Wohltmann, Hans-Werner; Winkler, Roland C.

Working Paper

Solution of RE Models with Anticipated Shocks and Optimal Policy

Economics working paper / Christian-Albrechts-Universität Kiel, Department of Economics, No. 2007,32

Provided in cooperation with:
Christian-Albrechts-Universität Kiel (CAU)

Solution of RE Models with Anticipated Shocks and Optimal Policy

by Hans-Werner Wohltmann and Roland Winkler

Christian-Albrechts-Universität Kiel

Department of Economics

Economics Working Paper

No 2007-32

Solution of RE Models with Anticipated Shocks and Optimal Policy

Hans-Werner Wohltmann* and Roland Winkler**

Christian-Albrechts-University of Kiel

December 20, 2007

Abstract

The purpose of this paper is to solve linear dynamic rational expectations models with anticipated shocks by using the generalized Schur decomposition method. We also determine the optimal unrestricted and restricted policy responses to temporary as well as permanent shocks which both are anticipated by the public. In particular, our method is useful for the analysis of optimal monetary policy in New Keynesian dynamic general equilibrium models.

JEL classification: C32, C61, E52

Keywords: Anticipated Shocks, Optimal Monetary Policy, Rational Expectations, Generalized Schur Decomposition

***Corresponding author:** University of Kiel, Department of Economics, Olshausenstr. 40, D-24098 Kiel, Germany, Phone: ++49-431-880-1449, Fax: ++49-431-880-3150, E-mail: wohltmann@economics.uni-kiel.de

**E-mail: winkler@economics.uni-kiel.de

1 Introduction

The purpose of this paper is to solve linear dynamic rational expectations models with *anticipated* shocks by using the generalized Schur decomposition method. We also determine the optimal unrestricted and restricted policy responses to temporary as well as permanent shocks which both are anticipated by the public.

Our paper is closely related to the work of Söderlind (1999), who also uses the generalized Schur decomposition method to solve linear rational expectations models with optimal policy. Our approach differs in one important respect, namely the possibility to deal with anticipated shocks. In this case, the occurrence of all future shocks is known exactly at the time when the solution of the model is computed. Thus, our RE model is deterministic. In deterministic RE models the concept of rational expectations is equivalent to perfect foresight. Söderlind (1999), on the other hand, only considers stochastic models with white noise shocks which are, by definition, unpredictable. Our method contains unanticipated shocks as a borderline case and can therefore be seen as a generalization of the work by Söderlind (1999).

The paper is organized as follows: Section 2 discusses optimal policies in RE models with anticipated temporary shocks. We first determine the optimal unrestricted policy under precommitment and calculate the minimal value of the intertemporal loss function. We then consider optimal simple rules and show how the Schur decomposition can be used in this case to solve the model. Section 3 deals with permanent anticipated shocks while section 4 presents a short discussion of the well known stochastic case with i.i.d. shocks.

2 The Model

In this paper we discuss the following linear expectational difference equations

$$A \begin{pmatrix} w_{t+1} \\ \mathrm{E}_t\, v_{t+1} \end{pmatrix} = B \begin{pmatrix} w_t \\ v_t \end{pmatrix} + C u_t + D \nu_{t+1} \tag{1}$$

where w_t is an $n_1 \times 1$ vector of predetermined variables with w_0 given, v_t an $n_2 \times 1$ vector of non-predetermined variables, u_t an $m \times 1$ vector of policy instruments and ν_{t+1} an $r \times 1$ vector of exogenous shocks. The matrices A and B are $n \times n$ (where $n = n_1 + n_2$), while the matrices C and D are $n \times m$ and $n \times r$ respectively. We allow the matrix A to be singular which is the case if static (intratemporal) equations are included among the dynamic relationships. The vector w of backward-looking variables can include exogenous variables following autoregressive processes. $\mathrm{E}_t\, v_{t+1}$ denotes rational (model consistent) expectations of v_{t+1} formed at time t. Equation (1) could represent a New Keynesian macroeconomic model with forward-looking expectations where the economy is being subjected to supply-side and demand-side shocks (see, for example, Clarida, Galí and Gertler (1999) or Walsh (2003)). We assume that the shocks are anticipated by the public in advance and take the following form

$$\nu_t = \begin{cases} \overline{\nu} & \text{for} \quad t = T > 0 \\ 0 & \text{for} \quad t \neq T \end{cases} \tag{2}$$

where $\overline{\nu} = (\overline{\nu}_1, \ldots, \overline{\nu_r})'$ is a constant non-zero $r \times 1$ vector. It is assumed that at time $t = 0$ the public anticipates a shock of the form (2) to take place at some future date $T > 0$. A macroeconomic example is the credible announcement of the OPEC that a temporary increase in the price of crude oil (p_O) will take place at some future date $T > 0$ where p_O follows the autoregressive $AR(1)$ process

$$p_{O,t} = \beta p_{O,t-1} + \kappa_t \quad (0 \leq \beta < 1) \tag{3}$$

with the one-unit price shock

$$\kappa_t = \begin{cases} 1 & \text{for} \quad t = T > 0 \\ 0 & \text{for} \quad t \neq T \end{cases} \tag{4}$$

Then $p_{O,t}$ would be a predetermined variable $w_{j,t}$ while κ_t would be part of the general shock vector ν_t. Since the shocks are anticipated by the public we have $\mathrm{E}_t \, \nu_{t+1} = \nu_{t+1}$. For notational convenience, define the $n \times 1$ vector k_t by

$$k_t = \begin{pmatrix} w_t \\ v_t \end{pmatrix} \tag{5}$$

Define further an $n_3 \times 1$ target vector s_t by

$$s_t = \tilde{A} k_t + \tilde{B} u_t \tag{6}$$

where the matrices $\tilde{A}$ and $\tilde{B}$ are $n_3 \times n$ and $n_3 \times m$ respectively. Assume that the policy maker´s welfare loss at time t is given by

$$J_t = \frac{1}{2} \mathrm{E}_t \sum_{i=0}^{\infty} \lambda^i \{ s'_{t+i} W_1 s_{t+i} + u'_{t+i} W_2 u_{t+i} \} \tag{7}$$

where W_1 and W_2 are symmetric and non-negative definite and λ is a discount factor with $0 < \lambda \leq 1$. We can rewrite J_t as

$$J_t = \frac{1}{2} \mathrm{E}_t \sum_{i=0}^{\infty} \lambda^i \{ k'_{t+i} \tilde{W} k_{t+i} + 2 k'_{t+i} P u_{t+i} + u'_{t+i} R u_{t+i} \} \tag{8}$$

where $\tilde{W} = \tilde{A}' W_1 \tilde{A}$ and $R = W_2 + \tilde{B}' W_1 \tilde{B}$ are symmetric and non-negative definite and $P = \tilde{A}' W_1 \tilde{B}$.

2.1 Optimal Policy with Precommitment

In the following the policy maker´s optimal policy rule at time $t = 0$ is developed. It is assumed that the policy maker is able to commit to such a rule. From the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2} E_0 \sum_{t=0}^{\infty} \lambda^t \{k_t' \tilde{W} k_t + 2k_t' P u_t + u_t' R u_t \\ + 2\rho_{t+i}' [Bk_t + Cu_t + D\nu_{t+1} - Ak_{t+1}]\} \tag{9}$$

with the $n \times 1$ multiplier ρ_{t+1} we get the first-order conditions with respect to ρ_{t+1}, k_t and u_t:

$$\begin{pmatrix} A & 0_{n\times m} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times m} & \lambda B' \\ 0_{m\times n} & 0_{m\times m} & -C' \end{pmatrix} \begin{pmatrix} k_{t+1} \\ u_{t+1} \\ \rho_{t+1} \end{pmatrix} \\ = \begin{pmatrix} B & C & 0_{n\times n} \\ -\lambda \tilde{W} & -\lambda P & A' \\ P' & R & 0_{m\times n} \end{pmatrix} \begin{pmatrix} k_t \\ u_t \\ \rho_t \end{pmatrix} + \begin{pmatrix} D \\ 0_{n\times r} \\ 0_{m\times r} \end{pmatrix} \nu_{t+1} \tag{10}$$

To solve the system of equations (10) expand the state and costate vector k_t and ρ_t as $(w_t', v_t')'$ and $(p_{wt}', p_{vt}')'$ respectively and reorder the rows of the $(2n+m)\times 1$ vector $(k_t', u_t', \rho_t')'$ by placing the predetermined vector p_{vt} after w_t. Since v_t is forward-looking with freely chosen initial value v_0, the corresponding Lagrange multiplier p_{vt} is predetermined with initial value $p_{v0} = 0$. Reorder the columns of the $(2n + m) \times (2n + m)$ matrices in (10) according to the reordering of $(k_t', u_t', \rho_t)'$ and write the result as

$$F \begin{pmatrix} \tilde{w}_{t+1} \\ \tilde{v}_{t+1} \end{pmatrix} = G \begin{pmatrix} \tilde{w}_t \\ \tilde{v}_t \end{pmatrix} + \begin{pmatrix} D \\ 0_{n\times r} \\ 0_{m\times r} \end{pmatrix} \nu_{t+1} \tag{11}$$

where

$$\tilde{w}_t = \begin{pmatrix} w_t \\ p_{vt} \end{pmatrix} \tag{12}$$

and

$$\tilde{v}_t = \begin{pmatrix} v_t \\ u_t \\ p_{wt} \end{pmatrix} \tag{13}$$

The $n \times 1$ vector $\tilde{w}_t$ contains the 'backward-looking' variables of (10) while the $(n + m) \times 1$ vector $\tilde{v}_t$ contains the 'forward-looking' variables.

Equation (10) implies that the $(2n + m) \times (2n + m)$ matrix F is singular. To solve equation (11) we apply the generalized Schur decomposition method [Söderlind (1999), Klein (2000)]. The decomposition of the square matrices F and G are given by

$$F = \overline{Q}' S \overline{Z}', \quad G = \overline{Q}' T \overline{Z}' \tag{14}$$

or equivalently

$$QFZ = S, \quad QGZ = T \tag{15}$$

where Q, Z, S and T are square matrices of complex numbers, S and T are upper triangular and Q and Z are unitary, i.e.

$$Q \cdot \overline{Q}' = \overline{Q}' \cdot Q = I_{(2n+m)\times(2n+m)} = Z \cdot \overline{Z}' = \overline{Z}' \cdot Z \tag{16}$$

where the non-singular matrix $\overline{Q}'$ is the transpose of $\overline{Q}$, which denotes the complex conjugate of Q. $\overline{Z}'$ is the transpose of the complex conjugate of Z. The matrices S and T can be arranged in such a way that the block with the stable generalized eigenvalues (the ith diagonal element of T divided by the ith diagonal element of S) comes first. Premultiply both sides of equation (11) with Q and define auxiliary variables $\tilde{z}_t$ and $\tilde{x}_t$ by

$$\begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} = \overline{Z}' \begin{pmatrix} \tilde{w}_t \\ \tilde{v}_t \end{pmatrix} \tag{17}$$

Partition the triangular matrices S and T conformably with $\tilde{z}$ and $\tilde{x}$ and set

$$Q \begin{pmatrix} D \\ 0_{n\times r} \\ 0_{m\times r} \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \tag{18}$$

where Q_1 is $n \times r$ and Q_2 is $(n+m) \times r$. We then obtain the equivalent system

$$\begin{pmatrix} S_{11} & S_{12} \\ 0_{(n+m)\times n} & S_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_{t+1} \\ \tilde{x}_{t+1} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0_{(n+m)\times n} & T_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \nu_{t+1} \tag{19}$$

where the $n \times n$ matrix S_{11} and the $(n+m)\times(n+m)$ matrix T_{22} are invertible while S_{22} is singular. The square matrix T_{11} may also be singular. The lower block of equation (19) contains the unstable generalized eigenvalues and must be solved forward. Since

$$\tilde{x}_{t+s} = T_{22}^{-1} S_{22} \tilde{x}_{t+s+1} - T_{22}^{-1} Q_2 \nu_{t+s+1} \quad (s = 0, 1, 2, \ldots) \tag{20}$$

the unique stable solution for $\tilde{x}_t$ is given by

$$\begin{aligned} \tilde{x}_t &= -\sum_{s=0}^{\infty} (T_{22}^{-1} S_{22})^s T_{22}^{-1} Q_2 \, \mathrm{E}_t \, \nu_{t+s+1} \\ &= \begin{cases} -(T_{22}^{-1} S_{22})^{T-1-t} T_{22}^{-1} Q_2 \overline{\nu} & \text{for} \quad 0 \le t < T \\ 0 & \text{for} \quad t \ge T \end{cases} \end{aligned} \tag{21}$$

The upper block of (19) contains the stable generalized eigenvalues and can be solved backward. Since

$$\tilde{z}_{t+1} = S_{11}^{-1} T_{11} \tilde{z}_t + S_{11}^{-1}(T_{12}\tilde{x}_t - S_{12}\tilde{x}_{t+1}) + S_{11}^{-1} Q_1 \nu_{t+1} \tag{22}$$

the general solution is given by

$$\tilde{z}_t = (S_{11}^{-1}T_{11})^t K + \sum_{s=0}^{t-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}(T_{12}\tilde{x}_s - S_{12}\tilde{x}_{s+1} + Q_1\nu_{s+1})$$
$$= \begin{cases} (S_{11}^{-1}T_{11})^t K + \sum_{s=0}^{t-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}(T_{12}\tilde{x}_s - S_{12}\tilde{x}_{s+1}) & \text{for} \quad 0 \leq t < T \\ \\ (S_{11}^{-1}T_{11})^t K + \sum_{s=0}^{T-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}(T_{12}\tilde{x}_s - S_{12}\tilde{x}_{s+1}) & \\ \quad +(S_{11}^{-1}T_{11})^{t-T}S_{11}^{-1}Q_1\overline{\nu} & \text{for} \quad t \geq T \end{cases} \tag{23}$$

where $\tilde{x}_s$ is defined in (21). The constant K can be determined using the initial value of the predetermined vector $\tilde{w}$. Premultiply equation (17) with Z and partition the matrix Z conformably with $\tilde{z}$ and $\tilde{x}$. We then obtain

$$\begin{pmatrix} \tilde{w}_t \\ \tilde{v}_t \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} \tag{24}$$

and therefore

$$\tilde{w}_0 = Z_{11}\tilde{z}_0 + Z_{12}\tilde{x}_0 \tag{25}$$

with

$$\tilde{w}_0 = (w_0', 0_{n_2\times 1}')' \tag{26}$$
$$\tilde{z}_0 = K \tag{27}$$

and

$$\tilde{x}_0 = -(T_{22}^{-1}S_{22})^{T-1}T_{22}^{-1}Q_2\overline{\nu} \tag{28}$$

where $T > 0$ is assumed.[1] Equation (25) implies

$$K = Z_{11}^{-1}\tilde{w}_0 - Z_{11}^{-1}Z_{12}\tilde{x}_0 \tag{29}$$

provided the inverse Z_{11}^{-1} exists. A necessary condition is that the dynamic system (11) has the saddle path property, i.e., that the number of backward-looking variables ($n_1 + n_2 = n$) coincides with the number of stable generalized eigenvalues [Söderlind (1999), Klein (2000)]. If Z_{11} is invertible, equation (24) implies

$$\tilde{v}_t = Z_{21}\tilde{z}_t + Z_{22}\tilde{x}_t = Z_{21}(Z_{11}^{-1}\tilde{w}_t - Z_{11}^{-1}Z_{12}\tilde{x}_t) + Z_{22}\tilde{x}_t = N\tilde{w}_t + \hat{Z}\tilde{x}_t \tag{30}$$

where

$$N = Z_{21}Z_{11}^{-1}, \quad \hat{Z} = Z_{22} - Z_{21}Z_{11}^{-1}Z_{12} \tag{31}$$

[1] In the special case $T = 0$ (unanticipated shocks) we have $\tilde{x}_0 = 0$ and $\tilde{z}_t = (S_{11}^{-1}T_{11})^t K + (S_{11}^{-1}T_{11})^t S_{11}^{-1}Q_1\overline{\nu}$ implying $\tilde{z}_0 = K + S_{11}^{-1}Q_1\overline{\nu}$ and $K = Z_{11}^{-1}\tilde{w}_0 - S_{11}^{-1}Q_1\overline{\nu}$ with $w_0 \neq 0$. By contrast, the initial value w_0 can be normalized to zero if $T > 0$.

Write equation (30) as

$$\begin{pmatrix} v_t \\ u_t \\ p_{w\,t} \end{pmatrix} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \\ N_{31} & N_{32} \end{pmatrix} \begin{pmatrix} w_t \\ p_{v\,t} \end{pmatrix} + \begin{pmatrix} \hat{Z}_1 \\ \hat{Z}_2 \\ \hat{Z}_3 \end{pmatrix} \tilde{x}_t \tag{32}$$

Assume the invertibility of the $n_2 \times n_2$ matrix N_{12}. Then the optimal policy rule under commitment can be written as

$$\begin{aligned} u_t &= N_{21}w_t + N_{22}p_{v\,t} + \hat{Z}_2\tilde{x}_t \\ &= N_{21}w_t + N_{22}N_{12}^{-1}(v_t - N_{11}w_t - \hat{Z}_1\tilde{x}_t) + \hat{Z}_2\tilde{x}_t \\ &= N_{22}N_{12}^{-1}v_t + (N_{21} - N_{22}N_{12}^{-1}N_{11})w_t + (\hat{Z}_2 - N_{22}N_{12}^{-1}\hat{Z}_1)\tilde{x}_t \end{aligned} \tag{33}$$

where $\tilde{x}_t$ is given by (21). For $t < T$ u_t depends on the auxiliary variable $\tilde{x}_t$ while for $t \geq T$ u_t is only a linear function of the predetermined state variables w_t and p_{vt} where p_{vt} can be substituted by the original state variables v_t and w_t.

Minimal Value of the Loss Function
To determine the minimal value of the loss function J_t at time $t = 0$ we express J_t as function of $\tilde{w}$ and $\tilde{v}$. The loss function (8) can be written as

$$J_t = \frac{1}{2}\sum_{i=0}^{\infty}\lambda^i(k'_{t+i}, u'_{t+i})H\begin{pmatrix} k_{t+i} \\ u_{t+i} \end{pmatrix} = \frac{1}{2}\sum_{i=0}^{\infty}\lambda^i(w'_{t+i}, v'_{t+i}, u'_{t+i})H\begin{pmatrix} w_{t+i} \\ v_{t+i} \\ u_{t+i} \end{pmatrix} \tag{34}$$

where the $(n+m) \times (n+m)$ matrix H is given by

$$H = \begin{pmatrix} \tilde{W} & P \\ P' & R \end{pmatrix} \tag{35}$$

with $H = H'$. Define the $n_1 \times n$ matrix $\tilde{D}_1$ and the $(n_2 + m) \times (n + m)$ matrix $\tilde{D}_2$ by

$$\tilde{D}_1 = (I_{n_1 \times n_1}, 0_{n_1 \times n_2}) \tag{36}$$

and

$$\tilde{D}_2 = (I_{(n_2+m)\times(n_2+m)}, 0_{(n_2+m)\times n_1}) \tag{37}$$

respectively. Then

$$w = \tilde{D}_1\begin{pmatrix} w \\ p_v \end{pmatrix} = \tilde{D}_1\tilde{w} \tag{38}$$

$$\begin{pmatrix} v \\ u \end{pmatrix} = \tilde{D}_2\begin{pmatrix} v \\ u \\ p_w \end{pmatrix} = \tilde{D}_2\tilde{v} \tag{39}$$

and

$$\begin{pmatrix} w \\ v \\ u \end{pmatrix} = \tilde{D}\begin{pmatrix} \tilde{w} \\ \tilde{v} \end{pmatrix} \tag{40}$$

with

$$\tilde{D} = \begin{pmatrix} \tilde{D}_1 & 0_{n_1\times(n+m)} \\ 0_{(n_2+m)\times n} & \tilde{D}_2 \end{pmatrix}$$
$$= \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} & 0_{n_1\times(n_2+m)} & 0_{n_1\times n_1} \\ 0_{(n_2+m)\times n_1} & 0_{(n+m)\times n_2} & I_{(n_2+m)\times(n_2+m)} & 0_{(n_2+m)\times n_1} \end{pmatrix} \quad (41)$$

which is an $(n+m)\times(2n+m)$ matrix. The loss function J_t can now be rewritten as

$$J_t = \frac{1}{2}\sum_{i=0}^{\infty}\lambda^i(\tilde{w}'_{t+i}, \tilde{v}'_{t+i})\tilde{D}'H\tilde{D}\begin{pmatrix}\tilde{w}_{t+i} \\ \tilde{v}_{t+i}\end{pmatrix} = J_t^{(1)} + J_t^{(2)} \quad (42)$$

where

$$J_t^{(1)} = \frac{1}{2}\sum_{i=0}^{T-1}\lambda^i(\tilde{w}'_{t+i}, \tilde{v}'_{t+i})\tilde{D}'H\tilde{D}\begin{pmatrix}\tilde{w}_{t+i} \\ \tilde{v}_{t+i}\end{pmatrix} \quad (43)$$

and

$$J_t^{(2)} = \frac{1}{2}\sum_{i=T}^{\infty}\lambda^i(\tilde{w}'_{t+i}, \tilde{v}'_{t+i})\tilde{D}'H\tilde{D}\begin{pmatrix}\tilde{w}_{t+i} \\ \tilde{v}_{t+i}\end{pmatrix} \quad (44)$$

We want to calculate $J_t^{(2)}$ at first. Since

$$\tilde{v}_t = N\tilde{w}_t \quad (N = Z_{21}Z_{11}^{-1}) \quad \text{for} \quad t \geq T \quad (45)$$

we get for $t \geq T$

$$\begin{pmatrix}\tilde{w}_t \\ \tilde{v}_t\end{pmatrix} = \tilde{N}\tilde{w}_t \quad (46)$$

where

$$\tilde{N} = \begin{pmatrix} I_{n\times n} \\ N \end{pmatrix} \quad (47)$$

is a $(2n+m)\times n$ matrix. $J_t^{(2)}$ can be the rewritten as

$$J_t^{(2)} = \frac{1}{2}\sum_{i=T}^{\infty}\lambda^i\tilde{w}'_{t+i}\tilde{N}'\tilde{D}'H\tilde{D}\tilde{N}\tilde{w}_{t+i} = \frac{1}{2}\sum_{i=T}^{\infty}\lambda^i\tilde{w}'_{t+i}H^*\tilde{w}_{t+i} \quad (48)$$

with

$$H^* = \tilde{N}'\tilde{D}'H\tilde{D}\tilde{N} \quad (49)$$

H^* is a symmetric $n\times n$ matrix. From (23) and (24) we obtain for $t \geq T$

$$\tilde{w}_t = Z_{11}\tilde{z}_t = Z_{11}[(S_{11}^{-1}T_{11})^t K + \sum_{s=0}^{T-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}(T_{12}\tilde{x}_s - S_{12}\tilde{x}_{s+1})$$
$$+ (S_{11}^{-1}T_{11})^{t-T}S_{11}^{-1}Q_1\overline{\nu}] \quad (50)$$

which can be written as

$$\tilde{w}_t = Z_{11} M^{t-T} \tilde{K} \quad (t \geq T) \tag{51}$$

with

$$M = S_{11}^{-1} T_{11} \tag{52}$$

(which is not invertible in general),

$$\tilde{K} = M^T K + S_{11}^{-1} Q_1 \overline{\nu} + \sum_{s=0}^{T-1} M^{T-s-1} S_{11}^{-1} (T_{12} \tilde{x}_s - S_{12} \tilde{x}_{s+1}) \tag{53}$$

and

$$\tilde{x}_s = -(T_{22}^{-1} S_{22})^{T-1-s} T_{22}^{-1} Q_2 \overline{\nu} \quad \text{for} \quad 0 \leq s < T \tag{54}$$

Inserting (51) in (48) we obtain

$$\begin{aligned} J_t^{(2)} &= \frac{1}{2} (M^t \tilde{K})' \lambda^T \left(\sum_{i=T}^{\infty} \lambda^{i-T} (Z_{11} M^{i-T})' H^* (Z_{11} M^{i-T}) \right) M^t \tilde{K} \\ &= \frac{1}{2} \lambda^T \varphi_t' V^* \varphi_t = \frac{1}{2} \lambda^T \operatorname{trace}(V^* \varphi_t \varphi_t') \end{aligned} \tag{55}$$

where

$$\varphi_t = M^t \tilde{K} \tag{56}$$

and V^* is the (convergent) geometric sum of matrices

$$V^* = \sum_{i=T}^{\infty} \lambda^{i-T} (Z_{11} M^{i-T})' H^* (Z_{11} M^{i-T}) \tag{57}$$

V^* is $n \times n$ and satisfies the Lyapunov equation [Currie and Levine (1993)]

$$\begin{aligned} V^* &= Z_{11}' H^* Z_{11} + \sum_{i=T+1}^{\infty} \lambda^{i-T} (Z_{11} M^{i-T})' H^* (Z_{11} M^{i-T}) \\ &= Z_{11}' H^* Z_{11} + \sum_{i=T}^{\infty} \lambda^{i+1-T} (Z_{11} M^{i+1-T})' H^* (Z_{11} M^{i+1-T}) \\ &= Z_{11}' H^* Z_{11} + \lambda M' V^* M \end{aligned} \tag{58}$$

To solve for V^*, we use the matrix identities [Rudebusch and Svensson (1999), Klein (2000)]

$$\operatorname{vec}(A + B) = \operatorname{vec}(A) + \operatorname{vec}(B) \tag{59}$$

and

$$\operatorname{vec}(ABC) = [C' \otimes A] \operatorname{vec}(B) \tag{60}$$

where vec(A) denotes the vector of stacked column vectors of the matrix A, and $\otimes$ denotes the Kronecker product of matrices. We then obtain the equation

$$\operatorname{vec}(V^*) - [\lambda M' \otimes M'] \operatorname{vec}(V^*) = \operatorname{vec}(Z'_{11} H^* Z_{11}) \tag{61}$$

with the solution

$$\operatorname{vec}(V^*) = [I_{n\times n} - \lambda M' \otimes M']^{-1} \operatorname{vec}(Z'_{11} H^* Z_{11}) \tag{62}$$

where

$$\operatorname{vec}(Z'_{11} H^* Z_{11}) = [Z'_{11} \otimes Z'_{11}] \operatorname{vec}(H^*) \tag{63}$$

with

$$\operatorname{vec}(H^*) = [(\tilde{D}\tilde{N})' \otimes (\tilde{D}\tilde{N})'] \operatorname{vec}(H) \tag{64}$$

For $t = 0$ we get

$$J_0^{(2)} = \frac{1}{2}\lambda^T \operatorname{trace}(V^* \varphi_0 \varphi_0') = \frac{1}{2}\lambda^T \operatorname{trace}(V^* \tilde{K}\tilde{K}') \tag{65}$$

The next step is the calculation of the finite sum $J_t^{(1)}$ for $t = 0$. Since

$$\begin{pmatrix} \tilde{w} \\ \tilde{v} \end{pmatrix} = Z \begin{pmatrix} \tilde{z} \\ \tilde{x} \end{pmatrix} \tag{66}$$

we obtain

$$J_0^{(1)} = \frac{1}{2}\sum_{i=0}^{T-1} \lambda^i (\tilde{w}_i', \tilde{v}_i') \tilde{D}' H \tilde{D} \begin{pmatrix} \tilde{w}_i \\ \tilde{v}_i \end{pmatrix} = \frac{1}{2}\sum_{i=0}^{T-1} \lambda^i (\tilde{z}_i', \tilde{x}_i') Z' \tilde{D}' H \tilde{D} Z \begin{pmatrix} \tilde{z}_i \\ \tilde{x}_i \end{pmatrix} \tag{67}$$

where $\tilde{z}_i$ and $\tilde{x}_i$ are defined in (23) and (54) respectively.

The optimal unrestricted policy under commitment yields a loss given by

$$J_0 = J_0^{(1)} + J_0^{(2)} \tag{68}$$

where

$$J_0^{(2)} = \frac{1}{2}\lambda^T \operatorname{trace}(V^* \varphi_0 \varphi_0') = \frac{1}{2}\lambda^T \operatorname{trace}(V^* \tilde{K}\tilde{K}') \tag{69}$$

In the special case $T = 0$ (unanticipated shocks) we have

$$J_0 = J_0^{(2)} = \frac{1}{2}\tilde{K}' V^* \tilde{K} \tag{70}$$

where

$$\tilde{K} = K \Big|_{T=0} + S_{11}^{-1} Q_1 \overline{v} = Z_{11}^{-1} \tilde{w}_0 - S_{11}^{-1} Q_1 \overline{v} + S_{11}^{-1} Q_1 \overline{v} = Z_{11}^{-1} \tilde{w}_0 \tag{71}$$

Then

$$J_0 = \frac{1}{2}\tilde{w}_0' Z_{11}^{-1'} V^* Z_{11}^{-1} \tilde{w}_0 = \frac{1}{2}\tilde{w}_0' V \tilde{w}_0' = \frac{1}{2}\operatorname{trace}(V \tilde{w}_0 \tilde{w}_0') \tag{72}$$

where

$$\tilde{w}_0\tilde{w}_0' = \begin{pmatrix} w_0 \\ p_{v\,0} \end{pmatrix} (w_0', p_{v\,0}') = \begin{pmatrix} w_0 w_0' & 0_{n_1\times n_2} \\ 0_{n_2\times n_2} & 0_{n_2\times n_2} \end{pmatrix} \tag{73}$$

and $V = Z_{11}^{-1'} V^* Z_{11}^{-1}$ satisfies the matrix equation

$$\begin{aligned} V = Z_{11}^{-1'} V^* Z_{11}^{-1} &= H^* + \lambda Z_{11}^{-1'} M' V^* M Z_{11}^{-1} \\ &= H^* + \lambda Z_{11}^{-1'} M' Z_{11}' Z_{11}'^{-1} V^* Z_{11}^{-1} Z_{11} M Z_{11}^{-1} = H^* + \lambda \Gamma' V \Gamma \end{aligned} \tag{74}$$

with

$$\Gamma = Z_{11} M Z_{11}^{-1} \quad (M = S_{11}^{-1} T_{11}) \tag{75}$$

2.2 Optimal Simple Rule

The policy maker could alternatively commit to a suboptimal simple rule of the form

$$u_t = \Lambda k_t + \Psi\, \mathrm{E}_t\, k_{t+1} \tag{76}$$

where the constant matrices Λ and Ψ are $m\times n$. Assuming rational expectations and exogenous shocks of the form (2) which are anticipated in $t = 0$ we get the dynamic system

$$\begin{pmatrix} A & 0_{n\times m} \\ \Psi & 0_{m\times m} \end{pmatrix} \begin{pmatrix} k_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} B & C \\ -\Lambda & I_{m\times m} \end{pmatrix} \begin{pmatrix} k_t \\ u_t \end{pmatrix} + \begin{pmatrix} D \\ 0_{m\times r} \end{pmatrix} \nu_{t+1} \tag{77}$$

The generalized Schur decomposition yields the system of equations

$$F \begin{pmatrix} \tilde{w}_{t+1} \\ \tilde{v}_{t+1} \end{pmatrix} = G \begin{pmatrix} \tilde{w}_t \\ \tilde{v}_t \end{pmatrix} + \begin{pmatrix} D \\ 0_{m\times r} \end{pmatrix} \nu_{t+1} \tag{78}$$

where $\tilde{w} = w$ is an $n_1\times 1$ vector, $\tilde{v} = (v', u')'$ is an $(n_2+m)\times 1$ vector and where the square matrices F and G are $(n+m)\times(n+m)$ with the decomposition

$$QFZ = S, \quad QGZ = T \tag{79}$$

Q, Z, S and T are $(n+m)\times(n+m)$ matrices. Since

$$\begin{pmatrix} \tilde{w} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{x} \end{pmatrix} \tag{80}$$

the matrices Z_{11}, Z_{12}, Z_{21} and Z_{22} are now $n_1\times n_1$, $n_1\times(n_2+m)$, $(n_2+m)\times n_1$ and $(n_2+m)\times(n_2+m)$ respectively. The auxiliary variables $\tilde{z}$ and $\tilde{x}$ satisfy the system of equations

$$\begin{pmatrix} S_{11} & S_{12} \\ 0_{(n_2+m)\times n_1} & S_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_{t+1} \\ \tilde{x}_{t+1} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0_{(n+m)\times n_1} & T_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \nu_{t+1} \tag{81}$$

where S_{11} and T_{11} are $n_1 \times n_1$ matrices, S_{22} and T_{22} are $(n_2+m) \times (n_2+m)$ and S_{12} and T_{12} are $n_1 \times (n_2+m)$. The matrices Q_1 and Q_2 are $n_1 \times r$ and $(n_2+m) \times r$ respectively with

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = Q \begin{pmatrix} D \\ 0_{m\times r} \end{pmatrix} \tag{82}$$

The solution of (81) is given by (21) and (23). For $t \geq T$ we get

$$\tilde{v}_t = N\tilde{w}_t = Nw_t \tag{83}$$

where $N = Z_{21}Z_{11}^{-1}$ is now an $(n_2+m) \times n_1$ matrix.

The loss function (42) simplifies to

$$J_t = \frac{1}{2}\sum_{i=0}^{\infty} \lambda^i (w'_{t+i}, \tilde{v}'_{t+i}) H \begin{pmatrix} w_{t+i} \\ \tilde{v}_{t+i} \end{pmatrix} \tag{84}$$

since $\tilde{D}_1 = I_{n_1\times n_1}$, $\tilde{D}_2 = I_{(n_2+m)\times(n_2+m)}$ and therefore $\tilde{D} = I_{(n+m)\times(n+m)}$ (cf. (41)). J_t can be partitioned via (42). $J_t^{(2)}$ can be written as (48) with

$$H^* = \tilde{N}'H\tilde{N} \tag{85}$$

and

$$\tilde{N} = \begin{pmatrix} I_{n_1\times n_1} \\ N \end{pmatrix} \tag{86}$$

The value of the loss function J_0 for given matrices Λ and Ψ is given by $J_0 = J_0^{(1)} + J_0^{(2)}$, where $J_0^{(1)}$ and $J_0^{(2)}$ are defined in (67) and (69) respectively. The minimization of J_0 with respect to the coefficients of the matrices Λ and Ψ yields an optimal simple rule of the form (76). The loss under such a policy rule is greater than the loss under the unrestricted optimal policy under commitment.

3 Anticipated Permanent Shocks

Up to now we have discussed the solution method in case of anticipated *temporary* shocks. Let us now discuss the the case of anticipated *permanent* shocks which take the following form:

$$\nu_t = \begin{cases} \overline{\nu}_0 & \text{for} \quad 0 \leq t < T \\ \overline{\nu}_1 \ (\neq \overline{\nu}_0) & \text{for} \quad t \geq T \end{cases} \tag{87}$$

Such a shock could be a permanent increase in the price of crude oil taking place at time $t = T$ which the public anticipates at time $t = 0$. The Schur decomposition (19) can again be used to solve the dynamic system in case of permanent anticipated shocks. The steady state system of (19) is given by

$$\begin{pmatrix} S_{11} & S_{12} \\ 0_{(n+m)\times n} & S_{22} \end{pmatrix} \begin{pmatrix} \overline{\tilde{z}} \\ \overline{\tilde{x}} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0_{(n+m)\times n} & T_{22} \end{pmatrix} \begin{pmatrix} \overline{\tilde{z}} \\ \overline{\tilde{x}} \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \overline{\nu} \tag{88}$$

where

$$\bar{\tilde{z}} = \begin{cases} \bar{\tilde{z}}_0 & \text{for} \quad 0 \le t < T \\ \bar{\tilde{z}}_1 & \text{for} \quad t \ge T \end{cases} \tag{89}$$

and

$$\bar{\tilde{x}} = \begin{cases} \bar{\tilde{x}}_0 & \text{for} \quad 0 \le t < T \\ \bar{\tilde{x}}_1 & \text{for} \quad t \ge T \end{cases} \tag{90}$$

The dynamics of the Schur decomposition can be written in the form

$$\tilde{x}_t = T_{22}^{-1} S_{22} \tilde{x}_{t+1} - T_{22}^{-1} Q_2 \nu_{t+1} \tag{91}$$

$$\tilde{z}_{t+1} = S_{11}^{-1} T_{11} \tilde{z}_t + S_{11}^{-1} (T_{12} \tilde{x}_t - S_{12} \tilde{x}_{t+1}) + S_{11}^{-1} Q_1 \nu_{t+1} \tag{92}$$

Since

$$\lim_{n \to \infty} (T_{22}^{-1} S_{22})^n = 0 \tag{93}$$

we get

$$\lim_{t \to \infty} \tilde{x}_t = \bar{\tilde{x}}_1 \tag{94}$$

Equation (94) already holds for $t \ge T$, i.e.,

$$\tilde{x}_t = \bar{\tilde{x}}_1 \qquad \text{for } t \ge T \tag{95}$$

which follows from the general solution formula (21): For $t \ge T$ we have $\nu_{t+s+1} = \bar{\nu}_1$ and therefore

$$\tilde{x}_t = -\left(\sum_{s=0}^{\infty} (T_{22}^{-1} S_{22})^s T_{22}^{-1} Q_2 \right) \bar{\nu}_1 \tag{96}$$

Let $\tilde{\Lambda}$ be the geometric sum of matrices

$$\tilde{\Lambda} = - \sum_{s=0}^{\infty} (T_{22}^{-1} S_{22})^s T_{22}^{-1} Q_2 \tag{97}$$

We then obtain the matrix equation

$$\begin{aligned} \tilde{\Lambda} &= -T_{22}^{-1} Q_2 - \sum_{s=1}^{\infty} (T_{22}^{-1} S_{22})^s T_{22}^{-1} Q_2 \\ &= -T_{22}^{-1} Q_2 - \sum_{s=0}^{\infty} (T_{22}^{-1} S_{22})^{s+1} T_{22}^{-1} Q_2 \\ &= -T_{22}^{-1} Q_2 - (T_{22}^{-1} S_{22}) \sum_{s=0}^{\infty} (T_{22}^{-1} S_{22})^s T_{22}^{-1} Q_2 \\ &= -T_{22}^{-1} Q_2 + (T_{22}^{-1} S_{22}) \tilde{\Lambda} \end{aligned} \tag{98}$$

with the solution

$$\tilde{\Lambda} = -(I - T_{22}^{-1}S_{22})^{-1}T_{22}^{-1}Q_2 = (S_{22} - T_{22})^{-1}Q_2 \tag{99}$$

Equation (96) now implies

$$\tilde{x}_t = \tilde{\Lambda}\overline{\nu}_1 = (S_{22} - T_{22})^{-1}Q_2\overline{\nu}_1 = \overline{\tilde{x}}_1 \qquad (t \geq T) \tag{100}$$

where the formula for $\overline{\tilde{x}}_1$ directly follows from the lower block of the steady state system (88) or from equation (91).

The solution formula for $\tilde{x}_t$ over the anticipation phase $0 \leq t < T$ can be either derived by backward iteration or from the general solution (21). Equation (91) implies for $t = T - 1$

$$\tilde{x}_{T-1} = T_{22}^{-1}S_{22}\overline{\tilde{x}}_1 - T_{22}^{-1}Q_2\overline{\nu}_1 \tag{101}$$

and for $t = T - 2$

$$\begin{aligned}\tilde{x}_{T-2} &= T_{22}^{-1}S_{22}\tilde{x}_{T-1} - T_{22}^{-1}Q_2\overline{\nu}_0 \\ &= (T_{22}^{-1}S_{22})^2\overline{\tilde{x}}_1 - (T_{22}^{-1}S_{22})T_{22}^{-1}Q_2\overline{\nu}_1 - T_{22}^{-1}Q_2\overline{\nu}_0\end{aligned} \tag{102}$$

For $t = T - 3$ we get

$$\begin{aligned}\tilde{x}_{T-3} &= T_{22}^{-1}S_{22}\tilde{x}_{T-2} - T_{22}^{-1}Q_2\overline{\nu}_0 \\ &= (T_{22}^{-1}S_{22})^3\overline{\tilde{x}}_1 - (T_{11}^{-1}S_{22})^2T_{22}^{-1}Q_2\overline{\nu}_1 - (T_{22}^{-1}S_{22})T_{22}^{-1}Q_2\overline{\nu}_0 - T_{22}^{-1}Q_2\overline{\nu}_0\end{aligned} \tag{103}$$

and for $t = T - n$

$$\tilde{x}_{T-n} = (T_{22}^{-1}S_{22})^n\overline{\tilde{x}}_1 - (T_{22}^{-1}S_{22})^{n-1}T_{22}^{-1}Q_2\overline{\nu}_1 - \sum_{j=2}^{n}(T_{22}^{-1}S_{22})^{n-j}T_{22}^{-1}Q_2\overline{\nu}_0 \quad {}^{2} \tag{104}$$

We therefore obtain for $0 \leq t < T$

$$\begin{aligned}\tilde{x}_t &= (T_{22}^{-1}S_{22})^{T-t}\overline{\tilde{x}}_1 - (T_{22}^{-1}S_{22})^{T-t-1}T_{22}^{-1}Q_2\overline{\nu}_1 \\ &\quad - \sum_{j=2}^{T-t}(T_{22}^{-1}S_{22})^{T-t-j}T_{22}^{-1}Q_2\overline{\nu}_0\end{aligned} \tag{105}$$

where

$$\overline{\tilde{x}}_1 = (S_{22} - T_{22})^{-1}Q_2\overline{\nu}_1 = -(I - T_{22}^{-1}S_{22})^{-1}T_{22}^{-1}Q_2\overline{\nu}_1 \tag{106}$$

[2] Note that $\sum_{j=2}^{n}(T_{22}^{-1}S_{22})^{n-j}T_{22}^{-1}Q_2\overline{\nu}_0 = (I - \tilde{M})^{-1}(I - \tilde{M}^{n-1})T_{22}^{-1}Q_2\overline{\nu}_0$ where $\tilde{M} = T_{22}^{-1}S_{22}$ and $n \geq 2$.

An equivalent representation of the solution formula for $\tilde{x}_t$ over the interval $0 \leq t < T$ follows from (21):

$$\begin{aligned}\tilde{x}_t &= -\sum_{s=0}^{\infty}(T_{22}^{-1}S_{22})^s T_{22}^{-1}Q_2\nu_{t+s+1} \\ &= -\sum_{s=0}^{T-t-2}(T_{22}^{-1}S_{22})^s T_{22}^{-1}Q_2\overline{\nu}_0 \\ &\quad -\sum_{s=T-t-1}^{\infty}(T_{22}^{-1}S_{22})^s T_{22}^{-1}Q_2\overline{\nu}_1 \qquad (0 \leq t < T) \end{aligned} \tag{107}$$

where

$$\begin{aligned} -\sum_{s=0}^{T-t-2}(T_{22}^{-1}S_{22})^s T_{22}^{-1}Q_2\overline{\nu}_0 &= -\sum_{j=2}^{T-t}(T_{22}^{-1}S_{22})^{T-t-j} T_{22}^{-1}Q_2\overline{\nu}_0 \\ &= (I - T_{22}^{-1}S_{22})^{-1}(I-(T_{22}^{-1}S_{22})^{T-t-1})T_{22}^{-1}Q_2\overline{\nu}_0 \end{aligned} \tag{108}$$

and

$$-\sum_{s=T-t-1}^{\infty}(T_{22}^{-1}S_{22})^s T_{22}^{-1}Q_2\overline{\nu}_1 = -(I - T_{22}^{-1}S_{22})^{-1}(T_{22}^{-1}S_{22})^{T-t-1}T_{22}^{-1}Q_2\overline{\nu}_1 \tag{109}$$

The show that the right-hand side of (109) equals the sum of the first two expressions on the r.h.s. of (105), rewrite this sum as follows:

$$\begin{aligned} &(T_{22}^{-1}S_{22})^{T-t}\overline{\tilde{x}}_1 - (T_{22}^{-1}S_{22})^{T-t-1}T_{22}^{-1}Q_2\overline{\nu}_1 \\ &= -\big[(T_{22}^{-1}S_{22})^{T-t}(I - T_{22}^{-1}S_{22})^{-1} + (T_{22}^{-1}S_{22})^{T-t-1}\big]T_{22}^{-1}Q_2\overline{\nu}_1 \\ &= -(T_{22}^{-1}S_{22})^{T-t-1}\big[T_{22}^{-1}S_{22}(I - T_{22}^{-1}S_{22})^{-1} + I\big]T_{22}^{-1}Q_2\overline{\nu}_1 \\ &= -(T_{22}^{-1}S_{22})^{T-t-1}\big[T_{22}^{-1}S_{22}(I - T_{22}^{-1}S_{22})^{-1} \\ &\quad + (I - T_{22}^{-1}S_{22})(I - T_{22}^{-1}S_{22})^{-1}\big]T_{22}^{-1}Q_2\overline{\nu}_1 \\ &= -(T_{22}^{-1}S_{22})^{T-t-1}\big[T_{22}^{-1}S_{22} + (I - T_{22}^{-1}S_{22})\big](I - T_{22}^{-1}S_{22})^{-1}T_{22}^{-1}Q_2\overline{\nu}_1 \\ &= -(T_{22}^{-1}S_{22})^{T-t-1}(I - T_{22}^{-1}S_{22})^{-1}T_{22}^{-1}Q_2\overline{\nu}_1 \end{aligned} \tag{110}$$

(110) is equivalent to (109) if and only if

$$(T_{22}^{-1}S_{22})^{T-t-1}(I - T_{22}^{-1}S_{22})^{-1} = (I - T_{22}^{-1}S_{22})^{-1}(T_{22}^{-1}S_{22})^{T-t-1} \quad \Leftrightarrow \tag{111}$$

$$(I - T_{22}^{-1}S_{22})(T_{22}^{-1}S_{22})^{T-t-1} = (T_{22}^{-1}S_{22})^{T-t-1}(I - T_{22}^{-1}S_{22}) \quad \Leftrightarrow \tag{112}$$

$$(T_{22}^{-1}S_{22})^{T-t-1} - (T_{22}^{-1}S_{22})^{T-t} = (T_{22}^{-1}S_{22})^{T-t-1} - (T_{22}^{-1}S_{22})^{T-t} \tag{113}$$

It is obvious that equation (113) holds so that the solution formula (107) is equivalent to (105).

Consider now the first subsystem of the Schur decomposition, equation (92). The general solution is given by (23) with the constant K defined in (29). For $t < T$ we have

$$\begin{aligned}\tilde{z}_t = (S_{11}^{-1}T_{11})^t K + \sum_{s=0}^{t-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}(T_{12}\tilde{x}_s - S_{12}\tilde{x}_{s+1}) \\ + \sum_{s=0}^{t-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}Q_1\overline{\nu}_0 \qquad (0 \le t < T)\end{aligned} \tag{114}$$

where

$$\begin{aligned}\sum_{s=0}^{t-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}Q_1\overline{\nu}_0 &= \left(\sum_{k=0}^{t-1}(S_{11}^{-1}T_{11})^k\right)S_{11}^{-1}Q_1\overline{\nu}_0 \\ &= (I - S_{11}^{-1}T_{11})^{-1}(I - (S_{11}^{-1}T_{11})^t)S_{11}^{-1}Q_1\overline{\nu}_0\end{aligned} \tag{115}$$

For $t \ge T$ we get

$$\begin{aligned}\tilde{z}_t = (S_{11}^{-1}T_{11})^t K + \sum_{s=0}^{T-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}T_{12}\tilde{x}_s \\ + \sum_{s=T}^{t-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}T_{12}\overline{\tilde{x}}_1 - \sum_{s=0}^{T-2}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}S_{12}\tilde{x}_{s+1} \\ - \sum_{s=T-1}^{t-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}S_{12}\overline{\tilde{x}}_1 + \sum_{s=0}^{T-2}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}Q_1\overline{\nu}_0 \\ + \sum_{s=T-1}^{t-1}(S_{11}^{-1}T_{11})^{t-s-1}S_{11}^{-1}Q_1\overline{\nu}_1 \qquad (t \ge T)\end{aligned} \tag{116}$$

Let $M = S_{11}^{-1}T_{11}$. Then[3]

$$\begin{aligned}\sum_{s=T}^{t-1}M^{t-s-1}S_{11}^{-1}T_{12}\overline{\tilde{x}}_1 &= \left(\sum_{k=0}^{t-T-1}M^k\right)S_{11}^{-1}T_{12}\overline{\tilde{x}}_1 \\ &= (I-M)^{-1}\big(I - M^{t-T}\big)S_{11}^{-1}T_{12}\overline{\tilde{x}}_1\end{aligned} \tag{117}$$

$$\begin{aligned}\sum_{s=T-1}^{t-1}M^{t-s-1}S_{11}^{-1}S_{12}\overline{\tilde{x}}_1 &= \left(\sum_{k=0}^{t-T}M^k\right)S_{11}^{-1}T_{12}\overline{\tilde{x}}_1 \\ &= (I-M)^{-1}\big(I - M^{t-T+1}\big)S_{11}^{-1}T_{12}\overline{\tilde{x}}_1\end{aligned} \tag{118}$$

[3] Note that

$$\sum_{k=0}^{n-1}M^k = (I-M)^{-1}(I-M^n)$$

$$\sum_{k=m}^{n-1}M^k = \sum_{k=0}^{n-1}M^k - \sum_{k=0}^{m-1}M^k = (I-M)^{-1}(M^m - M^n)$$

$$\sum_{s=0}^{T-2} M^{t-s-1} S_{11}^{-1} Q_1 \overline{\nu}_0 = \Big(\sum_{k=t-T+1}^{t-1} M^k \Big) S_{11} Q_1 \overline{\nu}_0$$
$$= (I-M)^{-1} \big(M^{t-T+1} - M^t \big) S_{11}^{-1} Q_1 \overline{\nu}_0 \tag{119}$$

$$\sum_{s=T-1}^{t-1} M^{t-s-1} S_{11}^{-1} Q_1 \overline{\nu}_1 = (I-M)^{-1} \big(I - M^{t-T+1} \big) S_{11}^{-1} Q_1 \overline{\nu}_1 \tag{120}$$

Inserting (117) to (120) in (116) yields for $t \geq T$

$$\begin{aligned}
\tilde{z}_t = M^t K &+ \sum_{s=0}^{T-1} M^{t-s-1} S_{11}^{-1} T_{12} \tilde{x}_s + (I-M)^{-1} \big(I - M^{t-T} \big) S_{11}^{-1} T_{12} \overline{\tilde{x}}_1 \\
&- \sum_{s=0}^{T-2} M^{t-s-1} S_{11}^{-1} S_{12} \tilde{x}_{s+1} - (I-M)^{-1} \big(I - M^{t-T+1} \big) S_{11}^{-1} S_{12} \overline{\tilde{x}}_1 \\
&+ (I-M)^{-1} \big(M^{t-T+1} - M^t \big) S_{11}^{-1} Q_1 \overline{\nu}_0 \\
&+ (I-M)^{-1} \big(I - M^{t-T+1} \big) S_{11}^{-1} Q_1 \overline{\nu}_1 \qquad (t \geq T)
\end{aligned} \tag{121}$$

Since $M = S_{11}^{-1} T_{11}$ is a stable matrix, i.e.,

$$\lim_{t \to \infty} M^t = 0 \tag{122}$$

$\tilde{z}_t$ converges towards its steady state value

$$\begin{aligned}
\overline{\tilde{z}}_1 &= (I-M)^{-1} S_{11}^{-1} T_{12} \overline{\tilde{x}}_1 - (I-M)^{-1} S_{11}^{-1} S_{12} \overline{\tilde{x}}_1 + (I-M)^{-1} S_{11}^{-1} Q_1 \overline{\nu}_1 \\
&= (I-M)^{-1} S_{11}^{-1} \big((T_{12} - S_{12}) \overline{\tilde{x}}_1 + Q_1 \overline{\nu}_1 \big) \\
&= (S_{11} - T_{11})^{-1} \big((T_{12} - S_{12}) \overline{\tilde{x}}_1 + Q_1 \overline{\nu}_1 \big)
\end{aligned} \tag{123}$$

The formula for $\overline{\tilde{z}}_1$ also results from the steady state system (88) and the dynamic equation (92).

Combining (121) and (123) yields for $t \geq T$

$$\begin{aligned}
\tilde{z}_t - \overline{\tilde{z}}_1 = M^t K &+ \sum_{s=0}^{T-1} M^{t-s-1} S_{11}^{-1} T_{12} \tilde{x}_s - \sum_{s=0}^{T-2} M^{t-s-1} S_{11}^{-1} S_{12} \tilde{x}_{s+1} \\
&- (I-M)^{-1} M^{t-T} S_{11}^{-1} T_{12} \overline{\tilde{x}}_1 \\
&+ (I-M)^{-1} \big(M^{t-T+1} - M^t \big) S_{11}^{-1} Q_1 \overline{\nu}_0 \\
&+ (I-M)^{-1} M^{t-T+1} S_{11}^{-1} \big(S_{12} \overline{\tilde{x}}_1 - Q_1 \overline{\nu}_1 \big)
\end{aligned} \tag{124}$$

Note that similar to (111) we have

$$(I-M)^{-1} M^{t-T} = M^{t-T} (I-M)^{-1} \tag{125}$$

(125) is equivalent to

$$\begin{aligned}
(S_{11}^{-1} T_{11})^{t-T} (I - S_{11}^{-1} T_{11}) &= (I - S_{11}^{-1} T_{11}) (S_{11}^{-1} T_{11})^{t-T} \qquad \Leftrightarrow \\
(S_{11}^{-1} T_{11})^{t-T} - (S_{11}^{-1} T_{11})^{t+1-T} &= (S_{11}^{-1} T_{11})^{t-T} - (S_{11}^{-1} T_{11})^{t+1-T}
\end{aligned} \tag{126}$$

For $t \geq T$ we therefore get

$$\tilde{z}_t - \overline{\tilde{z}}_1 = M^{t-T}\tilde{K} \tag{127}$$

where

$$\begin{aligned}\tilde{K} = M^T K &+ \sum_{s=0}^{T-1} M^{T-s-1} S_{11}^{-1} T_{12} \tilde{x}_s - \sum_{s=0}^{T-2} M^{T-s-1} S_{11}^{-1} S_{12} \tilde{x}_{s+1} \\ &- (I-M)^{-1} S_{11}^{-1} T_{12} \overline{\tilde{x}}_1 + \big(M - M^T\big)(I-M)^{-1} S_{11}^{-1} Q_1 \overline{\nu}_0 \\ &+ M(I-M)^{-1} S_{11}^{-1} (S_{12} \overline{\tilde{x}}_1 - Q_1 \overline{\nu}_1)\end{aligned} \tag{128}$$

In order to determine the minimal value of the loss function J_t, replace in (42) $\tilde{v}_t$ and $\tilde{w}_t$ by $\hat{\tilde{v}}_t$ and $\hat{\tilde{w}}_t$ respectively, where

$$\hat{\tilde{v}}_t = \begin{cases} \tilde{v}_t - \overline{\tilde{v}}_0 & \text{for} \quad 0 \leq t < T \\ \tilde{v}_t - \overline{\tilde{v}}_1 & \text{for} \quad t \geq T \end{cases} \tag{129}$$

and

$$\hat{\tilde{w}}_t = \begin{cases} \tilde{w}_t - \overline{\tilde{w}}_0 & \text{for} \quad 0 \leq t < T \\ \tilde{w}_t - \overline{\tilde{w}}_1 & \text{for} \quad t \geq T \end{cases} \tag{130}$$

with

$$\overline{\tilde{w}} = Z_{11}\overline{\tilde{z}} + Z_{12}\overline{\tilde{x}} \tag{131}$$

and

$$\overline{\tilde{v}} = Z_{21}\overline{\tilde{z}} + Z_{22}\overline{\tilde{x}} \tag{132}$$

Then $J_0 = J_0^{(1)} + J_0^{(2)}$ where

$$J_0^{(1)} = \frac{1}{2}\sum_{i=0}^{T-1} \lambda^i (\hat{\tilde{z}}_i', \hat{\tilde{x}}_i') Z' \tilde{D}' H \tilde{D} Z \begin{pmatrix} \hat{\tilde{z}}_i \\ \hat{\tilde{x}}_i \end{pmatrix} \tag{133}$$

with

$$\hat{\tilde{z}} = \tilde{z} - \overline{\tilde{z}}, \qquad \hat{\tilde{x}} = \tilde{x} - \overline{\tilde{x}} \tag{134}$$

and

$$J_0^{(2)} = \frac{1}{2}\lambda^T \operatorname{trace}(V^* \tilde{K}\tilde{K}') \tag{135}$$

with V^* defined by (58) and $\tilde{K}$ given by (128).

4 The Stochastic Case

Assume now that ν_{t+1} is an $r \times 1$ vector of white noise disturbances independently distributed with covariance matrix $\Sigma_{\nu\nu} = E(\nu_t \nu_t')$. The i.i.d shocks are, by definition, unpredictable ($T = 0$) and occur at time $t = 0$. Since $\mathrm{E}_t(\nu_{t+1}) = 0_{r\times 1}$, equation (11) implies

$$F \cdot \mathrm{E}_t \begin{pmatrix} \tilde{w}_{t+1} \\ \tilde{v}_{t+1} \end{pmatrix} = G \begin{pmatrix} \tilde{w}_t \\ \tilde{v}_t \end{pmatrix} \tag{136}$$

The Schur decomposition yields the system of equations

$$\begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \mathrm{E}_t \begin{pmatrix} \tilde{z}_{t+1} \\ \tilde{x}_{t+1} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} \tag{137}$$

where

$$\begin{pmatrix} \tilde{w}_t \\ \tilde{v}_t \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} \tag{138}$$

and $\tilde{x}_t = 0$ for all $t \geq T = 0$. Partition the matrices A and B in equation (1) conformably with w_t and v_t, i.e.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{139}$$

Equation (1) then implies

$$A_{11} w_{t+1} + A_{12} \mathrm{E}_t v_{t+1} = B_{11} w_t + B_{12} v_t + C_1 u_t + D_1 \nu_{t+1} \tag{140}$$

and

$$A_{11} \mathrm{E}_t w_{t+1} + A_{12} \mathrm{E}_t v_{t+1} = B_{11} w_t + B_{12} v_t + C_1 u_t \tag{141}$$

where

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \qquad D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \tag{142}$$

From (140) and (141) we get

$$A_{11}(w_{t+1} - \mathrm{E}_t w_{t+1}) = D_1 \nu_{t+1} \tag{143}$$

so that

$$w_{t+1} - \mathrm{E}_t w_{t+1} = A_{11}^{-1} D_1 \nu_{t+1} \tag{144}$$

holds (provided A_{11}^{-1} exists). The corresponding equation for the costate vector p_v is given by [Backus and Driffill (1986)]

$$p_{v,t+1} - \mathrm{E}_t p_{v,t+1} = 0_{n_2 \times 1} \tag{145}$$

Equations (137) and (138) and the definition of $\tilde{w}_t = (w_t', p_{vt}')'$ then imply

$$\tilde{w}_{t+1} - \mathrm{E}_t\,\tilde{w}_{t+1} = Z_{11}(\tilde{z}_{t+1} - \mathrm{E}_t\,\tilde{z}_{t+1}) = Z_{11}(\tilde{z}_{t+1} - S_{11}^{-1}T_{11}\tilde{z}_t) = \begin{pmatrix} A_{11}^{-1}D_1\nu_{t+1} \\ 0_{n_2\times 1} \end{pmatrix} \tag{146}$$

and therefore

$$\tilde{z}_{t+1} = (S_{11}^{-1}T_{11})\tilde{z}_t + Z_{11}^{-1}\begin{pmatrix} A_{11}^{-1}D_1\nu_{t+1} \\ 0_{n_2\times 1} \end{pmatrix} = (S_{11}^{-1}T_{11})\tilde{z}_t + Z_{11}^{-1}\begin{pmatrix} A_{11}^{-1}D_1 \\ 0_{n_2\times r} \end{pmatrix}\nu_{t+1} \tag{147}$$

The solution of the VAR(1) process (147) has the general form

$$\tilde{z}_t = (S_{11}^{-1}T_{11})^t K + \sum_{s=0}^{t-1}(S_{11}^{-1}T_{11})^{t-s-1}Z_{11}^{-1}\begin{pmatrix} A_{11}^{-1}D_1 \\ 0_{n_2\times r} \end{pmatrix}\nu_{s+1} \tag{148}$$

where

$$K = \tilde{z}_0 = Z_{11}^{-1}\tilde{w}_0 = Z_{11}^{-1}\begin{pmatrix} w_0 \\ 0_{n_2\times 1} \end{pmatrix} \tag{149}$$

Since $\mathrm{E}_0\,\nu_{s+1} = 0$ the expected time path of $\tilde{z}_t$ is given by

$$\mathrm{E}_0\,\tilde{z}_t = (S_{11}^{-1}T_{11})^t Z_{11}^{-1}\tilde{w}_0 \tag{150}$$

Premultiply (147) with Z_{11} and use $\tilde{w}_t = Z_{11}\tilde{z}_t$ to obtain the VAR(1) process

$$\tilde{w}_{t+1} = \Gamma\tilde{w}_t + \begin{pmatrix} A_{11}^{-1}D_1 \\ 0_{n_2\times r} \end{pmatrix}\nu_{t+1} \tag{151}$$

where

$$\Gamma = Z_{11}(S_{11}^{-1}T_{11})Z_{11}^{-1} \tag{152}$$

Then

$$\tilde{w}_t = \Gamma^t\tilde{w}_0 + \sum_{s=0}^{t-1}\Gamma^{t-s-1}\begin{pmatrix} A_{11}^{-1}D_1 \\ 0_{n_2\times r} \end{pmatrix}\nu_{s+1} \tag{153}$$

and the expected future path of $\tilde{w}_t$ is given by

$$\mathrm{E}_0\,\tilde{w}_t = \Gamma^t\tilde{w}_0 = \Gamma^t\begin{pmatrix} A_{11}^{-1}D_1 \\ 0_{n_2\times r} \end{pmatrix}\nu_0 \tag{154}$$

The solution of the forward-looking vector $\tilde{v}_t$ follows from

$$\tilde{v}_t = Z_{21}\tilde{z}_t = Z_{21}Z_{11}^{-1}\tilde{w}_t = N\tilde{w}_t \qquad (N = Z_{21}Z_{11}^{-1}) \tag{155}$$

by inserting the solution time path of $\tilde{w}_t$.

To determine the minimal value of the loss function J_0 set

$$\varepsilon_{t+1} = \begin{pmatrix} A_{11}^{-1} D_1 \\ 0_{n_2 \times r} \end{pmatrix} \nu_{t+1} \tag{156}$$

According to (34), (42), (48) and (153) we then obtain

$$\begin{aligned}
J_0 &= \frac{1}{2}\,\mathrm{E}_0 \sum_{i=0}^{\infty} \lambda^i (w_i', v_i', u_i') H \begin{pmatrix} w_i \\ v_i \\ u_i \end{pmatrix} \\
&= \frac{1}{2}\,\mathrm{E}_0 \sum_{i=0}^{\infty} \lambda^i (\tilde{w}_i', \tilde{v}_i') \tilde{D}' H \tilde{D} \begin{pmatrix} \tilde{w}_i \\ \tilde{v}_i \end{pmatrix} \\
&= \frac{1}{2}\,\mathrm{E}_0 \sum_{i=0}^{\infty} \lambda^i \tilde{w}_i' \tilde{N}' \tilde{D}' H \tilde{D} \tilde{N} \tilde{w}_i \\
&= \frac{1}{2}\,\mathrm{E}_0 \sum_{i=0}^{\infty} \lambda^i \tilde{w}_i' H^* \tilde{w}_i \\
&= \frac{1}{2} \sum_{i=0}^{\infty} \lambda^i \Big\{ (\Gamma^i \tilde{w}_0)' H^* (\Gamma^i \tilde{w}_0) + 2\,\mathrm{E}_0 (\Gamma^i \tilde{w}_0)' H^* \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1} \varepsilon_{s+1} \big) \\
&\quad + \mathrm{E}_0 \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1} \varepsilon_{s+1} \big)' H^* \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1} \varepsilon_{s+1} \big) \Big\} \\
&= \frac{1}{2} \tilde{w}_0' \big(\sum_{i=0}^{\infty} \lambda^i \Gamma^{i\prime} H^* \Gamma^i \big) \tilde{w}_0 \\
&\quad + \frac{1}{2} \sum_{i=0}^{\infty} \lambda^i\, \mathrm{E}_0 \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1} \varepsilon_{s+1} \big)' H^* \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1} \varepsilon_{s+1} \big)
\end{aligned} \tag{157}$$

where we have used

$$\mathrm{E}_0\, \varepsilon_{s+1} = 0 \tag{158}$$

Set

$$V = \sum_{i=0}^{\infty} \lambda^i \Gamma^{i\prime} H^* \Gamma^i \tag{159}$$

Then V satisfies the matrix equation (cf. (74))

$$V = H^* + \lambda \Gamma' V \Gamma \tag{160}$$

and

$$\frac{1}{2} \tilde{w}_0' \big(\sum_{i=0}^{\infty} \lambda^i \Gamma^{i\prime} H^* \Gamma^i \big) \tilde{w}_0 = \frac{1}{2} \tilde{w}_0' V \tilde{w}_0 = \frac{1}{2}\,\mathrm{trace}(V \tilde{w}_0 \tilde{w}_0') \tag{161}$$

To calculate the last expression in (157) note that

$$
\begin{aligned}
& \mathrm{E}_0 \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1} \varepsilon_{s+1}\big)' H^* \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1} \varepsilon_{s+1}\big) \\
&= \mathrm{E}_0 (\Gamma^{i-1}\varepsilon_1 + \Gamma^{i-2}\varepsilon_2 + ... + \Gamma^0 \varepsilon_i)' H^* (\Gamma^{i-1}\varepsilon_1 + \Gamma^{i-2}\varepsilon_2 + ... + \Gamma^0 \varepsilon_i) \\
&= \mathrm{E}_0 (\Gamma^{i-1}\varepsilon_1)' H^* (\Gamma^{i-1}\varepsilon_1) + \mathrm{E}_0 (\Gamma^{i-2}\varepsilon_2)' H^* (\Gamma^{i-2}\varepsilon_2) + ... + \mathrm{E}_0 (\Gamma^0\varepsilon_i)' H^* (\Gamma^0\varepsilon_i) \\
&= \mathrm{E}_0\, \varepsilon_i' (\Gamma^{0'} H^* \Gamma^0 + \Gamma' H^* \Gamma + ... + \Gamma^{i-2'} H^* \Gamma^{i-2} + \Gamma^{i-1'} H^* \Gamma^{i-1}) \varepsilon_i \\
&= \mathrm{E}_0\, \varepsilon_i' \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1'} H^* \Gamma^{i-s-1}\big) \varepsilon_i \qquad (162)
\end{aligned}
$$

since $\mathrm{E}_0(\varepsilon_i' \varepsilon_j) = 0$ for $i \neq j$ and the covariance matrix

$$
\mathrm{E}_0(\varepsilon_i \varepsilon_i') = \mathrm{E}_0(\varepsilon_j \varepsilon_j') = \Sigma_{\varepsilon\varepsilon} \qquad (163)
$$

is independent of i and j. We then obtain

$$
\begin{aligned}
& \frac{1}{2} \sum_{i=0}^{\infty} \lambda^i\, \mathrm{E}_0 \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1} \varepsilon_{s+1}\big)' H^* \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1} \varepsilon_{s+1}\big) \\
&= \frac{1}{2} \sum_{i=0}^{\infty} \lambda^i\, \mathrm{E}_0\, \varepsilon_i' \big(\sum_{s=0}^{i-1} \Gamma^{i-s-1'} H^* \Gamma^{i-s-1}\big) \varepsilon_i \\
&= \frac{1}{2} \sum_{i=0}^{\infty} \lambda^i\, \mathrm{E}_0\, \varepsilon_i' \big(\Gamma^{0'} H^* \Gamma^0 + \Gamma' H^* \Gamma + \ldots + \Gamma^{i-2'} H^* \Gamma^{i-2} + \Gamma^{i-1'} H^* \Gamma^{i-1}\big) \varepsilon_i \\
&= \frac{1}{2} \lambda\, \mathrm{E}_0\, \varepsilon_1' \Gamma^{0'} H^* \Gamma^0 \varepsilon_1 \\
&\quad + \frac{1}{2} \lambda^2\, \mathrm{E}_0\, \varepsilon_2' \big(\Gamma^{0'} H^* \Gamma^0 + \Gamma' H^* \Gamma\big) \varepsilon_2 \\
&\quad + \frac{1}{2} \lambda^3\, \mathrm{E}_0\, \varepsilon_3' \big(\Gamma^{0'} H^* \Gamma^0 + \Gamma' H^* \Gamma + \Gamma^{2'} H^* \Gamma^2\big) \varepsilon_3 + \\
&\quad \vdots \\
&\quad + \frac{1}{2} \lambda^n\, \mathrm{E}_0\, \varepsilon_n' \big(\Gamma^{0'} H^* \Gamma^0 + \Gamma' H^* \Gamma + \Gamma^{2'} H^* \Gamma^2 + \ldots + \Gamma^{n-1'} H^* \Gamma^{n-1}\big) \varepsilon_n \\
&\quad + \ldots \\
&= \frac{1}{2} \lambda\, \mathrm{trace}(H^* \Sigma_{\varepsilon\varepsilon}) \\
&\quad + \frac{1}{2} \lambda^2\, \mathrm{trace}\,\big((H^* + \Gamma' H^* \Gamma) \Sigma_{\varepsilon\varepsilon}\big) \\
&\quad + \frac{1}{2} \lambda^3\, \mathrm{trace}\,\big((H^* + \Gamma' H^* \Gamma + \Gamma^{2'} H^* \Gamma^2) \Sigma_{\varepsilon\varepsilon}\big) + \\
&\quad \vdots \\
&\quad + \frac{1}{2} \lambda^n\, \mathrm{trace}\,\big((H^* + \Gamma' H^* \Gamma + \Gamma^{2'} H^* \Gamma^2 + \ldots + \Gamma^{n-1'} H^* \Gamma^{n-1}) \Sigma_{\varepsilon\varepsilon}\big) + \ldots
\end{aligned}
$$

$$= \frac{1}{2}\lambda\,\text{trace}\left((H^* + \lambda H^* + \lambda^2 H^* + \ldots + \lambda^{n-1}H^* + \ldots)\Sigma_{\varepsilon\varepsilon}\right)$$
$$+ \frac{1}{2}\lambda\,\text{trace}\left((\lambda\Gamma' H^*\Gamma + \lambda^2\Gamma' H^*\Gamma + \ldots + \lambda^{n-1}\Gamma' H^*\Gamma' + \ldots)\Sigma_{\varepsilon\varepsilon}\right)$$
$$+ \frac{1}{2}\lambda\,\text{trace}\left((\lambda^2\Gamma^{2\prime} H^*\Gamma^2 + \lambda^3\Gamma^{2\prime} H^*\Gamma^2 + \ldots + \lambda^{n-1}\Gamma^{2\prime} H^*\Gamma^2 + \ldots)\Sigma_{\varepsilon\varepsilon}\right)$$
$$+ \ldots + \frac{1}{2}\lambda\,\text{trace}\left((\sum_{i=n}^{\infty}\lambda^i\Gamma^{n\prime}H^*\Gamma^n)\Sigma_{\varepsilon\varepsilon}\right) + \ldots$$
$$= \frac{1}{2}\lambda\,\text{trace}\left((\frac{1}{1-\lambda}H^* + \frac{\lambda}{1-\lambda}\Gamma' H^*\Gamma + \frac{\lambda^2}{1-\lambda}\Gamma^{2\prime}H^*\Gamma^2\right.$$
$$\left. + \ldots + \frac{\lambda^n}{1-\lambda}\Gamma^{n\prime}H^*\Gamma^n + \ldots)\Sigma_{\varepsilon\varepsilon}\right)$$
$$= \frac{1}{2}\frac{\lambda}{1-\lambda}\,\text{trace}\left((\sum_{i=0}^{\infty}\lambda^i\Gamma^{i\prime}H^*\Gamma^i)\Sigma_{\varepsilon\varepsilon}\right)$$
$$= \frac{1}{2}\frac{\lambda}{1-\lambda}\,\text{trace}(V\Sigma_{\varepsilon\varepsilon}) \tag{164}$$

with V defined in (160). The optimal value of the loss function J_0 in the stochastic case (with $T = 0$) is then given by

$$J_0 = \frac{1}{2}\,\text{trace}(V\tilde{w}_0\tilde{w}_0') + \frac{1}{2}\frac{\lambda}{1-\lambda}\,\text{trace}(V\Sigma_{\varepsilon\varepsilon}) \tag{165}$$

Note that (165) is a generalization of (72) where we have assumed that the shock in $t = 0$ is deterministic ($\Sigma_{\varepsilon\varepsilon} = 0$). The formula (165) holds for a discount factor λ with $0 < \lambda < 1$.[4] The right-hand side of (165) is not defined in the special case $\lambda = 1$. If the discount factor λ approaches unity we must scale the intertemporal loss function J_0 by the factor $(1-\lambda)$ [Rudebusch and Svensson (1999)]. Equation (165) then implies

$$(1-\lambda)J_0 = \frac{1}{2}(1-\lambda)\,\text{trace}(V\tilde{w}_0\tilde{w}_0') + \frac{1}{2}\lambda\,\text{trace}(V\Sigma_{\varepsilon\varepsilon}) \tag{166}$$

The scaled intertemporal loss function $(1-\lambda)J_0$ converges if λ approaches unity. (166) implies

$$\lim_{\lambda\to 1}(1-\lambda)J_0 = \frac{1}{2}\,\text{trace}(V\Sigma_{\varepsilon\varepsilon}) \tag{167}$$

Note that in case $T = 0$ and $\lambda = 1$ the r.h.s. of (167) equals the r.h.s of (72) if $w_0 w_0' = \Sigma_{\varepsilon\varepsilon}$. In this special case the stochastic and deterministic case are equivalent. If the off-diagonal elements of W_1 and W_2 in the loss function (7) are equal to zero, then the limit value of $(1-\lambda)J_0$ can be expressed as

$$\lim_{\lambda\to 1}(1-\lambda)J_0 = \frac{1}{2}\,\text{E}(L_t) \tag{168}$$

[4]In the deterministic case, where $\Sigma_{\varepsilon\varepsilon} = 0$, (165) also holds for $\lambda = 1$.

where $\mathrm{E}(L_t)$ is the unconditional mean of the period loss function

$$L_t = (s_t', u_t') \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} = \sum_{i=1}^{n_3} w_{ii,1} s_{i,t}^2 + \sum_{i=1}^{m} w_{ii,2} u_{i,t}^2 \tag{169}$$

Then

$$\mathrm{E}(L_t) = \sum_{i=1}^{n_3} w_{ii,1} \operatorname{Var} s_{i,t} + \sum_{i=1}^{m} w_{ii,2} \operatorname{Var} u_{i,t} \tag{170}$$

The period loss function can also be written as

$$L_t = Y_t' H Y_t \tag{171}$$

where $Y_t' = (k_t', u_t')$ and H defined in (35). Then the unconditional period loss also fulfills

$$\mathrm{E}(L_t) = \mathrm{E}(Y_t' H Y_t) = \operatorname{trace}(H \Sigma_{YY}) \tag{172}$$

where Σ_{YY} is the unconditional covariance matrix of the vector Y.

5 Summary

In this paper, we present a method to solve linear dynamic rational expectations models with anticipated shocks and optimal policy by using the generalized Schur decomposition method. We determine the optimal unrestricted and restricted policy responses to anticipated temporary and permanent shocks. In particular, our method can be applied to analyze optimal monetary policy in New Keynesian dynamic general equilibrium models. Our approach allows also the evaluation of the widely discussed case of unpredictable shocks and can therefore be seen as a generalization of the methods summarized by Söderlind (1999).

References

Backus, D., J. Driffill (1986), The Consistency of Optimal Policy in Stochastic Rational Expectations Models. Discussion Paper No. 124, CEPR, London.

Clarida, R., J. Galí and M. Gertler (1999), The Science of Monetary Policy: A New Keynesian Perspective. *Journal of Economic Literature* 37, 1661–1707.

Currie, D., P. Levine (1993), *Rules, Reputation and Macroeconomic Policy Coordination*. Cambrigde University Press, Cambridge.

Klein, P. (2000), Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model. *Journal of Economic Dynamics & Control* 24, 1405–1423.

Rudebusch, G., L. E. O. Svensson (1999), Policy Rules for Inflation Targeting. In *Monetary Policy Rules*, edited by J. B. Taylor. University of Chicago Press.

Söderlind, P. (1999), Solution and Estimation of RE Macromodels with Optimal Policy. *European Economic Review* 43, 813–823.

Walsh, C. E. (2003), *Monetary Theory and Policy*. Second Edition. MIT Press, Cambrigde (Mass.).