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# HOURS WORKED: LONG-RUN TRENDS 

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#### Abstract

For 200 years the average number of hours worked per worker declined, both in the market place and at home. Technological progress is the engine of such transformation. Three mechanisms are stressed: (i) The rise in real wages and its corresponding wealth effect; (ii) The enhanced value of time off from work, due to the advent of time-using leisure goods; (iii) The reduced need for housework, due to the introduction of time-saving appliances.

These mechanisms are incorporated into a model of household production. The notion of Edgeworth-Pareto complementarity/substitutability is key to the analysis. Numerical examples link theory and data.

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## 1 Facts

Over the course of the last century there was a precipitous drop in the average length of the workweek, both in the marketplace and at home. In 1830 the average workweek in the market place was 70 hours. This had plunged to just 41 hours by 2002. At the same time there was a 9 -fold gain in real wages. Figure 1 shows the descent in the length of the market workweek and the leap forward in real wages. Likewise, the amount of time spent on housework has dropped. A famous study of Middletown, Indiana, documented in 1924 that 87 percent of housewives spent more than 4 hours per day on housework - see Figure 2. None spent less than 1 hour. By 1999 only 14 percent toiled more than 4 hours per day in the home, while 33 percent spent less than 1 hour.

This decline in hours worked, both in the market and home, was met by a rise in leisure. One measure of the increase in leisure is the uptrend in the share of personal consumption expenditure spent on recreation. This rose from 3 percent in 1900 to 8.5 percent in 2001, as Figure 3 illustrates. Additionally, the amount of time that a person needs to work in order to buy the goods used in leisure has fallen by at least 2.2 percent a year - real wages grew at an annual rate of 1.65 percent over the 1901 to 1988 period. This price decline neglects the fact that many new forms of leisure goods have become available over time, or that old forms have gotten better. As the workweek - or the time spent on work both in the market and at home - was dropping, more and more women were entering into the market place to work. This may seem a little paradoxical. Only 4 percent of married women worked in 1890 as compared with 49 percent in 1980 - again, see Figure 2.


Figure 1: The fall in the market workweek and the gain in real wages


Figure 2: The ascent of female labor-force participation and the reduction in housework


Figure 3: The increase in recreation's share of expenditure and the decline in the time price of leisure goods

What can explain these facts? The answer is nothing mysterious: technological progress. Three channels of effect will be stressed here. First, technological progress increases wages. On the one hand, an increase in real wages should motivate more work effort since the price of consumption goods in terms of forgone leisure has fallen. On the other hand, for a given level of work effort a rise in wages implies that individuals are wealthier. People may desire to use some of this increase in living standards to enjoy more leisure. Second, the value of not working has also risen due to the advent of many new leisure goods. Leisure goods by their very nature are time using. Think about the impact of the following products: radio, 1919; monopoly, 1934; television, 1947; videocassette recorder, 1979; Nintendo and Trivial Pursuit, 1984. Third, other types of new household goods have reduced the need for housework. These household goods are time saving. Examples are: electric stove, 1900; iron, 1908; frozen food, 1930; clothes dryer, 1937; Tupperware, 1947; dishwasher, 1959; disposable diaper (Pampers), 1961; microwave oven, 1971; food processor, 1975. Some goods can be both time using or time saving depending on the context: the telephone, 1876; IBM PC, 1984. A model will now be developed to analyze the channels through which technological progress can affect hours worked in the market and time spent at home.

## 2 Analysis

Setup.- Let tastes be represented by

$$
U(c)+V(n), \text { with } U_{1}, V_{1}>0 \text { and } U_{11}, V_{11}<0
$$

Here the utility functions $U$ and $V$ are taken to have the standard properties, while $c$ and $n$ represent the consumption of a market and nonmarket good. Now, suppose that the nonmarket good is produced in line with the constant-returns-to-scale production function

$$
n=H(l, d)=d H\left(\frac{l}{d}, 1\right), \text { with } H_{1}, H_{2}>0 \text { and } H_{11}, H_{22}<0
$$

where $H$ has standard properties, $d$ represents purchased household inputs, and $l$ is time spent in household production. The idea that nonmarket goods are produced by inputs of time and goods, just as are market ones, was introduced in classic work on household production theory by Becker (1965) and Reid (1934). Assume for simplicity that there is some indivisibility associated with $d$. The household must use the quantity $d=\delta$. [This assumption is innocuous. Greenwood, Seshadri and Yorukoglu (2005, Section 6) and Vandenbroucke (2005) illustrate how it can easily be relaxed.] This fixed quantity of the household input sells at the price $q$, which is measured in terms of time. Last, an individual has one unit of time that he can divide between working in the market or using at home. The market wage rate is $w$.

Now, define the function

$$
X(l, d)=V\left(d H\left(\frac{l}{d}, 1\right)\right)
$$

Household time, $l$, and purchased household inputs, $d$, will be described as Edgeworth-Pareto complements in utility when $X_{12}>0$ and substitutes when $X_{12}<0-$ cf., Pareto (1971, Eq. 63 and 64). When $l$ and $d$ are Edgeworth-Pareto complements in utility an increase in $d$ raises the marginal utility from $l$, or $X_{1}$, and likewise more $l$ increases the marginal utility from $d$, or $X_{2}$.

The individual's optimization problem is

$$
W(w, q)=\max _{l}\{U(w(1-l)-q w)+X(l, \delta)\}
$$

The upshot of this maximization problem is summarized by the first- and second-order conditions written below.

$$
\begin{align*}
w U_{1}(w(1-l)-q w) & =X_{1}(l, \delta) \\
& =V_{1}\left(\delta H\left(\frac{l}{\delta}, 1\right)\right) H_{1}\left(\frac{l}{\delta}, 1\right) \tag{1}
\end{align*}
$$

and

$$
\Sigma \equiv w^{2} U_{11}+X_{11}<0
$$

The lefthand side of (1) represents the marginal cost of an extra unit of time spent at home. An extra unit of time spent at home results in a loss of wages in the amount $w$. This is worth $w U_{1}(w(1-l)-q w)$ in terms of forgone utility. The righthand side gives the marginal benefit derived from spending an extra unit of time at home, $X_{1}(l, \delta)$. The solution for $l$ is portrayed in Figure 4.

Effect of Technological Progress in Household Goods.- Now, suppose that there is technological progress in household goods. In particular, let this be manifested by an increase in the amount of home inputs, $\delta$, that can be purchased for $q$ forgone units of time. How will this affect the amount of time spent spent at home? It is easy to calculate that

$$
\frac{d l}{d \delta}=-\frac{X_{12}}{\Sigma} \gtreqless 0 \text { as } X_{12} \gtreqless 0 .
$$



Figure 4: The determination of time spent at home, $l$

Therefore, time spent on household activities will rise or fall depending on whether time and goods are complements or substitutes in household utility. When time and purchased inputs are complements in utility an extra unit of $d$ raises the worth of staying at home. So, time spent at home should rise. Leisure goods, such as the television, fall into this category. Such goods have contributed to the decline in work (either in the market place or at home) by both men and women. A detailed account of how this mechanism can contribute to the long-run decline in hours worked is provided by Vandenbroucke (2005). This case is shown in Figure 4 by a rightward shift in the marginal benefit curve from MB to MB ", resulting in time spent at home rising from $l^{*}$ to $l^{\prime \prime}$. The opposite is true when they are substitutes. This is portrayed in the figure by the leftward movement in the marginal benefit curve from MB
to MB'. Time-saving household appliances, such as the microwave oven, are an example of this case. Such products have reduced the need for housework and have contributed to the increase in market work by women. Greenwood, Seshadri and Yorukoglu (2005) show how the increase in female labor-force participation can be explained along these lines. Therefore, technological advance in household products is consistent with the long-run decline in the market workweek (leisure goods) and the rise in female labor-force participation (time-saving appliances and goods).

When are two goods Edgeworth-Pareto complements or substitutes? From (1) the marginal benefit of time spent at home, $X_{1}(l, \delta)$, is the product of two terms, the marginal utility from nonmarket goods, $V_{1}\left(\delta H\left(\frac{l}{\delta}, 1\right)\right)$, and the marginal product of household time, $H_{1}\left(\frac{l}{\delta}, 1\right)$. The marginal utility of housework is decreasing in $\delta$, while the marginal product of household time is increasing in it. Thus, the net effect of an increase in $\delta$ will depend upon whether the former falls faster with an increase in $\delta$ than the latter rises. Specifically,

$$
X_{12}=-V_{11} H_{1}^{2}(l / \delta)-V_{1} H_{11} l / \delta^{2}+V_{11} H H_{1},
$$

so that

$$
X_{12} \lesseqgtr 0 \text { as } \frac{-(l / \delta) H_{11}}{H_{1}} \lesseqgtr \frac{-n V_{11}}{V_{1}} \frac{\delta\left(H-H_{1} l / \delta\right)}{n} .
$$

In other words, whether or not $X_{12} \lesseqgtr 0$ depends on whether the elasticity of the marginal product of labor with respect to the time-goods ratio, $-(l / \delta) H_{11} / H_{1}$, is smaller or larger than the elasticity of marginal utility with respect to the home good, $-n V_{11} / V_{1}$, weighted by share of purchased inputs in output, $\delta\left(H-H_{1} l / \delta\right) / n$. Thus, for example, $l$ and $\delta$ are likely to be substitutes in utility when: (i) the responsiveness of the marginal product of $l / \delta$
is small with respect to a change in $\delta$; (ii) the marginal utility of home goods declines fast with more consumption; (iii) when purchased inputs are important in production.

Example 1 (The impact of leisure goods on hours worked) Let $U(c)=\phi \ln (c)$ and $V(n)=$ $(1-\phi) \ln (n)$. Represent the household technology by the constant-elasticity-of-substitution production function $H(l, \delta)=\left(\delta^{\rho}+l^{\rho}\right)^{1 / \rho}$. The household's budget constraint is $c=w(1-l-q)$. Given this setup, the first-order condition (1) can be rewritten as

$$
\begin{equation*}
\frac{\phi}{1-\phi}=\frac{1-l-q}{\delta^{\rho}+l^{\rho}} l^{\rho-1} . \tag{2}
\end{equation*}
$$

Observe that a change in wages, $w$, does not affect hours worked in the market, $1-l$. The length of the workweek in the 1890s was about 42 percent above that of the 1990s. In 1995 the typical worker spent about $1 / 3$ of his available time working in the market. So, set $1-l_{1995}=1 / 3$ and $1-l_{1895}=1.42 \times 1 / 3$. Let $\delta_{1895}=0.1$. The share of leisure goods in expenditures, $s$, is given by $s=q /(1-l)$. Costa (1997) reports that this share was 2 percent in the 1890 s, and 6 percent in the 1990 s. Thus, the time-price $q$ is given by $q_{t}=\left(1-l_{t}\right) s_{t}$, for $t=1895$ and 1995. Finally, pick $\rho=-0.6$, which implies an elasticity of substitution between leisure time and leisure goods of 0.63. Proceed now in two steps. First, use (2) to back out the value of $\phi$ that is consistent with $l=l_{1895}, q=q_{1895}$, and $\delta=\delta_{1895}$. This results in $\phi=0.19$. Second, use this equation to find the value of $\delta_{1995}$ that is in agreement with $l=l_{1995}, q=q_{1995}$, and $\phi=0.19$. This leads to $\delta_{1995}=0.69$. Voilà, an example has now been constructed where the change in market hours matches exactly the corresponding figure in the U.S. data. Additionally, the share of expenditure spent on leisure is in line with the data. In physical units, households in 1995 had 6.90 times more leisure goods than did households
in 1895. This number depends upon the elasticity of substitution between leisure time and leisure goods. The higher the degree of complementarity (or the smaller is $\rho$ ), the less is the required increase in $\delta$.

Remark 1 An example can be constructed in very similar fashion to show that labor-saving household inputs (or the case of Edgeworth-Pareto substitutes) can account for the rise in female labor-force participation. The interested reader is referred to Greenwood and Seshadri (2005, Example 5, p. 1256).

Effect of an Increase in Wages.- How will rising wages impact on hours worked? It's easy to calculate that

$$
\frac{d l}{d w}=\frac{U_{1}+w(1-l-q) U_{11}}{\Sigma} \gtreqless 0 \text { as } U_{1} \lesseqgtr-w(1-l-q) U_{11} .
$$

On the one hand, a boost in wages increases the opportunity cost of staying at home. This should lower time spent at home and is represented by the substitution effect term, $U_{1} / \Sigma<0$. On the other hand, higher wages make the individual wealthier. The individual should use some of this extra wealth to increase his time spent at home. This income effect is shown by the term, $w(1-l-q) U_{11} / \Sigma>0$. Thus, time spent at home can rise or fall with wages depending on whether the income effect dominates the substitution effect. In general, then, anything can happen, as the following two specialized cases for $U$ make clear.

1. Let $U(c)=\ln c$, the macroeconomist's favorite utility function. Here, $U_{1}=1 / c$ and $w(1-l-q) U_{11}=-1 / c$. Therefore, the substitution and income effects from a change
in wages exactly cancel out. Long-run changes in wages have no impact on hours worked, $1-l$.
2. Suppose $U(c)=\ln (c-\mathfrak{c})$, where $\mathfrak{c}>0$ is some subsistence level of consumption. Now, $U_{1}=1 /(c-\mathfrak{c})$ and $w(1-l-q) U_{11}=-c /(c-\mathfrak{c})^{2}$. Therefore, $d l / d w=-\mathfrak{c} /\left[(c-\mathfrak{c})^{2} \Sigma\right]>$ 0 . Consequently, rising wages lead to a fall in hours worked, $1-l$. The intuition is simple. At low levels of wages an individual must work hard to meet his subsistence level of consumption, $\boldsymbol{c}$. Covering the subsistence level of consumption becomes easier as wages rise and this allows an individual to let up on his work effort. Thus, this form for the utility function is in accord with a long-run decline in hours worked. Additionally, it is consistent with the observation reported in Vandenbroucke (2005) that unskilled workers labored longer hours in 1900 than did skilled ones, while today they work about the same.

Can an increase in wages explain the decline in the workweek? The answer is yes, as the following example makes clear.

Example 2 (The impact of rising wages on hours worked) $\operatorname{Let} U(c)=\ln (c-\mathfrak{c})$ and $V(n)=\alpha n$. Represent the household technology by $H(l, d)=l$. Equation (1) appears as

$$
\begin{equation*}
1-l=\frac{1}{\alpha}+\frac{\mathfrak{c}}{w}, \tag{3}
\end{equation*}
$$

which gives a very simple solution for hours worked, $1-l$. Let the time period for this example be 1830 to 1990. The real wage rate in 1990 (actually in 1988) was 9.15 times the wage rate of $1830-$ Williamson(1995). So, set $w_{1830}=1$ and $w_{1990}=9.15$. Following the
discussion in Example 1, fix hours worked in 1830 and 1990, or $1-l_{1830}$ and $1-l_{1990}$, using the equations $1-l_{1830}=1.65 \times 1 / 3$ and $1-l_{1990}=1 / 3$. Employing these restrictions in conjunction with (3) leads to a system of two equations in the two unknown parameters $\alpha$ and c. Specifically, one obtains

$$
1-l_{1830}=\frac{1}{\alpha}+\frac{\mathfrak{c}}{w_{1830}},
$$

and

$$
1-l_{1990}=\frac{1}{\alpha}+\frac{\mathfrak{c}}{w_{1990}} .
$$

Solving yields $\alpha=3.26$ and $\mathfrak{c}=0.24$. The subsistence level of consumption, $\mathfrak{c}$, amounts to 44 percent of consumption in 1830, and 8 percent in 1990.

The last century saw the advent of labor income taxation. So, perhaps the previous example should have focused on the rise of after-tax wages. This is easy to do.

Example 3 (The effect of higher labor income taxation on hours worked) Take the setup from Example 2 with one modification, to wit the introduction of labor income taxation. In particular, suppose that wages are taxed at rate $\tau$. A fraction $\theta$ of the revenue the government receives is rebated back to the worker via lump-sum transfer payments, $t$. The rest goes into worthless government spending on goods and services, $g$ - or equivalently one could assume that it enters into the consumer's utility function in a separable manner. Hence, the worker's budget constraint reads $c=(1-\tau) w(1-l)+t$, while the government's appears as $g+t=\tau w(1-l)$. The first-order condition for this setting is

$$
\frac{(1-\tau) w}{c-\mathfrak{c}}=\alpha
$$

Combining the worker's and government's budget constraints yields $c=[1-\tau(1-\theta)] w(1-l)$. Using this fact in the above first-order condition results in

$$
\begin{equation*}
1-l=\frac{1-\tau}{\alpha[1-\tau(1-\theta)]}+\frac{\mathfrak{c}}{w[1-\tau(1-\theta)]} . \tag{4}
\end{equation*}
$$

Observe that when $\mathfrak{c}=0$ and $\theta=0$ (no rebate) an increase in the tax rate will have no impact on hours worked, because the substitution and income effects exactly cancel out. When $\mathfrak{c}=0$ and $\theta=1$ (full rebate) higher taxes will dissuade hours worked since only the substitution effect is operational. Alternatively, if $\mathfrak{c}>0$ and $\theta=0$ (no rebate) then it transpires that $a$ rise in taxes will cause hours worked to move up. Here the negative income effect from the increase in government spending, which will result in more hours being worked, outweighs the substitution effect. Therefore, in general the effect of labor income taxation on hours worked is ambiguous. The result will depend on how the government uses the revenue it raises, and the functional forms and parameter values used for tastes and technology.

Take labor income taxes to be 0 in 1830. Assume a rate of 30 percent in 1990, in line with numbers reported by Mulligan (2002). Fix $\theta=0.33$, its value for 1990 as measured by the National Income Product Accounts. By following the procedure in Example 2, it can be deduced that the observed fall in hours worked is obtained when $\alpha=2.86$ and $\mathfrak{c}=0.20$. Furthermore, it can be inferred that the rise in wages accounts for 93 percent of the fall, while the increase in taxes explains the remaining 7 percent. ${ }^{1}$

[^0]All of the above examples are intended solely only as illustrations of some secular forces that potentially influence hours worked. A quantitative assessment of the impact that taxes have on hours worked will depend upon the particulars of the model used. A serious study is conducted in Prescott (2004).

The real world seems to have had two conflicting trends: a decline in market work, and a rise in female-labor participation. A more general model could be consistent with both of these facts. To see this, imagine a framework with two types of labor, male and female. There is a division of labor in the home. Men work primarily in the market. Females do housework and, time permitting, market work. Households purchase both time-saving and time-using household inputs. Female labor-force participation would rise as labor-saving goods economized on the amount of housework that had to be done. Simultaneously, the market workweek could decline, either due to the introduction of leisure goods or an income effect associated with a rise in wages. The value of leisure would rise for both men and women. Interestingly, Aguiar and Hurst (2005) document a dramatic increase in leisure for both men and women over the period 1965 to 2003. They construct various measures of leisure. They all showed a gain over the period under study. The narrowest definition rose by 6.4 hours a week for men and 3.8 hours for women, after adjusting for demographic changes in the population. This measure included time spent on activities such as entertainment, recreation, and relaxing. Their preferred measure increased by 7.9 hours a week for men and 6.0 hours for women. This broader definition also included activities such as eating, The first term in brackets is a measure of the change in hours worked, $\left(1-l^{\prime}\right)-(1-l)$, due to the shift in wages from $w$ to $w^{\prime}$, while the second term gives the change due to a movement in taxes from $\tau$ to $\tau^{\prime}$.
sleeping, personal care, and child care. Another manifestation of the rise in the value of leisure is the increase in the fraction of life spent retired. Kopecky (2005) relays that a twenty-year-old man in 1850 could expect to spend about 6 percent of his life retired, while one in 1990 should enjoy about 30 percent of his life in retirement. She shows how the trend toward enjoying more retirement can be analyzed in much the same way as the decline in the workweek.

## 3 Data Sources

- Figure 1: The average weekly hours data for 1830 to 1880 comes from Whaples (1990, Table 2.1). The source for the period 1890 to 1970 is the Historical Statistics of the United States: Colonial Times to 1970 (Series D765 and D803). The data for rest of the sample, or the years 1970 to 2002 , is taken from the Statistical Abstract of the United States. The wage data is provided in Williamson (1995, Table A1.1).
- Figure 2: The source for time spent on housework in Middletown is Caplow, Hicks and Wattenberg (2001, p. 37). The numbers for female labor-force participation are taken from the Statistical Abstract of the United States.
- Figure 3: Recreation's share of expenditure for the years 1900 to 1929 is contained in Lebergott (1996, Table A.1). After 1929 the data is taken from the Statistical Abstract of the United States. The source for the time price of leisure goods is Kopecky (2005).


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[^0]:    ${ }^{1}$ Represent the righthand side of (4) by $L(w, \tau)$. Then,

    $$
    \begin{aligned}
    &\left(1-l^{\prime}\right)-(1-l)=\left[L\left(w^{\prime}, \tau^{\prime}\right)-L\left(w, \tau^{\prime}\right)+L\left(w^{\prime}, \tau\right)-L(w, \tau)\right] / 2 \\
    &+\left[L\left(w^{\prime}, \tau^{\prime}\right)-L\left(w^{\prime}, \tau\right)+L\left(w, \tau^{\prime}\right)-L(w, \tau)\right] / 2
    \end{aligned}
    $$

