Omega\bar{b}$ is invertible, we solve the system $\bar{b} = (I - \Omega\bar{b})^{-1}\delta$ for the MSV solution, with $\bar{b}$ a 3×3 matrix in this case. Using Mathematica, one can verify that the MSV $\bar{b}$ solution takes the form given in (31), with $b_{xx} = \varphi_x\varphi_r^{-1}b_{xr}$, $b_{x\pi} = \varphi_\pi\varphi_r^{-1}b_{xr}$, $b_{\pi x} = \varphi_x\varphi_r^{-1}b_{\pi r}$, and

[18] This explains the (triangular) determinate region in the left hand panel of Figure 1 involving values of $\varphi_\pi < 1$ which violates the Taylor principle.

$b_{\pi\pi} = \varphi_\pi \varphi_r^{-1} b_{\pi r}$. The two nonlinear equations determining b_{xr} and $b_{\pi r}$ are given by

$$b_{xr} = \varphi_r[b_{xr}\varphi_r + \sigma\{b_{\pi r}(\beta\varphi_\pi + \varphi_r) - \varphi_r\}]E^{-1}, \tag{57}$$

$$b_{\pi r} = \varphi_r[\kappa\varphi_r(b_{xr} - \sigma) + b_{\pi r}\{\kappa\sigma\varphi_r + \beta(\varphi_r - \sigma\varphi_x)\}]E^{-1}, \tag{58}$$

where $E \equiv \varphi_r - (\kappa\varphi_\pi + \varphi_x)b_{xr} - \{\beta\varphi_\pi + \sigma(\kappa\varphi_\pi + \varphi_x)\}b_{\pi r}$.

7.4. APPENDIX D (*E*-stability of Inertial Lagged Data Rule)[19] . The matrix $\Omega + \Omega\bar{b} - I$ has one eigenvalue of -1, and the remaining two are given by solutions to $\eta^2 + \eta a_1 + a_2 = 0$, where

$$a_1 = 1 - \beta - \kappa\sigma - \varphi_r^{-1}[(\varphi_x + \kappa\varphi_\pi)b_{xr} + \{\sigma\varphi_x + (\beta + \kappa\sigma)\varphi_\pi\}b_{\pi r}], \tag{59}$$

$$a_2 = -[(1-\beta)\varphi_x + \kappa\varphi_\pi]\varphi_r^{-1}b_{xr} - \sigma(\varphi_x + \kappa\varphi_\pi)\varphi_r^{-1}b_{\pi r} - \kappa\sigma. \tag{60}$$

Hence, the necessary and sufficient conditions for $\Omega + \Omega\bar{b} - I$ to have eigenvalues with negative real parts are that $a_1 > 0$ and $a_2 > 0$. We next look at the 9×9 matrix $\bar{b}' \otimes \Omega + I \otimes \Omega\bar{b} - I$. Using Mathematica, one can verify that five of the eigenvalues are -1 and two of the remaining four are given by solutions to

$$\varphi_r^{-1}[(\varphi_x + \kappa\varphi_\pi)b_{xr} + \{\sigma\varphi_x + (\beta + \kappa\sigma)\varphi_\pi\}b_{\pi r}] - 1 = -\beta - \kappa\sigma - a_1, \tag{61}$$

where the right-hand equality above uses the expression (59). We conclude that $a_1 > 0$ implies that the eigenvalues (61) are negative, as required for E-stability. The final two eigenvalues of $\bar{b}' \otimes \Omega + I \otimes \Omega\bar{b} - I$ are given by the solution of $\eta^2 + \eta c_1 + c_2 = 0$, where

$$c_1 = -\varphi_r^{-1}[b_{xr}\{(2 + \beta + \kappa\sigma)\varphi_x + \kappa\varphi_\pi\} \tag{62}$$

$$+b_{\pi r}\{\sigma\varphi_x + (1 + 2\beta + 2\kappa\sigma)\varphi_\pi\} + \varphi_r\{(1 + \beta + \kappa\sigma)\varphi_r - 2\}], \tag{63}$$

$$c_2 = \varphi_r^{-2}[2\beta(\varphi_x b_{xr} + \varphi_\pi b_{\pi r})^2 + 3\beta(\varphi_x b_{xr} + \varphi_\pi b_{\pi r})\varphi_r^2$$

$$+\varphi_r^2\{\beta\varphi_r^2 - (1 + \beta + \kappa\sigma)\varphi_r + 1\} - b_{xr}\varphi_r\{\kappa\varphi_\pi + (2 + \beta + \kappa\sigma)\varphi_x\} \tag{64}$$

$$-b_{\pi r}\varphi_\pi(1 + 2\beta + 2\kappa\sigma)\varphi_r - \sigma\varphi_x\varphi_r b_{\pi r}]. \tag{65}$$

For E-stability we require $c_1 > 0$ and $c_2 > 0$. We finally look at the matrix $\rho\Omega + \Omega\bar{b} - I$ which has one eigenvalue equal to -1 and the remaining two given by the solutions to

[19] A Mathematica program which computes these E-stability conditions is available from the authors on request.

$\eta^2 + \eta a_{1\rho} + a_{2\rho} = 0$, where

$$\begin{aligned} a_{1\rho} &= 2 - \rho(1+\beta+\kappa\sigma) - (\varphi_x + \kappa\varphi_\pi)\varphi_r^{-1} b_{xr} - \\ &\quad \{\sigma\varphi_x + (\beta+\kappa\sigma)\varphi_\pi\}\varphi_r^{-1} b_{\pi r} \\ &= a_1 + (1-\rho)(1+\beta+\kappa\sigma), \end{aligned} \tag{66}$$

$$\begin{aligned} a_{2\rho} &= (1-\rho)(1-\beta\rho) - \rho\kappa\sigma - \{(1-\beta\rho)\varphi_x + \kappa\varphi_\pi\}\varphi_r^{-1} b_{xr} \\ &\quad -\{\sigma(\varphi_x + \kappa\varphi_\pi) + \beta(1-\rho)\varphi_\pi\}\varphi_r^{-1} b_{\pi r}, \end{aligned} \tag{67}$$

and for E-stability we require both $a_{1\rho} > 0$ and $a_{2\rho} > 0$. The right hand equality in (66) uses the expression for a_1 from (59) which, therefore, shows that $a_1 > 0$ implies that $a_{1\rho} > 0$ (since $0 < \rho < 1$). In summary, the necessary and sufficient conditions for E-stability given in (23), (24), and (25) reduce to the coefficients a_1, a_2, c_1, c_2, and $a_{2\rho}$ all being positive.

Details for Proposition 7. We first note that

$$a_1 - a_2 = 1 - \beta - \beta\varphi_r^{-1}(\varphi_x b_{xr} + \varphi_\pi b_{\pi r}) > 1 - \beta - \beta\varphi_r^{-1}(1-\varphi_r)\varphi_r \tag{68}$$

where the right hand inequality in (68) follows from the solution being stationary and $\varphi_r \geq 1$, i.e., (the right hand inequality in) condition (35). Hence, $\varphi_r \geq 1$ implies that $a_1 > a_2$ from (68) and hence $a_2 > 0$ implies $a_1 > 0$. Similarly, comparing term by term, it can be checked that $a_{2\rho} > a_2$ since $b_{xr} < 0$, $b_{\pi r} < 0$ and $0 < \rho < 1$. So $a_2 > 0$ also implies that $a_{2\rho} > 0$. The required necessary and sufficient conditions for E-stability have now reduced to $a_2 > 0, c_1 > 0$, and $c_2 > 0$. We now examine c_2. Since $\varphi_r > 0$, the sign of c_2 is determined by the expression within parentheses in (64). The first two terms within this parentheses can be combined together as

$$2\beta(\varphi_x b_{xr} + \varphi_\pi b_{\pi r})^2 + 3\beta(\varphi_x b_{xr} + \varphi_\pi b_{\pi r})\varphi_r^2 = \beta(\varphi_x b_{xr} + \varphi_\pi b_{\pi r})[3\varphi_r^2 + 2(\varphi_x b_{xr} + \varphi_\pi b_{\pi r})]. \tag{69}$$

We show that the expression (69) is positive since each of the individual terms in parentheses on the right hand side of (69) is negative. The first term, $\beta(\varphi_x b_{xr} + \varphi_\pi b_{\pi r})$, in (69) is negative by condition (35) when $\varphi_r \geq 1$. The second term in (69) is also negative since

$$\varphi_x b_{xr} + \varphi_\pi b_{\pi r} < -\frac{3}{2}\varphi_r^2 < (1-\varphi_r)\varphi_r, \tag{70}$$

where the final inequality in (70) again uses (35). The inequalities $b_{xr} < 0$ and $b_{\pi r} < 0$ then imply that the final three terms within the parentheses in (64) are positive. Hence, a sufficient condition for $c_2 > 0$ is that $g(\varphi_r) \equiv \beta\varphi_r^2 - (1+\beta+\kappa\sigma)\varphi_r + 1 \geq 0$. Since $g(0) > 0$ and $g(1) < 0$, $g(\varphi_r) = 0$ has two positive roots, one between 0 and 1, and the other more than 1. The root exceeding one is given by

$$\varphi_r^+ \equiv 2^{-1}\beta^{-1}[1+\beta+\kappa\sigma+\sqrt{(1+\beta+\kappa\sigma)^2-4\beta}]. \tag{71}$$

In addition, $g(\varphi_r) > 0$ for all $\varphi_r > \varphi_r^+$ since $g(\infty) = \infty$. This proves that $c_2 > 0$ when $\varphi_r \geq \varphi_r^+$.

Now $c_1 > 0$ iff the expression within the parentheses in (62) is *negative*. The first two terms of this parentheses can be grouped together as

$$\begin{aligned}
& b_{xr}\{(2+\beta+\kappa\sigma)\varphi_x+\kappa\varphi_\pi\}+b_{\pi r}\{\sigma\varphi_x+(1+2\beta+2\kappa\sigma)\varphi_\pi\} \qquad (72)\\
= \; & (2+\beta+\kappa\sigma)(\varphi_x b_{xr}+\varphi_\pi b_{\pi r})+(\beta+\kappa\sigma-1)\varphi_\pi b_{\pi r}+\sigma\varphi_x b_{\pi r}+\kappa\varphi_\pi b_{xr}\\
< \; & (2+\beta+\kappa\sigma)(1-\varphi_r)\varphi_r+[\sigma\varphi_x+(\beta+\kappa\sigma-1)\varphi_\pi]b_{\pi r}+\kappa\varphi_\pi b_{xr}
\end{aligned}$$

where the final inequality uses condition (35). Using this we can conclude the following about the expression within the parentheses of c_1

$$\begin{aligned}
& b_{xr}\{(2+\beta+\kappa\sigma)\varphi_x+\kappa\varphi_\pi\}+\\
& b_{\pi r}\{\sigma\varphi_x+(1+2\beta+2\kappa\sigma)\varphi_\pi\}+\varphi_r\{(1+\beta+\kappa\sigma)\varphi_r-2\}\\
< \; & (2+\beta+\kappa\sigma)(1-\varphi_r)\varphi_r+\\
& [\sigma\varphi_x+(\beta+\kappa\sigma-1)\varphi_\pi]b_{\pi r}+\kappa\varphi_\pi b_{xr}+\varphi_r\{(1+\beta+\kappa\sigma)\varphi_r-2\}\\
= \; & \varphi_r(\beta+\kappa\sigma-\varphi_r)+[\sigma\varphi_x+(\beta+\kappa\sigma-1)\varphi_\pi]b_{\pi r}+\kappa\varphi_\pi b_{xr}. \qquad (73)
\end{aligned}$$

If $\sigma\varphi_x + (\beta+\kappa\sigma-1)\varphi_\pi \geq 0$ and $\varphi_r \geq \beta+\kappa\sigma$, then the above expression is negative provided $b_{xr} < 0$ and $b_{\pi r} < 0$. This proves that $c_1 > 0$. The only remaining condition required is $a_2 > 0$ which is given in the proposition.

7.5. APPENDIX E (MSV Solution of Forward Rule). We first consider the nature of the MSV solution. Equations (16) involve three equations in the three unknowns b_x, b_π, and b_r. Assuming that $Det[I-\Omega\bar{b}] = 1 - b_x\varphi_x - b_\pi\varphi_\pi \neq 0$, the third equation determines b_r once b_x and b_π are known from the first two equations. The equation for b_r is given by (43). The first two equations (which can be verified using Mathematica) are

$$b_x = \varphi_r[b_x+(b_\pi-1)\sigma][1-b_x\varphi_x-b_\pi\varphi_\pi]^{-1}, \tag{74}$$

$$b_\pi = \varphi_r[\kappa(b_x-\sigma)+(\beta+\kappa\sigma)b_\pi][1-b_x\varphi_x-b_\pi\varphi_\pi]^{-1}. \tag{75}$$

These two equations yield the following simultaneous system in b_x and b_π:

$$\varphi_x b_x^2 + (b_\pi \varphi_\pi + \varphi_r - 1) b_x + (b_\pi - 1)\sigma\varphi_r = 0, \quad (76)$$

$$\varphi_\pi b_\pi^2 + [b_x \varphi_x + (\beta + \kappa\sigma)\varphi_r - 1] b_\pi + \kappa\varphi_r (b_x - \sigma) = 0. \quad (77)$$

One can solve for b_x in terms of b_π from equation (77) which yields

$$b_x = [\kappa\sigma\varphi_r + \{1 - (\beta + \kappa\sigma)\varphi_r\} b_\pi - b_\pi^2 \varphi_\pi](\kappa\varphi_r + b_\pi\varphi_x)^{-1} \quad (78)$$

and substituting equation (78) into equation (76) yields a (cubic) polynomial in b_π whose roots yield the MSV solutions for b_π. Once b_π is determined, b_x can be determined from (78) and finally b_r from (43).

We next examine what type of MSV solutions can be stationary with the forward rule. Stationarity requires that $|b_r| < 1$.[20] We consider three mutually exclusive cases for stationarity, namely

$$0 < b_x\varphi_x + b_\pi\varphi_\pi < 1, \quad (79)$$

$$b_x\varphi_x + b_\pi\varphi_\pi > 1, \quad (80)$$

$$b_x\varphi_x + b_\pi\varphi_\pi < 0. \quad (81)$$

Under case (79), stationarity is ruled out when $\varphi_r \geq 1$ since $b_r > 1$ from (43). Case (80), i.e., $b_x\varphi_x + b_\pi\varphi_\pi > 1$, is permissible only when at least one of b_x or b_π is *positive* (when $\varphi_x, \varphi_\pi > 0$) at the MSV solution. Furthermore, $b_x\varphi_x + b_\pi\varphi_\pi > 1$ implies that $b_r < 0$ by (43) and stationarity requires that

$$b_x\varphi_x + b_\pi\varphi_\pi > 1 + \varphi_r \quad (82)$$

Note that condition (82) cannot *a priori* be ruled out for a stationary MSV solution even when $\varphi_r \geq 1$. The final case, condition (81), is permissible only when at least one of b_x or b_π is *negative* at the MSV solution. In addition, $b_x\varphi_x + b_\pi\varphi_\pi < 0$ implies that $b_r > 0$ from (43) and stationarity is equivalent to the requirement that

$$b_x\varphi_x + b_\pi\varphi_\pi < 1 - \varphi_r\ . \quad (83)$$

These results are collected in Proposition 9.

[20] In matrix form, the $\bar{b}$ solution for the forward rule (5) has only zeros in the first two columns and the third column has b_x, b_π, and b_r, respectively. Hence, two of the eigenvalues of $\bar{b}$ are 0 and the third is b_r.

Details for the case when $\varphi_x = 0$. When $\varphi_x = 0$, we substitute (78) into (76) and the cubic polynomial in b_π simplifies to

$$p(b_\pi\varphi_\pi) \equiv (b_\pi\varphi_\pi)^3 + (b_\pi\varphi_\pi)^2 d_1 + (b_\pi\varphi_\pi)d_2 + d_3 = 0 \tag{84}$$

where $d_1 = (1+\beta+\kappa\sigma)\varphi_r - 2$, $d_2 = 1+\beta\varphi_r^2 - [1+\beta+\kappa\sigma(\varphi_\pi+1)]\varphi_r$, and $d_3 = \kappa\sigma\varphi_\pi\varphi_r$. The characteristic polynomial, (84), evaluated at $b_\pi\varphi_\pi = (1-\varphi_r)$, yields

$$p(1-\varphi_r) = \kappa\sigma\varphi_r^2(\varphi_r+\varphi_\pi-1) \tag{85}$$

so that $p(1-\varphi_r) > 0$ for all $\varphi_r + \varphi_\pi > 1$. This means that there exists a negative root b_π which satisfies (83) since $p(-\infty) = -\infty$. If the solution is determinate (say) under the conditions given in Proposition 4, then this is also the uniquely stationary solution. On the other hand, the characteristic polynomial, (84), evaluated at $b_\pi\varphi_\pi = (1+\varphi_r)$, yields

$$p(1+\varphi_r) = \varphi_r^2[\{\kappa\sigma + 2(1+\beta)\}\varphi_r + 2(1+\beta) - \kappa\sigma(\varphi_\pi - 1)]. \tag{86}$$

From (86), observe that $p(1+\varphi_r) < 0$ when

$$\{\kappa\sigma + 2(1+\beta)\}\varphi_r + 2(1+\beta) < \kappa\sigma(\varphi_\pi - 1), \tag{87}$$

that is, precisely when condition (8) in Proposition 4 is violated (with $\varphi_x = 0$). This shows that when $\varphi_r + \varphi_\pi > 1$ and condition (87) is satisfied, there exist two stationary solutions for b_π, one with $b_\pi < 0$ satisfying condition (83) and the other with $b_\pi > 0$ satisfying condition (82).

Note that equation (74) implies that

$$b_x = b_r[b_x + (b_\pi - 1)\sigma] \tag{88}$$

which can be rearranged to give

$$b_x(1-b_r) = \sigma b_r(b_\pi - 1). \tag{89}$$

The inequality $b_\pi < 0$ implies that

$$0 < b_r = \varphi_r[1 - b_\pi\varphi_\pi]^{-1} < 1 \tag{90}$$

which in turn implies that $b_x < 0$. We note that

$$b_x = \sigma\varphi_r(1-b_\pi)[b_\pi\varphi_\pi + \varphi_r - 1]^{-1}. \tag{91}$$

7.6. APPENDIX F (*E*-stability of Forward Rule). We look at the three pairs of matrices required for checking E-stability.[21] We first start with the 9×9 matrix $\bar{b}' \otimes \Omega + I \otimes \Omega\bar{b} - I$ which must have eigenvalues with negative real parts for E-stability. Using Mathematica, one can verify that five of the eigenvalues are -1 and two of the remaining four are given by

$$b_x\varphi_x + b_\pi\varphi_\pi - 1. \tag{92}$$

A necessary condition for E-stability is, therefore, $b_x\varphi_x + b_\pi\varphi_\pi < 1$ which is equivalent to $b_r > 0$. This proves Proposition 8.

The final two eigenvalues of $\bar{b}' \otimes \Omega + I \otimes \Omega\bar{b} - I$ are given by the solutions to the characteristic polynomial $\eta^2 + \eta c_1 + c_2 = 0$, where

$$\begin{aligned} c_1 &= [(1 - b_x\varphi_x - b_\pi\varphi_\pi)^2 + (1 - b_x\varphi_x - b_\pi\varphi_\pi) - \{1 + \beta - \kappa\sigma(\varphi_\pi - 1) - \sigma\varphi_x\}\varphi_r]X_a; \\ X_a &\equiv (1 - b_x\varphi_x - b_\pi\varphi_\pi)^{-1}; \end{aligned} \tag{93}$$

$$c_2 = [(1 - b_x\varphi_x - b_\pi\varphi_\pi)^3 + \beta\varphi_r^2 + \varphi_r X_r](1 - b_x\varphi_x - b_\pi\varphi_\pi)^{-2}; \tag{94}$$

$$\begin{aligned} X_r \equiv\ & b_\pi[b_\pi\varphi_\pi(\sigma\varphi_x - \varphi_\pi) + \varphi_\pi\{2 + \beta + \kappa\sigma(1 - \varphi_\pi) - \sigma\varphi_x\} - \sigma\varphi_x] \\ & + b_x[b_x\varphi_x\{\kappa\varphi_\pi - (\beta + \kappa\sigma)\varphi_x\} + \varphi_x(1 + 2\beta + 2\kappa\sigma - \kappa\sigma\varphi_\pi - \sigma\varphi_x) - \kappa\varphi_\pi] \\ & + b_x b_\pi[\kappa\varphi_\pi^2 + \sigma\varphi_x^2 - (1 + \beta + \kappa\sigma)\varphi_x\varphi_\pi] - 1 - \beta + \kappa\sigma(\varphi_\pi - 1) - \sigma\varphi_x(\beta\varphi_r - 1). \end{aligned}$$

Necessary and sufficient conditions for the above polynomial to have negative real parts are that $c_1 > 0$ and $c_2 > 0$.

For E-stability we also need the eigenvalues of $\Omega + \Omega\bar{b} - I$ to have negative real parts. One eigenvalue of this matrix is -1 and the remaining two are given by the solutions to the characteristic polynomial $\eta^2 + \eta a_1 + a_2 = 0$, where

$$a_1 = (1 - b_\pi\varphi_\pi - b_x\varphi_x) - \beta + \kappa\sigma(\varphi_\pi - 1) + \sigma\varphi_x, \tag{95}$$

$$\begin{aligned} a_2 = \ & b_x[\varphi_x(\beta + \kappa\sigma - 1) - \kappa\varphi_\pi] - \sigma\varphi_x b_\pi \\ & + \sigma[\kappa(\varphi_\pi - 1) + (1 - \beta)\varphi_x]. \end{aligned} \tag{96}$$

The necessary and sufficient conditions for the above polynomial to have negative real parts are that $a_1 > 0$ and $a_2 > 0$.

[21] A Mathematica program which computes these E-stability conditions is available from the authors on request.

Finally, one also needs the eigenvalues of $\rho\Omega + \Omega\bar{b} - I$ to have negative real parts. One eigenvalue of this matrix is -1 and the remaining two are given by the solutions to the characteristic polynomial $\eta^2 + \eta a_{1\rho} + a_{2\rho} = 0$ where

$$\begin{aligned} a_{1\rho} &= (2 - b_\pi\varphi_\pi - b_x\varphi_x) - \rho[1 + \beta - \kappa\sigma(\varphi_\pi - 1) - \sigma\varphi_x], \quad (97) \\ a_{2\rho} &= 1 - \rho[1 + \beta - \kappa\sigma(\varphi_\pi - 1) - \sigma\varphi_x] + \beta\rho^2(1 - \sigma\varphi_x) + \\ &\quad b_x[\varphi_x\{\rho(\beta + \kappa\sigma) - 1\} - \rho\kappa\varphi_\pi] - b_\pi[(1 - \rho)\varphi_\pi + \rho\sigma\varphi_x], \quad (98) \end{aligned}$$

so that for E-stability one requires $a_{1\rho} > 0$ and $a_{2\rho} > 0$.

We conclude that the necessary and sufficient conditions for E-stability of any MSV solution in the case of forward rules requires that all of the coefficients $c_1, c_2, a_1, a_2, a_{1\rho}$, and $a_{2\rho}$ are positive *and* that $b_x\varphi_x + b_\pi\varphi_\pi < 1$.

Details for Proposition 10. We consider E-stability of the unique MSV solution (when $\varphi_x = 0$) which exists under the conditions given in Proposition 4, i.e., when $\varphi_r \geq 1$ and condition (8) is satisfied. As proved in Appendix E, this MSV solution has $b_\pi < 0$, $b_x < 0$, $0 < b_r < 1$, and satisfies (83). Note that condition (83) implies that the eigenvalue (92) is negative.

We first examine the coefficients c_1, c_2 in (93) and (94) involved in the eigenvalues of $\bar{b}' \otimes \Omega + I \otimes \Omega\bar{b} - I$. Consider c_1 in (93) first. Since $b_\pi < 0$, $X_a \equiv (1 - b_\pi\varphi_\pi)^{-1} > 0$, and E-stability requires the expression in parentheses of c_1 to be positive. This expression simplifies (when $\varphi_x = 0$) to

$$\begin{aligned} & 2 + (b_\pi\varphi_\pi)^2 - 3b_\pi\varphi_\pi - \varphi_r\{1 + \beta - \kappa\sigma(\varphi_\pi - 1)\} \\ > \; & 2 + (\varphi_r - 1)^2 + 3(\varphi_r - 1) - \varphi_r\{1 + \beta - \kappa\sigma(\varphi_\pi - 1)\} \\ = \; & \varphi_r^2 + \varphi_r - \varphi_r\{1 + \beta - \kappa\sigma(\varphi_\pi - 1)\} = \varphi_r[\varphi_r - \beta + \kappa\sigma(\varphi_\pi - 1)]. \quad (99) \end{aligned}$$

The first inequality above uses the fact that b_π satisfies (83), i.e., $b_\pi\varphi_\pi < 1 - \varphi_r$. Equation (99) shows that $\varphi_r \geq \beta + \kappa\sigma$ suffices to make $c_1 > 0$ for all $\varphi_\pi > 0$.

Next we turn to c_2. Since $(1 - b_\pi\varphi_\pi)^{-2} > 0$ by $b_\pi < 0$, E-stability requires the expression in parentheses of c_2 in (94) to be positive. This expression simplifies, after some manipulation, to (when $\varphi_x = 0$)

$$\begin{aligned} & \beta\varphi_r^2 + (1-b_\pi\varphi_\pi)^3 + (1-b_\pi\varphi_\pi)\varphi_r[\kappa\varphi_\pi(\sigma - b_x) + b_\pi\varphi_\pi - (1+\beta+\kappa\sigma)] \\ = \; & \beta\varphi_r^2 + \varphi_r^3 b_r^{-3} + \varphi_r^2 b_r^{-1}[\kappa\varphi_\pi(\sigma - b_x) + b_\pi\varphi_\pi - (1+\beta+\kappa\sigma)] \\ = \; & \varphi_r^2 b_r^{-1}[(\varphi_r b_r^{-2} - \beta)(1-b_r) + \kappa\sigma(\varphi_\pi - 1) - \kappa\varphi_\pi b_x], \end{aligned} \tag{100}$$

where we have used the value of $b_r = \varphi_r(1-b_\pi\varphi_\pi)^{-1}$ at the MSV solution from (90) and eliminated $b_\pi\varphi_\pi$ in the final line (100). If $\varphi_\pi \geq 1$, then $c_2 > 0$ for all $\varphi_r \geq \beta$ since $0 < b_r < 1$, and $b_x < 0$.

We consider further the situation when $\varphi_\pi < 1$. For this we substitute the value of b_x from (89) at the MSV solution in the final term of (100), i.e., $-\kappa\varphi_\pi b_x$, and write this in terms of b_r. Before doing this, we first note from (89) that

$$\begin{aligned} b_x(1-b_r) &= \sigma b_r(b_\pi - 1) = \sigma b_r(\varphi_\pi^{-1} - \varphi_r\varphi_\pi^{-1}b_r^{-1} - 1) \\ &= \sigma\varphi_\pi^{-1}[b_r(1-\varphi_\pi) - \varphi_r], \end{aligned} \tag{101}$$

where we have manipulated $b_r = \varphi_r(1-b_\pi\varphi_\pi)^{-1}$ (obtained from (90)) to get the final expression on the right hand side above in terms of b_r. Using (101), we finally obtain

$$-\kappa\varphi_\pi b_x = \kappa\sigma(1-b_r)^{-1}[(\varphi_\pi - 1)b_r + \varphi_r]. \tag{102}$$

Using (102), the expression within parentheses in (100) simplifies to

$$\begin{aligned} & (\varphi_r b_r^{-2} - \beta)(1-b_r) + \kappa\sigma(\varphi_\pi - 1) - \kappa\varphi_\pi b_x \\ = \; & (\varphi_r b_r^{-2} - \beta)(1-b_r) + \kappa\sigma(\varphi_\pi - 1) + \kappa\sigma(1-b_r)^{-1}[(\varphi_\pi - 1)b_r + \varphi_r] \\ = \; & \varphi_r[\kappa\sigma(1-b_r)^{-1} + (1-b_r)b_r^{-2}] + \kappa\sigma\varphi_\pi(1-b_r)^{-1} - \kappa\sigma(1-b_r)^{-1} - \beta(1-b_r), \end{aligned} \tag{103}$$

where the first two terms in the third line of (103) has grouped together terms involving φ_r and φ_π. Then $c_2 > 0$ iff the expression in the third line of (103) is positive. This will be so iff

$$\varphi_r[\kappa\sigma b_r^2 + (1-b_r)^2](1-b_r)^{-1}b_r^{-2} > \kappa\sigma(1-b_r)^{-1} - \kappa\sigma\varphi_\pi(1-b_r)^{-1} + \beta(1-b_r), \tag{104}$$

that is, iff (after multiplying both sides of the above equation by $(1-b_r)$),

$$\varphi_r[\kappa\sigma b_r^2 + (1-b_r)^2]b_r^{-2} > \kappa\sigma(1-\varphi_\pi) + \beta(1-b_r)^2, \tag{105}$$

$$\varphi_r > [\kappa\sigma b_r^2(1-\varphi_\pi) + \beta b_r^2(1-b_r)^2][\kappa\sigma b_r^2 + (1-b_r)^2]^{-1}. \tag{106}$$

Comparing the terms within the two parentheses in the right hand side of (106), the right hand expression in (106) is less than 1 since $0 < \beta$, $b_r < 1$ and φ_π is assumed to be less than 1. This proves that a sufficient condition for $c_2 > 0$, for all $\varphi_\pi > 0$, is $\varphi_r \geq 1$.

We next turn to the eigenvalues of $\Omega + \Omega\bar{b} - I$ which need to have negative real parts. When $\varphi_x = 0$, a_1 and a_2, defined in (95), and (96), reduce respectively to

$$\begin{aligned} a_1 &= 1 - b_\pi\varphi_\pi - \beta + \kappa\sigma(\varphi_\pi - 1), && (107) \\ a_2 &= \kappa\sigma(\varphi_\pi - 1) - \kappa\varphi_\pi b_x. && (108) \end{aligned}$$

We first examine a_2. From (108), observe that $a_2 > 0$ when $\varphi_\pi \geq 1$ since $b_x < 0$ at the MSV solution. We now prove that $a_2 > 0$ even when $\varphi_\pi < 1$. From (108), when $\varphi_\pi < 1$, $a_2 > 0$ iff

$$-\kappa\varphi_\pi b_x > \kappa\sigma(1 - \varphi_\pi), \tag{109}$$

that is, iff

$$\kappa\sigma(1 - b_r)^{-1}[(\varphi_\pi - 1)b_r + \varphi_r] > \kappa\sigma(1 - \varphi_\pi), \tag{110}$$

where we have used (102) in (110). Inequality (110) is equivalent to

$$(1 - b_r)^{-1}[\varphi_r(1 - \varphi_\pi)^{-1} - b_r] > 1. \tag{111}$$

Since $\varphi_\pi < 1$ and $\varphi_r \geq 1$, (111) is satisfied and hence, $a_2 > 0$ for all $\varphi_\pi > 0$.

We next turn to a_1. From (107), $a_1 > 0$ when $\varphi_\pi \geq 1$ since $b_\pi < 0$ at the MSV solution and $0 < \beta < 1$. We now prove that $a_1 > 0$ even when $\varphi_\pi < 1$. From (107), when $\varphi_\pi < 1$, $a_1 > 0$ iff

$$1 - b_\pi\varphi_\pi > \beta + \kappa\sigma(1 - \varphi_\pi), \tag{112}$$

that is, iff

$$\varphi_r > [\beta + \kappa\sigma(1 - \varphi_\pi)]b_r \tag{113}$$

where in moving from (112) to (113), we have used the value of b_r in (90) above. From (113), it is clear that since $0 < b_r < 1$, a sufficient condition for $a_1 > 0$ for all $\varphi_\pi > 0$, is that $\varphi_r \geq \beta + \kappa\sigma$.

Finally, we turn to the eigenvalues of $\rho\Omega + \Omega\bar{b} - I$ which need to have negative real parts. The coefficient $a_{1\rho}$, defined in (97), reduces to (when $\varphi_x = 0$)

$$a_{1\rho} = 2 - b_\pi\varphi_\pi - \rho[1 + \beta - \kappa\sigma(\varphi_\pi - 1)] = 2 - \rho(1 + \beta) - b_\pi\varphi_\pi + \rho\kappa\sigma(\varphi_\pi - 1) \tag{114}$$

which is positive when $\varphi_\pi \geq 1$ since $0 < \beta, \rho < 1$ and $b_\pi < 0$ at the MSV solution. We now show that $a_{1\rho} > 0$ when $\varphi_r \geq \beta + \kappa\sigma$ even when $\varphi_\pi < 1$. For this note that we can write $a_{1\rho}$ as

$$a_{1\rho} = 2 - b_\pi\varphi_\pi - \rho[1 + \beta - \kappa\sigma(\varphi_\pi - 1)] = a_1 + (1 + \beta)(1 - \rho) - (1 - \rho)\kappa\sigma(\varphi_\pi - 1) \quad (115)$$

where we have used the expression of a_1 from (107) in the right hand equality of (115). From (115), and since $0 < \rho < 1$, $a_1 > 0$ implies that $a_{1\rho} > 0$ when $\varphi_\pi < 1$. Since it was proved above that $a_1 > 0$ when $\varphi_r \geq \beta + \kappa\sigma$, for all $\varphi_\pi > 0$, it follows, therefore, that $a_{1\rho} > 0$ under the same condition.

We now turn to $a_{2\rho}$, defined in (98), which simplifies (when $\varphi_x = 0$) to

$$\begin{aligned} a_{2\rho} &= 1 - \rho[1 + \beta - \kappa\sigma(\varphi_\pi - 1)] + \beta\rho^2 - \rho\kappa\varphi_\pi b_x - (1 - \rho)b_\pi\varphi_\pi \\ &= (1 - \rho)(1 - \beta\rho) - (1 - \rho)b_\pi\varphi_\pi + \rho[\kappa\sigma(\varphi_\pi - 1) - \kappa\varphi_\pi b_x] \\ &= (1 - \rho)(1 - \beta\rho) - (1 - \rho)b_\pi\varphi_\pi + \rho a_2, \end{aligned} \quad (116)$$

where we have used the value of a_2 from (108). Since $a_2 > 0$ was proved before (for all $\varphi_\pi > 0$ when $\varphi_r \geq 1$), it follows from (116) that $a_{2\rho} > 0$ also since $0 < \beta, \rho < 1$, and $b_\pi < 0$.

7.7. APPENDIX G (Details for the Infinite Horizon Model). We first consider the scenario when agents know the form of the interest rate rule (5). Substituting this known rule into the output gap equation (46) yields

$$\begin{aligned} x_t &= \hat{E}_t\left\{\sum_{T=t}^{\infty}\beta^{T-t}[(1 - \beta - \sigma\varphi_x)x_{T+1} + \sigma(1 - \varphi_\pi)\pi_{T+1} - \sigma\varphi_r r_{T-1} + r_T^n]\right\} \quad (117) \\ &= -\sigma\varphi_r(r_{t-1} + \beta r_t) + \\ &\quad \sum_{T=t}^{\infty}\beta^{T-t}[(1 - \beta - \sigma\varphi_x)\hat{E}_t x_{T+1} + \sigma(1 - \varphi_\pi)\hat{E}_t\pi_{T+1} - \sigma\varphi_r\hat{E}_t r_{T+1}] + r_t^n \\ &= -\sigma\varphi_r r_{t-1} - \beta\sigma\varphi_r(\varphi_\pi\hat{E}_t\pi_{t+1} + \varphi_x\hat{E}_t x_{t+1} + \varphi_r r_{t-1}) + \\ &\quad \sum_{T=t}^{\infty}\beta^{T-t}[(1 - \beta - \sigma\varphi_x)\hat{E}_t x_{T+1} + \sigma(1 - \varphi_\pi)\hat{E}_t\pi_{T+1} - \sigma\varphi_r\hat{E}_t r_{T+1}] + r_t^n \end{aligned}$$

Substituting this value of x_t into (47) yields the matrix system

$$y_t = \sum_{T=t}^{\infty}(A_T\hat{E}_t y_{T+1}) + B_\delta\hat{E}_t y_{t+1} + \Lambda y_{t-1} + \chi r_t^n, \quad (118)$$

$$
\begin{aligned}
A_T &= \begin{bmatrix} \beta^{T-t}(1-\beta-\sigma\varphi_x) & \beta^{T-t}\sigma(1-\varphi_\pi) & -\beta^{T-t}\sigma\varphi_r \\ \beta^{T-t}\kappa(1-\beta-\sigma\varphi_x)+ & \beta^{T-t}\kappa\sigma(1-\varphi_\pi)+ & -\kappa\sigma\varphi_r\beta^{T-t} \\ (\alpha\beta)^{T-t}\kappa\alpha\beta & (\alpha\beta)^{T-t}(1-\alpha)\beta & \\ 0 & 0 & 0 \end{bmatrix}, \\
B_\delta &= \begin{bmatrix} -\beta\sigma\varphi_r\varphi_x & -\beta\sigma\varphi_r\varphi_\pi & 0 \\ -\kappa\beta\sigma\varphi_r\varphi_x & -\kappa\beta\sigma\varphi_r\varphi_\pi & 0 \\ \varphi_x & \varphi_\pi & 0 \end{bmatrix}, \ \Lambda = \begin{bmatrix} 0 & 0 & -\sigma\varphi_r(1+\beta\varphi_r) \\ 0 & 0 & -\kappa\sigma\varphi_r(1+\beta\varphi_r) \\ 0 & 0 & \varphi_r \end{bmatrix}, \ \chi = \begin{bmatrix} 1 \\ \kappa \\ 0 \end{bmatrix}.
\end{aligned}
$$

Substituting the forecasts (49) into (118) yields the ALM (50)

$$
\begin{aligned}
y_t &= X_a + X_b y_{t-1} + X_c r_t^n; \\
X_a &\equiv [\sum_{T=t}^{\infty} A_T(I-b)^{-1}(I-b^{T-t+2}) + B_\delta(I-b)^{-1}(I-b^2)]a, \\
X_b &\equiv \sum_{T=t}^{\infty} A_T b^{T-t+2} + B_\delta b^2 + \Lambda, \\
X_c &\equiv \sum_{T=t}^{\infty} A_T b^{T-t+1} c + B_\delta bc + \chi.
\end{aligned}
$$

The mapping from the PLM to the ALM can be analyzed for E-stability as before.

The second scenario is when the agents do not know the interest rate rule (5) so that they have to use the PLM for the interest rate to form forecasts rather than the actual rule (5). The equation for output is then

$$
x_t = -\sigma(a_r + b_r r_{t-1} + c_r r_t^n) + \left\{ \sum_{T=t}^{\infty} \beta^{T-t}[(1-\beta)\hat{E}_t x_{T+1} + \sigma\hat{E}_t\pi_{T+1} - \sigma\beta\hat{E}_t r_{T+1}] \right\} + r_t^n \tag{119}
$$

Substituting this value of x_t into the (47) yields the system in this case

$$
y_t = \sum_{T=t}^{\infty}(A_T\hat{E}_t y_{T+1}) + B_a + B_\delta\hat{E}_t y_{t+1} + \Lambda y_{t-1} + \chi r_t^n \tag{120}
$$

with the matrices

$$
\begin{aligned}
A_T &= \begin{bmatrix} \beta^{T-t}(1-\beta) & \beta^{T-t}\sigma & -\beta^{T-t+1}\sigma \\ \beta^{T-t}\kappa(1-\beta)+(\alpha\beta)^{T-t}\kappa\alpha\beta & \beta^{T-t}\kappa\sigma+(\alpha\beta)^{T-t}(1-\alpha)\beta & -\kappa\sigma\beta^{T-t+1} \\ 0 & 0 & 0 \end{bmatrix}, \\
B_a &= \begin{bmatrix} -\sigma a_r \\ -\kappa\sigma a_r \\ 0 \end{bmatrix}, \ B_\delta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varphi_x & \varphi_\pi & 0 \end{bmatrix}, \ \Lambda = \begin{bmatrix} 0 & 0 & -\sigma b_r \\ 0 & 0 & -\kappa\sigma b_r \\ 0 & 0 & \varphi_r \end{bmatrix}, \ \chi = \begin{bmatrix} 1-\sigma c_r \\ \kappa-\kappa\sigma c_r \\ 0 \end{bmatrix}.
\end{aligned}
$$

Substituting the forecasts (49) into (120) yields an ALM of the form (50). The mapping from the PLM to the ALM can be analyzed for E-stability as before.