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Solving Linear Difference Systems with Lagged Expectations by a Method of Undetermined Coefficients*

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Abstract

This paper proposes a solution method to solve linear difference models with lagged expectations in the form of

$$AE_tS_{t+1} + \sum_{i=1}^{N} \Lambda_i E_{t-i}S_{t+1} = BS_t + \sum_{i=1}^{N} \Gamma_i E_{t-i}S_t + CZ_t,$$

where N is the order of lagged expectations. Variables with lagged expectations expand the model's state space greatly when N is large; and getting the system into a canonical form solvable by the traditional methods involves substantial manual work (e.g., arranging the state vector S_t and the associated coefficient matrices to accommodate $E_{t-i}S_t$, $i = 1, 2, ..., N$), which is prone to human errors. Our method avoids the need of expanding the state space of the system and shifts the burden of analysis from the individual economist/model solver toward the computer. Hence it can be a very useful tool in practice, especially in testing and estimating economics models with a high order of lagged expectations. Examples are provided to demonstrate the usefulness of the method. We also discuss the implications of lagged expectations on the equilibrium properties of indeterminate DSGE models, such as the serial correlation properties of sunspots shocks in these models.

Keywords: Undetermined Coefficients, Linear Difference System, Lagged Expectations, Labor Hoarding, Sticky Information, Indeterminacy, Serially Correlated Sunspots.

JEL Codes: C63, C68, E30, E40.

*Matlab programs used in this paper are available at our web sites. We thank John McAdams for excellent research assistance. The views expressed in the paper and any errors that may remain are the authors' alone. Correspondence: Yi Wen, Research Department, Federal Reserve Bank of St. Louis, St. Louis, MO, 63144. Phone: 314-444-8559. Fax: 314-444-8731. Email: yi.wen@stls.frb.org.

1 Introduction

The standard methods of solving linear rational expectations models deal with the linear difference system,

$$AE_tS_{t+1} = BS_t + CZ_t, \tag{1}$$

where S is a vector of endogenous variables, Z is a vector of exogenous forcing variables, and E_t denotes the expectation operator based on information available up to period t. Linear rational expectations models with lagged expectations take the form,

$$AE_tS_{t+1} + \sum_{i=0}^{N} \Lambda_i E_{t-i}S_{t+1} = BS_t + \sum_{i=1}^{N} \Gamma_i E_{t-i}S_t + CZ_t, \tag{2}$$

which differs from system (1) by the lagged expected variables, $E_{t-i}S_t$ $(i > 0)$. Thus, system (2) is more general than system (1).

Linear difference systems with lagged expectations are becoming increasingly popular as a way to characterize economic behaviors in macroeconomics. For example, lagged expectations exist in DSGE models when some choice variables in period t must be determined $N > 0$ periods in advance, or be based on information available N periods earlier, before the current economic shocks are realized in period t. This type of model includes the labor hoarding model of Burnside, Eichenbaum, and Rebelo (1993) and the sticky-information model of Mankiw and Reis (2002).

To directly apply the traditional solution methods, such as that of Blanchard and Kahn (1980), King and Watson (1998), Christiano (2002), or Klein (2000) and Sims (2002), to solve system (2) requires expanding the state vector S_t to include $E_{t-i}S_t$ $(i = 1, 2, ..., N)$ as additional state variables. This is not a trivial task, especially when N is large. For example, in the sticky-information model of Mankiw and Reis (2002), $N = \infty$. Notice that the state variables with lagged expectations are not predetermined variables, hence they must also be solved for in equilibrium. An example of solving a DSGE sticky-information model directly by traditional method can be found in Andrés, López-Salido, and Nelson (2005). Because of the inconvenience of traditional method, these authors have to restrict the order of lagged expectations to be $N = 4$. This arbitrary truncation may suffer from accuracy problems and can severely distort a model's true underling dynamics.

This paper proposes an alternative method to solve linear difference systems with lagged expectations such as system (2). The solution method is a combination of the method of undetermined coefficients discussed in Taylor (1986) and the traditional methods based on Blanchard and Kahn (1980). Our method treats the order of lagged expectations N as a free parameter and is easy to implement. Under this method the dimension of the state space in system (2) is exactly the same as that in system (1). Thus there is no need to expand the state space of system (1) in the presence of lagged expectations. This method is also simpler than that used by Taylor because in our method there is no need to determine the entire infinite sequence of coefficients in the $MA(\infty)$ representations of the decision rules.

The key of the method is to convert any lagged expected variables, $E_{t-i}S_t$, into i-step ahead forecast errors, $S_t - E_{t-i}S_t$. Since in equilibrium, S_t always has the moving average representation, $S_t = \sum_{j=0}^{\infty} \Phi_j \boldsymbol{\varepsilon}_{t-j}$, where $\boldsymbol{\varepsilon}$ is a vector of $i.i.d$ innovations in the exogenous forcing processes Z_t, the i-step ahead forecast error is given by $S_t - E_{t-i}S_t = \sum_{j=0}^{i-1} \Phi_j \boldsymbol{\varepsilon}_{t-j}$, which is a finite moving average process and is much easier to handle than the expected variables $E_{t-i}S_{t+l}$. Thus, system (2) can be converted into system (1) easily with just a finite number of undetermined coefficient matrices $\{\Phi_j\}_{j=0}^{N}$. This is an easier task (especially when N is large) since getting the system (2) into a form solvable directly by the traditional methods may involve substantial manual work, as expanding the state vector S_t and changing the associated coefficient matrices to accommodate $E_{t-i}S_t$ $(i = 1, 2, ..., N)$ cannot be done mechanically by computers and is hence prone to human errors, whereas our method shifts the burden of analysis from the individual economist/model solver toward the computer.

In this paper, we also consider several examples to illustrate the usefulness of our method in solving DSGE models with lagged expectations. In particular, we show how serially correlated sunspots shocks (in contrast to the $i.i.d$ sunspots shocks) can be introduced into the Benhabib-Farmer (1994) model via lagged expectations. This example refutes a popular notion in the indeterminacy literature that introducing predetermined variables or dynamic adjustment costs in capital or labor necessarily eliminates sunspots equilibria in the Benhabib-Farmer model.

2 The Method of Undetermined Coefficients

Let $X_t = [x_{1t}, x_{2t}, ..., x_{pt}]'$ be a vector of non-predetermined variables in period t, let $Y_t = [y_{1t}, y_{2t}, ...y_{q_t}]'$ be a vector of predetermined variables in period t, and let $Z_t = [z_{1t}, z_{2t}, ..., z_{mt}]'$ be a vector of exogenous forcing variables with the law of motion given by $Z_t = \boldsymbol{\rho} Z_{t-1} + \boldsymbol{\varepsilon}_t$, where $\boldsymbol{\rho}$ is a stable $m \times m$ matrix and $\boldsymbol{\varepsilon}$ is an $m \times 1$ vector of othorgonal $i.i.d.$ innovations with realization in the beginning of period t. Let $S_t = [X_t', Y_t', Z_t']'$ and let $k = p + q + m$. A rational expectations DSGE model with lagged expectations can always be reduced to the following log-linear difference system:

$$AE_tS_{t+1} + \sum_{i=1}^{N} \Lambda_i E_{t-i}S_{t+1} = BS_t + \sum_{i=1}^{N} \Gamma_i E_{t-i}S_t, \tag{3}$$

where $\{A, B, \Lambda_i, \Gamma_i\}$ are $k \times k$ coefficient matrices.

The corresponding system without lagged expectations is given by

$$\tilde{A} E_t S_{t+1} = \tilde{B} S_t, \tag{3'}$$

where $\tilde{A} = \left(A + \sum_{i=1}^{N} \Lambda_i\right), \tilde{B} = \left(B + \sum_{i=1}^{N} \Gamma_i\right)$. Thus, a solution exists for system (3) if and only if a solution exists for system (3′). In other words, the eigenvalues of system (3) and system (3′) are identical. This can be seen by taking expectations on both sides of (3) based on E_{t-N} to get

$$\tilde{A} E_{t-N} S_{t+1} = \tilde{B} E_{t-N} S_t. \tag{3''}$$

If S_t^* is a stationary solution for equation (3), then $E_{t-N} S_t^*$ must also be stationary and hence it must constitute a solution for equation (3″). On the other hand, if $E_{t-N} S_t^*$ is a stationary solution for equation (3″), then S_t^* must also be stationary and hence it must constitutes a solution for equation (3).

An implication of this is that letting some variables be predetermined in a DSGE model, such as via labor hoarding or sticky information, will not affect the equilibrium properties of the model in terms of the existence and multiplicity of equilibria. For example, if the original model without labor hoarding (or sticky-information) has a unique equilibrium, then introducing labor hoarding (or sticky information) has no effect on the uniqueness of the equilibrium. Similarly, if the model without labor hoarding (or sticky information) is indeterminate, then introducing labor hoarding or sticky-information will not eliminate indeterminacy. This also implies that linear sunspots shocks to expectation errors do not have to be *i.i.d* processes in indeterminate DSGE models once lagged expectations are introduced. Linear sunspots shocks can be serially correlated processes under lagged expectations. The order of the serial correlations of sunspots are determined by the order of lagged expectations in the indeterminate model. An example is provided in the next section.

Our method of undetermined coefficients constitutes the following steps:

Step 1. Replace the lagged expected variables ($E_{t-i} S_t$) by the forecast errors ($S_t - E_{t-i} S_t$) with undetermined coefficients $\{\Phi_j\}$.

Since the solution for S_t can always be expressed as a moving average process, $S_t = \sum_{i=0}^{\infty} \Phi_i \varepsilon_{t-i}$, where Φ_i are undetermined $k \times m$ matrices, we have $S_t - E_{t-i} S_t = \sum_{j=0}^{i-1} \Phi_j \varepsilon_{t-j}$, for $i = 1, 2, ..., N$.[1] Adding well defined zeros, $(S_{t+1} - S_{t+1})$ and $(S_t - S_t)$, to Equation (3) and re-arranging terms, we have

$$\tilde{A} E_t S_{t+1} + \sum_{i=1}^{N} \Lambda_i \left(E_{t-i} S_{t+1} - E_t S_{t+1}\right) = \tilde{B} S_t + \sum_{i=1}^{N} \Gamma_i \left(E_{t-i} S_t - S_t\right), \tag{4}$$

[1] Note that the forecast errors for the exogenous forcing vector Z_t are known, hence we only need to determine a subset of the elements in Φ_i.

where $\tilde{A} = \left(A + \sum_{i=1}^{N} \Lambda_i\right), \tilde{B} = \left(B + \sum_{i=1}^{N} \Gamma_i\right)$. Substituting out the j-step ahead forecast errors by their corresponding moving average processes with undetermined coefficients gives

$$\begin{aligned} & \tilde{A}E_t S_{t+1} - \left(\sum_{i=1}^{N} \Lambda_i\right) \Phi_1 \boldsymbol{\varepsilon}_t - \left(\sum_{i=2}^{N} \Lambda_i\right) \Phi_2 \boldsymbol{\varepsilon}_{t-1} - \dots - \Lambda_N \Phi_N \boldsymbol{\varepsilon}_{t-N+1} \\ = \quad & \tilde{B} S_t - \left(\sum_{i=1}^{N} \Gamma_i\right) \Phi_0 \boldsymbol{\varepsilon}_t - \left(\sum_{i=2}^{N} \Gamma_i\right) \Phi_1 \boldsymbol{\varepsilon}_{t-1} - \dots - \Gamma_N \Phi_{N-1} \boldsymbol{\varepsilon}_{t-N+1}. \end{aligned} \tag{4'}$$

Notice that we only need to determine the finite sequence of undetermined coefficients, $\{\Phi_0, \Phi_1, ..., \Phi_N\}$, instead of the infinite sequence of undetermined coefficients, $\{\Phi_i\}_{i=0}^{\infty}$. Define $\tilde{\Lambda}_j = \sum_{i=j}^{N} \Lambda_i$ and $\tilde{\Gamma}_j = \sum_{i=j}^{N} \Gamma_i$. Equation (4′) becomes

$$\tilde{A}E_t S_{t+1} = \tilde{B} S_t + \left[\Omega_\lambda \Psi_1 - \Omega_\gamma \Psi_0\right] \epsilon_t, \tag{4''}$$

where

$$\Omega_\lambda \equiv \begin{bmatrix} \tilde{\Lambda}_1 & \tilde{\Lambda}_2 & \cdots & \tilde{\Lambda}_N \end{bmatrix}_{k \times Nk}, \quad \Omega_\gamma \equiv \begin{bmatrix} \tilde{\Gamma}_1 & \tilde{\Gamma}_2 & \cdots & \tilde{\Gamma}_N \end{bmatrix}_{k \times Nk},$$

$$\Psi_1 \equiv \begin{bmatrix} \Phi_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Phi_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \Phi_N \end{bmatrix}_{Nk \times Nm}, \quad \Psi_1 \equiv \begin{bmatrix} \Phi_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Phi_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \Phi_{N-1} \end{bmatrix}_{Nk \times Nm}$$

$$\epsilon_t \equiv \left[\boldsymbol{\varepsilon}_t', \boldsymbol{\varepsilon}_{t-1}', ..., \boldsymbol{\varepsilon}_{t-N+1}'\right]_{Nm \times 1}';$$

where the subscript outside each square parentheses denotes the dimension of the matrix. The white noise vector ϵ_t has the following law of motion:

$$\epsilon_{t+1} = \Theta \epsilon_t + \begin{bmatrix} I_m \\ \mathbf{0} \end{bmatrix} \boldsymbol{\varepsilon}_{t+1}, \tag{5}$$

where I_m is an $m \times m$ identity matrix and

$$\Theta \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ I_m & \mathbf{0} & \ddots & \ddots & \vdots \\ \mathbf{0} & I_m & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & I_m & \mathbf{0} \end{bmatrix}_{Nm \times Nm},$$

which has the property of $\Theta^j = \mathbf{0}$ for $j \geq Nm$.

Step 2. Solve equation $(4'')$.

Notice that equation $(4'')$ is a standard system without lagged expectations. Hence we can solve the system with standard methods such as that proposed by Blanchard and Kahn (1980), King and Watson (1998), Christiano (2002), or Klein (2000) and Sims (2002).[2] The solution is a set of decision rules of the form

$$X_t = H_y \begin{bmatrix} Y_t \\ Z_t \end{bmatrix} + H_\epsilon(\Psi)\epsilon_t, \tag{7}$$

$$\begin{bmatrix} Y_{t+1} \\ Z_{t+1} \end{bmatrix} = M_y \begin{bmatrix} Y_t \\ Z_t \end{bmatrix} + M_\epsilon(\Psi)\epsilon_t + \begin{bmatrix} \mathbf{0} \\ I_m \end{bmatrix} \boldsymbol{\varepsilon}_{t+1}, \tag{8}$$

where $\Psi \equiv (\Omega_\lambda \Psi_1 - \Omega_\gamma \Psi_0)$. It can be shown that $H_\epsilon(\Psi)$ and $M_\epsilon(\Psi)$ depend linearly on the undetermined coefficient matrix Ψ and that H_y and M_y do not depend on Ψ.

Step 3. Solve for $\{\Phi_0, \Phi_1, ..., \Phi_N\}$.

Denote as the expanded state vector $Q_t \equiv \begin{bmatrix} Y_t' & Z_t' & \epsilon_t' \end{bmatrix}'$. The equilibrium decision rules and laws of motion (7)-(8) can be expressed as

$$X_t = H(\Psi)Q_t \tag{7'}$$

$$Q_{t+1} = M(\Psi)Q_t + G\boldsymbol{\varepsilon}_{t+1}. \tag{8'}$$

Hence, the forecast errors are given by

$$\begin{aligned} Q_t - E_{t-i}Q_t &= G\boldsymbol{\varepsilon}_t + M(\Psi)G\boldsymbol{\varepsilon}_{t-1} + ... + M^{i-1}(\Psi)G\boldsymbol{\varepsilon}_{t-i+1} \\ &= \sum_{j=0}^{i-1} M^j(\Psi)G\boldsymbol{\varepsilon}_{t-j}, \end{aligned} \tag{9}$$

$$\begin{aligned} X_t - E_{t-i}X_t &= H(\Psi)\left[Q_t - E_{t-i}Q_t\right] \\ &= H\sum_{j=0}^{i-1} M^j(\Psi)G\boldsymbol{\varepsilon}_{t-j}, \end{aligned} \tag{10}$$

for $i = 0, 1, 2, ..., N-1$. Equation systems (9) and (10) can be stacked into the following form after leaving out the exogenous variables ϵ_t from the bottom rows in equation system (9):

$$\begin{bmatrix} X_t \\ Y_t \\ Z_t \end{bmatrix} - E_{t-i} \begin{bmatrix} X_t \\ Y_t \\ Z_t \end{bmatrix} = \sum_{j=0}^{i-1} P_j(\Psi)\boldsymbol{\varepsilon}_{t-j}. \tag{11}$$

[2] Our method is completely different from that of Christiano (2002). Christiano's method of undetermined coefficients does not involve transforming variables with lagged expectations ($E_{t-i}S_t$) to the corresponding forecast errors ($S_t - E_{t-i}S_t$).

By definition (4) we also have

$$\begin{bmatrix} X_t \\ Y_t \\ Z_t \end{bmatrix} - E_{t-i} \begin{bmatrix} X_t \\ Y_t \\ Z_t \end{bmatrix} = \sum_{j=0}^{i-1} \Phi_j \boldsymbol{\varepsilon}_{t-j}. \tag{12}$$

Recall that $\Psi = (\Omega_\lambda \Psi_1 - \Omega_\gamma \Psi_0)$ is a super matrix with Φ_j $(j = 0, 1, ..., N)$ as its elements. Clearly, the equivalence of representation (11) and representation (12) constitutes $N \times m$ equations with $(N+1) \times m$ unknowns in $\{\Phi_j\}_{j=0}^{N}$. An extra equation is found by updating equations (11) and (12) by one period forward and then taking expectation based on E_t. In particular, for $i = 1, 2, ...N+1$, term-by-term comparison between (11) and (12) for all of the coefficients of $\boldsymbol{\varepsilon}_{t-i}$ suggests that

$$\begin{aligned} P_0(\Omega_\lambda \Psi_1 - \Omega_\gamma \Psi_0) &= \Phi_0 \\ P_1(\Omega_\lambda \Psi_1 - \Omega_\gamma \Psi_0) &= \Phi_1 \\ \vdots &= \vdots \\ P_N(\Omega_\lambda \Psi_1 - \Omega_\gamma \Psi_0) &= \Phi_{N-1} \\ P_{N+1}(\Omega_\lambda \Psi_1 - \Omega_\gamma \Psi_0) &= \Phi_N, \end{aligned} \tag{13}$$

where the last equation in (13) is derived by updating (11) and (12) by one period forward and then taking expectation based on E_t. System (13) can be compactly expressed as

$$P(\Psi) = \Psi. \tag{14}$$

The solution for the sequence $\{\Phi_i\}_{i=0}^{N}$ can thus be found as a fixed point for equation (14). Although analytical solutions for equation (14) exist, it can also be solved numerically using standard packages in Gauss or Matlab. In particular, since (14) is a linear mapping, the existence of an analytical solution and the fast speed of convergence under numerical iterations are guaranteed. The linear property of the function $P(\Psi)$ is established by the following proposition:

Proposition 1 *$P_j(\Psi)$ is linear in Ψ for all j.*

Proof. See the Appendix. ■

In the next section we consider several examples to illustrate the usefulness of our method of undetermined coefficients for solving linear difference systems with lagged expectations.

3 Examples

3.1 The Mankiw-Reis Model in General Equilibrium

DSGE models with sticky information have also been studied by Keen (2004) and Trabandt (2005), among others.[3] A version of the model is described as follows. A representative household in the model chooses consumption (c), labor (n), investment (i), money holdings (M), and bond holdings (B) to solve

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\gamma} - 1}{1-\gamma} - a \frac{n_t^{1+\gamma_n}}{1+\gamma_n} \right)$$

subject to

$$\left(M_{t+1} + B_{t+1}\right)/P_t + (c_t + i_t) \leq w_t n_t + q_t k_t + \pi_t + \left(M_t + X_t + R_{t-1} B_t\right)/P_t, \tag{15}$$

$$c_t + i_t \leq \left(M_t + X_t\right)/P_t, \tag{16}$$

where $i_t = k_{t+1} - (1-\delta)k_t$ denotes investment. The budget constraint (15) implies that the household begins in period t with initial levels of money, M_t, bonds, B_t, capital, k_t, real profits (dividends), π_t, and the wage and rental payment, $w_t n_t + q_t k_t$, from firms, where R_t is the gross nominal interest rate, w_t is the real wage rate, P_t is the aggregate price level, and q_t is the real rental rate of capital. X_t represents money injection from the monetary authority. The total income is allocated to the next period money demand, M_{t+1}, bond holdings, B_{t+1}, consumption, c_t, and investment, i_t. All purchases are subject to a cash-in-advance constraint (16).

Denote $\{\lambda_t, \mu_t\}$ as the Lagrangian multipliers for the budget constraint (15) and the CIA constraint (16), the first order conditions of the household with respect to $\{c_t, n_t, k_{t+1}, M_{t+1}, B_{t+1}\}$ are given, respectively, by

$$c_t^{-\gamma} = \lambda_t + \mu_t$$

$$a n^{\gamma_n} = \lambda_t w_t$$

$$\lambda_t + \mu_t = \beta(1-\delta) E_t \left(\lambda_{t+1} + \mu_{t+1}\right) + \beta E_t \lambda_{t+1} q_{t+1}$$

$$\lambda_t / P_t = \beta E_t \left(\lambda_{t+1} + \mu_{t+1}\right) / P_{t+1}$$

$$\lambda_t / P_t = \beta R_t E_t \lambda_{t+1} / P_{t+1}$$

Aggregate output is produced competitively using intermediate goods according to the technology,

[3] Keen (2004), however, does not discuss how his model is solved. Although Trabandt (2005) does discuss his solution technique, his method is very different from ours.

$$y = \left(\int_0^1 y(i)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}},$$

where $\sigma > 1$ measures the elasticity of substitution among the intermediate goods, $y(i)$. Let $p(i)$ denote the price of intermediate good i, the demand for intermediate goods is given by $y(i) = \left(\frac{p(i)}{P}\right)^{-\sigma} y_t$, and the relationship between the final goods price and intermediate goods prices is given by $P = \left(\int_0^1 p(i)^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}$.

Each intermediate good i is produced by a single monopolistically competitive firm according to the technology, $y(i) = k(i)^{\alpha} n(i)^{1-\alpha}$. Intermediate good firms face perfectly competitive factor markets, and hence have the following factor demand functions for $\{k, n\}$: $q_t = mc_t \alpha \frac{y_t(i)}{k_t(i)}, w_t = mc_t(1-\alpha)\frac{y_t(i)}{n_t(i)}$; where $mc = \left(\frac{q_t}{\alpha}\right)^{\alpha} \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}$ denotes real marginal cost. In each period, a fraction $(1-\theta)$ of firms update information about the state of the economy and set their optimal prices accordingly. The rest continue to set their prices based on old information. A firm who updated its information j periods ago maximizes its profits in the current period t by solving

$$\max_{p_t(j)} E_{t-j} \left(\frac{p_t(j)}{P_t} - mc_t \right) y_t(j),$$

subject to the demand constraint, $y_t(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\sigma} y_t$. The optimal price is given by $p_t(j) = \frac{\sigma E_{t-j}(P_t^{\sigma} y_t mc_t)}{(\sigma-1)E_{t-j}(P_t^{\sigma-1} y_t)}$. Following Mankiw and Reis, the aggregate price level is given by $P = (1-\theta)\left[\sum_{j=0}^{\infty} \theta^j p(j)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$. Log-linearizing the above two price-equations and re-arranging terms (see Mankiw and Reis, 2002, for details), we can obtain what Mankiw and Reis called the "sticky information Phillips Curve":

$$\pi_t = \frac{1-\theta}{\theta} mc_t + (1-\theta) \sum_{j=0}^{\infty} \theta^j E_{t-1-j}(\pi_t + \Delta mc_t)], \tag{17}$$

which can be re-arranged into an equation without lagged expectations but involving only forecast errors:

$$mc_t = \theta mc_{t-1} + (1-\theta) \sum_{j=1}^{\infty} \theta^j \left[\pi_t + \Delta mc_t - E_{t-j}(\pi_t + \Delta mc_t)\right] \tag{18}$$

In a symmetric equilibrium, $k(i) = k, n(i) = n$, and $y(i) = y$. Market clearing in the goods and the asset markets implies $c + i = y, M_{t+1} = M_t + X_t$, and $B_{t+1} = B_t = 0$. The equilibrium dynamics of the model can be approximated by the following system of log-linearized equations (after substituting out $\{\lambda, \mu, w, q, y\}$ using the first-order conditions):

$$-\gamma c_t = \beta(1-\delta)E_t(-\gamma c_{t+1}) + (1-\beta(1-\delta))\,E_t\left((\gamma_n+1)\,n_{t+1} - k_{t+1}\right)$$

$$(\gamma_n+\alpha)\,n_t - mc_t - \alpha k_t = E_t\left(-\gamma c_{t+1} - \pi_{t+1}\right)$$

$$mc_t = \theta mc_{t-1} + (1-\theta)\sum_{j=1}^{\infty}\theta^j\left[\pi_t + \Delta mc_t - E_{t-j}(\pi_t + \Delta mc_t)\right]$$

$$s_c c_t + s_i\left(\frac{1}{\delta}k_{t+1} - \frac{1-\delta}{\delta}k_t\right) = \alpha k_t + (1-\alpha)\,n_t$$

$$\alpha k_t + (1-\alpha)\,n_t - (\alpha k_{t-1} + (1-\alpha)\,n_{t-1}) = x_t - \pi_t$$

$$R_t = (\gamma_n+\alpha)\,n_t - mc_t - \alpha k_t - E_t\left((\gamma_n+\alpha)\,n_{t+1} - mc_{t+1} - \alpha k_{t+1} - \pi_{t+1}\right)$$

These equations can be arranged into a form similar to system (2) (after adding three identities, $k_t = k_t, n_t = n_t, mc_t = mc_t$, and the law of motion for money growth, $x_{t+1} = \rho x_t + \varepsilon_{t+1}$ into the system):

$$AE_tS_{t+1} = BS_t + G\sum_{i=1}^{\infty}\theta^i\left[S_t - E_{t-i}S_t\right], \tag{19}$$

where $S_{t+1} = [k', c', \pi', n', mc', R', k, n, mc, x']$ (note: $z' \equiv z_{t+1}$) and G is a 10×10 matrix with zeros everywhere except that the fourth row is given by the row vector

$$(1-\theta)\left[\begin{array}{cccccccccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{array}\right].$$

Letting $S_t = \sum_{i=0}^{\infty}\Phi_i\boldsymbol{\varepsilon}_{t-i}$, Equation(19) can be truncated to

$$\begin{aligned} AE_tS_{t+1} &= BS_t + G\sum_{i=1}^{N}\theta^i\Phi_{i-1}\varepsilon_{t-i} \\ &= BS_t + G\Omega\Psi\epsilon_t. \end{aligned} \tag{20}$$

Since A is singular, we use the method of King and Watson (1998) to solve Equation (20). Analysis shows that $N = 20$ gives very good results. For example, setting $N = 50$ does not lead to noticeable difference in the impulse responses of the model.

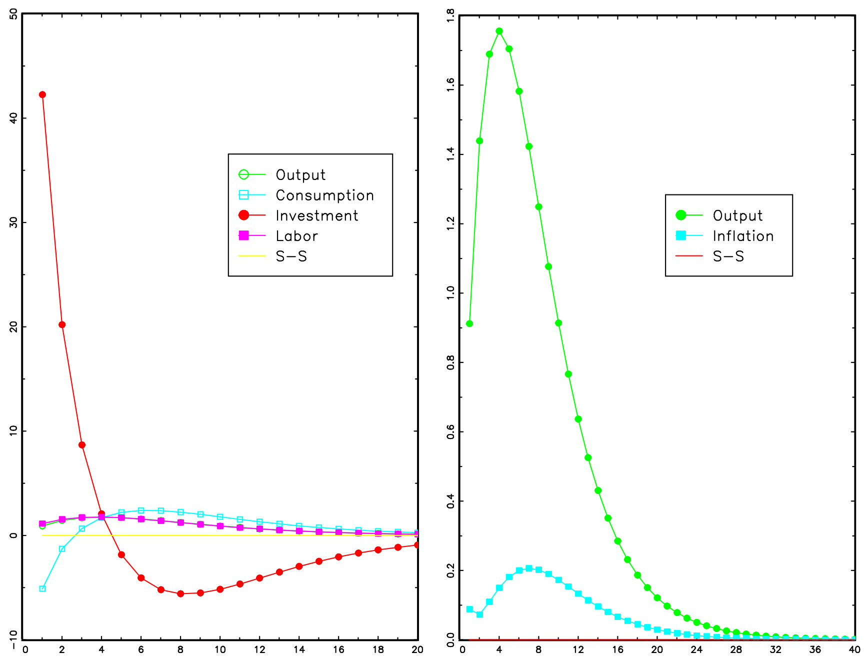

Figure 1. Impulse Responses to a Money-Growth Shock.

Impulse Responses. The time period is a quarter. In order to preserve the output-inflation dynamics of the Mankiw-Reis (2002) model, we set the risk aversion coefficient $\gamma = 0.005$, and the inverse labor supply elasticity $\gamma_n = 0.05$. These parameter values are needed because larger values tend to worsen the persistence of inflation and its lagged relationship to output in the Mankiw-Reis model. For the same reason, we also set $\theta = 0.8$, which implies that it takes the average price level five quarters to fully adjust after an initial shock. We find that the larger θ is, the more inflation lags output. We set the time discounting factor $\beta = 0.99$, the capital's share parameter $\alpha = 0.2$,[4] the rate of depreciation $\delta = 0.025$, the elasticity of substitution parameter $\sigma = 10$ (implying a markup of about 10%) as in Mankiw and Reis. We assume that money growth follows an $AR(1)$ process with persistence parameter $\rho = 0.6$, which is consistent with Mankiw and Reis and much of the sticky-price literature.[5] The impulse responses of the model to a money growth shock are graphed in Figure 1, which is generated by setting $N = 30$.

The left window in Figure 1 shows that after a correlated money growth shock, consumption drops first before it increases. Investment, however, increases a lot in the initial periods despite the cash-in-advance constraint. Labor and output have similar volatilities in responding to the shock. The right window in Figure 1 shows that both output and inflation are hump-shaped and highly persistent after the shock. Most importantly, inflation lags output by 3-4 quarters. This pattern of output-inflation dynamics has been cited by the existing literature as the litmus test for monetary

[4] A large value of α tends to destroy the hump-shaped and lagged inflation dynamics. The smaller α is, the closer the results are to those in Mankiw and Reis (2002).

[5] A larger value of ρ also tends to generate more lagged inflation relative to output.

models (see, e.g., Fuhrer and Moore, 1995, Mankiw and Reis, 2002, Christiano, Eichenbaum, and Evans, 2005, and Wang and Wen, 2005).

3.2 A Labor Hoarding Model

Consider the labor hoarding model of Burnside et al. (1993) and Burnside and Eichenbaum (1996) in which a representative agent chooses sequences of consumption (c), probability to work (n), effort to work (e), capital utilization rate (u), and next-period capital stock (k) to solve

$$\max_{\{n\}} E_{t-N}\left\{\max_{\{c,u,e,k'\}} E_t\left\{\sum_{t=0}^{\infty}\beta^t\left[\begin{array}{c}\theta_t\log c_{t+s}+n_{t+s}\log\left(T-\xi-e_{t+s}f\right)\\ +\left(1-n_{t+s}\right)\log T\end{array}\right]\right\}\right\}$$

subject to

$$c_{t+s}+g_{t+s}+k_{t+1+s}-\left(1-\delta u_{t+s}^{\phi}\right)k_{t+s}\leq A_t\left(u_{t+s}k_{t+s}\right)^{\alpha}\left(e_{t+s}n_{t+s}\right)^{1-\alpha},$$

where T is time endowment in each period, ξ is the cost of time from going to work and f is the length of working hours per shift. Since the size of the labor force is normalized to one, n also represents the employment rate. The E_{t-N} operator indicates that employment is always determined $N\geq 0$ periods in advance based on information available in period $t-N$. The key difference between this model and the model studied by Burnside and Eichenbaum is that the number of labor hoarding periods N in this model is a free parameter, hence it can be made arbitrarily large. As will become clear shortly, the size of N has no effect on the model's state space under our method of undetermined coefficients.

If the time period is a quarter, than the labor hoarding period N may be around $1-4$ quarters based on the U.S. data. However, if the time period is a month or a day, then N can become quite large. The literature has shown that due to consistence in parameter calibrations, quarterly time series generated from a daily model by aggregation may differ substantially from those generated from the corresponding quarterly model. This is especially the case for employment dynamics (see, e.g., Aadland, 2001; and Aadland and Huang, 2004). As pointed out by Aadland and Huang (2004), in reality, actual decisions by economic agents are likely to be made at time intervals that are more frequent than the intervals at which economic data are sampled. Hence, in practice, it may be necessary for researchers to solve labor hoarding models at daily or monthly frequency and then convert the times series into quarterly or annual frequency, which would involve solving models with a large value of N.

If $N>0$, adjusting employment stock is not instantaneous in the model. But the effort level e (or utilization rate of labor) and the utilization rate of capital can be adjusted instantaneously, reflecting the idea of factor hoarding (Burnside et al., 1993). The rate of capital depreciation,

δu_t^{ϕ}, is time dependent in this model, reflecting costs associated with the capital utilization rate ($\phi > 1$). The random variable θ_t represents aggregate impulses shifting the marginal utilities of agents' consumption by creating urges to consume, g_t is shocks to government spending, and A_t is shocks to technology. All shocks follow stationary AR(1) processes:

$$\log\theta_t = (1-\rho_\theta)\bar{\theta} + \rho_\theta \log\theta_{t-1} + \varepsilon_{\theta t}, \quad \varepsilon_{\theta t} \sim N(0,\sigma_\theta^2);$$

$$\log g_t = (1-\rho_g)\bar{g} + \rho_g \log g_{t-1} + \varepsilon_{gt}, \quad \varepsilon_{gt} \sim N(0,\sigma_g^2);$$

$$\log A_t = (1-\rho_a)\bar{A} + \log A_{t-1} + \varepsilon_{at} \qquad \varepsilon_{at} \sim N(0,\sigma_a^2);$$

where the innovations $\{\varepsilon_{\theta t}, \varepsilon_{gt}, \varepsilon_{at}\}$ are assumed to be othorgonal to each other.

The first-order conditions with respect to $\{n, c, u, e, k\}$ are given respectively by:

$$E_{t-N}\left\{\log T - \log\left(T - \xi - e_t f\right) - (1-\alpha)\,\lambda_t A_t \left(u_t k_t\right)^{\alpha} e_t^{1-\alpha} n_t^{-\alpha}\right\} = 0$$

$$\frac{\theta_t}{c_t} = \lambda_t$$

$$A_t \alpha u_t^{\alpha-1} k_t^{\alpha} \left(e_t n_t\right)^{1-\alpha} = \phi\delta u_t^{\phi-1} k_t$$

$$\frac{f n_t}{T - \xi - e_t f} = (1-\alpha)\,\lambda_t A_t \left(u_t k_t\right)^{\alpha} e_t^{-\alpha} n_t^{1-\alpha}$$

$$\lambda_t = \beta E_t\left\{\lambda_{t+1}\left[\alpha A_{t+1} u_{t+1}^{\alpha} k_{t+1}^{\alpha-1}\left(e_{t+1} n_{t+1}\right)^{1-\alpha} + 1 - \delta u_{t+1}^{\phi}\right]\right\}$$

$$c_t + g_t + k_{t+1} - \left(1 - \delta u_t^{\phi}\right) k_t = A_t \left(u_t k_t\right)^{\alpha} \left(e_t n_t\right)^{1-\alpha}.$$

The model's equilibrium is characterized by linear approximations around the steady state. Using circumflex variables to denote log deviations from steady state values, where the steady state refers to the model economy's stationary point in the absence of random shocks (e.g., all variables including the exogenous shocks are constant), the log-linearized first-order conditions are given by:[6]

$$E_{t-N}\left\{\hat{A}_t + \hat{\lambda}_t + \alpha\left(\hat{u}_t + \hat{k}_t\right) - \alpha\left(\hat{e}_t + \hat{n}_t\right)\right\} = 0 \tag{21}$$

$$\hat{\theta}_t - \hat{c}_t = \hat{\lambda}_t$$

$$\hat{A}_t + (1-\alpha)\left(\hat{e}_t + \hat{n}_t - \hat{k}_t\right) = (\phi - \alpha)\,\hat{u}_t$$

[6] Note that the first-order conditions with respect to n_t and e_t imply that in the steady state,

$$\frac{1}{\log T - \log\left(T - \xi - \bar{e} f\right)} = \frac{\bar{e} f}{T - \xi - \bar{e} f}.$$

$$\pi\hat{e}_t = \hat{A}_t + \hat{\lambda}_t + \alpha\left(\hat{u}_t + \hat{k}_t\right) - \alpha\left(\hat{e}_t + \hat{n}_t\right)$$

$$\hat{\lambda}_t = E_t\left\{\hat{\lambda}_{t+1} + \frac{\eta}{1-\alpha}\hat{A}_{t+1} - \eta\left(\hat{u}_{t+1} + \hat{k}_{t+1}\right) + \eta\left(\hat{e}_{t+1} + \hat{n}_{t+1}\right)\right\}$$

$$s_c\hat{c}_t + s_g\hat{g}_t + \frac{s_i}{\bar{\delta}}\hat{k}_{t+1} = \hat{A}_t + \left(\alpha + s_i\frac{1-\bar{\delta}}{\bar{\delta}}\right)\hat{k}_t + \left(\alpha - s_i\phi\right)\hat{u}_t + (1-\alpha)\left(\hat{e}_t + \hat{n}_t\right);$$

where $\pi \equiv \frac{\bar{e}f}{T-\xi-\bar{e}f}, \eta \equiv \left(1-\beta\left(1-\bar{\delta}z\right)\right)(1-\alpha)$, and $s_c + s_i + s_g = 1$ represent the steady-state ratios of consumption, investment and government spending with respect to output, and $\bar{\delta}$ is the steady-state capital depreciation rate. The important steady-state relationships that help determine the steady-state values and the elasticity of depreciation cost (ϕ) are implied by the first order conditions of the model, which are given respectively by

$$\frac{\bar{k}}{\bar{y}} = \frac{\beta\alpha}{1-\beta\left(1-\bar{\delta}\right)}, \quad s_i = \bar{\delta}\frac{\bar{k}}{\bar{y}}, \quad s_c = 1 - s_i - s_g, \quad \phi = \frac{1-\beta\left(1-\bar{\delta}\right)}{\beta\bar{\delta}}, \quad \bar{\delta} = \delta\bar{u}^{\phi}.$$

To solve the model, denote $Z_t = [\hat{c}_t, \hat{n}_t, \hat{e}_t, \hat{u}_t]'$ as the control vector, $S_t = \left[\hat{k}_t, \hat{\lambda}_t, \hat{A}_t, \hat{\theta}_t, \hat{g}_t\right]'$ as the expanded state vector, and denote the forecast errors as $Z_t - E_{t-N}Z_t = \sum_{j=0}^{N-1}\Phi_j^z\varepsilon_{t-j} \equiv \Psi^z\epsilon_t$, and $S_t - E_{t-N}S_t = \sum_{j=0}^{N-1}\Phi_j^s\varepsilon_{t-j} \equiv \Psi^s\epsilon_t$. In equation (21), replace all variables with lagged expectations by their corresponding forecast errors (notice that $\hat{n}_t - E_{t-N}\hat{n}_t = 0$ because labor is determined in period $t-N$). Equation (21) then can be written as:

$$A_1 Z_t = B_1 S_t + \left(A_1\Psi^z - B_1\Psi^s\right)\epsilon_t, \tag{22}$$

where $A_1 = \begin{pmatrix} 0 & \alpha & \alpha & -\alpha \end{pmatrix}$, and $B_1 = \begin{pmatrix} \alpha & 1 & 1 & 0 & 0 \end{pmatrix}$. The first-order conditions can then be represented by the following linear difference system

$$Z_t = H_s S_t + H_\epsilon(\Psi)\epsilon_t, \tag{23}$$

$$E_t S_{t+1} = M_s S_t + M_\epsilon(\Psi)\epsilon_t; \tag{24}$$

which can be solved by standard methods as discussed in the previous section.

Impulse Responses. The time period is a quarter. In calibrating the parameter values for a quarterly model, we follow Burnside and Eichenbaum (1996) by setting $T = 1,369$ per quarter, $\xi = 60$, and $f = 324.8$ (implying a steady-state effort level $\bar{e} = 1$). We also set the discounting factor $\beta = 0.99$, the capital's elasticity $\alpha = 0.36$, the steady-state government-spending to output ratio $s_g = 0.2$, and the steady-state quarterly rate of capital depreciation $\bar{\delta} = 0.025$ (implying 10 percent a year and $\phi \approx 1.4$). These parameter values imply $\frac{\bar{k}}{\bar{y}} = 8.5$ (in a quarter or 2.1 in a year) and $s_i \approx 0.2$. There is no need to pin down the steady-state capital utilization rate since δ can always be chosen so that $\bar{u}$ matches the data. We assume $\rho_\theta = \rho_g = \rho_a = 0.9$.

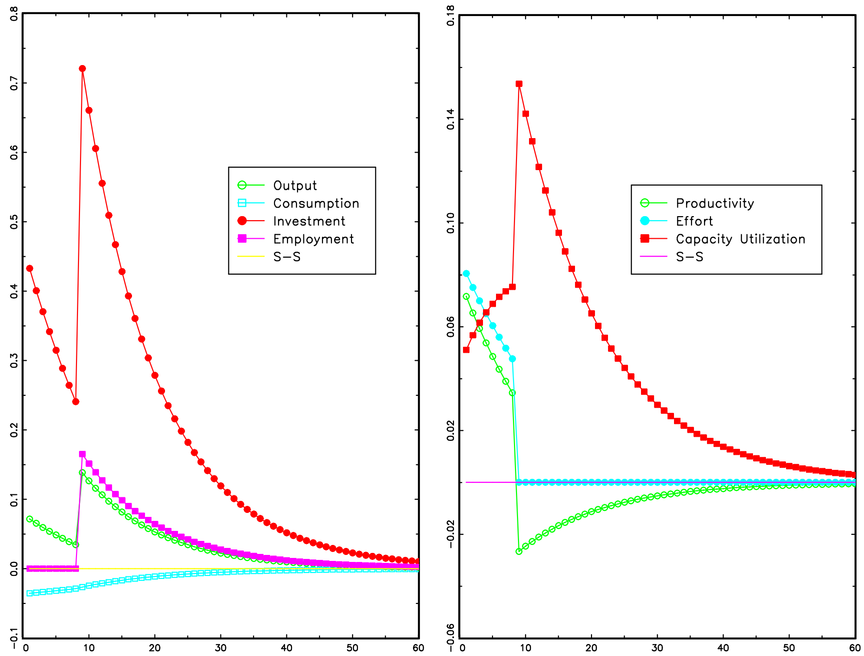

Figure 2. Impulse Responses to a Government-Spending Shock.

As an example, the impulse responses of the model economy to a one-standard-deviation shock to government spending are graphed in Figure 2, where the labor hoarding period is set at $N =$ 10 quarters. The left window shows that the volatility of the economy is substantially smaller during the period of labor hoarding than that after employment becomes flexible ten quarters later. Other than that, the model behaves similar to a standard RBC model. The right window shows that labor's productivity is procyclical during the period of labor hoarding, which is due to the movements in effort and capacity utilization. But as soon as labor becomes flexible ten quarters later after the shock, the effort level returns to zero and productivity becomes countercyclical. Figure 3 reports the impulse responses when $N = 50$. It shows that labor becomes essentially constant even after the hoarding period has ended, and its role in the business cycle is replaced by effort and capacity utilization during the periods of labor hoarding. This tends to reduce the volatility of output even further compared to the case of $N = 10$. Thus, the longer the labor hoarding period is, the less volatile the economy is.

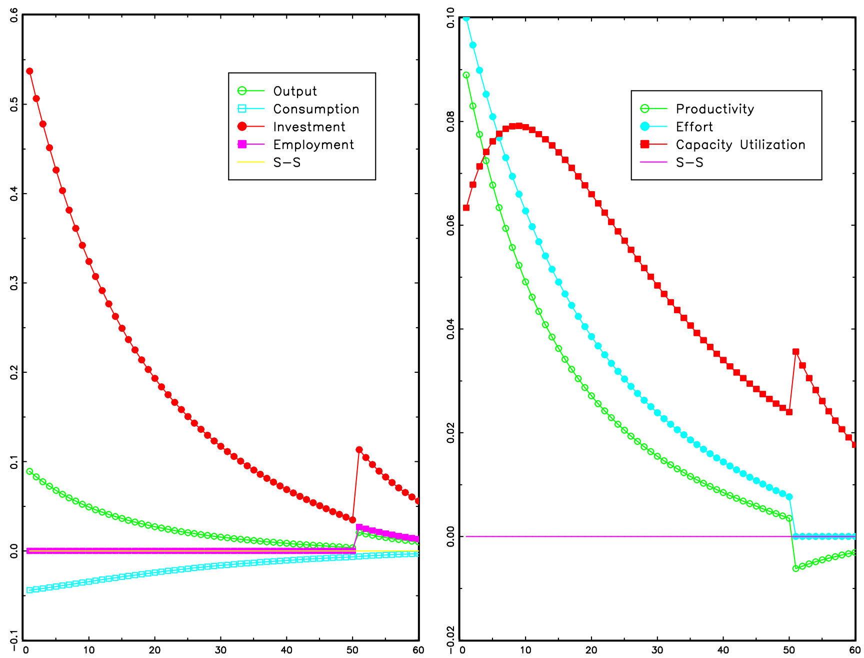

Figure 3. Impulse Responses to a Government-Spending Shock ($N = 50$).

3.3 Serially Correlated Sunspots in the Benhabib-Farmer Model

This is based on the one-sector RBC model studied by Benhabib and Farmer (1994). A representative agent in the model chooses sequences of consumption (c), hours (n), and next-period capital stock (k') to solve

$$\max_{\{n\}} E_{t-N} \left\{ \max_{\{c,k'\}} \; E_t \left\{ \sum_{s=0}^{\infty} \beta^s \left(\log c_{t+s} - a \frac{n_{t+s}^{1+\gamma}}{1+\gamma} \right) \right\} \right\}$$

subject to

$$c_{t+s} + k_{t+s+1} - (1-\delta) k_{t+s} = \Phi_{t+s} k_{t+s}^{\alpha} n_{t+s}^{1-\alpha},$$

where Φ represents production externalities and is defined as a function of average output in the economy which individuals take as parametric:

$$\Phi = \left[k^{\alpha} n^{1-\alpha} \right]^{\eta}, \quad \eta \geq 0.$$

The first-order conditions of the model are summarized by

$$E_{t-N} \left\{ a n_t^{\gamma} - \frac{1}{c_t} (1-\alpha) k_t^{\alpha(1+\eta)} n_t^{(1-\alpha\_(1+\eta)} \right\} = 0,$$

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left[\alpha k_{t+1}^{\alpha(1+\eta)-1} n_{t+1}^{(1-\alpha)(1+\eta)} + 1 - \delta \right] \right\},$$

$$c_t + k_{t+1} - (1-\delta)k_t = k_t^{\alpha(1+\eta)} n_t^{(1-\alpha)(1+\eta)}.$$

Since labor is assumed to be determined N periods in advance as in the labor hoarding model, it is known in period t. Thus the log-linearized first-order conditions of the model can be represented by the following system of linear equations:

$$n_t = A E_{t-N} S_t \tag{25}$$

$$E_t S_{t+1} + B_1 E_{t-N} S_{t+1} = B_2 S_t + B_3 E_{t-N} S_t, \tag{26}$$

where $S_t = [k_t, c_t]'$ denotes an expanded state vector and $\{A, B_1, B_2, B_3\}$ denote coefficient matrices. As discussed in Farmer (1999) for the case of $N = 0$, the model is indeterminate if η is sufficiently large, hence it permits fluctuations driven by self-fulfilling expectations even in the absence of fundamental shocks. Farmer also shows that in the case of $N = 0$, the sunspots shocks to expectations must be *i.i.d.* processes. To see this, let $N = 0$ and rearrange equation (26) (assuming $I + B_1$ is invertible) as

$$S_{t+1} = M S_t + C\varepsilon_{t+1}, \tag{27}$$

where $\varepsilon_{t+1} = c_{t+1} - E_t c_{t+1}$ denotes the forecast errors that satisfy $E_t \varepsilon_{t+1} = 0$. In equilibrium, the economy can be subject to sunspots shocks if the matrix B has all of its eigenvalues lying inside the unit circle. This will happen if and only if η is large enough. In this case, given the current state of the economy (k_t), any initial value of consumption (c_t) can constitute an equilibrium. Hence the forecast error does not need be related to fundamental shocks and can thus represent shocks to agents' expectations.

Since fluctuations driven by self-fulfilling expectations in the Benhabib-Farmer model has to do with the indeterminacy of the initial values of consumption or labor given the initial value of the capital stock in each period, one may think naively that if at least one of these endogenous control variables is predetermined, then indeterminacy can no longer arise in the model. This intuition is false. For example, let labor be predetermined in this model by $N > 0$ periods in advance (the same results hold if we let consumption be predetermined by $N > 0$ periods in advance). Let

$$S_t - E_{t-N} S_t = \sum_{j=1}^{N} \Phi_j \varepsilon_{t-j+1}, \quad \Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \Phi_j = 0 \text{ for } j \leq 0;$$

to represent the forecast errors of S_t. Equation (26) then becomes

$$\begin{aligned} (I + B_1) E_t S_{t+1} &= (B_2 + B_3) S_t + B_1 (E_t S_{t+1} - E_{t-N} S_{t+1}) - B_3 (S_t - E_{t-N} S_t) \\ &= (B_2 + B_3) S_t + B_1 \sum_{j=1}^{N} \Phi_{j+1} \varepsilon_{t-j+1} - B_3 \sum_{j=1}^{N} \Phi_j \varepsilon_{t-j+1}, \end{aligned}$$

which can be written as

$$E_t S_{t+1} = M S_t + \sum_{j=1}^{N} C_j \varepsilon_{t-j+1},$$

where C_j are $2 \times N$ coefficient matrices that depend on the undetermined coefficients Φ_i. Since $E_t S_{t+1} = S_{t+1} - \Phi_0 \varepsilon_{t+1}$, the above equation can be written as

$$S_{t+1} = M S_t + \sum_{j=0}^{N} C_j \varepsilon_{t-j+1}, \tag{28}$$

with $C_0 = \Phi_0$. Notice that this representation reduces to the Benhabib-Famer model when $N = 0$. It can be seen that the conditions for indeterminacy in this model are the same as those in the Benhabib-Farmer model, since the matrix M is independent of N. Thus, if both of the eigenvalues of M lie inside the unit circle due to a large enough value of the externality parameter η, any path of consumption that satisfies

$$c_t = M_2 S_{t-1} + \sum_{j=0}^{N} C_{2j} \varepsilon_{t-j+1} \tag{29}$$

constitutes an equilibrium, where M_2 and C_2j are the second rows of the corresponding matrices. The undetermined coefficients $\{\Phi_j\}_{j=1}^{N}$ can be solved by the method proposed in the previous section.

Impulse Responses. Following the existing literature, we calibrate the model by setting the discount factor $\beta = 0.99$, the capital's share $\alpha = 0.3$, the inverse elasticity of labor supply $\gamma = 0$ (indivisible labor), and the rate of capital depreciation $\delta = 0.025$. The minimum degree of the externality η required for indeterminacy under this parameterization is 0.49. We choose $\eta = 0.75$. This value of η implies a degree of aggregate returns to scale at 1.75, which, based on recent empirical studies is obviously too high.[7] We set the variance of the sunspots shocks to $\sigma_\varepsilon^2 = 1$ and the period of labor hoarding $N = 10$ quarters. The curves with solid circles in Figure 4 show the impulse responses of output, consumption, investment, and employment to a one-standard-deviation sunspots shock to consumption. Notice that in the initial $N = 10$ periods during which labor is predetermined, the higher consumption level generated by the self-fulfilling expectations is sustained by a lower level of savings (investment). This is optimal because the representative agent anticipates high rates of of investment in the future by working harder later. This is also feasible because of increasing returns to scale, which enables the agent to pay back any amount of debt in savings by working harder 10 periods later. Thus, in the model, the longer that labor is hoarded

[7] See Wen (1998) for a modified model that can give rise to indeterminacy with nearly constant returns to scale under very mild externalities. Also see Lubik and Schorfheide (2003, 2004) for how to test indeterminate DSGE models.

(predetermined), the more volatile labor (as well as output and investment) are going to be later after the hoarding period has ended. This is confirmed by the curves with empty circles in Figure 4, which are generated by setting $N = 20$ quarters. Therefore, unlike the previous labor-hoarding model with constant returns to scale, in this model variables are more volatile the longer the labor hoarding period is. As $N \to \infty$, the variance of the model also approaches infinity at the point when the labor hoarding period has ended.[8]

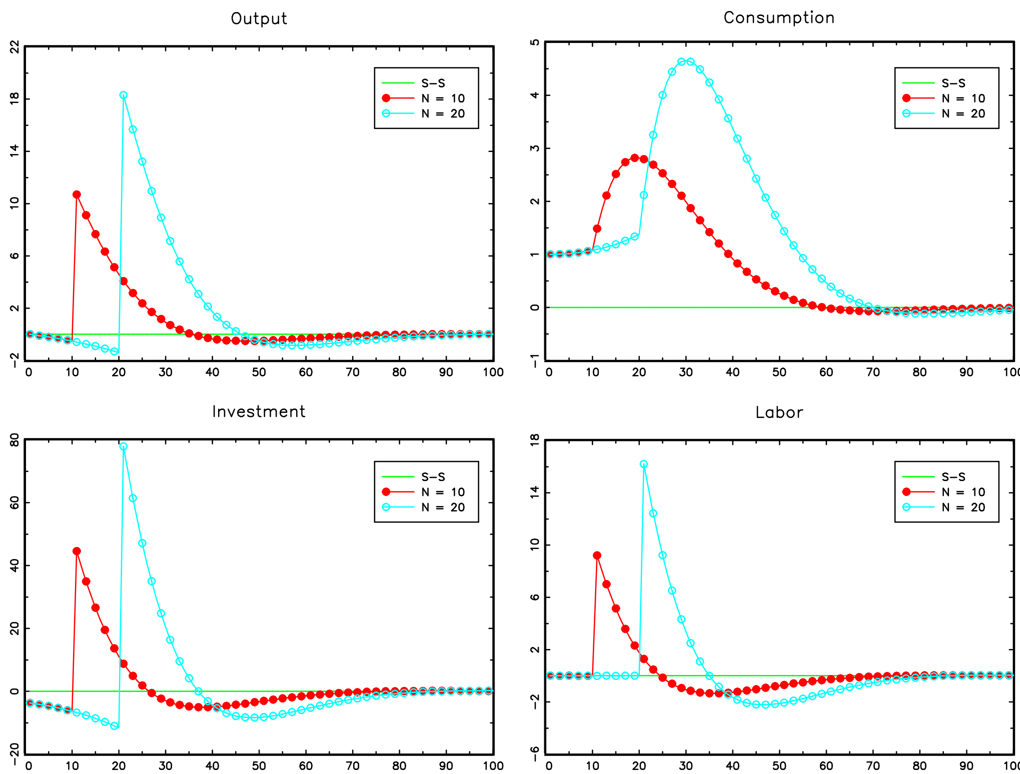

Figure 4. Impulse Responses to a Sunspots Shock.

4 Conclusion

In this paper, a method of solving linear difference models with lagged expectations is proposed. The method does not require expanding the model's state space nor solving for the entire infinite sequence of undetermined $MA(\infty)$ coefficients. By transforming variables with lagged expectations into their forecast errors, the model's state space is maintained at its minimum and there is only a finite order of undetermined coefficients to be solved. The transformed model can then be solved by the traditional methods of Blanchard and Kahn (1980), King and Watson (1998), Christiano (2002), and Klein (2000) or Sims (2002). Without the transformation, solving the model directly

[8] This, however, does not necessarily make the variance of the sample infinity because the sample has zero variance for the entire period during which labor is fixed.

by the traditional methods can be very difficult when the order of lagged expectations is large. Several examples are provided in the paper to demonstrate the usefulness of our solution method. These examples are also of independent interest to researchers in the business-cycle literature.

Appendix

This appendix outlines proofs for Proposition 1. Since equation ($4''$) is a linear difference system in S_t with ϵ_t as the forcing variable, and since $\tilde{A}$ and $\tilde{B}$ are both independent of Ψ, the solution of S_t must also be a linear function of the coefficient of ϵ_t. In addition, this implies that the endogenous dynamics of S_t in the absence of ϵ_t (i.e., the eigenvalues and eigenvectors of the system) do not depend on Ψ. Hence the coefficient matrices H_y and M_y in equations (7) and (8) must be independent of Ψ, while $H_\epsilon(\Psi)$ and $M_\epsilon(\Psi)$ in equations (7) and (8) must be linear in Ψ. Denote $\hat{Y}_t = [Y', Z']'$, $G_0 = [\mathbf{0}', I_m]'$, and rewrite equation (8) as

$$\hat{Y}_t = M_y \hat{Y}_{t-1} + M_\epsilon \epsilon_{t-1} + G_0 \varepsilon_t.$$

Also rewrite equation (5) as

$$\epsilon_t = \Theta \epsilon_{t-1} + I_0 \varepsilon_t.$$

Based on these two equations, if we denote $e_x^j = E_{t-j} x_t - E_{t-j-1} x_t$ as the one-step ahead forecast error of variable X_t based on information $j-1$ periods ago, then we can compute the forecast errors of $\hat{Y}_t$ as

$$\begin{aligned} e_y^0 \left(= \hat{Y}_t - E_{t-1}\hat{Y}_t \right) &= G_0 \varepsilon_t \\ e_y^1 \left(= E_{t-1}\hat{Y}_t - E_{t-2}\hat{Y}_t \right) &= \left[M_y G_0 + M_\epsilon I_0 \right] \varepsilon_{t-1} \\ e_y^2 \left(= E_{t-2}\hat{Y}_t - E_{t-3}\hat{Y}_t \right) &= \left[M_y \left[M_y G_0 + M_\epsilon I_0 \right] + M_\epsilon \Theta I_0 \right] \varepsilon_{t-2} \\ &\vdots \\ e_y^j &= \left[M_y e_y^{j-1} + M_\epsilon \Theta^{j-1} I_0 \right] \varepsilon_{t-j}. \end{aligned}$$

Notice the recursive nature of these one-step ahead forecast errors. Given that we can express $\hat{Y}_t = \sum_{j=0}^{\infty} \Phi_j^y \varepsilon_{t-j}$, thus we have the relationships,

$$\Phi_0^y = G_0$$

and

$$\Phi_j^y = M_y \Phi_{j-1}^y + M_\epsilon(\Psi) \Theta^{j-1} I_0; \quad \text{for } j \geq 1.$$

Also, based on equation (7), we have $e_x^j = H_y e_y^j + H_\epsilon(\Psi)\Theta^j I_0 \varepsilon_{t-j}$, which implies

$$\Phi_j^x = H_y \Phi_j^y + H_\epsilon(\Psi)\Theta^j I_0.$$

Combining the above expressions for Φ_j^y and Φ_j^x, we can see that $P_j(\Psi)$ in equations (11) and (13) is linear in Ψ for all j. Hence the undetermined coefficients in $S_t - E_{t-N} S_t = \sum_{j=0}^{N-1} \Phi_j \varepsilon_{t-j}$, where $\Phi_j = \left[\Phi_j^x, \Phi_j^y\right]'$, can either be solved recursively in a linear fashion or be determined by solving a simultaneous linear equation system composed of $\Phi_j = P_j(\Psi)$ for all j in terms of the linear mapping, $P(\Psi) = \Psi$.■

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