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CONSISTENT SIMPLE SUM AGGREGATION OVER ASSETS

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ABSTRACT

This paper discusses the issue of consistent simple sum aggregation over assets within the context of expected utility maximizing investors. The first part of the paper extends the Hicks and Leontief aggregation theorems of consumer choice theory to the portfolio choice problem. Next, necessary and sufficient conditions for consistent simple sum aggregation are derived for Merton's (1973) continuous-time trading model of investor behavior. Results relating to the construction of consistent rate of return indices for simple sum composite assets are also presented.

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I. Introduction

Recently, a number of empirical studies have sought to model the portfolio behavior of selected investor groups by estimating their demands for various types of assets.[1] These demand equations reflect numerous simplifying assumptions intended to render the specification and estimation tasks less formidable. One common simplification involves aggregating over assets in order to reduce the dimensionality of the universe of assets to be considered and, therefore, the number of parameters to be estimated in the model. Quantity indices in these studies are usually taken to be simple sum aggregates over assets.[2]

Implicitly, the justification for simple sum aggregation in these studies involves decomposing the universe of assets into a tractable number of mutually exclusive groups (e.g., common stocks, long-term corporate bonds, mortgages, etc.) and then assuming that for the purposes of allocating funds among these groups the investor treats each dollar invested in one of the groups as if it is being invested in a single "composite asset" whose probability distribution of rate of return is some function of the joint distribution of the rates of return of the assets in the group.[3] Demand equations for simple sum quantity aggregates of assets (e.g., the amounts of dollars invested in long-term corporate bonds, equities, etc.) may then be formally expressed as functions of the joint probability distribution of rate of return indices for these aggregates.

In this literature seldom is any attention given to the conditions which must prevail in order for the above assumption to be valid for a given simple sum aggregation scheme.[4] That is, given a particular set of asset groupings, when can one formulate the investor choice problem in terms of simple sum "composite assets" in such a way that its solution generates the same allocation of funds among asset groups as does the investor's original choice problem formulated in terms of the individual assets? Equivalently, when is it valid to "assume" that an investor treats each dollar invested in one of the groups as a whole as if it is being invested in a single asset whose rate of return depends only upon the joint probability distribution of the rates of return of the members of the group? A related issue concerns how the rate of return indices for these composite assets should be constructed.

The primary purpose of this paper is to provide answers to the above questions. Briefly, the plan of the paper is as follows. In Section II the expected utility maximization framework of this paper is reviewed and the concepts of consistent simple sum aggregation and consistent rate of return index are defined.

Two asset aggregation theorems analogous to the Hicks and Leontief aggregation theorem of consumer choice theory are presented in Section III.

Section IV concerns itself with the issues of consistent aggregation and consistent rate of return indices within the mean-variance context of Merton's (1973) continuous-time portfolio choice model. Therein, necessary and sufficient conditions for consistent simple sum aggregation over assets within this framework are presented. One implication of our results is that a group of assets need not be perfect substitutes in order for simple sum aggregation over these assets to be consistent.[5] Closed-form expressions for consistent rate of return indices are also derived.

II. Consistent Simple Sum Aggregation and Consistent Rate of Return Indices

Consider a von Neumann-Morgenstern investor whose portfolio choice problem consists of allocating wealth W_o among "k" assets at the beginning of a period so as to maximize the expected utility of end-of-period wealth. Mathematically, his task can be formulated as the following constrained optimization problem:

(1) $$\underset{A_1,\ldots,A_k}{\text{maximize}} \quad E\left\{U[\sum_{i=1}^{k} A_i \tilde{R}_i]\right\}$$
$$\text{subject to: } \sum_{i=1}^{k} A_i = W_o$$

where A_i = the amount of dollars invested in the i^{th} asset at the beginning of the period,

$E\{\cdot\}$ = the expectation operator,

$U[\cdot]$ = the investor's von Neumann-Morgenstern (v-M) utility function (assumed to display nonsatiation and risk aversion), and

$\tilde{R}_i$ = the gross real after-tax rate of return on the i^{th} asset during the period.

If the joint probability distribution of the $\tilde{R}_i$ is denoted by $P(R_1,\ldots,R_k)$, then the solution to the portfolio choice problem (1) takes the form:

(2) $$A_i^* = A_i^*[P(R_1,\ldots,R_k);\ W_o;\ U] \qquad \text{for } i=1,\ldots,k.$$

The optimal allocation of funds depends upon the joint probability distribution of rates of return, the level of initial wealth and the form of the investor's v-M utility function.

It will prove convenient below to rewrite (1) and (2) in vector notation:

(1') $$\underset{\underline{A}}{\text{maximize}} \quad E\{U[\underline{A}^T\underline{\tilde{R}}]\}$$

$$\text{subject to: } \underline{1}^T\underline{A}=W_o$$

(2') $$\underline{A}^* = \underline{A}^*[P(\underline{R});W_o;U]$$

where an underscore (_) denotes a vector quantity and $\underline{1}$ denotes an appropriately dimensioned column vector of ones.

In the real world the number of assets available to an investor, "k", may be very large. Consequently, it is usually desirable to aggregate over assets, creating a fewer number of composite assets, in order to reduce the dimensionality of problem (1) and thereby make the empirical implementation of the theory more tractable.

To this end, partition the universe of assets into, say, "s" mutually exclusive and exhaustive groups of assets where s<k. Conformably partition and re-order the elements of $\underline{A}$ and $\underline{\tilde{R}}$ so that

(3) $$\underline{A} = (\underline{A}_1^T,\ldots,\underline{A}_s^T)^T \quad \text{and} \quad \underline{\tilde{R}}=(\underline{\tilde{R}}_1^T,\ldots,\underline{\tilde{R}}_s^T)^T.$$

We shall say that, relative to the above aggregation scheme, simple sum aggregation is consistent if we can find a set of rate of return indices $\phi_i[\underline{\tilde{R}}_i;P(\underline{R}_i)]$ for i=1,...,s and a v-M utility function $U^*[\cdot]$ such that

(4) $$\underline{1}^T\underline{A}_i^*[P(\underline{R}_i..,\underline{R}_s);W_o;U] = \alpha_i[P(\phi_1,..,\phi_s);W_o;U^*] \qquad \text{for } i=1,..,s$$

where the α_i are the solutions to the following surrogate portfolio choice problem expressed in terms of simple sum composite assets:

(5) $\underset{\alpha_1,\dots,\alpha_s}{\text{maximize}} \quad E\{U^*[\sum_{i=1}^{s} \alpha_i \tilde{\phi}_i]\}.$

subject to: $\sum_{i=1}^{s} \alpha_i = W_o$

If conditions (4) and (5) hold, then the $\tilde{\phi}_i$ are said to be consistent rate of return indices for this aggregation scheme.

In words, a particular simple sum aggregation scheme is consistent if the investor acts as if his allocations to the various asset groups are being generated by his solving a portfolio choice problem in terms of simple sum composite assets. The joint probability distribution of these composite assets is given by $P(\phi_1,\dots,\phi_s)$ where each $\tilde{\phi}$ is a function of $\underline{\tilde{R}}_i$ parameterized by the marginal probability distribution of $\underline{\tilde{R}}_i$, $P(\underline{R}_i)$. We do not require the utility function in the surrogate (composite asset) choice problem, U^*, to be the same as in the original problem, U.

When a simple sum aggregation scheme is consistent each asset group as a whole can be treated as a single asset for the purposes of modelling the investor's allocation of funds among asset groups. The appropriate rate of return index is given by the random variable $\tilde{\phi}_1[\cdot]$ while the quantity index for the group is simply the amount of dollars invested in the group as a whole.

If a particular aggregation scheme is not consistent in the above sense then the paradigm of portfolio behavior discussed in the introduction is an invalid characterization of the portfolio allocation process. Hence, if a proposed simple sum aggregation scheme consisting of equities, bonds and mortgages is not consistent then, in a strict sense, it is meaningless to talk of the investor's demand for equities vis-a-vis bonds and mortgages. Equivalently, in the absence of consistent simple sum aggregation a system of demand equations for these composite assets derivable from expected utility maximization will not exist.

III. The Hicks and Leontief Asset Aggregation Theorems

For general v-M expected utility maximizers two asset aggregation theorems may be proved which are analogous to the Hicks and Leontief aggregation theorems of consumer choice theory:

Theorem 1: (Hicks Aggregation Theorem) If short sales are precluded[6] and a limited liability asset exists then the i^{th} group of assets admit consistent simple sum aggregation if they are characterized by constant relative rates of return (i.e., if $\underline{\tilde{R}}_i = \underline{d}\,\tilde{R}_i$ for some constant positive vector $\underline{d}$ and scalar random variable $\tilde{R}_i$). In this instance, the consistent rate of return index for this group, $\tilde{\phi}_i$, is $\tilde{R}_1 \cdot d_{max}$ where d_{max} is the largest element of vector $\underline{d}$.

Theorem 2: (Leontief Aggregation Theorem) The i^{th} group of assets admit consistent simple sum aggregation if the investor is constrained to hold these assets in constant relative proportions (i.e., if $\underline{A}_i^*$ is constrained to be of the form $\underline{A}_i^* = \alpha_i \underline{g}$ for some vector $\underline{g}$ satisfying $\underline{1}^T \underline{g}=1$). In this instance, the consistent rate of return index for this group is given by $\tilde{\phi}_i = \underline{g}^T \underline{\tilde{R}}_i$

Proofs may be found in the appendix.

To the author's knowledge, no major investor class is legally, morally or otherwise constrained to hold any group of assets in fixed relative proportions. Hence, the Leontief Aggregation Theorem is unlikely to have any practical relevence.

The Hicks Aggregation Theorem, on the other hand, is potentially quite useful and is implicitly invoked in many empirical asset demand studies. For example, an aggregation scheme which treats all long-term corporate bonds as a single homogeneous asset may be "approximately" consistent to the extent that the holding period returns on all such bonds are thought by market participants to be highly correlated.

A more subtle application of the Hicks Theorem is found in a recent structural study of the U.S. equity market by the author.[7] Therein it is argued that it is the policy of most institutional investors to hold well diversified portfolios of equities. In well diversified portfolios, moreover, most of the risk is likely to be systematic: the actual return on such a portfolio will mimic that of the market equity portfolio up to a factor of proportionality given by the portfolio's "Beta" coefficient. The returns for all feasible portfolios, then, are likely to be highly correlated with that of the market in a subjective ex ante sense. Hence, the conditions of the Hicks Theorem are likely to be "approximately" fulfilled and simple sum aggregation over equities is approximately consistent.

IV. Consistent Simple Sum Aggregation in a Mean-Variance Model

Less restrictive simple sum aggregation conditions than those embodied in the Hicks and Leontief Theorems can be obtained within the framework of Merton's (1973) continuous-time consumption-saving-portfolio choice model. If the instantaneous vector of rates of return follows a Gaussian diffusion process with instantaneous mean vector $\underline{\bar{r}}$ and variance-covariance matrix Ω, and if security trading is costless and permitted to take place continuously in time, then the investor's portfolio choice problem at each instant essentially involves solving the following mean-variance problem for the vector of portfolio shares $\underline{h}$ $(\equiv \frac{1}{W_o} \cdot \underline{A})$:[8]

(6) maximize $\underline{h}^T\underline{\bar{r}} - (\rho/2)\ \underline{h}^T\Omega\underline{h}$

subject to: $\underline{1}^T\underline{h}=1$

where "ρ" is the investor's coefficient of relative risk aversion. Below, for simplicity, we shall assume that the investor displays constant relative risk aversion and that Ω is nonsingular.

As Friend, Landskronner and Losq (1976) and Friedman and Roley (1980) have shown, problem (6) can also be motivated by assuming that trading takes place discretely provided that holding periods are sufficiently short. Jones (1979) has also shown that Lintner's (1972) lognormal securities market model of investor behavior essentially reduces to (6).

Solving problem (6), the optimal share vector, $\underline{h}^*$, is found to be:

(7) $\underline{h}^* = \frac{1}{\rho} B\ \underline{\bar{r}} + \underline{b}$

where $B = [\Omega^{-1} - (1/\underline{1}^T\Omega^{-1}\underline{1})\Omega^{-1}\underline{1}\,\underline{1}^T\Omega^{-1}]$ and $\underline{b} = (1/\underline{1}^T\Omega^{-1}\underline{1})\Omega^{-1}\underline{1}$.

Turning our attention now to the issue of consistent simple sum aggregation within this model, partition the set of assets into two groups and also partition $\underline{h}$, $\underline{\bar{r}}$ and Ω conformably. Thus,

(8) $\underline{h} \equiv (\underline{h}_1^T, \underline{h}_2^T)^T$, $\underline{\bar{r}} \equiv (\underline{\bar{r}}_1^T, \underline{\bar{r}}_2^T)^T$, and $\Omega \equiv \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^T & \Omega_{22} \end{pmatrix}$.

We seek to know when the first subgroup of assets admit consistent aggregation.

The answer to this question is revealed in the following theorem which is proved in the appendix:[9]

Theorem 3: A necessary and sufficient condition for the first group of assets to admit consistent simple sum aggregation is that the covariance matrix Ω_{12} is of the form:

(9) $\Omega_{12} = \underline{c}_1 \underline{1}^T + \underline{1}\,\underline{c}_2^T$

for some fixed vectors $\underline{c}_1$ and $\underline{c}_2$. In this case, the unique consistent rate of return index for the first group of assets is given by:

(10) $\tilde{\phi}_1 = (\underline{1}^T G_1^{-1} \tilde{\underline{r}}_1)/(\underline{1}^T G_1^{-1} \underline{1})$

where $G_1 = [\Omega_{11} - \underline{c}_1 \underline{1}^T - \underline{1}\,\underline{c}_1^T]$.[10]

Notice that the conditions for consistent aggregation pertain only to the structure of the covariance matrix Ω_{12} and are independent of the structure of $\bar{\underline{r}}_1$, $\bar{\underline{r}}_2$, Ω_{11}, Ω_{22} and ρ. Moreover, the unique consistent rate of return index $\tilde{\phi}_1$ for the first group is a linear combination of the rates of return of the assets in the group with weights depending on the variance-covariance matrix of the first group, Ω_{11}, and one of the vectors parameterizing the covariance matrix Ω_{12}, $\underline{c}_1$.[11]

When the aggregation restriction (9) obtains the optimal vector of composite asset demands is given by:

$$(11)\quad \hat{\underline{h}}^* \equiv \begin{pmatrix} \underline{1}^T\underline{h}_1^* \\ \underline{h}_2^* \end{pmatrix} = \frac{1}{\rho}\hat{B}\,\bar{\hat{\underline{r}}} + \hat{\underline{b}}$$

where $\hat{B} = [\hat{\Omega} - (1/\underline{1}^T\hat{\Omega}^{-1}\underline{1})\,\hat{\Omega}^{-1}\,\underline{1}\underline{1}^T\hat{\Omega}^{-1}]$, $\hat{\underline{b}} = (1/\underline{1}^T\hat{\Omega}^{-1})\hat{\Omega}^{-1}\underline{1}$,

and $\bar{\hat{\underline{r}}}$ and $\hat{\Omega}$ are the mean vector and variance-covariance matrix associated with the composite asset yield vector $\tilde{\hat{\underline{r}}} \equiv (\tilde{\phi}_1, \tilde{\underline{r}}_2)$. This system, then, has the same general form as (7). Thus, when consistent simple sum aggregation obtains, the investor allocates funds among the first group as a whole and the remaining assets individually as if each dollar allocated to an asset in the first group is being invested in a single asset with yield $\tilde{\phi}_1$.

The above theorem also refutes the claim that simple sum aggregation presumes that the assets of the group to be aggregated are perfect substitutes. It is clear that a covariance matrix of the form given in (9) does not imply that the assets in the first group are perfect substitutes for one another provided that Ω_{11} is nonsingular.

The conditions for consistent aggregation embodied in (9) are quite restrictive. One model of asset returns, however, which generates this structure is the following variance-components or factor-model:

$$(12)\quad \begin{aligned} \tilde{\underline{r}}_1 &= \bar{\underline{r}}_1 + \underline{c}_1\tilde{\varepsilon}_1 + \underline{1}\tilde{\varepsilon}_2 + \tilde{\underline{\gamma}}_1 \\ \tilde{\underline{r}}_2 &= \bar{\underline{r}}_2 + \underline{1}\tilde{\varepsilon}_1 + \underline{c}_2\tilde{\varepsilon}_2 + \tilde{\underline{\gamma}}_2 \end{aligned}$$

where $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, $\tilde{\underline{\gamma}}_1$ and $\tilde{\underline{\gamma}}_2$ are independently distributed scalar and vector random variables with zero means and variance-covariance matrices given by σ_1^2, σ_2^2, Σ_{11} and Σ_{22} respectively. Assets of the first and second groups share two common factors or forcing elements. One of these forcing elements affects all rates of return in the first group identically and the other element affects all rates of return in the second group identically as well. Under these assumptions the covariance matrix between $\underline{\tilde{r}}_1$ and $\underline{\tilde{r}}_2$ is of the form given in (9) and simple sum aggregation over the first group of assets is consistent.

Above we have assumed that all assets are risky. While this is the most realistic assumption in a world with uncertain inflation and no indexed bonds it may be of interest to determine the conditions for consistent simple sum aggregation when a riskless asset exists. In this case, let r_f denote the real rate of return on the riskless asset and $\underline{h}$, $\underline{\bar{r}}$ and Ω the portfolio share vector, mean vector and variance-covariance matrix for the risky assets respectively. The portfolio choice problem in the presence of a riskless asset is then

$$(11) \quad \underset{h_f,\underline{h}}{\text{maximize}} \quad h_f \cdot r_f + \underline{h}^T \underline{\bar{r}} - (\rho/2)\, \underline{h}^T \Omega \underline{h} \,.$$

$$\text{subject to:} \quad h_f + \underline{1}^T \underline{h} = 1$$

As before, partition the risky assets into two non-empty mutually exclusive groups and also partition $\underline{h}$, $\underline{\bar{r}}$ and Ω comformably as in (8). Then we may prove the following result:[12]

<u>Theorem</u> 4: A necessary and sufficient condition for the first subgroup or risky assets to admit consistent simple sum aggregation is that the covariance matrix Ω_{12} takes the special form:[13]

(12) $\Omega_{12} = \underline{1}\,\underline{c}_2^T$

for some fixed vector $\underline{c}_2$.[14] In this case, the unique consistent rate of return index for the first subgroup of assets is

(13) $\phi_1 = (\underline{1}^T\Omega_{11}^{-1}\,\tilde{\underline{r}}_1)/(\underline{1}^T\Omega_{11}^{-1}\,\underline{1})$[15]

Again, the aggregation conditions are independent of the structure of ρ, $\bar{\underline{r}}_1$, $\bar{\underline{r}}_2$, Ω_{11} and Ω_{22}. Moreover, it is evident from Theorem 4 that aggregation across risky assets requires more stringent conditions on the covariance matrix Ω_{12} when a riskless asset exists than when one does not exist.

If the restriction that the second group of assets is non-empty is relaxed then we obtain the following related result:

<u>Theorem</u> 5: If a riskless asset exists then all risky assets admit consistent simple sum aggregation and the unique consistent rate of return index for this aggregate is given by:

(14) $\tilde{\phi} = (\underline{1}^T\Omega^{-1}\tilde{\underline{r}})\,/\,(\underline{1}^T\Omega^{-1}\underline{1})$

which is, incidentally, induced by the minimum variance portfolio consisting of risky assets.[16]

This result is essentially equivalent to the well known separation theorem of mean-variance analysis when a riskless asset exists. However, unlike Theorem 5, the standard separation result does not instruct us how to construct or interpret a consistent rate of return index for the risky assets.

V. <u>Summary</u>

Above we have set forth conditions on the probability distribution of rates of return which justify simple sum aggregation over sets of assets.

These conditions tend to be rather stringent: a group of assets admit consistent simple sum aggregation if their rates of return are perfectly correlated or, in a continuous-time trading environment, if the covariances between the rates of return of assets inside and outside the group assume a specific structure. In this latter case we have demonstrated that perfect substitutability between assets is not a necessary condition for simple sum aggregation to be consistent. We have also described the construction of consistent rate of return indices when a simple sum aggregation scheme is consistent.

Footnotes

1. See Friedman (1977), Jones (1979) and Roley (1977).

2. That is, the quantity index for a group of assets is taken to be the amount of dollars invested in the group as a whole.

3. This is essentially equivalent to the efficiency of two-stage budgeting in the theory of commodity aggregation. See Blackorky et al. (1975) and Green (1964).

4. Superficial discussions of this issue may be found in Leijonhufvud (1968) and Tobin (1961).

5. Barnett (1979) suggests that perfect substitutability is necessary if simple sum aggregation is to be consistent.

6. Unless short sales are precluded the investor will perceive that unlimited profit opportunities through arbitrage exist.

7. See Jones (1979).

8. The validity of this formulation of the investor choice problem does not require $\underline{\bar{r}}$ and Ω to be constant over time. Both parameters may, in fact, be generated by their own diffusion processes provided that the forecast errors in predicting instantaneous changes in $\underline{\bar{r}}$ are uncorrelated with instantaneous forecast errors associated with predicting actual rates of return. See section 5 of Merton (1973).

9. Notice that the following aggregation theorem is symmetric in terms of the first and second asset groups. Hence, if the first group admits consistent aggregation then so too does the second group.

10. An intuitive explanation of Theorem 3 is as follows. From the theory of consumer choice, a group of commodities admit consistent aggregation if and only if the utility function is groupwise separable. Similarly, in terms of problem (6) consistent aggregation obtains if and only if the expected utility function $\underline{h}^T\underline{\bar{r}} - (\rho/2)\,\underline{h}^T\Omega\underline{h}$ is separable in terms of $\underline{h}_1$ and $\underline{h}_2$. It is shown in the appendix that this type of separability obtains if and only if Ω_{12} takes the form given in (9).

11. It is interesting, and important, to note that the portfolio share vector which induces the rate of return index $\tilde{\phi}_1, \underline{g}_1 = (1/\underline{1}^T G_1^{-1}\underline{1})G_1^{-1}\underline{1}$, does not correspond to the optimal allocation of relative shares among assets of the first group. That is, the optimal allocation of funds among assets of the first group is not $W_1^* \cdot \underline{g}_1$ where W_1^* is the optimal allocation of funds to the first group as a whole. Rather, the optimal relative share vector for the first group is readily shown to be:

$$\underline{g}_1^* = \frac{1}{\rho \cdot (W_1^*/W_o)} \quad Q_1(\underline{\bar{r}}_1 - \rho\underline{c}_1) + \underline{g}_1$$

where $Q_1 = [G_1^{-1} - (1/\underline{1}^T G_1^{-1}\underline{1})G_1^{-1}\underline{1}\,\underline{1}^T\, G_1^{-1}]$.

12. This result is proven in the appendix.

13. Notice that this restriction, unlike that in Theorem 3, is not symmetric with respect to the first and second groups of assets. Thus, if a riskless asset exists, even if the first group of assets admit consistent aggregation the second group may not.

14. It follows from (12), then, that if the second group of assets contains only one asset then the first group admits consistent aggregation.

15. The portfolio share vector which induces $\tilde{\phi}_1$ (i.e., $\underline{g}=(1/\underline{1}^T\Omega_{11}^{-1}\underline{1})\Omega_{11}^{-1}\underline{1}$) is interestingly the minimum variance portfolio constructed only of assets from the first group.

16. Again, it is important to point out that the portfolio share vector which induces $\tilde{\phi}$ (i.e., $\underline{g} = (1/\underline{1}^T\Omega^{-1}\underline{1})\Omega^{-1}\underline{1}$) is not the optimal share vector of risky assets obtained by solving the portfolio choice problem (11).

Appendix

Proof of Theorem 1:

Step 1: Let us rewrite the portfolio choice problem as

(A1) $$\underset{\underline{A}_1,\ A_2}{\text{maximize}}\ E\{U[(\underline{A}_1^T\underline{d})\ \tilde{R}_1 + \underline{A}_2^T\underline{\tilde{R}}_2]\}$$

$$\text{subject to: } \underline{1}^T\underline{A}_1 + \underline{1}^T\underline{A}_2 \leq W_o;\ \underline{A}_1,\ \underline{A}_2 \geq 0$$

where the first group of assets are those to be aggregated.

Step 2: Let d_{max} be the largest element of the vector $\underline{d}$ and let "j" be the corresponding position index. It is clear that if a limited liability asset exists then investors will invest a positive amount in the "j" asset of the first group and nothing in the remaining elements of the group, Hence, if α_1 is the amount invested in the j^{th} asset then (A1) is equivalent to the problem

(A2) $$\underset{\alpha_1,\ \underline{A}_2}{\text{maximize}}\ E\{U[\alpha_1(\tilde{R}_1\cdot d_{max}) + \underline{A}_2^T\underline{\tilde{R}}_2]\}\ .$$

$$\text{subject to: } \alpha_1 + \underline{1}^T\underline{A}_2 \leq W_o;\ \alpha_1,\ \underline{A}_2 \geq 0$$

Visual inspection reveals that the optimal $\underline{A}_2$ is the same in both problems. Therefore, the first group of assets admit consistent simple sum aggregation and the consistent rate of return index for this group is $\tilde{R}_1\cdot d_{max}$.

Proof of Theorem 2:

Step 1: The investor's choice problem is

(A3) $$\underset{\underline{A}_1,\ \underline{A}_2}{\text{maximize}}\ E\{U[\underline{A}_1^T\underline{\tilde{R}}_1 + \underline{A}_2^T\underline{\tilde{R}}_2]\}$$

$$\text{subject to: } \underline{1}^T\underline{A}_1 + \underline{1}^T\underline{A}_2 \leq W_o;\quad \underline{A}_1 = \alpha_1\ \underline{g} \text{ for some } \alpha_1$$

where the first group of assets are those to be aggregated.

Step 2: Problem (A3) may be rewritten in the equivalent form:

(A4) $$\underset{\alpha_1,\ \underline{A}_2}{\text{maximize}}\ E\{U[\alpha_1\ (\underline{g}^T\underline{\tilde{R}}_1) + \underline{A}_2^T\underline{\tilde{R}}_2]\}$$

$$\text{subject to: } \alpha_1 + \underline{1}^T\underline{A}_2 \leq W_o$$

where $\alpha_1 \equiv \underline{1}^T\underline{A}_1$. Visual inspection of (A3) and (A4) reveals that the optimal values of $\underline{A}_2$ in both problems are identical. Hence, simple sum aggregation over the first group of assets is consistent and the rate of return index for this group is $\tilde{\phi}_1 \equiv \underline{g}^T\underline{\tilde{R}}_1$.

Proof of Theorem 3:

Necessity

Step 1: Temporarily assume that Ω is fixed and that aggregation obtains for all $\underline{\bar{r}}$. Then the equilibrium portfolio shares take the form:

(A5) $$\begin{pmatrix} \underline{h}_1^* \\ \underline{h}_2^* \end{pmatrix} = (1/\rho)\cdot\begin{pmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{pmatrix}\cdot\begin{pmatrix} \underline{\bar{r}}_1 \\ \underline{\bar{r}}_2 \end{pmatrix} + \begin{pmatrix} \underline{b}_1 \\ \underline{b}_2 \end{pmatrix}$$

where

(A6) $$B \equiv \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{pmatrix} = [\Omega^{-1} - (1/\underline{1}^T\Omega^{-1}\underline{1})\Omega^{-1}\underline{1}\underline{1}^T\Omega^{-1}]$$

and

(A7) $$b \equiv \begin{pmatrix} \underline{b}_1 \\ \underline{b}_2 \end{pmatrix} = (1/\underline{1}^T\Omega^{-1}\underline{1})\Omega^{-1}\underline{1}\quad .$$

The important point to note about (A5) is that B is independent of $\underline{\bar{r}}$. Hence, a necessary condition for the first group of assets to admit aggregation with respect to Ψ^* (the class of random rate of return vectors having a fixed variance-covariance matrix Ω) is that B_{12} and B_{11} satisfy

(A8) $$B_{12}^T = G \cdot \underline{1}^T B_{11}$$

for some matrix G. This is because otherwise we would not be able to write $\underline{1}^T \underline{h}_1^*$ and $\underline{h}_2^*$ in the form

(A9) $$\underline{1}^T \underline{h}_1^* = f[g(\bar{\underline{r}}_1), \bar{\underline{r}}_2]$$

$$\underline{h}_2^* = q[g(\bar{\underline{r}}_1), \bar{\underline{r}}_2]$$

for some functions f, g, and q.

Step 2: It may be verified from the wealth constraint that (A8) implies that B_{12} takes the form

(A10) $$B_{12}^T = \underline{ed}^T$$

for some vectors $\underline{e}$ and $\underline{d}$ with the normalization $\underline{d}^T \underline{1} = 1$. Substituting this result into (A5) we solve for:

(A11) $$\underline{h}_2^* = (1/\rho) \cdot [\underline{e} \cdot (\underline{d}^T \bar{\underline{r}}_1) + B_{22} \cdot \bar{\underline{r}}_2] + \underline{b}_2 \ .$$

Step 3: From the fact that $\underline{1}^T B = \underline{0}^T$ we know that

(A12) $$\underline{e} = -B_{22} \cdot \underline{1} \ .$$

Substituting this expression for $\underline{e}$ into (A11) we find that

(A13) $$\underline{h}_2^* = (1/\rho) \cdot [-B_{22} \cdot (\underline{d}^T \bar{\underline{r}}_1) \underline{1} + B_{22} \bar{\underline{r}}_2] + \underline{b}_2$$

$$= (1/\rho) \cdot [B_{22} \cdot (\bar{\underline{r}}_2 - (\underline{d}^T \bar{\underline{r}}_1) \underline{1})] + \underline{b}_2 \ .$$

This relation implies that any perturbation of $\bar{\underline{r}}_1$ by the amount

(A14) $$[I - (1/\underline{d}^T \underline{d}) \cdot \underline{dd}^T] \cdot \underline{x}$$

for any vector $\underline{x}$ leaves $\underline{h}_2^*$ unchanged.

Step 4: Rewrite problem (6) in the form

$$\text{(A15)}\quad \underset{\underline{h}_1,\ \underline{h}_2}{\text{maximize}}\quad \underline{h}_1^T\bar{\underline{r}}_1 + \underline{h}_2^T\bar{\underline{r}}_2 - (\rho/2)\cdot(\underline{h}_1^T\Omega_{11}\underline{h}_1 + 2\underline{h}_1^T\Omega_{12}\underline{h}_2 + \underline{h}_2^T\Omega_{22}\underline{h}_2)$$

$$\text{subject to:}\quad \underline{1}^T\underline{h}_1 + \underline{1}^T\underline{h}_2 = 1$$

and note that the first-order conditions are

$$\text{(A16)}\quad \underline{0} = \bar{\underline{r}}_i - \rho\Omega_{ii}\underline{h}_i - \rho\Omega_{ij}\underline{h}_j - \underline{1}\lambda \qquad \text{for } i,j = 1,2$$

$$0 = \underline{1}^T\underline{h}_1 + \underline{1}^T\underline{h}_2 - 1$$

where λ is the Lagrangian multiplier associated with the wealth constraint. These equations may be solved to obtain

$$\text{(A17)}\quad \underline{h}_2^* = (1/\rho)\cdot B_2\bar{\underline{r}}_2 - B_2\Omega_{12}\underline{h}_1^* + [(\underline{1}^T\underline{h}_2)(\underline{1}^T\Omega_{22}^{-1}\underline{1})]\Omega_{22}^{-1}\underline{1}$$

where

$$B_2 = [\Omega_{22}^{-1} - (1/\underline{1}^T\Omega_{22}^{-1}\underline{1})\Omega_{22}^{-1}\underline{1}\underline{1}^T\Omega_{22}^{-1}] .$$

Substituting for $\underline{h}_1^*$ equation (A17) becomes

$$\text{(A18)}\quad h_2^* = (1/\rho)\cdot B_2\bar{\underline{r}}_2 - B_2\Omega_{12}^T\cdot[(1/\rho)\cdot B_1\bar{\underline{r}}_1 - B_1\Omega_{12}\underline{h}_2^*]$$

$$+ [(1 - \underline{1}^T\underline{h}_2^*)/(\underline{1}^T\Omega_{11}^{-1}\underline{1})\Omega_{11}^{-1}\underline{1} + [(\underline{1}^T\underline{h}_2^*)/(\underline{1}^T\Omega_{22}\underline{1})\Omega_{22}^{-1}\underline{1} .$$

Step 5: From Steps 3 and 4 we conclude that it must be true that

$$\text{(A19)}\quad B_2\Omega_{12}^TB_1\cdot[I - (1/\underline{d}^T\underline{d})\cdot\underline{dd}^T]\cdot\underline{x} = \underline{0}$$

for all $\underline{x}$ or more simply

$$\text{(A20)}\quad B_2\Omega_{12}^TB_1\cdot[I - (1/\underline{d}^T\underline{d})\cdot\underline{dd}^T] = 0 .$$

Step 6: We also note from (A16) that given $\bar{\underline{r}}_2$, $\underline{1}^T\underline{h}_2^*$ uniquely determines λ^*. Now, solving (A16) for $\underline{1}^T\underline{h}_1^*$ we get

$$\text{(A21)} \qquad \underline{1}^T\underline{h}_1^* = (1/\rho)\cdot\underline{1}^T\Omega_{11}^{-1}\bar{\underline{r}}_1 - \underline{1}^T\Omega_{11}^{-1}\Omega_{12}\underline{h}_2^* - (1/\rho)\cdot(\underline{1}^T\Omega_{11}^{-1}\underline{1})\lambda^* .$$

From Step 3 we know that replacing $\bar{\underline{r}}_1$ by

$$\bar{\underline{r}}_1 + [\ I - (1/\underline{d}^T\underline{d})\cdot\underline{dd}^T]\cdot\underline{x}$$

for any $\underline{x}$ should leave $(\underline{1}^T\underline{h}_1^*)$, $\underline{h}_2^*$ and λ^* unchanged. Hence, from (A20) we conclude that

$$\text{(A22)} \qquad \underline{1}^T\Omega_{11}^{-1}\cdot[I - (1/\underline{d}^T\underline{d})\cdot\underline{dd}^T] \doteq 0$$

or

$$\text{(A23)} \qquad \underline{d} = (1/\underline{1}^T\Omega_{11}^{-1}1)\cdot\Omega_{11}^{-1}\underline{1} \quad .$$

Therefore

$$\text{(A24)} \qquad [I - (1/\underline{d}^T\underline{d})\cdot\underline{dd}^T] = [I - (1/\underline{1}^T\Omega_{11}^{-1}\underline{1})\Omega_{11}^{-1}\underline{11}^T\Omega_{11}^{-1}] \equiv V_1$$

so that (A20) implies that

$$\text{(A25)} \qquad V_1\cdot B_1\cdot\Omega_{12}\cdot B_2 = 0 \quad .$$

Step 7: Relation (A25) may be rewritten

$$\text{(A26)} \qquad V_1\cdot\Omega_{11}^{-1}\ \Omega_{12}\cdot B_2 = 0 \quad .$$

Also, since the dimensionality of the null space of V_1 is unity this space is spanned by $\Omega_{11}^{-1}\underline{1}$. Hence, it must be true that

$$\text{(A27)} \qquad \Omega_{11}^{-1}\Omega_{12}\cdot B_2 = \Omega_{11}^{-1}\underline{1c}_2^T$$

for some vector $\underline{c}_2$ or equivalently

(A28) $$\Omega_{12}\cdot B_2 = \underline{1}\underline{c}_2^T$$

for some vector $\underline{c}_2$. By the structure of B_2, however, this implies that Ω_{12} has the form

(A29) $$\Omega_{12} = \underline{c}_1\underline{1}^T + \underline{1}\underline{c}_2^T$$

for some vectors $\underline{c}_1$ and $\underline{c}_2$.

Step 8: The proof of necessity is completed upon noting that the class of all random return vectors having unrestricted $\bar{\underline{r}}$, Ω_{11} and Ω_{22} includes Ψ^* (defined in Step 1) as a subset.

Sufficiency

Step 1: When (A29) obtains note that problem (A15) may be rewritten in the form:

(A30) $$\underset{\underline{h}_1,\ \underline{h}_2}{\text{maximize}} \quad \underline{h}_1^T(\bar{\underline{r}}_1 - \rho\underline{c}_1) + \underline{h}_2^T\cdot(\bar{\underline{r}}_2 - \rho\underline{c}_2) - (\rho/2)\underline{h}_1^T\cdot(\Omega_{11} - \underline{1}\underline{c}_1^T - \underline{c}_1\underline{1}^T)\underline{h}_1 - (\rho/2)\underline{h}_2^T\cdot(\Omega_{22} - \underline{1}\underline{c}_2^T - \underline{c}_2\underline{1}^T)\underline{h}_2$$

$$\text{subject to: } \underline{1}^T\underline{h}_1 + \underline{1}^T\underline{h}_2 = 1 \ .$$

The first-order conditions for this problem are

(A31) $$\underline{0} = (\bar{\underline{r}}_1 - \rho\underline{c}_1) - \rho\cdot(\Omega_{11} - \underline{1}\underline{c}_1^T - \underline{c}_1\underline{1}^T)\underline{h}_1 - \underline{1}\lambda \ ,$$

(A32) $$\underline{0} = (\bar{\underline{r}}_2 - \rho\underline{c}_2) - \rho\cdot(\Omega_{22} - \underline{1}\underline{c}_2^T - \underline{c}_2\underline{1}^T)\underline{h}_2 - \underline{1}\lambda \ ,$$

(A33) $$\underline{0} = \underline{1}^T\underline{h}_1 + \underline{1}^T\underline{h}_2 - 1.$$

Let us define G_i by

$$G_i = [\Omega_{ii} - \underline{1}\underline{c}_i^T - \underline{c}_i\underline{1}^T]$$

for i = 1,2 and let

$$Q_i = [G_i^{-1} - (1/\underline{1}^T G_i^{-1}\underline{1})G_i^{-1}\underline{1}\underline{1}^T G_i^{-1}]$$

for i = 1,2.

The first-order conditions (A31) - (A33) may be solved for

$$\text{(A34)} \qquad \underline{h}_1^* = (1/\rho)\cdot Q_1\cdot(\underline{\bar{r}}_1 - \rho\underline{c}_1) + [(\underline{1}^T\underline{h}_1^*)/(\underline{1}^T G_1^{-1}\underline{1})]G_1^{-1}\underline{1} \quad .$$

Step 2: Substituting expression (A34) into (A30) and simplifying yields $(\underline{1}^T\underline{h}_1) = h_1^*$ and $\underline{h}_2^*$ as the solution to the "composite asset" portfolio choice problem:

$$\text{(A35)} \qquad \underset{h_1,\,\underline{h}_2}{\text{maximize}} \quad h_1\cdot[(\underline{1}^T G_1^{-1}\cdot(\underline{\bar{r}}_1 - \rho\underline{c}_1))/(\underline{1}^T G_1^{-1}\underline{1})] + \underline{h}_2^T\cdot(\underline{\bar{r}}_2 - \rho\underline{c}_2)$$

$$- (\rho/2)\cdot(h_1^2)/(\underline{1}^T G_1^{-1}\underline{1}) - (\rho/2)\cdot\underline{h}_2^T G_2\underline{h}_2$$

$$\text{subject to:} \quad h_1 + \underline{1}^T\underline{h}_2 = 1 \quad .$$

But problem (A35) is equivalent to optimizing over the holdings of $\underline{h}_2$ and a single risky asset with rate of return

$$\text{(A36)} \qquad \tilde{\phi}_1 = (\underline{1}^T G_1^{-1}\underline{\tilde{r}}_1)/(\underline{1}^T G_1^{-1}\underline{1})$$

where $\tilde{\phi}_1$ is clearly linear homogeneous in $\underline{\tilde{r}}_1$. Hence, we conclude that if condition (A29) obtains then the first group of assets may be aggregated with the associated rate of return index given in (A36).

Proof of Theorem 4:

Necessity

Step 1: The vector of asset demands is

$$\text{(A37)}\qquad \begin{pmatrix} \underline{h}^* \\ h_F^* \end{pmatrix} = (1/\rho)\cdot\begin{pmatrix} \Omega^{-1} & -\Omega^{-1}\underline{1} \\ -\underline{1}^T\Omega^{-1} & \underline{1}^T\Omega^{-1}\underline{1} \end{pmatrix}\cdot\begin{pmatrix} \underline{\bar{r}} \\ r_F \end{pmatrix} + \begin{pmatrix} \underline{0} \\ 1 \end{pmatrix} .$$

Write

$$\text{(A38)}\qquad \Omega^{-1} = \begin{pmatrix} \Omega^{11} & \Omega^{12} \\ \Omega^{21} & \Omega^{22} \end{pmatrix} .$$

If a consistent yield index for the first group exists then Ω^{21} must be of the form:

$$\text{(A39)}\qquad \Omega^{21} = \underline{q}\,\underline{1}^T\,\Omega^{11}$$

for some vector $\underline{q}$.

Step 2: Using the formula for the inverse of a partitioned matrix we compute that

$$\text{(A40)}\qquad \Omega^{21} = -\Omega_{22}^{-1}\Omega_{12}^T\cdot(\Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{12}^T)^{-1} = -\Omega_{22}^{-1}\Omega_{12}^T\Omega^{11} .$$

Step 3: Combining (A39) and (A40) yields

$$\text{(A41)}\qquad \Omega_{12} = \underline{1}\underline{c}_2^T$$

for some vector $\underline{c}_2$ as was to be shown.

Sufficiency

Step 1: Proof of sufficiency is exactly analogous to the proof of sufficiency in Theorem 1 with $\underline{c}_1 = \underline{0}$ and will be omitted here.

Proof of Theorem 5:

Step 1: From (A37)

$$\text{(A42)}\quad \begin{pmatrix} \underline{1}^T\underline{h}^* \\ h_F^* \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} \underline{1}^T\Omega^{-1} & -\underline{1}^T\Omega^{-1}\underline{1} \\ -\underline{1}^T\Omega^{-1} & \underline{1}^T\Omega^{-1}\underline{1} \end{pmatrix}\begin{pmatrix} \bar{\underline{r}} \\ r_F \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 2: Suppose that the investor has only two assets in which to invest. The riskless asset and one with yield $\tilde{\phi}$ as defined in the statement of the theorem. Then it is easily verified that the optimal portfolio proportions are:

$$\text{(A43)}\quad \begin{pmatrix} h_\phi^* \\ h_f^* \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} 1/\sigma_\phi^2 & -1/\sigma_\phi^2 \\ -1/\sigma_\phi^2 & 1/\sigma_\phi^2 \end{pmatrix}\begin{pmatrix} \bar{\phi} \\ r_f \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $\bar{\phi}$ and σ_ϕ^2 are the mean and variance respectively of $\tilde{\phi}$. The reader may verify that the h_f^* in (A42) and (A43) are identical.

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