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When Does Determinacy Imply Expectational Stability?

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Abstract

In the recent literature on monetary and fiscal policy design, adoption of policies that induce both determinacy and learnability of equilibrium has been considered fundamental to economic stabilization. We study the connections between determinacy of rational expectations equilibrium, and expectational stability or learnability of that equilibrium, in a general class of purely forward-looking models. We ask what types of economic assumptions drive differences in the necessary and sufficient conditions for the two criteria. We apply our result to a relatively general New Keynesian model. Our framework is sufficiently flexible to encompass lags in information, a cost channel for monetary policy, and either Euler equation or infinite horizon approaches to learning. We are able to isolate conditions under which determinacy does and does not imply learnability, and also conditions under which long horizon forecasts make a clear difference to conclusions about expectational stability. The sharpest result is that informational delays break equivalence connections between determinacy and learnability.

Keywords: Long horizon expectations, multiple equilibria, learnability of equilibrium, E-stability.

JEL codes: E3, E4, D8.

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1 Introduction

1.1 Overview

In the recent literature on monetary and fiscal policy design, there has been considerable interest in the promotion of policies that can deliver both a determinate, or unique, rational expectations equilibrium, and also learnability, or stability, of that equilibrium. Policies that generate equilibria which are both unique and stable are viewed as preferable to those that might allow either a multiplicity of equilibria, or instability, or both. The determinacy criterion is defined according to Blanchard and Kahn (1980), and the learnability, or expectational stability,[1] criterion is defined according to Evans and Honkapohja (2001).

In some parts of this literature, there appears to be a tight connection between determinacy and learnability. The discussion in Woodford (2003a, 2003b) as well as in Bullard and Mitra (2002) highlights cases where the conditions for determinacy of equilibrium are the same as the conditions for expectational stability. Woodford (2003a, p. 1180) states, "Thus both criteria ... amount in this case to a property of the eigenvalues of [a matrix] A, and the conditions required for satisfaction of both criteria are related, though not identical." McCallum (2007a) presents results suggesting that, for a large class of models of interest for macroeconomists, expressed as linear systems of expectational difference equations, determinacy implies E-stability.[2] Yet, an examination of the "general linear model" case in Evans and Honkapohja (2001) makes it plain that determinacy does not imply learnability. And, all of the above authors stress that there is no general presumption that determinacy implies learnability.[3] Still, there seems to be a close relationship between the criteria in many applications, and this is a puzzle we would like to help resolve. In particu-

[1] We use the terms *expectational stability*, *E-stability*, and *learnability* interchangably in this paper. The connections between the expectational stability condition and local convergence of systems under real time recursive learning are discussed extensively in Evans and Honkapohja (2001).

[2] Recently, McCallum (2007b) has argued that "well-formulated" models, which meet a certain technical condition, the learnable rational expectations equilibrium is always unique, even when the rational expectations equilibrium itself may not be unique.

[3] Bullard and Mitra (2002) in particular provide one example where the conditions required for the two criteria do not coincide.

lar, we would like to better understand the nature of the relationship between determinacy and learnability in economic terms.

Meanwhile, Preston (2005, 2006) and Woodford (2003b) have suggested that introducing learning into certain microfounded environments creates situations where long-horizon forecasts can matter for learning dynamics.[4] In the *infinite horizon approach* to learning,[5] the fundamental equations describing the evolution of the state of the economy are altered under learning relative to the case under rational expectations. Appropriately taking those changes in the dynamic system into account can, but does not always, change the conclusions one would draw concerning the learnability of a particular rational expectations equilibrium. For the baseline Bullard and Mitra (2002) findings, the infinite horizon approach yields the same conclusions, and Preston (2005, p. 86) notes, "That these results concur with the results [under the infinite horizon approach] is not necessarily to be expected." Indeed, in subsequent work, such as Preston (2006), the learnability conditions differ under the two approaches. We would like to understand more about the economics of the relationship between determinacy and learnability in the infinite horizon learning environment as well.

In this paper we consider a general class of purely forward-looking models where, in equilibrium, current endogenous variables are determined by infinite horizon expectations (as in most microfounded macroeconomic models). On the one hand, we study under what conditions determinacy and the different learning approaches deliver the same stability conditions. On the other hand, we investigate what aspects of the model might cause learnability and determinacy to be governed by differing conditions. To this end, we apply our more general results to a standard and widely-studied macroeconomic model. It is already known that determinacy does not imply E-stability in general; what is not well understood is the nature of economic models in which the two sets of conditions diverge. To understand the economic aspects we need to perform our analysis inside of a known model framework, which is why we commit to using a fairly general version of the New Keynesian (NK) macroeconomic model. This also

[4] Also see Marcet and Sargent (1989).

[5] Contrasting with the *one-step-ahead* (a.k.a, *Euler equation*) approach to learning standard in Evans and Honkapohja (2001).

facilitates our inclusion of the infinite horizon approach, which is inherently an issue that depends on the microfoundations.

1.2 What we do

We first study a general class of purely forward-looking models with infinite horizon expectations and informational delays. This class of models have a finite horizon representation which relies on the assumption of rational expectations, as discussed in Preston (2005). Within this class of models, we investigate the connection between determinacy and E-stability under infinite horizon (IH) and finite horizon (FH) learning. We then study a generalized NK macroeconomic model which includes certain features which will help us delineate between the conditions for determinacy and those for learnability in a variety of circumstances. We allow for informational delays, as explained below, and also we allow for the cost channel of monetary policy suggested recently by Ravenna and Walsh (2006). With the model environment in place, we turn to calculating determinacy and expectational stability conditions. We characterize situations in which determinacy implies E-stability and situations in which it does not, under both Euler equation and infinite horizon learning.

1.3 Main findings

We first present two propositions that discuss the general model. Proposition 1 isolates conditions under which the Euler equation and infinite horizon learning yield the same expectational stability conditions. Proposition 2 provides conditions under which determinacy implies expectational stability. The results suggest that differences in the stability conditions under alternative approaches are related to the rate at which the future is discounted and the existence of information delays. Models that have a reduced-form representation with a unique discount factor in each behavioral equation and do not have information delays imply the same stability conditions. Models that have a representation implying heterogeneous discount factors or display information delays might not yield the same stability conditions.

Propositions 3 and 4 focus on the NK model. It is shown that determinacy implies E-stability when there are no information delays, but determinacy does not imply E-stability when there are information delays. Furthermore, Euler and infinite horizon learning yield different stability conditions. This sharp result suggests that for models that have some type of informational delay, a wedge will be driven between the determinacy conditions and the learnability conditions. Since information delays of some type are probably the most realistic case, this is the most general conclusion of the paper.

1.4 Organization

In the next section we develop the general infinite horizon model and review and characterize the E-stability conditions. We then discuss the propositions indicating when determinacy and learnability conditions will coincide, and when they will not. In section 3 we discuss the application to the NK model. We summarize our findings in the concluding section; there are also two appendices to the paper which contain some of the development of the model.

2 A "general model"

2.1 Informational delays

A key aspect of our approach is that we allow for differing information sets to be available to agents at the time expectations are formed and decisions are made as we consider the microfoundations of the model. This means that date t expectations may be formed either with information available at date t, or with information available at date $t-1$, and that the equations we derive to represent the equilibrium dynamics will be equally valid in either case. To accomplish this, we use the operator $\hat{E}_{t-\ell}$, where $\ell \in \{0,1\}$, which we think of as an information lag or a "delay" when $\ell = 1$, and which is just the standard $\hat{E}_t$ when $\ell = 0$. The hat indicates that expectations may not initially be rational.

Many models in macroeconomics assume rational expectations and have $t-$dating of the expectations operator, E_t, and this has come to be thought of as the standard case. For many purposes under the rational expectations assumption it may not be too important, although it is rarely analyzed in the literature. In an environment with learning, the dating of the expectations operator may be more critical, and Evans and Honkapohja (2001) have suggested that the $t-1$ dating of the expectations operator may be more natural when learning is explicitly considered. This is because the general equilibrium has date t quantities and prices being determined at date t, but the agents are supposed to be forming expectations using the date $t-1$ data in their recursive algorithms. This type of simultaneity is a constant companion in standard economic theory but does not make very much sense if we think more explicitly about the microfoundations of how the expectations are being formed. Even under the rational expectations assumption, the case for information lags has been made. McCallum (1999) has argued that actual policymakers rarely have contemporaneous information available when making decisions, especially concerning variables like GDP, and that operational interest rate rules would involve a reaction to readings on endogenous variables at least one period in the past. In an influential paper, Rotemberg and Woodford (1999) estimated a DSGE model with rational expectations but dated their expectations operator at $t-2$ to provide a better fit to the data.

2.2 The model

The general class of models we have described can be expressed in matrix notation as

$$A_0 Y_t = A_1 \hat{E}_{t-\ell} Y_t + \sum_{d=1}^{N} A_{2,d} \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta_d^{T-t} Y_{T+1} + A_3 X_{t-1} + A_4 \epsilon_t, \qquad (1)$$

where Y_t denotes a $n-$dimensional vector of endogenous variables and X_t denotes a $k-$dimensional vector of shocks which evolve according to

$$X_t = H X_{t-1} + \epsilon_t. \qquad (2)$$

The matrix H is assumed to be diagonal with elements $0 \leq h_{i,i} < 1$, for $i = 1...k$. The model equations imply $N \leq n$ discount factors, β_d, associated with the future evolution of the endogenous variables. For an example of a model consistent with this representation, see Preston (2005). The discount factors emerge from the decisions associated with different behavioral equations in the model. Finally, all matrices A are conformable.

The model is purely forward-looking, an assumption that is made in order to maintain analytical tractability. Of course such an assumption excludes many widely-used models that include, for example, capital accumulation, habit formation and more generally, lags in agent's decision rules. We leave the analysis of these more complicated frameworks for future research. A discussion on the model's limitations can be found in Section 4.

2.3 E-stability

We assume that agents do not have rational expectations at the outset but form expectations using adaptive learning rules. The methodology to analyze the convergence properties of the agent's learning process has been developed by Marcet and Sargent (1989) and Evans and Honkapohja (2001). Evans and Honkapohja (2001) discuss the tight connection between the convergence properties of real time learning algorithms, such as recursive least squares, and the concept of E-stability that is used here.

As is standard in the literature, agents are endowed with a perceived law of motion (PLM) for Y_t. In this paper we focus on a PLM which is consistent with the Minimum State Variable solution of (1), which takes the form

$$Y_t = a + bX_{t-\ell}, \tag{3}$$

where a denotes the intercept vector, b denotes a $n \times k$ matrix of coefficients and ℓ takes values of zero (no delays) or one (one period delay). Agents use (3) to form expectations: the discounted infinite sum in equation (1) can then be expressed as

$$\begin{aligned} \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta_d^{T-t} Y_{T+1} &= \sum_{T=t}^{\infty} \beta_d^{T-t} a + \sum_{T=t}^{\infty} b \beta_d^{T-t} H^{(T-t+1)} X_{T-\ell} \\ &= \frac{1}{1-\beta_d} a + b \left(1 - \beta_d H\right)^{-1} H X_{t-\ell}. \end{aligned} \tag{4}$$

Substituting (4) in (1) we obtain the actual law of motion (ALM) of Y_t under learning dynamics

$$Y_t = \tilde{A}_1 \left(a + bX_{t-\ell}\right) + \sum_{d=1}^{N} \tilde{A}_{2,d} \left(\frac{1}{1-\beta_d} a + b \left(1 - \beta_d H\right)^{-1} H X_{t-\ell} \right) + A_3 X_{t-1} + A_4 \epsilon_t, \quad (5)$$

where $\tilde{A}_i = A_0^{-1} A_i$ for $i = 1, 2$, and 3. Equation (5) defines the following mapping between the PLM and the ALM

$$T\left(a, b\right) = \left(\tilde{A}_1 a + \sum_{d=1}^{N} \tilde{A}_{2,d} \frac{1}{1-\beta_d} a, \quad \tilde{A}_1 b + \sum_{d=1}^{N} \tilde{A}_{2,d} b \left(1 - \beta_d H\right)^{-1} H + \tilde{A}_3 \right). \quad (6)$$

At the rational expectations equilibrium PLM and ALM coincide, that is $T\left(a, b\right) = \left(a, b\right)$. The notion of E-stability describes the interaction between PLM and ALM which arises during the learning process, where $T\left(a, b\right) \neq \left(a, b\right)$. It is determined by the following matrix differential equations

$$\frac{d}{d\tau} \left(a, b\right) = T\left(a, b\right) - \left(a, b\right) \quad (7)$$

where τ denotes "artificial" time. The rational expectations equilibrium is defined to be E-stable if equation (7), evaluated at the REE, is locally stable. The ODE intuitively describes a stylized learning rule under which the parameters a and b are adjusted towards their REE values. The local asymtotic stability of (7) depends on the eigenvalues of the Jacobian of (7). In the class of models described by (1), the problem of understanding the eigenvalues can be reduced to analyzing two separate matrices. The matrix governing expectational stability of the intercept is

$$M^{IH}(a) = \tilde{A}_1 + \sum_{d=1}^{N} \tilde{A}_{2,d} \frac{1}{1-\beta_d} - I_n. \quad (8)$$

For the matrix of coefficients b, after vectorizing $T(b)$ we obtain

$$M^{IH}\left(b\right) = \left(I_k \otimes \tilde{A}_1\right) + \sum_{d=1}^{N} \left\{ \left[\left(I_k - \beta_d H\right)^{-1} H \right]' \otimes \tilde{A}_{2,d} \right\} - I_k \otimes I_n. \quad (9)$$

E-stability obtains if and only if the real parts of the eigenvalues of $M^{IH}(a)$ and $M^{IH}\left(b\right)$ are negative. We stress that to obtain the case of contemporaneous expectations ($\ell = 0$) all that is needed is to re-define the matrix A_0 and set $\tilde{A}_1 = 0$. In the next section we use (8) and (9) to establish that E-stability conditions need not be the same as determinacy conditions.

2.4 An equivalence result

We first consider the connections between infinite horizon (IH) and finite horizon (FH) (*a.k.a.* one-period-ahead or Euler equation learning). The following proposition states the *sufficient* condition to have equivalence in the E-stability conditions under different learning approaches.

Proposition 1 *Consider the model (1) and assume that* $\beta_d = \beta_{d'}$ $\forall$ *d and* d'*. Then finite horizon and infinite horizon learning approaches deliver the same expectational stability conditions.*

Proof. The model in matrix representation becomes

$$Y_t = \tilde{A}_1 \hat{E}_{t-\ell} Y_t + \tilde{A}_2 \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} Y_{T+1} + A_3 X_{t-1} + A_4 \epsilon_t \tag{10}$$

where it is assumed that all decision rules use the same discount factor β. This implies that every row in $\tilde{A}_2$ must constrain at least one non-zero element. Otherwise, the corresponding equation would imply $\beta = 0$, violating the assumption. We now proceed to quasi-difference (10). First, we forward (10) one period and taking expectations

$$E_{t-\ell} Y_{t+1} = \tilde{A}_1 \hat{E}_{t-\ell} Y_{t+1} + \tilde{A}_2 \hat{E}_{t-\ell} \sum_{T=t+1}^{\infty} \beta^{T-t-1} Y_{T+1} + \hat{E}_{t-\ell} \tilde{A}_3 X_t + E_{t-\ell} A_4 \epsilon_t.$$

Second, we rewrite (10) as

$$Y_t = \tilde{A}_1 \hat{E}_{t-\ell} Y_t + \tilde{A}_2 \hat{E}_{t-\ell} Y_{T+1} + \tilde{A}_2 \hat{E}_{t-\ell} \sum_{T=t+1}^{\infty} \beta^{T-t} Y_{T+1} + A_3 X_{t-1} + A_4 \epsilon_t$$

and combining the two expressions we obtain,

$$\begin{aligned} Y_t = \tilde{A}_1 \hat{E}_{t-\ell} Y_t + \tilde{A}_2 \hat{E}_{t-\ell} Y_{t+1} + \beta \hat{E}_{t-\ell} Y_{t+1} \\ - \beta \tilde{A}_1 \hat{E}_{t-\ell} Y_{t+1} + \tilde{A}_3 \left(H - \beta I_k\right) X_{t-1} \\ + A_4 \left(E_{t-\ell} \epsilon_t - \epsilon_t\right). \end{aligned}$$

Re-arranging yields

$$Y_t = \tilde{A}_1 \hat{E}_{t-\ell} Y_t + \left[\tilde{A}_2 + \beta\left(I_n - \tilde{A}_1\right)\right] \hat{E}_{t-\ell} Y_{t+1} + \tilde{A}_3 \left(H - \beta I_k\right) X_{t-1} + A_4 \left(E_{t-\ell}\epsilon_t - \epsilon_t\right), \tag{11}$$

which defines the finite horizon representation of the model. This is the reduced-form representation of the model solution that is used to evaluate E-stability under the FH approach. The ALM (11) gives the following mappings

$$\begin{aligned} T_{FH}(a) &= \left[\tilde{A}_1 + \tilde{A}_2 + \beta\left(I_n - \tilde{A}_1\right)\right] a \\ &= \left[(1-\beta)\tilde{A}_1 + \tilde{A}_2 + \beta I_n\right] a \end{aligned}$$

$$\begin{aligned} T_{FH}(b) &= \tilde{A}_1 b + \left[\tilde{A}_2 + \beta\left(I_n - \tilde{A}_1\right)\right] bH \\ &= \tilde{A}_1 b - \tilde{A}_1 b\beta H + \tilde{A}_2 bH + \beta bH \end{aligned}$$

and the following associated Jacobian matrices, for the intercept

$$M^{FH}(a) = (1-\beta)\tilde{A}_1 + \tilde{A}_2 - (1-\beta) I_n \tag{12}$$

$$M^{FH}(a) = (1-\beta) M^{IH}(b) \tag{13}$$

where the last equality follows from (8), imposing $d = 1$. Hence, if $M^{FH}(a)$ has negative eigenvalues, so does $M^{IH}(b)$ and vice versa. For the regression coefficients we obtain (after vectorization) the following Jacobian

$$\begin{aligned} M^{FH}(b) &= I_k \otimes \tilde{A}_1 - \beta H \otimes \tilde{A}_1 + H \otimes \tilde{A}_2 + \beta H \otimes I_n - I_k \otimes I_n \\ &= (I_k - \beta H) \otimes \tilde{A}_1 + H \otimes \tilde{A}_2 - (I_k - \beta H) \otimes I_n \\ M^{FH}(b) &= M^{IH}(b) \cdot \left[(I_k - \beta H) \otimes I_n\right] \end{aligned}$$

where the first equality follows from the properties of the Kroneker product and the second equality follows from setting $N = 1$ (that is $\beta_d = \beta \; \forall \; d$) in (9), which gives

$$M^{IH}(b) = \left(I_k \otimes \tilde{A}_1\right) + \left[\left(I_k - \beta H\right)^{-1} H\right]' \otimes \tilde{A}_2 - I_k \otimes I_n,$$

and again is obtained using the properties of the Kroneker product.[6] The next step is to show that if $M^{IH}(b)$ has negative eigenvalues, then $M^{FH}(b)$ also has

[6] In particular, given a $k \times k$ matrix P and a $n \times n$ matrix Q

$$I_k P \otimes Q I_n = (I_k \otimes Q)(P \otimes I_n).$$

negative eigenvalues. The matrix $(I_k - \beta H) \otimes I_n$ is a $nk \times nk$ block diagonal matrix

$$(I_k - \beta H) \otimes I_n = \begin{bmatrix} (1-\beta h_{11}) I_n & ... & 0 \\ 0 & ... & 0 \\ 0 & ... & (1-\beta h_{kk}) I_n \end{bmatrix},$$

so that

$$M^{IH}(b) \cdot [(I_k - \beta H) \otimes I_n] = \begin{bmatrix} \bar{A}_{11} & ... & 0 \\ 0 & ... & 0 \\ 0 & ... & \bar{A}_{kk} \end{bmatrix}. \tag{14}$$

where

$$\bar{A}_{11} = (1-\beta h_{11}) \left[\tilde{A}_1 + (1-\beta h_{11})^{-1} h_{11} \tilde{A}_2 - I_n \right]$$

and

$$\bar{A}_{kk} = (1-\beta h_{kk}) \left[\tilde{A}_1 + (1-\beta h_{11})^{-1} h_{11} \tilde{A}_2 - I_n \right].$$

The eigenvalues of (14) are given by the eigenvalues of the sub-matrices

$$(1-\beta h_{ii}) \left[\tilde{A}_1 + (1-\beta h_{ii})^{-1} h_{ii} \tilde{A}_2 - I_n \right] \text{ for } i = 1...k.$$

We then obtain the eigenvalues as

$$\begin{aligned} eig\left((1-\beta h_{ii}) \left[\tilde{A}_1 + (1-\beta h_{ii})^{-1} h_{ii} \tilde{A}_2 - I_n \right] \right) &= \\ (1-\beta h_{ii})\, eig\left(\left[\tilde{A}_1 + (1-\beta h_{ii})^{-1} h_{ii} \tilde{A}_2 - I_n \right] \right) \text{ for } i &= 1...k, \end{aligned}$$

which proves the claim. Similarly, the reverse also holds true, given that

$$M^{IH}(b) = M^{FH}(b) \cdot [(I_k - \beta H) \otimes I_n]^{-1}$$

is well defined under the maintained assumptions about H. ■

2.5 When determinacy implies E-stability

The next proposition states the sufficient conditions under which determinacy implies expectational stability.

Proposition 2 *Consider the model (1) and assume* $\beta_d = \beta_{d'}$ $\forall$ d *and* d'*. If* $\tilde{A}_1 = \mathbf{0}$ *determinacy implies E-stability.*

Proof. By setting $\tilde{A}_1 = 0$ local determinacy depends on the eigenvalues of the matrix

$$M^D = \tilde{A}_2 + \beta I_n$$

which is obtained by setting $\tilde{A}_1 = \mathbf{0}$ in eq (11). In order to yield local determinacy all eigenvalues of M^D must be inside the unit circle. From (12) and (13) and after imposing $\tilde{A}_1 = \mathbf{0}$ we obtain

$$M^D = M^{FH}(a) + I_n = (1-\beta) M^{IH}(a) + I_n$$

where the second equality comes from Proposition 1. If the eigenvalues of M^D are inside the unit circle, the eigenvalues of $M^{FH}(a)$ –and $M^{IH}(a)$– have negative real parts. Hence E-stability is verified. Concerning the Jacobian for the shock coefficients, we have

$$\begin{aligned} eig\left(M^{FH}(b)\right) &= eig\left(H \otimes \tilde{A}_2 - (I_k - \beta H) \otimes I_n\right) \\ &= eig\left(H \otimes \tilde{A}_2\right) - eig\left((I_k - \beta H) \otimes I_n\right) \end{aligned}$$

which[7] implies that $M^{FH}(b) + I_n$ has eigenvalues with real parts less than one if $M^{FH}(a) + I_n$ has real eigenvalues less than one. ■

3 A simple monetary model

In the recent literature, problems of equilibrium selection have mostly been connected to monetary policy design. The choice of a particular policy rule to conduct monetary policy depends on the ability of such rule to stabilize expectations. Below we describe a simple new Keynesian model. It is simple enough to fit the class of model described above but it provides a useful starting point to discuss the connection between determinacy and E-stability.

[7] The second equality uses the following result. Let us define λ_i^* as an eigenvalue of the matrix $(1-\beta h_{ii}) I_n$, that is $\lambda_i^* = (1-\beta h_{ii})$, and λ_i an eigenvalue of $\tilde{A}_2 h_{ii} - (1-\beta h_{ii}) I_n$. Then the determinant

$$\left|\lambda I_n - \tilde{A}_2 h_{ii} + (1-\beta h_{ii}) I_n\right| = 0$$

implies

$$\left|(\lambda + \lambda^*) I_n - \tilde{A}_2 h_{ii}\right| = 0.$$

3.1 Households

There is a continuum of households that consume, save and supply labor in a homogeneous labor market. A household i maximizes utility over an infinite horizon

$$\hat{E}^i_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \left[U\left(C^i_T\right) - V\left(h^i_T\right)\right], \tag{15}$$

where the parameter $\beta \in (0,1)$ is the discount factor and the period utility functions $U(C_t)$ and $V(h_t)$ have standard properties. For tractability we assume that in the case of delays, $\ell = 1$, *only consumption decisions are predetermined.* This means that labor supply decisions and saving decisions are taken with date t information, independently of the value of ℓ. Consumption C_t is given as a composite of all goods produced in the economy,

$$C^i_t \equiv \left[\int_0^1 c^i_t(j)\, dj\right]^{\frac{\theta}{\theta-1}} \tag{16}$$

and has an associated price index

$$P_t \equiv \left[\int_0^1 p_t(j)^{1-\theta}\, dj\right]^{\frac{1}{1-\theta}}. \tag{17}$$

Households hold money that can be used to purchase the consumption good and can also be deposited at a financial intermediary. The budget constraint of household i is given by

$$M^i_{t+1} + P_t C^i_t \leq M^i_t - D^i_t + (1+i_t) D^i_t + W_t h^i_t + P_t \int \phi^f_t(j) dj + P_t \phi^M, \tag{18}$$

where M^i_t denotes money holdings at the beginning of the period and D_t denotes the amount on deposit at the financial intermediary, which pays the gross nominal interest rate $(1+i_t)$.[8] The variable W_t is the economywide nominal wage determined in a perfectly competitive labor market, and $\phi_t(j)$ and ϕ^M denote real profits from firms and the financial intermediary. Each agent i is assumed to have an equal share of each firm j. These assumptions guarantee that the households income profiles are identical, even in the case of incomplete markets. The household also faces the cash-in-advance constraint

$$P_t C^i_t \leq M^i_t + W_t h^i_t - D_t, \tag{19}$$

[8] We assume that money bears no interest.

which takes this form because households receive their wages at the beginning of the period.

Solving the household problem gives the optimal consumption decision given current beliefs[9] in log-linear deviations from the nonstochastic steady state

$$\hat{C}_t^i = \hat{E}_{t-\ell}^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)\hat{Y}_T^i - \beta\sigma(i_T - \pi_{T+1}) \right], \tag{20}$$

where σ denotes the intertemporal elasticity of substitution in consumption and $\hat{Y}_t$ denotes real income.[10] Details of the derivation can be found in the Appendix. The labor supply, decided with period-t information is

$$\frac{V_h\left(h_t^i\right)}{u_c\left(C_t^i\right)} = \frac{w_t}{P_t}. \tag{21}$$

Finally, in an equilibrium with a positive nominal interest rate, the cash in advance constraint will bind as

$$P_t C_t^i = M_t^i + W_t h_t^i - D_t.$$

3.2 Firms

Each firm j produces a differentiated good and has market power. They face a demand for their output given by

$$Y_t^D(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\theta} C_t$$

where C_t denotes aggregate consumption. Labor is the only input in the production function, which is assumed to be linear in order to simplify the analysis,

$$Y_t(j) = A_t h_t(j).$$

The labor market is viewed as economywide and perfectly competitive. Firms are subject to a cash constraint, which gives rise to a *cost channel* for monetary policy. In particular, the firms must anticipate their wage bill to the workers and

[9] Here we consider optimal decisions in the anticipated utility framework – see Marcet and Sargent (1989). Agents are boundedly rational in that they ignore the future updating in their beliefs.

[10] Following Preston (2002), for simplicity we assume that each agent forecasts her entire income and not its single components. This has no implication for our conclusions. More details are available in the Appendix.

therefore have to borrow funds from the financial intermediary in the amount corresponding to $h_t\,(j)\,W_t$.

We study a standard model of nominal pricing rigidities a là Calvo. Firms maximize their expected profits,[11]

$$\hat{E}_{t-\ell}\sum_{T=t}^{\infty}\alpha^{T-t}Q_{t,T}\phi_T(j),$$

where $Q_{t,T}$ is the stochastic discount factor and where real flow profits are

$$\phi_t(j)=\frac{p_t\,(j)}{P_t}Y_t^D\,(j)-(1+i_t)\,\frac{w_t}{P_t}h_t(j).$$

Financial intermediaries operate in a perfectly competitive markets for funds. Therefore the cost of borrowing for each firm is

$$i_t\frac{W_t}{P_t}h_t(j).$$

In order to simplify the analysis we assume that, independently of informational delays, firms can observe the current aggregate price level when deciding their optimal price.[12] When there are informational delays the optimal relative price depends on the expected current and future marginal cost. Similarly to households, firms choose their labor input (and the amount of funds to borrow) using current information—delays can occur only at the pricing stage. The first order condition for the optimal price becomes

$$\hat{E}_{t-\ell}\sum_{T=t}^{\infty}\alpha^{T-t}Q_{t,T}Y_T^D\,(P_T)^{\theta}\left[P_t^*\,(j)-\frac{\theta-1}{\theta}P_Ts_T\right]=0, \tag{22}$$

The optimal pricing decision after log-linearization is

$$\hat{p}_t^*(j)=\hat{E}_{t-\ell}\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[(1-\alpha\beta)\,\hat{s}_T+\alpha\beta\pi_{T+1}\right] \tag{23}$$

where $\hat{p}_t^*=\ln\left(P_t^*/P_t\right)$ and

$$s_t=\frac{w_t}{P_tA_t}\left(1+i_t\right), \tag{24}$$

[11] Discounted with the average household stochastic discount factor, which turns out to be the correct way of discounting the future stream of profits because agents are identical.

[12] The instability results in this paper do not depend on this assumption. Eusepi and Preston (2007a) consider the case where firms no not observe the current price level when deciding their price, in a simpler model.

is the real marginal cost of production, which is a function of the real wage and the opportunity cost of holding cash. Given homogeneous factor markets implies that each firm that chooses the optimal price will choose the same price $\hat{p}_t^*(j) = \hat{p}_t^*$. While Calvo pricing implies a distribution of prices across firms, nevertheless the model specification leads to a simple log-linear relation between the optimal price and inflation

$$\pi_t = \frac{1-\alpha}{\alpha}\hat{p}_t^*. \tag{25}$$

Finally, financial intermediaries make a profit of

$$P_t \phi_t^M = (1+i_t)\int W_t h(j)dj - (1+i_t)\int D_t^i di = (1+i_t)\,T_t \tag{26}$$

where T_t is a cash injection from the government to be defined below.

3.3 Monetary and fiscal policies

We assume that the fiscal authority operates a zero debt, zero spending fiscal policy. The fiscal variable T_t denotes a cash injection from the government which is equal to

$$T_t = M_{t+1} - M_t. \tag{27}$$

Monetary policy is described by a simple Taylor-type rule of the (log-linear) form

$$\hat{\imath}_t = \hat{\imath}_t^* + \phi\left[\hat{E}_{t-\ell}\pi_t + \bar{\lambda}\hat{E}_{t-\ell}x_t\right] \tag{28}$$

where the monetary authority reacts to *private sector* expectations about inflation and the output gap, denoted by x_t. The latter is defined as the log-deviation of output from its natural level, consistent with flexible prices.[13] For generality, the intercept $\hat{\imath}_t^*$ is time-varying.[14] The term in brackets defines the linear combination between expected output and inflation corresponding to the optimal discretionary targeting rule under rational expectations, where the coefficient $\bar{\lambda}$ on the output gap can be interpreted as a function of the central bank's relative preference for output stabilization.[15]

[13] Please see the appendix.

[14] Under optimal discretionary policy $\hat{\imath}_t^*$ is a linear combination of the underlying shocks. Alternatively, it can be interpreted as a monetary policy shock. The specific form of the intercept does not affect the stability and determinacy results.

[15] Details are provided in the appendix.

3.4 Equilibrium

Equilibrium in the goods market requires

$$y_t(j) = \left(\frac{p_t\,(j)}{P_t}\right)^{-\theta} C_t$$

for every j, and equilibrium in the labor market gives

$$\int h^i di = h_t = \int h(j)dj,$$

where $h_t^i = h_t$. We also have a market clearing condition for loanable funds of financial intermediaries,

$$W_t \int h(j)dj = \int D_t^i di + T_t.$$

3.5 Evolution of aggregate variables

3.5.1 The output gap

Combining the consumption decision rule (20) and market clearing conditions, aggregate demand can be expressed as

$$x_t = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)x_T - \sigma\beta(\hat{\imath}_T - \pi_{T+1}) + \beta\sigma r_T^n\right], \tag{29}$$

where x_t is the output gap, i_t is the deviation of the one-period nominal interest rate from the value consistent with inflation at target and output at potential, π_t is the deviation of inflation from target, and r_t^n is a disturbance term[16] describing the natural rate of interest in the economy. This is the equilibrium interest rate that arises under flexible prices consistent with the natural level of output. We assume this term is governed by the stochastic process

$$r_t^n = \rho r_{t-1}^n + \epsilon_t$$

where $0 < \rho < 1$ and $\epsilon_t \sim N\left(0, \sigma_\epsilon\right)$. We normalize the inflation target to zero. The parameter β is the discount factor of the representative household and, again, the parameter σ is the intertemporal elasticity of substitution in

[16] This is a function of the only exogenous shock A_t.

consumption. Quasi-differencing this expression we obtain the familiar forward-looking IS curve, including one extra term for information delays

$$x_t = (1-\beta)\hat{E}_{t-\ell}x_t - \beta\sigma\hat{E}_{t-\ell}\left(i_t + \pi_{t+1} - \hat{r}_t^n\right) + \beta\hat{E}_{t-\ell}x_{t+1}. \tag{30}$$

The aggregate demand (30) states output is a function of one-period-ahead expectations.

3.5.2 Inflation

Inflation dynamics is described by

$$\pi_t = \hat{E}_{t-\ell}\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left\{\xi\left[(\sigma^{-1}+\gamma)x_T + \hat{\imath}_T\right] + \beta(1-\alpha)\pi_{T+1}\right\}. \tag{31}$$

where $\xi = \frac{(1-\alpha\beta)(1-\alpha)}{\alpha} > 0$. Quasi-differencing and imposing rational expectations we obtain

$$\pi_t = \beta\hat{E}_{t-\ell}\pi_{t+1} + \xi\hat{E}_{t-\ell}\left[\left(\sigma^{-1}+\gamma\right)x_t + \hat{\imath}_t\right], \tag{32}$$

which, again, depends only on one-period-ahead forecasts.

In the next sections we describe the stability conditions under different approaches. Most papers in the adaptive learning literature use versions of equations (30) and (32) to evaluate the E-stability properties of different policy rules.[17] Below, we compare the E-stability conditions obtained using (30) and (32) with the E-stability conditions arise when using (29) and (31).

3.6 Finite vs. Infinite Horizon

Proposition 1 shows that the two approaches to modeling learning need not imply the same E-stability conditions. This is potentially true for the simple model presented above. In fact the model can be re-expressed in general form (1), where $N = 3$ and

$$\begin{aligned}\beta_1 &= \beta\\ \beta_2 &= \alpha\beta\\ \beta_3 &= 0.\end{aligned}$$

[17] See for example Bullard and Mitra (2004), among the others..

As shown in Preston (2007), assuming that the policy rule responds to expected inflation and output gap E-stability conditions are *stricter* than determinacy conditions.[18] More generally, Eusepi and Preston (2008) emphasize that communication of the policy rule has a crucial role in determining E-stability, whereas it does not affect determinacy, given that under rational expectations policy is fully understood. Interestingly, Proposition 1 can be interpreted as suggesting that differences in E-stability conditions depend on agents' *heterogeneity* in the way the future is discounted. Infinite horizon decisions *per se* do not alter the stability conditions. Assume that, (1) agents know the monetary policy rule, and (2) firms set prices according to the following 'sophisticated' decision rule[19]

$$\pi_t = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \left\{ \xi \left[(\sigma^{-1} + \gamma) x_T + \hat{\imath}_T \right] \right\} \tag{33}$$

obtained by iterating forward on (32) and imposing a transversality condition. This model translated in form (1) implies $N = 2$, where $\beta_1 = \beta_2 = \beta$. It follows directly from Proposition 1 that E-stability conditions are the same in both finite and infinite horizon approach. This simple result will be used in the Propositions below.

3.7 The role of information delays

We now consider the crucial role of information delays. In the sequel we assume that agents correctly understand the policy rule followed by the central bank. This simplifies the analysis and distinguishes our results from those found by Eusepi and Preston (2008) and Preston (2007). That is, failure to achieve E-stability is not connected to agents' information about the policy rule but to other features in the model; delays in decision rules and, more generally, the monetary transmission mechanism. We first consider the model with contemporaneous information and a standard Taylor rule and compare the results with the case of information delays. In the case of contemporaneous expectations we obtain the following proposition.

[18] The same representation obtain in a model with nominal rigidities a la Rotemberg – see Eusepi and Preston.

[19] We call it 'sophisticated' rule because it uses some elements of rational expectations – see Preston (2005).

Proposition 3 *In the model with contemporaneous expectations ($\ell = 1$) determinacy implies E-stability.*

Proof. Consider a pricing rule of the form

$$\pi_t = \hat{E}_{t-\ell} \sum_{T=t}^{\infty} (\vartheta\beta)^{T-t} \left\{ \xi \left[(\sigma^{-1} + \gamma) x_T + \hat{\imath}_T \right] + \beta(1-\vartheta)\pi_{T+1} \right\}. \tag{34}$$

Here we define a new parameter, $\vartheta \in \{\alpha, 1\}$. If $\vartheta = \alpha$, we obtain pricing equation (**31**), and if $\vartheta = 1$ we obtain the 'sophisticated' pricing equation (33). As stated above, Proposition 1 implies that E-stability conditions under the finite horizon learning are the same as obtained from the infinite horizon model with $\vartheta = 1$.

Part 1. Set $\ell = 1$. We first show the parameter restrictions that ensure expectational stability under infinite horizon learning. E-stability in this case depends on the eigenvalues of the matrix

$$M^{IH}(a) = (1-\beta)^{-1} \left(I_2 - \tilde{A}_1 \right)^{-1} \tilde{A}_{2,1} + (1-\vartheta\beta)^{-1} \left(I_2 - \tilde{A}_1 \right)^{-1} \tilde{A}_{2,2} - I_2.$$

where $\vartheta \in (0, 1]$. Local stability obtains if $M^{IH}(a)$ has eigenvalues with negative real values. This is verified if and only if the trace of the matrix is negative and the determinant is positive. Consider the determinant: it is positive if

$$\frac{\left[(\phi - 1) + (\phi - 1)\gamma\sigma \right] \xi + \sigma\phi\bar{\lambda} \left[(1-\beta) - \xi \right]}{\left(1 + \phi\gamma\sigma\xi + \sigma\phi\bar{\lambda} \right) (\beta - 1)(\vartheta\beta - 1)} > 0.$$

If $(1-\beta) - \xi > 0$, this condition is satisfied for any other parameter values. If $(1-\beta) - \xi < 0$, the determinant is positive provided

$$0 \leq \bar{\lambda} < \bar{\lambda}^* = \frac{\phi - 1}{\phi} \frac{\xi \left(\sigma^{-1} + \gamma \right)}{\xi - 1 + \beta}. \tag{35}$$

We can write the trace as

$$\frac{Tr(\vartheta, \beta)}{\left(1 + \phi\gamma\sigma\xi + \sigma\phi\bar{\lambda} \right) (\beta - 1)(\vartheta\beta - 1)}$$

where

$$\begin{aligned} Tr(\vartheta, \beta) = {} & -1 - \phi\xi - 2\sigma\phi\bar{\lambda} - \xi\gamma\sigma\vartheta\beta - \xi\vartheta\beta + \\ & \phi\xi\vartheta\beta\gamma\sigma + \phi\xi\gamma\sigma\beta - \beta^2 + \sigma\phi\bar{\lambda}\vartheta\beta + \\ & \left(-\sigma\phi\bar{\lambda}\beta^2 \right) - \xi\vartheta\beta\bar{\lambda}\phi\sigma + \xi - 2\phi\xi\gamma\sigma + \\ & \xi\phi\bar{\lambda}\sigma + 2\beta + 2\beta\sigma\phi\bar{\lambda} + \xi\gamma\sigma + \phi\xi\vartheta\beta. \end{aligned}$$

First, it is straightforward to show that

$$Tr(1,\beta) = \\ -\left[\xi\phi\gamma\sigma + (1-\beta) + (\phi-1)\,\xi + (\phi-1)\,\xi\gamma\sigma + ((2-\beta)-\xi)\,\sigma\phi\bar{\lambda}\right] < 0$$

provided the conditions for a positive determinant are satisfied. Second, consider

$$Tr(0,\beta) = -1 - \phi\xi\gamma\sigma\,(1-\beta) - 2\,(1-\beta)\,\sigma\phi\bar{\lambda} + \Psi\,(\beta)$$

where

$$\Psi\,(\beta) \equiv -\left(1+\sigma\phi\bar{\lambda}\right)\beta^2 + 2\beta + \xi\phi(1-\beta) \\ - [(\phi-1) + (\phi-1)\,\gamma\sigma]\,\xi + \xi\phi\bar{\lambda}\sigma.$$

It is easy to verify that $\Psi\,(0) < 0$ and $\Psi\,(1) < 0$ (provided the conditions for a positive determinant are satisfied). Given that $\Psi'\,(\beta)$ is linear[20] in β, this implies $\Psi\,(\beta) < 0$. Finally, given that $Tr(\vartheta,\beta)$ is linear in ϑ, we can conclude that $Tr(\vartheta,\beta) < 0\ \forall\vartheta \in (0,1]$. The above results imply that both IH and FH learning give E-stability if and only if (35) is satisfied or $(1-\beta) - \xi > 0$.

Part 2. Using Proposition 2, we conclude that local determinacy implies expectational stability.[21] ■

We now prove the second proposition concerning the model with delays.

Proposition 4 *In the model with delays ($\ell = 1$):*

1) Local determinacy does not imply expectational stability.

2) IH learning and FH learning imply different E-stability conditions.

Proof. As in the previous Proposition, we consider the model with pricing equation (34), assuming $\ell = 1$. The determinant of $M^{IH}(a)$ is

$$\frac{\beta([(\phi-1) + (\phi-1)\,\gamma\sigma]\,\xi + \sigma\phi\bar{\lambda}\,[(1-\beta)-\xi])}{(1-\beta)\,(1-\vartheta\beta)} > 0 \tag{36}$$

[20] That is

$$\Psi'(\beta) = -(1+\sigma\phi\bar{\lambda})\beta - \zeta\phi.$$

[21] We stress that the reverse does not hold: indeterminate equilibria can be E-stable in this model.

provided (35) holds. The trace is

$$\frac{\beta-1+\xi\phi}{1-\vartheta\beta}-\frac{\beta\sigma\phi\bar{\lambda}}{1-\beta}, \tag{37}$$

and it is negative provided

$$\bar{\lambda}>\bar{\lambda}_{*}=\frac{1-\beta}{1-\vartheta\beta}\frac{[\xi\phi-1+\beta]}{\beta\sigma\phi} \tag{38}$$

which adds an extra restriction on the parameters with respect to the one that guarantees determinacy. The extra restriction does not vanish once we set $\vartheta = 1$, but it is clear that the restriction is less stringent in this case. Also, from Proposition 1, inequality (38) holds also under the Euler approach (equivalent to the case where $\vartheta = 1$). Considering the coefficients on the shocks, the trace is

$$-\frac{\beta(1+\sigma\phi\bar{\lambda}-\rho)}{(1-\rho\beta)}+\frac{(\xi\phi+\rho\beta-1)}{(1-\vartheta\beta\rho)} \tag{39}$$

which is negative if the condition for the constant is satisfied. The determinant is

$$-\frac{\beta(\xi\phi+\rho\beta-1+\sigma\phi\bar{\lambda}\rho\beta-\sigma\phi\bar{\lambda}-\xi\phi\rho}{(1-\rho\beta)(1-\vartheta\beta\rho)} \\ \frac{-\rho^{2}\beta+\rho-\xi\phi\gamma\sigma-\xi\phi+\xi\rho\gamma\sigma+\xi\rho+\xi\rho\phi\bar{\lambda}\sigma)}{(1-\rho\beta)(1-\vartheta\beta\rho)} \tag{40}$$

which can be simplified to

$$\beta\left[\xi\left(\phi-\rho\right)+\xi\gamma\sigma\left(\phi-\rho\right)+\sigma\phi\bar{\lambda}\left(1-\rho\beta-\xi\rho\right)\right] \\ +\beta\left(1-\rho\right)\left(1-\beta\rho-\xi\phi\right)>0 \tag{41}$$

provided condition (38) is verified. ■

3.8 An Example

We now consider a calibrated version of the simple monetary model. We set parameters to values that are commonly used in the literature. In particular, we set $\beta = 0.99$, $\gamma = 0$ (infinitely elastic labor supply), $\sigma = 1$ and $\phi = 1.5$. We wish to show how determinacy and E-stability conditions change as we vary the degree of price rigidity. We measure the degree of price rigidity with α, the probability of not having an opportunity to set the price. We then calibrate ξ consistently with a given value of α.

The parameter $\bar{\lambda}$ governs the policymaker response to the output gap in the monetary policy rule. Figure 1 shows how $\bar{\lambda}^*$ from (35) and $\bar{\lambda}_*$ from (38) change as the degree of price rigidity changes. We plot both $\bar{\lambda}_*^{\vartheta=1}$, consistent with finite horizon learning, and $\bar{\lambda}_*^{\vartheta=\alpha}$ for infinite horizon learning. According to the above discussion, for a given α, expectational stability obtains for $\bar{\lambda}_* < \bar{\lambda} < \bar{\lambda}^*$.

The Figure shows that for the infinite horizon learning, the interval for $\bar{\lambda}$ consistent with determinacy and expectational stability is roughly constant as the degree of price flexibility increases, perhaps with some exception toward the right in the diagram. However, for finite horizon learning the interval for $\bar{\lambda}$ consistent with determinacy and expectational stability is generally smaller, and there would be no interval at all for a sufficiently high degree of price stickiness, that is, toward the left in the diagram. The unlabeled black lines in the diagram indicate the difference in the E-stability regions implied by the two alternative learning approaches.

4 Discussion

The "general model" described by (1) was chosen because it allows a clear characterization of the connection between determinacy and E-stability while maintaining analytical tractability. McCallum (2007a) and Ellison and Perlman (2008) explore the connection between E-stability and determinacy in models with endogenous lagged variables but focus only on the finite horizon approach. McCallum (2007a) does not consider the case of information delays, which are shown to be crucial in breaking the connection between determinacy and learnability. Ellison and Perlman (2008) discuss alternative information assumptions in expectations formation and find that they do not play a role in the stability analysis. The class of model they consider does not include informational delays in the agents decision rules which explains the different conclusions in the two papers.

The purely forward-looking class of models discussed here offers a rich set of results. Importantly, it stresses the importance of focusing *both* on determinacy

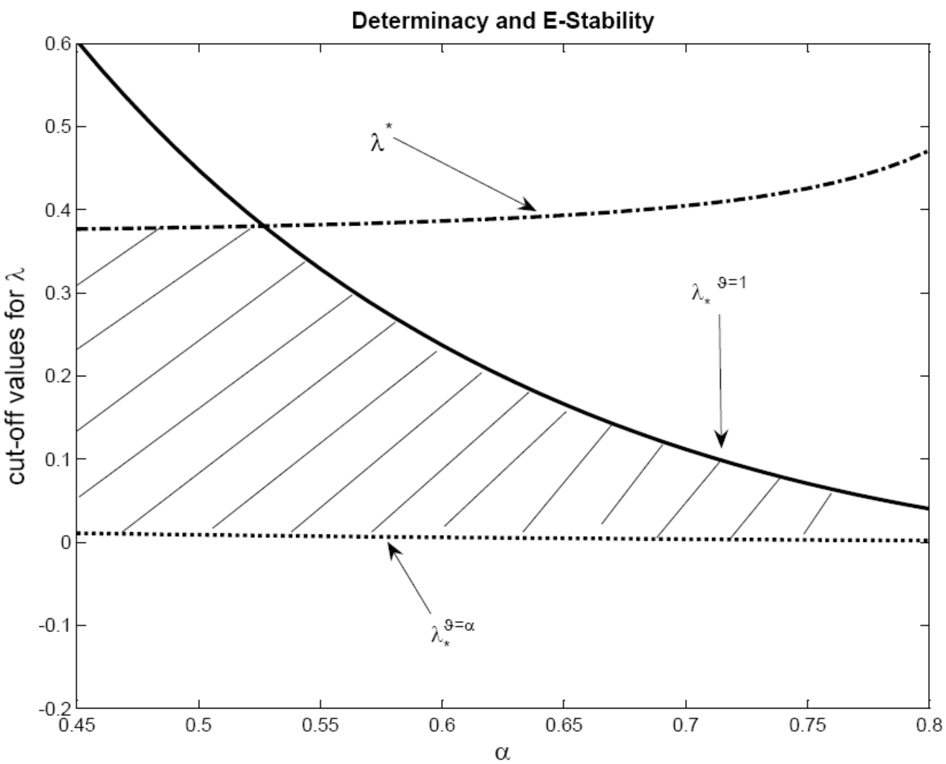

Figure 1: The relationship between determinacy and E-stability as a function of the degree of price stickiness α and the policymaker response to the output gap λ. Expectational stability requires a value of λ between λ_* and λ^*. IH learning, $\vartheta = \alpha$, creates a larger region than FH learning, $\vartheta = 1$.

and E-stability when evaluating the stability of rational expectations equilibria. Moreover, it emphasizes the importance of a microfounded approach to modeling learning dynamics. Concerning the absence of lag structures in our model, we can nevertheless point at two examples where a numerical exploration of the models accords with our results. Consider a standard real business cycle model with aggregate externalities, as in Benhabib and Farmer (1994). As shown in Eusepi and Preston (2008) the model has an infinite horizon behavioral equation and only one discount factor. Numerical exploration of the model shows that E-stability conditions under the infinite horizon and finite horizon approaches coincide, and determinacy implies E-stability. Also, a numerical exploration of NK model with endogenous capital accumulation and no information delays shows that E-stability conditions under infinite horizon and finite horizon are different.[22] Moreover, E-stability conditions under infinite horizon learning do not coincide with determinacy conditions. These are only examples, and they clearly do not offer proof in general that the results proposed in this paper extend to more complex models. However, they do offer some evidence that our proposed explanation for the break-down in the connection between determinacy and E-stability might survive increased model complexity.

5 Conclusions

We have studied a general class of models and investigated the connection between determinacy and E-stability. As an example we have applied our findings to a New Keynesian model generalized on certain dimensions in an attempt to delineate the differences between conditions for equilibrium determinacy and conditions for equilibrium learnability in terms of meaningful economic assumptions. One of the sharpest findings is that in models with informational lags, the connections between determinacy and learnability conditions are broken. In other situations, and certainly in some interesting baseline cases, we are able to show that determinacy does imply learnability and thus that in some practical settings, it is not necessary to analyze the two conditions separately. We are also

[22] In particular, Taylor rules that respond too aggressively to output deliver E-instability under the infinite horizon approach.

able to illustrate some interesting cases where there are and are not differences between the Euler equation and infinite horizon approaches to learnability.

An important caveat to this analysis is that there are no natural lags in the NK framework as we have analyzed it here, such as those that might come from time-to-build technologies or related concepts. The class of models considered in this paper is entirely forward-looking. Given that we have found that information lags are one important source of differences in conditions for determinacy versus learnability, one might suspect that in models with more realistic lag structures, conditions for determinacy will in general not be the same as conditions for learnability. We leave this as an interesting topic for future research.

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APPENDICES

A Derivation of the consumption rule

Combining the household's budget constraint and the cash-in-advance constraint (which holds with equality at a positive nominal interest rate), we obtain

$$M_{t+1}^i \geq (1+i_t)\, M_t^i - (1+i_t)\, P_t C_t^i + (1+i_t)\, W_t h_t^i + \Phi_t^i, \tag{42}$$

where Φ_t^i denotes the household shares in the profits of firms and financial intermediaries. Solving forward and making use of the tranversality condition we obtain

$$M_t^i \geq \sum_{j=0}^{\infty} R_{t,t+j} \left[P_{t+j} C_{t+j} + \left(i_{t+j} / \left(1+i_{t+j}\right)\right) M_{t+j+1}^i - T_{t+j}^i - P_{t+j} Y_{t+j}^i \right], \tag{43}$$

which must hold in all states and dates, where

$$R_{t,t+j} = \prod_{k=1}^{j} (1 + i_{t+k-1})^{-1} \tag{44}$$

and where $P_t Y_t^i = W_t h_t^i + P_t \int \left[\phi_t^f (j) dj + i_t W_t h_t \right] dj$. To obtain the expression above we use[23]

$$\frac{(1 + i_t) W_t h_t^i + \Phi_t^i}{(1 + i_t)} = \left[W_t h_t^i + P_t \frac{\int \phi_t^f (j) dj}{1 + i_t} + (i_t / (1 + i_t)) \int P_t C_t dj + T_t \right] - (i_t / (1 + i_t)) M_{t+1}^i, \tag{45}$$

where the second and third terms in brackets take into account that wages are anticipated by households at the beginning of the period, while revenues are transferred at the end of the period. Furthermore, for simplicity we assume each household understands that the present value of government tranfers must be equal to the present value of government liabilities held net of seigniorage,[24] so that the intertemporal budget constraint simplifies to

$$0 = \sum_{j=0}^{\infty} R_{t,t+j} \left[P_{t+j} C_{t+j} - P_{t+j} Y_{t+j} \right], \tag{46}$$

where for simplicity we assume each household has zero initial net wealth. Log-linearizing the intertemporal budget constraint we obtain

$$0 = \sum_{j=0}^{\infty} \beta^t \hat{C}_{t+j} - \sum_{j=0}^{\infty} \beta^t \hat{Y}_{t+j}. \tag{47}$$

The first order condition for consumption gives

$$U_c \left(C_t^i \right) = \beta \hat{E}_{t-\ell} \left[\frac{(1 + i_t) U_c \left(C_{t+1}^i \right)}{\Pi_{t+1}} \right]. \tag{48}$$

[23] The last expression can be simplified by using $P_t C_t = M_{t+1}$ in every period along with the condition that aggregate sales must be equal to aggregate consumption. Also notice that $C_t^i = C_t$ and $M_t^i = M_t$ in every period and for every household i.

[24] Eusepi and Preston (2007b) relax this assumption and study the implications of learning when agents have to forecast fiscal variables.

Linearization of the Euler equation yields

$$\hat{C}_t = \hat{E}_{t-\ell}\hat{C}_{t+1} - \sigma\hat{E}_{t-\ell}(\hat{\imath}_t - \pi_{t+1}),$$

and solving backward we obtain

$$\hat{C}_t = \hat{E}_{t-\ell}\hat{C}_T + \sigma\hat{E}_{t-\ell}\sum_{s=t}^{T-1}(i_s - \pi_{s+1}).$$

Taking the expectation of the budget constraint yields

$$\hat{E}_{t-\ell}\sum_{T=t}^{\infty}\beta^{T-t}\hat{C}_T = \hat{E}_{t-\ell}\sum_{T=t}^{\infty}\beta^{T-t}\hat{Y}_T, \tag{49}$$

and substituting for $E_{t-1}\hat{C}_T$ we obtain

$$\hat{E}_{t-\ell}\sum_{T=t}^{\infty}\beta^{T-t}\left[\hat{C}_t - \sigma\sum_{s=t}^{T-1}(i_s - \pi_{s+1})\right] = \hat{E}_{t-\ell}\sum_{T=t}^{\infty}\beta^{T-t}\hat{Y}_T$$

which implies

$$\hat{C}_t = \hat{E}_{t-\ell}\sum_{T=t}^{\infty}\beta^{T-t}\left[(1-\beta)\hat{Y}_T - \beta\sigma(i_T - \pi_{T+1})\right]. \tag{50}$$

B Marginal cost and the output gap

From the households' first order conditions, the labor supply is[25]

$$\frac{V_h\left(h_t^i\right)}{u_c\left(C_t^i\right)} = \frac{w_t}{P_t}. \tag{51}$$

Log-linearizing (51) and (24) and combining them gives a relation between output and the real marginal cost as

$$\hat{s}_t = \left(\sigma^{-1} + \gamma\right)\hat{y}_t - (1+\gamma)\hat{a}_t + \hat{\imath}_t, \tag{52}$$

where γ is the inverse of the elasticity of the labor supply, which depends on the disutility of hours worked, and σ is the intertemporal elasticity of substitution in consumption. The marginal cost also depends on the nominal interest rate, depending on the wage bill that needs to be anticipated by the firm.

[25] We stress the assumption that households take their labor supply decisions using current information about real wages.

Next, we define $\hat{y}_t^n$ as the natural level of output, that is, the equilibrium level of output under flexible prices and no information delays, which can be shown to be

$$y_t^n = \frac{(1+\gamma)}{(\sigma^{-1}+\gamma)}\hat{a}_t - \left(\sigma^{-1}+\gamma\right)^{-1}\hat{\imath}_t^n. \tag{53}$$

Using this expression we can rearrange the marginal cost equation to obtain

$$\hat{s}_t = \left(\sigma^{-1}+\gamma\right)\left(y_t - y_t^n\right) + \left(\hat{\imath}_t - \hat{\imath}_t^n\right). \tag{54}$$

We assume that in the flexible price equilibrium the nominal interest rate is set to zero, eliminating the labor supply distortion, as in Ravenna and Walsh (2006). Hence in the remainder of the paper we set $\hat{\imath}_t^n = 0$.

C Optimal policy under discretion

In this appendix we show that the monetary policy rule in the main text is of the same form as if the policymaker was pursuing a certain optimal policy. We consider optimal policy under *rational expectations*,[26] meaning that the central bank is assuming rational expectations of the private sector when it is deciding upon an optimal policy. The behavioral equations for output gap and inflation can be reduced to

$$x_t = -\sigma E_{t-\ell}\left(\hat{\imath}_t - \pi_{t+1}\right) + E_{t-\ell}x_{t+1} + r_t^n \tag{55}$$

and

$$\pi_t = \beta E_{t-\ell}\pi_{t+1} + E_{t-\ell}\xi\left(\left(\sigma^{-1}+\gamma\right)x_t + \hat{\imath}_t\right) \tag{56}$$

where $\xi = \frac{(1-\alpha\beta)(1-\alpha)}{\alpha}$. The central bank maximizes a quadratic loss function in inflation and output gap

$$L = \pi_t^2 + \lambda x_t^2 \tag{57}$$

which is consistent with the model's microfoundations. We consider the general case where λ is arbitrary.[27] The central bank chooses optimal policy to maximize

$$E_{t-\ell}\sum_{t=0}^{\infty}\beta^t L_t \tag{58}$$

[26] We could do this under learning but it would complicate the analysis a lot without making any difference for the issues we are discussing.

[27] This because we are going to show that the instability result is not only obtained in the case where the central bank maximizes the representative consumer's welfare function.

$$E_{t-\ell}\sum_{T=t}^{\infty}\beta^{t}\left(\pi_{T}^{2}+\lambda x_{T}^{2}\right) \tag{59}$$

subject to (55) and (56). The first order conditions are, for x_t:

$$-\lambda x_{t}-\xi\left(\sigma^{-1}+\gamma\right)E_{t-\ell}\psi_{2,t}+E_{t-\ell}\psi_{1,t}=0, \tag{60}$$

for π:

$$-\pi_{t}+E_{t-\ell}\psi_{2,t}=0, \tag{61}$$

and for i_t:

$$-\xi\zeta E_{t-\ell}\psi_{2,t}+\sigma E_{t-\ell}\psi_{1,t}=0. \tag{62}$$

Eliminating the multipliers, we obtain two conditions

$$-\lambda E_{t-\ell}x_{t}-\xi\left(1+\gamma\right)E_{t-\ell}\pi_{t}+\xi E_{t-\ell}\pi_{t} = 0, \tag{63}$$

$$\bar{\lambda}E_{t-\ell}x_{t}+E_{t-\ell}\pi_{t} = 0, \tag{64}$$

where $\bar{\lambda}=\frac{\lambda}{\xi\gamma}$. The central bank is responding to private sector forecasts under the assumption that the private sector forecasts are rational. This policy rule is of the same form as the one employed in the main text, under the assumption that the targeting rule is implemented using a Taylor-type rule.