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How Robust is Robust Control in the Time Domain?

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Abstract: By applying robust control the decision maker wants to make good decisions when his model is only a good approximation of the true one. Such decisions are said to be robust to model misspecification. In this paper it is shown that both a "probabilistically sophisticated" and a non-"probabilistically sophisticated" decision maker applying robust control in the time domain are indeed assuming a very special kind of "misspecification of the approximating model." This is true when unstructured uncertainty à la Hansen and Sargent is used or when uncertainty is related to unknown structural parameters of the model.

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1. Introduction

A characteristic "feature of most robust control theory", observes Bernhard (2002, p. 19), "is that the *a priori* information on the unknown model errors (or signals) is nonprobabilistic in nature, but rather is in terms of *sets of possible realizations*. Typically, though not always, the errors are bounded in some way. ... As a consequence, robust control aims at synthesizing control mechanisms that control in a satisfactory fashion (e.g., stabilize, or bound, an output) a *family* of models."[1] Then "standard control theory tells a decision maker how to make optimal decisions when his model is correct (whereas) robust control theory tells him how to make good decisions when his model approximates a correct one" (Hansen and Sargent,

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[1] Robust control has been a very popular area of research in the last two decades and shows no sign of fatigue. See, e.g., Giannoni (2002, 2007), Hansen and Sargent (2001, 2003, 2007, 2008), Hansen et al. (1999, 2002), Onatski and Stock (2002), Rustem (1992, 1994, 1998), Rustem and Howe (2002) and Tetlow and von zur Muehlen (2001a,b). However the use of the minimax approach in control theory goes back to the 60's as pointed out in Basar and Bernhard (1991, pp. 1–4).

2008, p. 25). In other words, by applying robust control the decision maker makes good decisions when it is statistically difficult to distinguish between his approximating model and the correct one using a time series of moderate size. "Such decisions are said to be robust to misspecification of the approximating model" (Hansen and Sargent, 2008, p. 27).

The main focus of the present work is to investigate the assumptions implicit in the "nonprobabilistic" nature of the *a priori* information used to derive the linear-quadratic robust control in discrete-time. This is a relevant point emphasized in Sims (2001) where, on page 52, it reads "once one understands the appropriate role for this tool (i.e. robust control or maxmin expected utility), it should be apparent that, whenever possible, its results should be compared to more direct approaches to assessing prior beliefs." Then he continues, "the results may imply prior beliefs that obviously make no sense … (or) they may … focus the minimaxing on a narrow, convenient, uncontroversial range of deviations from a central model." In the latter case "the danger is that one will be misled by the rhetoric of robustness to devoting less attention than one should to technically inconvenient, controversial deviations from the central model."[2]

Tucci (2006, p. 538) argues that "the true model in Hansen and Sargent (2008) … is observationally equivalent to a model with a time-varying intercept." Then he goes on showing that, when the same "malevolent" shock is used in both procedures, the robust control for a linear system with an objective function having desired paths for the states and controls set to zero applied by a "probabilistically sophisticated" decision maker is identical to the optimal control for a linear system with an intercept following a "Return to Normality" model and the same objective function *only when* the transition matrix in the law of motion of the parameters is zero.[3] He concludes that the decision maker applying robust control implicitly assumes "that today's malevolent shock is linearly uncorrelated with tomorrow's malevolent" shock" (p. 553). The goal of this paper is to see if this result holds in more general settings both for a "probabilistically sophisticated" and a non-"probabilistically sophisticated" decision maker.

The remainder of the paper is organized as follows. Section 2 presents a robust control problem with unstructured uncertainty à la Hansen and Sargent, i.e. a nonparametric

[2] See also Hansen and Sargent (2008, pp. 14–17).

set of additive mean-distorting model perturbations, where the decision maker is assumed to be "probabilistically sophisticated". An example of a non-"probabilistically sophisticated" decision maker, namely the case sometimes labeled robust filtering without commitment, is discussed in Section 3. In Section 4 both problems are reformulated as linear quadratic tracking control problems where the system equations have a time-varying intercept following a 'Return to Normality' model and their solutions are compared with those of the previous sections. Sometimes robust control is applied to situations where uncertainty is related to unknown structural parameters.[4] Then the optimizing model for monetary policy used in Giannoni (2002, 2007) is presented (Sect. 5). Section 6 reports some numerical results obtained using the permanent income model, a popular model in the robust control literature (see, e.g., Hansen and Sargent, 2001, 2003, 2008; Hansen et al. 1999, 2002), and Giannoni's structural model with uncertain parameters. The main conclusions are summarized in Section 7.

2. Robust control à la Hansen and Sargent: The standard case

Hansen and Sargent (2008, p. 140) consider a decision maker "who has a unique explicitly specified approximating model but concedes that the data might actually be generated by an unknown member of a set of models that surround the approximating model."[5] Then the linear system

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{u}_t + \mathbf{C}\boldsymbol{\varepsilon}_{t+1} \qquad \text{for } t = 0, ..., \infty, \tag{2.1}$$

with $\mathbf{y}_t$ the $n\times 1$ vector of state variables at time t, $\mathbf{u}_t$ the $m\times 1$ vector of control variables and $\boldsymbol{\varepsilon}_{t+1}$ an $l\times 1$ identically and independently distributed (*iid*) Gaussian vector process with mean zero and an identity contemporaneous covariance matrix, is viewed as an approximation to the true unknown model

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{u}_t + \mathbf{C}(\boldsymbol{\varepsilon}_{t+1} + \boldsymbol{\omega}_{t+1}) \qquad \text{for } t = 0, ..., \infty. \tag{2.2}$$

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3 By probabilistic sophistication is meant that "in comparing utility processes, all that matters are the induced distortions under the approximating model" (Hansen and Sargent, 2008, p. 406).

4 The reasons that may lead to prefer this formulation are discussed in Giannoni (2007, p. 182).

5 See Hansen and Sargent (2008, Ch. 2 and 7) for the complete discussion of robust control in the time domain.

The matrices of coefficients **A**, **B** and **C** are assumed known and $\mathbf{y}_0$ given. [6]

In Equation (2.2) the vector $\boldsymbol{\omega}_{t+1}$ denotes an "unknown" $l\times 1$ "process that can feed back in a possibly nonlinear way on the history of **y**, (i.e.) $\boldsymbol{\omega}_{t+1} = \mathbf{g}_t(\mathbf{y}_t, \mathbf{y}_{t-1}, \ ...)$ where $\{\mathbf{g}_t\}$ is a sequence of measurable functions" (Hansen and Sargent, 2008, pp. 26-27). It is introduced because the "*iid* random process ... ($\boldsymbol{\varepsilon}_{t+1}$) can represent only a very limited class of approximation errors and in particular cannot depict such examples of misspecified dynamics as are represented in models with nonlinear and time-dependent feedback of $\mathbf{y}_{t+1}$ on past states" (p. 26).[7] To express the idea that (2.1) is a good approximation of (2.2) the **ω**'s are restrained by

$$E_0\left[\sum_{t=0}^{\infty}\beta^{t+1}\boldsymbol{\omega}'_{t+1}\boldsymbol{\omega}_{t+1}\right]\le \eta_0 \qquad \text{with } 0<\beta<1 \qquad (2.3)$$

where E_0 denotes mathematical expectation evaluated with respect to model (2.2) and conditioned on $\mathbf{y}_0$ and η_0 measures the set of models surrounding the approximating model.[8]

"The decision maker's distrust of his model ... (2.1) makes him want good decisions over a set of models ... (2.2) satisfying ... (2.3)" write Hansen and Sargent (2008, p. 27). They consider two robust control problems: the constraint problem and the multiplier problem. [9] The *constraint robust control problem* is defined as

$$\max_{\mathbf{u}}\ \min_{\boldsymbol{\omega}}\ -E_0\left[\sum_{t=0}^{\infty}\beta^t r(\mathbf{y}_t,\mathbf{u}_t)\right], \qquad (2.4)$$

[6] It is assumed, see e.g. page 140 in Hansen and Sargent (2008), that the pair ($\sqrt{\beta}\mathbf{A},\mathbf{B}$) is stabilizable, i.e. the eigenvalues of $\mathbf{A}-\mathbf{B}\mathbf{F}_t$, where $\mathbf{F}_t$ is the 'feedback' matrix (i.e. $\mathbf{u}_t = -\mathbf{F}_t\,\mathbf{y}_t$), have absolute values strictly less than $1/\sqrt{\beta}$ where β is a discount factor between 0 and 1. The matrix **C** is sometimes called the "volatility matrix" because, given the assumptions on the ε's, it "determines the covariance matrix $\mathbf{C}'\mathbf{C}$ of random shocks impinging on the system" (p. 29).

[7] When Equation (2.2) "generates the data it is as though the errors in ... (2.1) were conditionally distributed as N $(\boldsymbol{\omega}_{t+1},\mathbf{I}_l)$ rather than as N $(\mathbf{0},\mathbf{I}_l)$ ... (so) we capture the idea that the approximating model is misspecified by allowing the conditional mean of the shock vector in the model that actually generates the data to feedback arbitrarily on the history of the state" (Hansen and Sargent, 2008, pp. 27).

[8] See Hansen and Sargent (2008, p. 11).

[9] See p. 32 and Chapters 6–8 in Hansen and Sargent (2008) for details.

with $r(\mathbf{y}_t, \mathbf{u}_t)$ the one-period loss function, subject to (2.2)-(2.3) where $\eta^* > \eta_0$ and η^* "measures the largest set of perturbations against which it is possible to seek robustness" (p. 32). The *multiplier robust control problem* is formalized as

$$\max_{\mathbf{u}} \min_{\boldsymbol{\omega}} -E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[r(\mathbf{y}_t, \mathbf{u}_t) - \theta\beta\boldsymbol{\omega}'_{t+1}\boldsymbol{\omega}_{t+1} \right] \right\} \tag{2.5}$$

subject to (2.2) with θ, $0 < \theta^* < \theta \leq \infty$, a penalty parameter restraining the minimizing choice of the $\{\boldsymbol{\omega}_{t+1}\}$ sequence The "breakdown point" θ^* represents "a lower bound on θ that is required to keep the objective of the two-person zero-sum game convex in ... ($\boldsymbol{\omega}_{t+1}$) and concave in $\mathbf{u}_t$" (p. 161).[10] Both problems can be reinterpreted as two-player zero-sum games where one player is the decision maker maximizing the objective function by choosing the sequence for $\mathbf{u}$ and the other player is a malevolent nature choosing a feedback rule for a model-misspecification process $\boldsymbol{\omega}$ to minimize the same criterion function.[11] For this reason, the *constraint* and the *multiplier robust control problem* are also referred to as the *constraint* and *multiplier game*, respectively.

Hansen and Sargent (2008, p. 139) notice that "constraint and multiplier games differ in how they parameterize a set of alternative specifications that surround an approximating model ... (the former) require that the discounted entropy of each alternative model relative to the approximating model not exceed a nonnegative parameter ... (η_0) ... (whereas the latter) restrict the discounted entropy implicitly via a penalty parameter θ."[12] However if the

10 As noted in Hansen and Sargent (2008, p. 40) "this lower bound is associated with the largest set of alternative models, as measured by entropy, against which it is feasible to seek a robust rule ... This cutoff value of θ ... is affiliated with a rule that is robust to the biggest allowable set of misspecifications." See also Ch. 7 in the same reference and Hansen and Sargent (2001) for a further discussion of the restrictions on the robustness parameter θ.

11 See Hansen and Sargent (2008, p. 35).

12 Hansen and Sargent (2008, pp. 12–14) argue that entropy is the most appropriate way to measure model misspecification. Let f_0 denote "the conditional distribution of next period's state ... (and) f ... an arbitrary alternative conditional distribution that puts positive probability on the same events as the approximating model f_0 (then) ... entropy $I(f_0, f)(\mathbf{y}_t)$ is ... the conditional expectation of the log-likelihood ratio evaluated with respect to the distorted model f" (Hansen and Sargent, 2008, pp. 41–42). An intertemporal measure of model misspecification is "$\mathrm{I}(f_\alpha, f) = E_f \sum_{t=0}^{\infty} \beta^t I(f_\alpha, f)(\mathbf{y}_t)$ where E_f is the mathematical expectation evaluated with respect to the distribution f ... (and the) decision maker confronts model misspecification by seeking a decision rule that will work well across a set of models for which $\mathrm{I}(f_\alpha, f) \leq \eta_0$" (p. 11). See also pages 30–31

parameters η_0 and θ are appropriately related the two "games have equivalent outcomes." *Equivalent* in the sense that if there exists a solution $\mathbf{u}^*$, $\boldsymbol{\omega}^*$ to the *multiplier robust control problem*, that $\mathbf{u}^*$ also solves the *constraint robust control problem* with $\eta_0 = \eta_0^* = E_0[\sum_{t=0}^{\infty} \beta^{t+1} \boldsymbol{\omega}_{t+1}^{*}{}' \boldsymbol{\omega}_{t+1}^{*}]$.[13] Then, in Appendix C of Ch. 7, two sets of formulae to compute the robust decision rule are provided and it is pointed out that the Riccati equation for the robust control problem (2.5) "is the Riccati equation associated with an ordinary optimal linear regulator problem (also known as the linear quadratic control problem) with controls $(\mathbf{u}_t' \;\; \boldsymbol{\omega}_{t+1}')'$ and penalty matrix on those controls appearing in the criterion function of $diag(\mathbf{R}, \; -\beta\theta\mathbf{I}_l)$" (p. 170).[14]

Therefore the robust rules for $\mathbf{u}_t$ and the worst-case shock $\boldsymbol{\omega}_{t+1}$ can be directly computed from the associated ordinary linear regulator problem. In particular, when the one-period loss function $r(\mathbf{y}_t, \mathbf{u}_t)$ is specified as[15]

$$\left(\mathbf{y}_t - \tilde{\mathbf{y}}_t^d\right)' \mathbf{Q}\left(\mathbf{y}_t - \tilde{\mathbf{y}}_t^d\right) + 2\left(\mathbf{y}_t - \tilde{\mathbf{y}}_t^d\right)' \widehat{\mathbf{W}}\left(\mathbf{u}_t - \mathbf{u}_t^d\right) + \left(\mathbf{u}_t - \mathbf{u}_t^d\right)' \mathbf{R}\left(\mathbf{u}_t - \mathbf{u}_t^d\right), \qquad (2.6)$$

with $\mathbf{Q}$ a positive semi-definite matrix, $\mathbf{R}$ a positive definite matrix, $\mathbf{W}$ an $n \times m$ array, $\mathbf{y}_t^d$ and $\mathbf{u}_t^d$ the desired values of the states and controls, respectively, for period t, the robust control

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and 61-62 in the same reference. Finally, it should be noticed that the Bellman equation for the multiplier robust control model can be written as in their Eqt. 2.6.1 on page 42, i.e.

$$-\mathbf{y}_t'\mathbf{P}_t\mathbf{y}_t - \mathbf{p}_t = \max_{\mathbf{u}} \; \min_{f} \; - E_0\left[r(\mathbf{y}_t, \mathbf{u}_t) + 2\theta\beta I(f_0, f)(\mathbf{y}_t) - \beta\mathbf{y}_{t+1}'\mathbf{P}_t\mathbf{y}_{t+1} - \beta\mathbf{p}_t \right],$$

when the desired paths for the states and controls are equal to 0 in the one-period utility function.

13 See Hansen and Sargent (2008, pp. 159–160).

14 This is due to the fact that the "Riccati equation for the optimal linear regulator emerges from first-order conditions alone, and that the first order conditions for (the max-min problem (2.5) subject to (2.2)) match those for an ordinary, i.e. non-robust, optimal linear regulator problem with joint control process $\{\mathbf{u}_t \, , \, \boldsymbol{\omega}_{t+1}\}$" (Hansen and Sargent, 2008, p. 43).

15 This is a minor generalization of the case discussed in Hansen and Sargent (2008, Ch. 2 and 7) where the desired values for the states and controls are 0 and there are no cross products between states and controls in the objective function. See their Ch. 4 and pages 167–168 for a transformation of the control problem that eliminates cross products between states and controls in the objective function.

rule is derived by extremizing, i.e. maximizing with respect to $\mathbf{u}_t$ and minimizing with respect to $\boldsymbol{\omega}_{t+1}$, the objective function[16]

$$-E_0\left[\sum_{t=0}^{\infty}\beta^t r(\mathbf{y}_t,\tilde{\mathbf{u}}_t)\right] \tag{2.7}$$

with

$$\begin{aligned} &r(\mathbf{y}_t,\tilde{\mathbf{u}}_t) = \\ &\left(\mathbf{y}_t-\mathbf{y}_t^d\right)'\mathbf{Q}\left(\mathbf{y}_t-\mathbf{y}_t^d\right)+2\left(\mathbf{y}_t-\mathbf{y}_t^d\right)'\tilde{\mathbf{W}}\left(\tilde{\mathbf{u}}_t-\tilde{\mathbf{u}}_t^d\right)+\left(\tilde{\mathbf{u}}_t-\tilde{\mathbf{u}}_t^d\right)'\tilde{\mathbf{R}}\left(\tilde{\mathbf{u}}_t-\tilde{\mathbf{u}}_t^d\right) \end{aligned} \tag{2.8}$$

subject to

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \tilde{\mathbf{B}}\tilde{\mathbf{u}}_t + \mathbf{C}\boldsymbol{\varepsilon}_{t+1} \qquad \text{for } t = 0, ..., \infty \tag{2.9}$$

where[17]

$$\tilde{\mathbf{R}} = \begin{bmatrix}\mathbf{R} & \mathbf{O}\\ \mathbf{O} & -\beta\theta\mathbf{I}_l\end{bmatrix},\ \tilde{\mathbf{u}}_t = \begin{bmatrix}\mathbf{u}_t\\ \boldsymbol{\omega}_{t+1}\end{bmatrix},\ \tilde{\mathbf{B}} = \begin{bmatrix}\mathbf{B} & \mathbf{C}\end{bmatrix},\ \tilde{\mathbf{u}}_t^d = \begin{bmatrix}\mathbf{u}_t^d\\ \mathbf{0}\end{bmatrix} \tag{2.10}$$

and $\tilde{\mathbf{W}} = [\mathbf{W} \quad \mathbf{O}]$ with **O** and **0** null arrays of appropriate dimension.

Setting $\boldsymbol{\varepsilon}_{t+1} = \mathbf{0}$ and writing the optimal value of (2.7) as $-\mathbf{y}_t'\mathbf{P}_t\mathbf{y}_t - 2\mathbf{y}_t'\mathbf{p}_t$,[18] the Bellman equation looks like[19]

[16] Indeed, as pointed out in Intriligator (1971, p. 342, fn. 4) Eqt. (2.7) is a functional dependent on the control trajectory whereas the solution to the problem is a function dependent on the parameters given by the initial state $\mathbf{y}_0$ and the initial time $t = 0$.

[17] The penalty matrix $\tilde{\mathbf{R}}$ implies that each component of the vector process $\boldsymbol{\omega}_{t+1}$ is penalized in the same way.

[18] Using the deterministic counterpart to (2.7) and (2.9) allows to simplify some formulas by dropping constants from the value function without affecting the formulas for the decision rules. As noted in Hansen and Sargent (2008, p. 33) "the certainty equivalence principle that applies to the linear quadratic dynamic programming model without concern for model misspecification ... fails to hold when there is concern about model misspecification." However, they continue, "it can be verified directly that precisely the same Riccati equations and the same decision rules for $\mathbf{u}_t$ and $\boldsymbol{\omega}_{t+1}$ emerge from solving the random version of the Bellman equation (for model (2.7)) ... as would from a version that sets $\boldsymbol{\varepsilon}_{t+1} \equiv 0$" in Eqt. (2.9). The concern for model misspecification, as they underline in the following page, makes the decision rule for $\mathbf{u}_t$ dependent upon the 'volatility matrix' **C**. Sometimes, the optimal value of (2.7) is called the optimal cost-to-go. See, e.g., Kendrick (1981) and Tucci (2004).

$$-\mathbf{y}_t'\mathbf{P}_t\mathbf{y}_t - 2\mathbf{y}_t'\mathbf{p}_t = \underset{\tilde{\mathbf{u}}}{ext} - \big[\mathbf{y}_t'\mathbf{Q}_t\mathbf{y}_t + \mathbf{u}_t'\mathbf{R}_t\mathbf{u}_t - \beta\theta\boldsymbol{\omega}_{t+1}'\boldsymbol{\omega}_{t+1} + 2\mathbf{y}_t'\mathbf{W}_t\mathbf{u}_t + 2\mathbf{y}_t'\mathbf{q}_t + 2\mathbf{u}_t'\mathbf{r}_t + \mathbf{y}_{t+1}'\mathbf{P}_{t+1}\mathbf{y}_{t+1} + 2\mathbf{y}_{t+1}'\mathbf{p}_{t+1}\big] \tag{2.11}$$

with $\mathbf{P}_{t+1} = \beta\mathbf{P}_t$, $\mathbf{Q}_t = \beta^t\mathbf{Q}$, $\mathbf{W}_t = \beta^t\mathbf{W}$, $\mathbf{R}_t = \beta^t\mathbf{R}$, $\mathbf{q}_t = -(\mathbf{Q}_t\mathbf{y}_t^d + \mathbf{W}_t\mathbf{u}_t^d)$ and $\mathbf{r}_t = -(\mathbf{R}_t\mathbf{u}_t^d + \mathbf{W}_t'\mathbf{y}_t^d)$.[20] Then expressing the right-hand side of (2.11) only in terms of $\mathbf{y}_t$ and $\tilde{\mathbf{u}}_t$ and extremizing it yields the optimal control for the decision maker

$$\mathbf{u}_t = -\left(\mathbf{R}_t + \mathbf{B}'\mathbf{P}_{t+1}\mathbf{B}\right)^{-1}\left[\left(\mathbf{B}'\mathbf{P}_{t+1}\mathbf{A} + \mathbf{W}_t'\right)\mathbf{y}_t + \mathbf{B}'\mathbf{P}_{t+1}\mathbf{C}\boldsymbol{\omega}_{t+1} + \mathbf{B}'\mathbf{p}_{t+1} + \mathbf{r}_t\right] \tag{2.12}$$

and the optimal control for the malevolent nature

$$\boldsymbol{\omega}_{t+1} = (\beta\theta\mathbf{I}_l - \mathbf{C}'\mathbf{P}_{t+1}\mathbf{C})^{-1}(\mathbf{C}'\mathbf{P}_{t+1}\mathbf{A}\mathbf{y}_t + \mathbf{C}'\mathbf{P}_{t+1}\mathbf{B}\mathbf{u}_t + \mathbf{C}'\mathbf{p}_{t+1}). \tag{2.13}$$

It follows that the θ-constrained worst-case controls are[21]

$$\mathbf{u}_t = -\left(\mathbf{R}_t + \mathbf{B}'\mathbf{P}_{t+1}^*\mathbf{B}\right)^{-1}\left[\left(\mathbf{B}'\mathbf{P}_{t+1}^*\mathbf{A} + \mathbf{W}_t'\right)\mathbf{y}_t + \mathbf{B}'\mathbf{p}_{t+1}^* + \mathbf{r}_t\right] \tag{2.14}$$

and[22]

$$\boldsymbol{\omega}_{t+1} = \left(\beta\theta\mathbf{I}_l - \mathbf{C}'\mathbf{P}_{t+1}\mathbf{C}\right)^{-1}\mathbf{C}'\mathbf{P}_{t+1}\left\{\left[\mathbf{A} - \mathbf{B}\left(\mathbf{R}_t + \mathbf{B}'\mathbf{P}_{t+1}^*\mathbf{B}\right)^{-1}\left(\mathbf{B}'\mathbf{P}_{t+1}^*\mathbf{A} + \mathbf{W}_t'\right)\right]\mathbf{y}_t - \mathbf{B}\left(\mathbf{R}_t + \mathbf{B}'\mathbf{P}_{t+1}^*\mathbf{B}\right)^{-1}\left(\mathbf{B}'\mathbf{p}_{t+1}^* + \mathbf{r}_t\right) + \mathbf{P}_{t+1}^{-1}\mathbf{p}_{t+1}\right\} \tag{2.15}$$

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19 The constant term appearing on the right-hand side and on the left-hand side of the equation have been dropped because they do not affect the solution of the optimization problem. See, e.g., Eqt. (2.5.3) in Hansen and Sargent (2008, Ch. 2).

20 When the desired paths for the states and controls are set to 0, $\mathbf{p}_t = \mathbf{q}_t = \mathbf{r}_t = 0$.

21 See, e.g., Eqs. (7.C.18)–(7.C.19) in Hansen and Sargent (2008, p. 169). As suggested on page 35 of the same reference Eqt. (2.11) "can be represented as"

$$-\mathbf{y}_t'\mathbf{P}_t\mathbf{y}_t - 2\mathbf{y}_t'\mathbf{p}_t = \max_{\mathbf{u}} - \left[\mathbf{y}_t'\mathbf{Q}_t\mathbf{y}_t + \mathbf{u}_t'\mathbf{R}_t\mathbf{u}_t + 2\mathbf{y}_t'\mathbf{W}_t\mathbf{u}_t + 2\mathbf{y}_t'\mathbf{q}_t + 2\mathbf{u}_t'\mathbf{r}_t + \mathbf{y}_{t+1}'\mathbf{P}_{t+1}^*\mathbf{y}_{t+1} + 2\mathbf{y}_{t+1}'\mathbf{p}_{t+1}^*\right]$$

subject to the approximating model (2.1) instead of the distorted model (2.2).

22 See, e.g., Eqt. (7.C.9) in Hansen and Sargent (2008, p. 168). As pointed out on page 139 of the same work, the "two-player zero-sum dynamic games ... (i.e.) an effectively static Stackelberg multiplier game in which a ... player at time 0 chooses a history-dependent sequence of controls … (and) a Markov perfect multiplier game in which both players choose sequentially … have identical outcomes" both when the $\boldsymbol{\omega}$-player chooses before the $\mathbf{u}$-player, at time 0 or in each period $t \geq 0$, and vice versa.

with[23]

$$\mathbf{P}^*_{t+1} = \mathbf{P}_{t+1} + \mathbf{P}_{t+1}\mathbf{C}(\beta\theta\mathbf{I}_l - \mathbf{C}'\mathbf{P}_{t+1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}_{t+1} \tag{2.16}$$
$$\mathbf{p}^*_{t+1} = \left[\mathbf{I}_n + \mathbf{P}_{t+1}\mathbf{C}(\beta\theta\mathbf{I}_l - \mathbf{C}'\mathbf{P}_{t+1}\mathbf{C})^{-1}\mathbf{C}'\right]\mathbf{p}_{t+1}. \tag{2.17}$$

The "robust" Riccati matrix $\mathbf{P}^*_{t+1}$ is always greater or equal to $\mathbf{P}_{t+1}$ because it is assumed that, in the "admissible" region, the parameter θ is large enough to make $(\beta\theta\mathbf{I}_l - \mathbf{C}'\mathbf{P}_{t+1}\mathbf{C})$ positive definite.[24] The two Riccati matrices are equal when $\theta = \infty$.[25] The same considerations apply to $\mathbf{p}^*_{t+1}$.

As noted in Hansen and Sargent (2008, p. 11), the first order conditions for problem (2.11) subject to (2.9) imply the matrix Riccati equation

$$\mathbf{P}_t = \mathbf{Q}_t + \mathbf{A}'\mathbf{P}_{t+1}\mathbf{A} - \left(\mathbf{A}'\mathbf{P}_{t+1}\tilde{\mathbf{B}} + \tilde{\mathbf{W}}_t\right)\left(\tilde{\mathbf{B}}'\mathbf{P}_{t+1}\tilde{\mathbf{B}} + \tilde{\mathbf{R}}_t\right)^{-1}\left(\mathbf{A}'\mathbf{P}_{t+1}\tilde{\mathbf{B}} + \tilde{\mathbf{W}}_t\right)', \tag{2.18}$$

where $\tilde{\mathbf{R}}_t = \beta^t\tilde{\mathbf{R}}$ and $\tilde{\mathbf{W}}_t = \beta^t\tilde{\mathbf{W}}$, and the Riccati equation for the $\mathbf{p}$ vector[26]

$$\mathbf{p}_t = \mathbf{q}_t + \mathbf{A}'\mathbf{p}_{t+1} - \left(\mathbf{A}'\mathbf{P}_{t+1}\tilde{\mathbf{B}} + \tilde{\mathbf{W}}_t\right)\left(\tilde{\mathbf{B}}'\mathbf{P}_{t+1}\tilde{\mathbf{B}} + \tilde{\mathbf{R}}_t\right)^{-1}\left(\tilde{\mathbf{B}}'\mathbf{p}_{t+1} + \tilde{\mathbf{r}}_t\right) \tag{2.19}$$

where $\tilde{\mathbf{R}}_t = \beta^t\tilde{\mathbf{R}}$, $\tilde{\mathbf{W}}_t = \beta^t\tilde{\mathbf{W}}$ and $\tilde{\mathbf{r}}_t = (\mathbf{r}'_t \quad \mathbf{0}'_l)'$. It is straightforward to show that the right-hand side of Eqs. (2.18)-(2.19) can be rewritten as

$$\mathbf{Q}_t + \mathbf{A}'\mathbf{P}_{t+1}\mathbf{A} - \left(\mathbf{A}'\mathbf{P}^*_{t+1}\mathbf{B} + \mathbf{W}_t\right)\left(\mathbf{R}_t + \mathbf{B}'\mathbf{P}^*_{t+1}\mathbf{B}\right)^{-1}\left(\mathbf{A}'\mathbf{P}^*_{t+1}\mathbf{B} + \mathbf{W}_t\right)'$$

and

$$\mathbf{q}_t + \mathbf{A}'\mathbf{p}^*_{t+1} - \left(\mathbf{A}'\mathbf{P}^*_{t+1}\mathbf{B} + \mathbf{W}_t\right)\left(\mathbf{B}'\mathbf{P}^*_{t+1}\mathbf{B} + \mathbf{R}_t\right)^{-1}\left(\mathbf{B}'\mathbf{p}^*_{t+1} + \mathbf{r}_t\right),$$

[23] See, e.g., Eqs. (2.5.6) on p. 35 and (7.C.10) on p. 168 in Hansen and Sargent (2008) where the quantity $\beta^{-1}\mathbf{P}^*_{t+1}$ is denoted by $\mathsf{D}(\mathbf{P})$.

[24] See, e.g., Theorem 7.6.1 (assumption v) in Hansen and Sargent (2008, p. 150).

[25] The parameter θ is closely related to the risk-sensitivity parameter, say σ, appearing in intertemporal preferences obtained recursively. Namely, it can be interpreted as minus the inverse of σ. See, e.g., Hansen and Sargent (2008, pp. 40–41, 45 and 225), Hansen et al. (1999) and the references therein cited.

[26] This recursion is not necessary when the desired paths are set to 0 as in Hansen and Sargent (2008, Ch. 2 and 7).

respectively. Equations (2.18) and (2.19) reduce to the usual Riccati equations of the linear quadratic tracking control problem when $\boldsymbol{\omega}_{t+1} = \mathbf{0}$.[27]

3. Robust filtering without commitment

The previous section has considered the case where the decision maker is "probabilistically sophisticated." Namely, a decision maker who cares only of the "induced distributions under the approximating model." As explained in Hansen and Sargent (2008, pp. 405-406) "this is a property of expected utility preferences ... preferences that are defined by using a single constraint or penalty make the decision maker indifferent between utility processes with identical induced distributions." However robust control can be applied also to situations where there are multiple penalty functions (i.e. more than one θ), in other words cases where the decision maker is not "probabilistically sophisticated."

For instance, Hansen and Sargent (2008, p. 383) "study a decision maker who does not observe parts of the state that help forecast variables he cares about." They assume that his "approximating model includes a recursive representation of the estimator of the hidden state that is derived by applying the ordinary (i.e. non robust) Kalman filter to the approximating state space model for states and measurements." Then "to obtain decision rules that are robust with respect to perturbations of the conditional distributions associated with the approximating model, the decision maker imagines a malevolent agent who perturbs the distribution of future states conditional on the entire state as well as the distribution of the hidden state conditional on the history of signals." This is sometimes referred to as the "robust filtering without commitment" problem.

The law of motion for the states in the approximating model is

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{u}_t + \mathbf{C}\boldsymbol{\varepsilon}_{t+1} \tag{3.1}$$

with

[27] See, e.g., Kendrick (1981, Ch. 2).

$$\mathbf{y}_{t+1} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}_{t+1}, \ \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}, \ \boldsymbol{\varepsilon}_{t+1} = \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \end{bmatrix}_{t+1}$$

where now the state vector is partitioned into two parts with $\mathbf{y}_1$ containing the n_1 observed variables and $\mathbf{y}_2$ the n_2 hidden state variables, with $n_1 + n_2 = n$, and $\mathbf{u}_t$ and $\boldsymbol{\varepsilon}_{t+1}$ are as in the previous section.[28] The decision maker ranks sequences of states and controls according to

$$-E_0\left[\sum_{t=0}^{\infty} \beta^t U\left(\mathbf{y}_{1,t}, \mathbf{y}_{2,t}, \mathbf{u}_t\right)\right] \tag{3.2}$$

with the one-period utility function U defined as

$$U\left(\mathbf{y}_{1,t}, \mathbf{y}_{2,t}, \mathbf{u}_t\right) =$$

$$\begin{bmatrix} \mathbf{y}_{1,t}' & \mathbf{y}_{2,t}' & \mathbf{u}_t' \end{bmatrix} \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{W}_1 \\ \mathbf{Q}_{12}' & \mathbf{Q}_{22} & \mathbf{W}_2 \\ \mathbf{W}_1' & \mathbf{W}_2' & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{1,t} \\ \mathbf{y}_{2,t} \\ \mathbf{u}_t \end{bmatrix} \equiv \begin{bmatrix} \mathbf{y}_t' & \mathbf{u}_t' \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{W} \\ \mathbf{W}' & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{y}_t \\ \mathbf{u}_t \end{bmatrix} \tag{3.3}$$

with the matrices **Q**, **R** and **W** as in the previous section and $\mathbf{y}_{1,0}$, the observed portion of the state vector at time 0, given.

Assuming that the decision maker believes that the distribution of the initial value of the unobserved part of the state is $\mathbf{y}_{2,0} \sim \mathrm{N}(\breve{\mathbf{y}}_{2,0}, \Delta_0)$ and taking into account that $\mathbf{y}_1$ is observed, the ordinary Kalman filter gives the projected value of $\mathbf{y}_{1,t+1}$ conditional on all the available information at time *t*, i.e. $E(\mathbf{y}_{1,t+1} \mid I_t)$, and the updated value of $\mathbf{y}_{2,t+1}$ conditional

[28] In Ch. 18, Hansen and Sargent (2008) consider the general case where at time t+1 the decision maker observes a vector **s** that includes $\mathbf{y}_1$ and possibly other signals about the hidden subset of the state. The law of motion relating these signals with the states and controls is written as $\mathbf{s}_{t+1} = \mathbf{D}_1\mathbf{y}_{1,t} + \mathbf{D}_2\mathbf{y}_{2,t} + \mathbf{H}\mathbf{u}_t + \mathbf{G}\boldsymbol{\varepsilon}_{t+1}$ and the relationship between $\mathbf{y}_1$ and **s** is $\mathbf{y}_{1,t+1} = \Pi_{\mathbf{s}}\mathbf{s}_{t+1} + \Pi_{\mathbf{y}_1}\mathbf{y}_{1,t} + \Pi_{\mathbf{u}}\mathbf{u}_t$. Then the arrays in (3.1) are defined as $\mathbf{A}_{11} = \Pi_{\mathbf{s}}\mathbf{D}_1 + \Pi_{\mathbf{y}_1}$, $\mathbf{A}_{12} = \Pi_{\mathbf{s}}\mathbf{D}_2$, $\mathbf{B}_1 = \Pi_{\mathbf{s}}\mathbf{H} + \Pi_{\mathbf{u}}$ and $\mathbf{C}_1 = \Pi_{\mathbf{s}}\mathbf{G}$. When the vector **s** is simply $\mathbf{y}_1$, the matrices $\Pi_{\mathbf{y}_1}$ and $\Pi_{\mathbf{u}}$ are null, $\Pi_{\mathbf{s}} = \mathbf{I}$ and $\mathbf{A}_{11} \equiv \mathbf{D}_1$, $\mathbf{A}_{12} \equiv \mathbf{D}_2$, $\mathbf{B}_1 \equiv \mathbf{H}$ and $\mathbf{C}_1 \equiv \mathbf{G}$.

on all the available information at time t+1, namely $\breve{\mathbf{y}}_{2,t+1} \equiv E(\mathbf{y}_{2,t+1} \mid I_{t+1})$.[29] Under the approximating model, $\mathbf{y}_{2,t}$ is distributed as $\mathrm{N}(\breve{\mathbf{y}}_{2,t}, \Delta_t)$, with $\Delta_t = E[(\mathbf{y}_{2,t} - \breve{\mathbf{y}}_{2,t})(\mathbf{y}_{2,t} - \breve{\mathbf{y}}_{2,t})']$, and the mean and variance of the state represent sufficient statistics for the distribution of the unobserved part of the state at time t.[30] Equation (3.1) is then rewritten with the system equations for $\mathbf{y}_2$ replaced by the associated ordinary Kalman filter updating equation and the law of motion for the observed subvector expressed in terms of the updated estimate of the hidden state and the discrepancy between this value and the true one, i.e.[31]

$$\breve{\mathbf{y}}_{t+1} = \mathbf{A}\breve{\mathbf{y}}_t + \mathbf{B}\mathbf{u}_t + \breve{\mathbf{C}}_1(\Delta_t)\boldsymbol{\varepsilon}_{t+1} + \breve{\mathbf{C}}_2(\Delta_t)\left(\mathbf{y}_{2,t} - \breve{\mathbf{y}}_{2,t}\right) \tag{3.4}$$

with[32]

$$\breve{\mathbf{y}}_{t+1} = \begin{bmatrix} \mathbf{y}_1 \\ \breve{\mathbf{y}}_2 \end{bmatrix}_{t+1}, \quad \breve{\mathbf{C}}_1(\Delta_t) = \begin{bmatrix} \mathbf{C}_1 \\ \Xi(\Delta_t)\mathbf{C}_1 \end{bmatrix}, \; \breve{\mathbf{C}}_2(\Delta_t) = \begin{bmatrix} \mathbf{A}_{12} \\ \Xi(\Delta_t)\mathbf{A}_{12} \end{bmatrix},$$

where $\Xi(\Delta_t) = (\mathbf{A}_{22}\Delta_t\mathbf{A}'_{12} + \mathbf{C}_2\mathbf{C}'_1)(\mathbf{A}_{12}\Delta_t\mathbf{A}'_{12} + \mathbf{C}_1\mathbf{C}'_1)^{-1}$, $\boldsymbol{\varepsilon}_{t+1} \sim \mathrm{N}(\mathbf{0}, \mathbf{I}_l)$ and $\mathbf{y}_{2,t} \sim \mathrm{N}(\breve{\mathbf{y}}_{2,t},\ \Delta_t)$.

In this approximating model appear two random vectors: $\boldsymbol{\varepsilon}_{t+1}$ and $\mathbf{y}_{2,t} - \breve{\mathbf{y}}_{2,t}$. Hansen and Sargent (2008, p. 386) “seek a decision rule that is robust to statistical perturbations of ...

[29] See Hansen and Sargent (2008, p. 386). As well known, see e.g. Hamilton, Eqt. 13.2.13, the updating value for $\mathbf{y}_{2,t+1}$ can be written as

$$\begin{aligned} &E(\mathbf{y}_{2,t+1} \mid I_{t+1}) = E(\mathbf{y}_{2,t+1} \mid I_t) + \{E[(\mathbf{y}_{2,t+1} - E(\mathbf{y}_{2,t+1} \mid I_t))(\mathbf{s}_{t+1} - E(\mathbf{s}_{t+1} \mid I_t))']\} \\ &\times\{E[(\mathbf{s}_{t+1} - E(\mathbf{s}_{t+1} \mid I_t))(\mathbf{s}_{t+1} - E(\mathbf{s}_{t+1} \mid I_t))']\}^{-1}(\mathbf{s}_{t+1} - E(\mathbf{s}_{t+1} \mid I_t)) \end{aligned}$$

when $\mathbf{s}$ is the vector of observed signals.

[30] In this case, the equation for updating the covariance estimate is $\Delta_{t+1} = \mathbf{A}_{22}\Delta_t\mathbf{A}'_{22} + \mathbf{C}_2\mathbf{C}'_2 - (\mathbf{A}_{22}\Delta_t\mathbf{A}'_{12} + \mathbf{C}_2\mathbf{C}'_1)(\mathbf{A}_{12}\Delta_t\mathbf{A}'_{12} + \mathbf{C}_1\mathbf{C}'_1)^{-1}(\mathbf{A}_{22}\Delta_t\mathbf{A}'_{12} + \mathbf{C}_2\mathbf{C}'_1)'$. As pointed out in Hansen and Sargent (2008, p. 387) “Δ_t evolves exogenously with respect to ... $(\mathbf{y}_1, \breve{\mathbf{y}}_2)$ so that given an initial condition Δ_0 a path $\{\Delta_t\}_{t=0}^{\infty}$ can be computed before observing anything else.”

[31] This equation corresponds to the first two rows of Eqt. (18.2.7) in Hansen and Sargent (2008, p. 387). As pointed out in footnote 5 on page 386 of the same reference, in the case of robust filtering with commitment the “approximate model ... (does) not include the law of motion for an estimate of the hidden state induced by applying the ordinary Kalman Filter.”

(3.4). (Namely, they) ... want to perturb at date t: ... the conditional distribution of the shock ... ($\boldsymbol{\varepsilon}_{t+1}$) which according to the approximating model is $N(\mathbf{0}, \mathbf{I}_l)$; and ... the distribution of the hidden state ... ($\mathbf{y}_{2,t}$) which according to the approximating model is $N(\breve{\mathbf{y}}_{2,t}, \Delta_t)$." Let $\boldsymbol{\omega}_{1,t}$ and $\boldsymbol{\omega}_{2,t}$ represent the perturbation to the distribution of $\boldsymbol{\varepsilon}_{t+1}$ and of the hidden state conditional on ($\mathbf{y}_{1,t}, \breve{\mathbf{y}}_{2,t}$), respectively.[33] Then the misspecified model is written as[34]

$$\breve{\mathbf{y}}_{t+1} = \mathbf{A}\breve{\mathbf{y}}_t + \mathbf{B}\mathbf{u}_t + \breve{\mathbf{C}}_1(\Delta_t)\left(\boldsymbol{\varepsilon}_{t+1} + \boldsymbol{\omega}_{1,t}\right) + \breve{\mathbf{C}}_2(\Delta_t)\left(\boldsymbol{\omega}_{2,t} + \mathbf{y}_{2,t} - \breve{\mathbf{y}}_{2,t}\right) \tag{3.5}$$

and the associated return function is

$$U\left(\mathbf{y}_{1,t}, \mathbf{y}_{2,t}, \mathbf{u}_t\right) - \theta_1 \left|\boldsymbol{\omega}_{1,t}\right|^2 - \theta_2 \boldsymbol{\omega}'_{2,t} \Delta_t^{-1} \boldsymbol{\omega}_{2,t} \tag{3.6}$$

where θ_1 and θ_2 penalize distortions $\boldsymbol{\omega}_{1,t}$ and $\boldsymbol{\omega}_{2,t}$, respectively.[35]

When $\mathbf{y}_t^d$ and $\mathbf{u}_t^d$ denote the vectors of desired values of the states and controls, respectively, for period t, Equation (3.6) can be rewritten as[36]

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[32] In the general case discussed in footnote 28 the second block in $\breve{\mathbf{C}}_1(\Delta_t)$ and $\breve{\mathbf{C}}_2(\Delta_t)$ is $\Xi(\Delta_t)\mathbf{G}$ and $\Xi(\Delta_t)\mathbf{D}_2$, respectively, with $\Xi(\Delta_t) = (\mathbf{A}_{22}\Delta_t\mathbf{D}'_2 + \mathbf{C}_2\mathbf{G}')(\mathbf{D}_2\Delta_t\mathbf{D}'_2 + \mathbf{G}\mathbf{G}')^{-1}$ and $\Delta_{t+1} = \mathbf{A}_{22}\Delta_t\mathbf{A}'_{22} + \mathbf{C}_2\mathbf{C}'_2 - \Xi(\Delta_t)(\mathbf{A}_{22}\Delta_t\mathbf{D}'_2 + \mathbf{C}_2\mathbf{G}')$.

[33] This is as though $\boldsymbol{\varepsilon}_{t+1}$ and $\mathbf{y}_{2,t} - \breve{\mathbf{y}}_{2,t}$, conditionally on ($\mathbf{y}_{1,t}, \breve{\mathbf{y}}_{2,t}$), are distributed as $N(\boldsymbol{\omega}_{1,t}, \mathbf{I}_l)$ and $N(\boldsymbol{\omega}_{2,t}, \Delta_t)$, respectively, rather than as $N(\mathbf{0}, \mathbf{I}_l)$ and $N(\mathbf{0}, \Delta_t)$, respectively, as in the approximating model. See footnote 7 above.

[34] See Hansen and Sargent (2008, p. 390). In the formal analysis of Ch. 18, they assume that "the decision maker distorts a model conditioned on the hidden state by applying an operator $\mathbf{T}^1$ and distorts a prior over models by applying an operator $\mathbf{T}^2$" (p. 384). In other words the "operator $\mathbf{T}^1$ systematically perturbs the distribution of ... ($\boldsymbol{\varepsilon}_{t+1}$) conditional on ... ($\mathbf{y}_{1,t}, \breve{\mathbf{y}}_{2,t}, \mathbf{y}_{2,t}$) and another operator $\mathbf{T}^2$ perturbs the distribution of ... ($\mathbf{y}_{2,t}$) conditional on ... ($\mathbf{y}_{1,t}, \breve{\mathbf{y}}_{2,t}$)" (p. 387).

[35] As underlined in Hansen and Sargent (2008, p. 387) in the case of robust filtering with commitment the "benchmark model ... (is repeatedly modified because) past distortions alter the current period reference model." On the other hand when applying robust filtering without commitment "each period the decision maker retains the same original benchmark model. By itself this diminishes the impact of robust filtering." They suggest to let "θ_2 to be smaller than θ_1 thereby giving the current period minimizing agent more flexibility to distort the distribution of the current hidden state."

[36] The penalty matrix $\mathbf{Q}$ corresponds to $\mathbf{\Pi}_{22}$ in Hansen and Sargent's (2008, Ch. 18) notation, $\tilde{\mathbf{R}}$ to $\mathbf{\Pi}_{11}$ with the blocks in the second row and column appearing in the third row and column and vice versa and $\tilde{\mathbf{W}}$ to $\mathbf{\Pi}_{21}$ with the second and third column inverted.

$$r\left(\breve{\mathbf{y}}_t,\tilde{\mathbf{u}}_t\right)=$$
$$\left(\breve{\mathbf{y}}_t-\mathbf{y}_t^d\right)'\mathbf{Q}\left(\breve{\mathbf{y}}_t-\mathbf{y}_t^d\right)+\left(\tilde{\mathbf{u}}_t-\tilde{\mathbf{u}}_t^d\right)'\tilde{\mathbf{R}}(\Delta_t)\left(\tilde{\mathbf{u}}_t-\tilde{\mathbf{u}}_t^d\right)+2\left(\breve{\mathbf{y}}_t-\mathbf{y}_t^d\right)'\tilde{\mathbf{W}}\left(\tilde{\mathbf{u}}_t-\tilde{\mathbf{u}}_t^d\right) \tag{3.7}$$

where[37]

$$\tilde{\mathbf{R}}(\Delta_t)=\begin{bmatrix}\mathbf{R} & \mathbf{E}^* \\ \mathbf{E}^{*\prime} & \Delta_\omega(\Delta_t)\end{bmatrix} \text{ with } \underset{m\times(l+n_2)}{\mathbf{E}^*}=\begin{bmatrix}\underset{m\times l}{\mathbf{O}} & \underset{m\times n_2}{\mathbf{W}_2'}\end{bmatrix},\ \Delta_\omega(\Delta_t)=\begin{bmatrix}-\theta_1\mathbf{I}_l & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_{22}-\theta_2\Delta_t^{-1}\end{bmatrix},$$

$\tilde{\mathbf{W}}=(\mathbf{W}\ \ \mathbf{M}^*)$ with $\mathbf{M}^*=[\mathbf{O}\ \ \mathbf{Q}_2]$, $\mathbf{O}$ being a null matrix of dimension $n\times l$ and $\mathbf{Q}_2$ the matrix of dimension $n\times n_2$ obtained deleting the first n_1 columns of matrix $\mathbf{Q}$ in (3.3), $\tilde{\mathbf{u}}_t^d = (\mathbf{u}_t^{d\prime}\ \ \mathbf{0}')'$ and $\mathbf{0}$ a null $(l + n_2)$-dimensional vector. As stressed in Hansen and Sargent (2008, p. 389) "assigning different values to θ ... lets the decision maker to focus more or less on misspecifications of one or the other of the two distributions being perturbed."

For the linear quadratic problem at hand, $(\mathbf{y}_1,\breve{\mathbf{y}}_2)$-contingent distortions $\boldsymbol{\omega}_{1,t}$ and $\boldsymbol{\omega}_{2,t}$ and the associated robust rule for $\mathbf{u}$ can be computed by solving the deterministic, certainty equivalent, problem[38]

$$\max_{\mathbf{u}_t}\ \min_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}\left[-\sum_{t=0}^{\infty}\beta^t r\left(\breve{\mathbf{y}}_t,\tilde{\mathbf{u}}_t\right)\right] \tag{3.8}$$

subject to

$$\breve{\mathbf{y}}_{t+1}=\mathbf{A}\breve{\mathbf{y}}_t+\tilde{\mathbf{B}}(\Delta_t)\tilde{\mathbf{u}}_t \tag{3.9}$$

where

[37] "In the special case that the decision maker conditions on an infinite history of signals and in which Δ_t has converged we can set $\Delta_{t+1}=\Delta_t$" (Hansen and Sargent, 2008, p. 301). Then the matrices $\tilde{\mathbf{R}}(\Delta_t)$ and $\Delta_\omega(\Delta_t)$ can be simply denoted by $\tilde{\mathbf{R}}$ and Δ_ω, respectively, and are constant over time.

[38] Equations (3.8)-(3.9) in the text correspond to Eqs. (18.2.16) and (18.2.17) in Hansen and Sargent (2008, Ch. 18).

$$\breve{\mathbf{y}}_{t+1} = \begin{bmatrix} \mathbf{y}_1 \\ \breve{\mathbf{y}}_2 \end{bmatrix}_{t+1}, \ \tilde{\mathbf{u}}_t = \begin{bmatrix} \mathbf{u} \\ \omega \end{bmatrix}_t, \omega_t = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_t, \ \tilde{\mathbf{B}}(\Delta_t) = \begin{bmatrix} \mathbf{B} & \breve{\mathbf{C}}_1(\Delta_t) & \breve{\mathbf{C}}_2(\Delta_t) \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \breve{\mathbf{C}}(\Delta_t) \end{bmatrix}$$

and the Gaussian random vectors with mean zero have been dropped as in the previous section.[39]

Writing the optimal value of (3.8) as $-\breve{\mathbf{y}}_t'\mathbf{P}_t\breve{\mathbf{y}}_t - 2\breve{\mathbf{y}}'\mathbf{p}_t$,[40] the Bellman equation looks like[41]

$$\begin{aligned} -\breve{\mathbf{y}}_t'\mathbf{P}_t\breve{\mathbf{y}}_t - 2\breve{\mathbf{y}}_t'\mathbf{p}_t = \underset{\mathbf{u},\omega}{ext} - \big[&\breve{\mathbf{y}}_t'\mathbf{Q}_t\breve{\mathbf{y}}_t + \mathbf{u}_t'\mathbf{R}_t\mathbf{u}_t + 2\mathbf{u}_t'\mathbf{E}_t^*\omega_t - \omega_t'\Delta_{\omega,t}(\Delta_t)\omega_t \\ &+2\breve{\mathbf{y}}_t'\mathbf{W}_t\mathbf{u}_t + 2\breve{\mathbf{y}}_t'\mathbf{M}_t^*\omega_t + 2\breve{\mathbf{y}}_t'\mathbf{q}_t + 2\mathbf{u}_t'\mathbf{r}_t + \breve{\mathbf{y}}_{t+1}'\mathbf{P}_{+1t}\breve{\mathbf{y}}_{t+1} + 2\breve{\mathbf{y}}_{t+1}'\mathbf{p}_{t+1}\big] \end{aligned} \tag{3.10}$$

with $\mathbf{P}_{t+1} = \beta\mathbf{P}_t$, $\mathbf{Q}_t = \beta^t\mathbf{Q}$, $\mathbf{R}_t = \beta^t\mathbf{R}$, $\mathbf{E}_t^* = \beta^t\mathbf{E}^*$, $\Delta_{\omega,t}(\Delta_t) = \beta^t\Delta_\omega(\Delta_t)$, $\mathbf{W}_t = \beta^t\mathbf{W}$, $\mathbf{M}_t^* = \beta^t\mathbf{M}^*$, $\mathbf{q}_t = -(\mathbf{Q}_t\breve{\mathbf{y}}_t^d + \mathbf{W}_t\mathbf{u}_t^d)$ and $\mathbf{r}_t = -(\mathbf{R}_t\mathbf{u}_t^d + \mathbf{W}_t'\breve{\mathbf{y}}_t^d)$. Then expressing the right-hand side of (3.10) only in terms of $\breve{\mathbf{y}}_t$ and $\tilde{\mathbf{u}}_t$ and extremizing it yields the optimal control for the decision maker, i.e.

$$\begin{aligned} \mathbf{u}_t &= -\left(\mathbf{R}_t + \mathbf{B}'\mathbf{P}_{t+1}\mathbf{B}\right)^{-1} \\ &\times\left[\left(\mathbf{B}'\mathbf{P}_{t+1}\mathbf{A} + \mathbf{W}_t'\right)\breve{\mathbf{y}}_t + \left(\mathbf{E}_t^* + \mathbf{B}'\mathbf{P}_{t+1}\breve{\mathbf{C}}(\Delta_t)\right)\omega_t + \mathbf{B}'\mathbf{p}_{t+1} + \mathbf{r}_t\right], \end{aligned} \tag{3.11}$$

and the optimal control for the malevolent nature

$$\omega_t = \Theta_t^{-1}\left[\left(\mathbf{M}_t^{*\prime} + \breve{\mathbf{C}}(\Delta_t)'\mathbf{P}_{t+1}\mathbf{A}\right)\breve{\mathbf{y}}_t + \left(\mathbf{E}_t^{*\prime} + \breve{\mathbf{C}}(\Delta_t)'\mathbf{P}_{t+1}\mathbf{B}\right)\mathbf{u}_t + \breve{\mathbf{C}}(\Delta_t)'\mathbf{p}_{t+1}\right] \tag{3.12}$$

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39 The modified certainty equivalence principle discussed in Hansen and Sargent (2008, Ch. 2) guarantees that omitting these terms does not affect the computations of $\omega_{1,t}$ and $\omega_{2,t}$. See also footnote 8 in Ch. 18 of the same reference. Finally it should be stressed that problem (3.8)–(3.9) allows to compute the decision rule **u** and the distortion $\omega_{2,t}$ that solves the general 'misspecification problem' stated in Eqt. (18.2.12) of the same chapter but it does not provide the distortion $\omega_{1,t}$ conditional on $(\mathbf{y}_1, \mathbf{y}_2, \breve{\mathbf{y}}_2)$ that the $\mathbf{T}^1$ operator introduced in footnote 34 computes. This computation requires additional steps that go beyond the scope of the present discussion (Hansen and Sargent, 2008, Sect. 18.2.9).

40 See footnote 18 above.

41 As in the previous instances, the constant term appearing on the right-hand side and on the left-hand side of the equation have been dropped. See footnote 19 above.

with $\boldsymbol{\Theta}_t = -[\boldsymbol{\Delta}_{\boldsymbol{\omega},t}(\boldsymbol{\Delta}_t) + \breve{\mathbf{C}}(\boldsymbol{\Delta}_t)'\mathbf{P}_{t+1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)]$. It follows that the (θ_1,θ_2)-constrained worst-case control vector is

$$\mathbf{u}_t = -\left\{\mathbf{R}_t + \mathbf{B}'\mathbf{P}_{t+1}\mathbf{B} + \left(\mathbf{E}_t^* + \mathbf{B}'\mathbf{P}_{t+1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)\right)\boldsymbol{\Theta}_t^{-1}\left(\mathbf{E}_t^* + \mathbf{B}'\mathbf{P}_{t+1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)\right)'\right\}^{-1}$$
$$\times\left\{\left[\mathbf{B}'\mathbf{P}_{t+1}\mathbf{A} + \mathbf{W}_t' + \left(\mathbf{E}_t^* + \mathbf{B}'\mathbf{P}_{t+1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)\right)\boldsymbol{\Theta}_t^{-1}\left(\mathbf{M}_t^{*\prime} + \breve{\mathbf{C}}(\boldsymbol{\Delta}_t)'\mathbf{P}_{t+1}\mathbf{A}\right)\right]\mathbf{y}_t \right. \quad (3.13)$$
$$\left. + \left(\mathbf{E}_t^* + \mathbf{B}'\mathbf{P}_{t+1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)\right)\boldsymbol{\Theta}_t^{-1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)'\mathbf{p}_{t+1} + \mathbf{B}'\mathbf{p}_{t+1} + \mathbf{r}_t\right\}.$$

In this general case, where the arrays $\mathbf{E}_t^*$ and $\mathbf{M}_t^*$ are not necessarily null matrices, the expressions in the numerator and denominator of Eq. (3.13) cannot be written as in (2.14) with

$$\mathbf{P}_{t+1}^* = \mathbf{P}_{t+1} + \mathbf{P}_{t+1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)\boldsymbol{\Theta}_t^{-1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)'\mathbf{P}_{t+1}.$$

In any case the relations

$$\mathbf{B}'\mathbf{P}_{t+1}\mathbf{B} + \left(\mathbf{E}_t^* + \mathbf{B}'\mathbf{P}_{t+1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)\right)\boldsymbol{\Theta}_t^{-1}\left(\mathbf{E}_t^* + \mathbf{B}'\mathbf{P}_{t+1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)\right)' \geq \mathbf{B}'\mathbf{P}_{t+1}\mathbf{B}, \quad (3.14a)$$

$$\mathbf{B}'\mathbf{P}_{t+1}\mathbf{A} + \left(\mathbf{E}_t^* + \mathbf{B}'\mathbf{P}_{t+1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)\right)\boldsymbol{\Theta}_t^{-1}\left(\mathbf{M}_t^{*\prime} + \breve{\mathbf{C}}(\boldsymbol{\Delta}_t)'\mathbf{P}_{t+1}\mathbf{A}\right) \geq \mathbf{B}'\mathbf{P}_{t+1}\mathbf{A} \quad (3.14b)$$

$$\left(\mathbf{E}_t^* + \mathbf{B}'\mathbf{P}_{t+1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)\right)\boldsymbol{\Theta}_t^{-1}\breve{\mathbf{C}}(\boldsymbol{\Delta}_t)'\mathbf{p}_{t+1} + \mathbf{B}'\mathbf{p}_{t+1} \geq \mathbf{B}'\mathbf{p}_{t+1} \quad (3.14c)$$

always hold because it is assumed that θ_1 and θ_2 are large enough to make $\boldsymbol{\Theta}_t$ positive definite.[42] The equality signs prevail when $\theta_1 = \theta_2 = \infty$.

As in the previous section, the first order conditions for problem (3.8) subject to (3.9) imply the matrix Riccati equation

[42] See Eqt. 18.2.19 in Hansen and Sargent (2008, p. 391).

$$\mathbf{P}_t = \mathbf{Q}_t + \mathbf{A}'\mathbf{P}_{t+1}\mathbf{A}$$
$$-\left[\mathbf{A}'\mathbf{P}_{t+1}\tilde{\mathbf{B}}(\Delta_t) + \tilde{\mathbf{W}}_t\right]\left[\tilde{\mathbf{B}}(\Delta_t)'\mathbf{P}_{t+1}\tilde{\mathbf{B}}(\Delta_t) + \tilde{\mathbf{R}}_t(\Delta_t)\right]^{-1}\left[\mathbf{A}'\mathbf{P}_{t+1}\tilde{\mathbf{B}}(\Delta_t) + \tilde{\mathbf{W}}_t\right]' \qquad (3.15)$$

where $\tilde{\mathbf{R}}_t(\Delta_t) = \beta^t\tilde{\mathbf{R}}(\Delta_t)$ and $\tilde{\mathbf{W}}_t = \beta^t\tilde{\mathbf{W}}$. The Riccati equation for the **p** vector looks like[43]

$$\mathbf{p}_t = \mathbf{q}_t + \mathbf{A}'\mathbf{p}_{t+1}$$
$$-\left[\mathbf{A}'\mathbf{P}_{t+1}\tilde{\mathbf{B}}(\Delta_t) + \tilde{\mathbf{W}}_t\right]\left[\tilde{\mathbf{B}}(\Delta_t)'\mathbf{P}_{t+1}\tilde{\mathbf{B}}(\Delta_t) + \tilde{\mathbf{R}}_t(\Delta_t)\right]^{-1}\left[\tilde{\mathbf{B}}(\Delta_t)'\mathbf{p}_{t+1} + \mathbf{r}_t\right]. \qquad (3.16)$$

Both Equation (3.15) and (3.16) reduce to the usual Riccati equations of the linear quadratic tracking control problem when $\boldsymbol{\omega}_t = \mathbf{0}$.

4. Optimal control of a linear system with time-varying parameters

Tucci (2006) argues that the model used by a "probabilistically sophisticated' decision maker to represent dynamic misspecification, i.e. Eqt. (2.2), is observationally equivalent to a model with a time-varying intercept. When this intercept is restricted to follow a 'Return to Normality' model,[44] and the symbols are as in Section 2, the latter takes the form

$$\mathbf{y}_{t+1} = \mathbf{A}_1\mathbf{y}_t + \mathbf{B}\mathbf{u}_t + \mathbf{C}\left(\boldsymbol{\alpha}_{t+1} + \boldsymbol{\varepsilon}_{t+1}\right) \qquad \text{for } t = 0, ..., \infty, \qquad (4.1)$$

with

$$\boldsymbol{\alpha}_{t+1} = \mathbf{a} + \boldsymbol{\nu}_{t+1} \qquad \text{for } t = 0, ..., \infty, \qquad (4.2a)$$
$$\boldsymbol{\nu}_{t+1} = \boldsymbol{\Phi}\boldsymbol{\nu}_t + \boldsymbol{\zeta}_{t+1} \qquad \text{for } t = 0, ..., \infty, \qquad (4.2b)$$

where **a** is the unconditional mean vector of $\boldsymbol{\alpha}_{t+1}$, $\boldsymbol{\Phi}$ the $l\times l$ transition matrix with eigenvalues strictly less than one in absolute value to guarantee stationarity and $\boldsymbol{\zeta}_{t+1}$ is a Gaussian *iid* vector process with mean zero and an identity covariance matrix. The matrix $\mathbf{A}_1$

43 Again, this recursion is not necessary when the desired paths are set to 0.
44 See, e.g., Harvey (1981).

is such that $\mathbf{A}_1\mathbf{y}_t + \mathbf{Ca}$ in (4.1) is equal to $\mathbf{Ay}_t$ in (2.2).[45] Obviously, the robust control formulation is more general than model (4.1)-(4.2) because in (2.2) the vector $\boldsymbol{\omega}_{t+1}$ can represent a very general, and possibly complicated, process.

Then the approach discussed in Kendrick (1981) and Tucci (2004) can be used to find the set of controls $\mathbf{u}_t$ which maximizes[46]

$$J = E_0\left[-\sum_{t=0}^{\infty} L_t\left(\mathbf{y}_t,\ \mathbf{u}_t\right)\right], \tag{4.3}$$

where

$$L_t\left(\mathbf{y}_t, \mathbf{u}_t\right) = \tag{4.4}$$

$$\beta^t\left[\left(\mathbf{y}_t - \mathbf{y}_t^d\right)'\mathbf{Q}\left(\mathbf{y}_t - \mathbf{y}_t^d\right) + 2\left(\mathbf{y}_t - \mathbf{y}_t^d\right)'\mathbf{W}\left(\mathbf{u}_t - \mathbf{u}_t^d\right) + \left(\mathbf{u}_t - \mathbf{u}_t^d\right)'\mathbf{R}\left(\mathbf{u}_t - \mathbf{u}_t^d\right)\right],$$

subject to (4.1)-(4.2). This control problem can be solved treating the stochastic parameters as additional state variables. If the same objective function used in the robust control problem is optimized, the expression in square bracket is identical to the one-period loss function defined in (2.6).

When the hyper-structural parameters $\mathbf{a}$ and $\boldsymbol{\Phi}$ are known, the original problem is restated in terms of an augmented state vector $\mathbf{z}_t$ as: find the controls $\mathbf{u}_t$ maximizing[47]

$$J = E_0\left[-\sum_{t=0}^{\infty} L_t\left(\mathbf{z}_t,\ \mathbf{u}_t\right)\right] \tag{4.5}$$

subject to[48]

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[45] When $\mathbf{a}$ is a null vector, $\mathbf{A}_1 \equiv \mathbf{A}$. If $\mathbf{a}$ is not zero, $\mathbf{A}_1$ is identical to $\mathbf{A}$ except for a column of 0's associated with the intercept and $\mathbf{Ca}$ is identical to the column of $\mathbf{A}$ associated with the intercept.

[46] Kendrick (1981, Ch. 10) analyzes the case where $\mathbf{a} = \mathbf{0}$ and the hyperstructural parameter $\boldsymbol{\Phi}$ is known. Tucci (2004) deals with the case where $\mathbf{a}$ and $\boldsymbol{\Phi}$ are estimated.

[47] See Kendrick (1981, Ch. 10) or Tucci (2004, Ch. 2).

$$\mathbf{z}_{t+1} = \mathbf{f}(\mathbf{z}_t, \mathbf{u}_t) + \boldsymbol{\varepsilon}^*_{t+1} \qquad \text{for } t = 0, ..., \infty, \tag{4.6}$$

with[49]

$$\mathbf{z}_t = \begin{bmatrix} \mathbf{y}_t \\ \boldsymbol{\alpha}_{t+1} \end{bmatrix}, \mathbf{f}\left(\mathbf{z}_t, \mathbf{u}_t\right) = \begin{bmatrix} \mathbf{A}_1\mathbf{y}_t + \mathbf{B}\mathbf{u}_t + \mathbf{C}\boldsymbol{\alpha}_{t+1} \\ \boldsymbol{\Phi}\boldsymbol{\alpha}_{t+1} + \left(\mathbf{I}_l - \boldsymbol{\Phi}\right)\mathbf{a} \end{bmatrix} \text{ and } \boldsymbol{\varepsilon}^*_t = \begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\zeta}_{t+1} \end{bmatrix}. \tag{4.7}$$

and the arrays $\mathbf{z}_t$ and $\mathbf{f}(\mathbf{z}_t, \mathbf{u}_t)$ having dimension $n+l$, i.e. the number of original states plus the number of stochastic parameters. For this 'augmented' control problem the L's in Eqt. (4.5) are defined as

$$\begin{aligned} &L_t\left(\mathbf{z}_t, \mathbf{u}_t\right) = \\ &\left(\mathbf{z}_t - \mathbf{z}^d_t\right)' \mathbf{Q}^*_t \left(\mathbf{z}_t - \mathbf{z}^d_t\right) + 2\left(\mathbf{z}_t - \mathbf{z}^d_t\right)' \mathbf{W}^*_t \left(\mathbf{u}_t - \mathbf{u}^d_t\right) + \left(\mathbf{u}_t - \mathbf{u}^d_t\right)' \mathbf{R}_t \left(\mathbf{u}_t - \mathbf{u}^d_t\right) \end{aligned} \tag{4.8}$$

with $\mathbf{Q}^*_t = \beta^t \mathbf{Q}^*$, $\mathbf{Q}^* = diag\,(\mathbf{Q},\ -\beta\theta\mathbf{I}_l)$, $\mathbf{W}^*_t = \beta^t [\mathbf{W}' \quad \mathbf{O}']'$ and $\mathbf{R}_t = \beta^t \mathbf{R}$.

By replacing $\mathbf{A}_1\mathbf{y}_t + \mathbf{C}\alpha_{t+1} with \mathbf{A}\mathbf{y}_t + \mathbf{C}\nu_{t+1}$ in (4.7), treating the vector of stochastic components $\boldsymbol{\nu}_{t+1}$ as additional state variables, setting $\boldsymbol{\varepsilon}^*_{t+1} = \mathbf{0}$ and using the deterministic counterpart to (4.5)-(4.8),[50] namely

$$\mathbf{z}_{t+1} = \mathbf{A}^*\mathbf{z}_t + \mathbf{B}^*\mathbf{u}_t \qquad \text{for } t = 0, ..., \infty, \tag{4.9}$$

with

$$\mathbf{z}_t = \begin{bmatrix} \mathbf{y}_t \\ \boldsymbol{\nu}_{t+1} \end{bmatrix}, \ \mathbf{A}^* = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{O} & \boldsymbol{\Phi} \end{bmatrix} \text{ and } \mathbf{B}^* = \begin{bmatrix} \mathbf{B} \\ \mathbf{O} \end{bmatrix}, \tag{4.10}$$

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[48] When the error term is assumed *iid* it is equivalent to write the system equations as in (4.6) or as in Tucci (2004, Ch. 2).

[49] Equations (4.2) are rewritten as $\boldsymbol{\alpha}_t - \mathbf{a} = \boldsymbol{\Phi}(\boldsymbol{\alpha}_{t-1} - \mathbf{a}) + \boldsymbol{\varepsilon}_t$ in (4.7). In Tucci (2006, p. 540), the symbol $\boldsymbol{\alpha}_t$ should be replaced by $\boldsymbol{\alpha}_{t+1}$ and $\boldsymbol{\omega}_t$ by $\boldsymbol{\omega}_{t+1}$.

[50] These preparatory steps are needed to keep the following discussion as close as possible to that carried out in section 2 and 3.

the optimal value of (4.5) can be written as $-\mathbf{z}_t'\mathbf{K}_t\mathbf{z}_t - 2\mathbf{z}_t'\mathbf{k}_t$ and it satisfies the Bellman equation[51]

$$\begin{aligned}-\mathbf{z}_t'\mathbf{K}_t\mathbf{z}_t - 2\mathbf{z}_t'\mathbf{k}_t = \max_{\mathbf{u}_t} -\Big[&\left(\mathbf{z}_t - \mathbf{z}_t^*\right)' \mathbf{Q}_t^* \left(\mathbf{z}_t - \mathbf{z}_t^*\right) + \left(\mathbf{u}_t - \mathbf{u}_t^*\right)' \mathbf{R}_t \left(\mathbf{u}_t - \mathbf{u}_t^*\right) \\ &+2\left(\mathbf{z}_t - \mathbf{z}_t^*\right)' \mathbf{W}_t^* \left(\mathbf{u}_t - \mathbf{u}_t^*\right) + \mathbf{z}_{t+1}'\mathbf{K}_{t+1}\mathbf{z}_{t+1} + 2\mathbf{z}_{t+1}'\mathbf{k}_{t+1}\Big]\end{aligned} \tag{4.11}$$

with $\mathbf{K}_{t+1} = \beta\mathbf{K}_t$. Again, expressing the right-hand side of (4.11) only in terms of $\mathbf{z}_t$ and $\mathbf{u}_t$ and maximizing it yields the optimal control in the presence of time-varying parameters (or tvp-control) , i.e.[52]

$$\begin{aligned}\mathbf{u}_t = &-\left(\mathbf{R}_t + \mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{B}\right)^{-1} \\ &\times\Big[\left(\mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{A} + \mathbf{W}_t'\right)\mathbf{y}_t + \mathbf{B}'\left(\mathbf{K}_{11,t+1}\mathbf{C} + \mathbf{K}_{12,t+1}\mathbf{\Phi}\right)\mathbf{\nu}_{t+1} + \left(\mathbf{B}'\mathbf{k}_{1,t+1} + \mathbf{r}_t\right)\Big].\end{aligned} \tag{4.12}$$

The matrices $\mathbf{K}_{11}$ and $\mathbf{K}_{12}$ in (4.12) denote the $n \times n$ North-West block and the $n \times l$ North-East block, respectively, of the Riccati matrix

$$\begin{aligned}\mathbf{K}_t = &\mathbf{Q}_t^* + \mathbf{A}^{*\prime}\mathbf{K}_{t+1}\mathbf{A}^* \\ &-\left(\mathbf{B}^{*\prime}\mathbf{K}_{t+1}\mathbf{A}^* + \mathbf{W}_t^*\right)' \left(\mathbf{R}_t + \mathbf{B}^{*\prime}\mathbf{K}_{t+1}\mathbf{B}^*\right)^{-1} \left(\mathbf{B}^{*\prime}\mathbf{K}_{t+1}\mathbf{A}^* + \mathbf{W}_t^*\right)\end{aligned} \tag{4.13}$$

Consequently they are defined as[53]

$$\begin{aligned}\mathbf{K}_{11,t} = &\mathbf{Q}_t + \mathbf{A}'\mathbf{K}_{11,t+1}\mathbf{A} \\ &-(\mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{A} + \mathbf{W}_t')'(\mathbf{R}_t + \mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{B})^{-1}(\mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{A} + \mathbf{W}_t')\end{aligned} \tag{4.14a}$$

and

$$\begin{aligned}\mathbf{K}_{12,t} = &(\mathbf{A}'\mathbf{K}_{11,t+1}\mathbf{C} + \mathbf{A}'\mathbf{K}_{12,t+1}\mathbf{\Phi}) \\ &-(\mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{A} + \mathbf{W}_t')'(\mathbf{R}_t + \mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{B})^{-1}\mathbf{B}'(\mathbf{K}_{11,t+1}\mathbf{C} + \mathbf{K}_{12,t+1}\mathbf{\Phi}).\end{aligned} \tag{4.14b}$$

[51] As in the previous sections, the constant term appearing on the right-hand side and on the left-hand side of the Bellman equation have been dropped because they do not affect the solution of the optimization problem.
[52] See also, e.g., Kendrick (1981, Ch. 2 and 10) and Tucci (2004, Ch. 2).
[53] See Tucci (2004, pp. 26–27).

It is apparent from (4.14b) that even when $\mathbf{K}_{12,t+1}$ is zero, it is sufficient that the same is not true for $\mathbf{K}_{11,t+1}$, a condition typically met, to have a non zero $\mathbf{K}_{12,t}$ matrix. This means that even when the terminal condition for $\mathbf{K}_{12}$ is a null matrix, this array will not vanish in all the time periods in the planning horizon except for the last one. Consequently, only the last control, namely the control applied at the 'final period minus 1' of the planning horizon, will be independent of the transition matrix characterizing the time-varying parameters.

The optimal control (4.12) is independent of the parameter θ which enters the $l\times l$ South-East block of $\mathbf{K}$, namely

$$\begin{aligned}\mathbf{K}_{22,t} &= -(\beta\theta\mathbf{I}_l - \mathbf{C}'\mathbf{K}_{11,t+1}\mathbf{C}) + 2\mathbf{\Phi}'\mathbf{K}_{21,t+1}\mathbf{C} + \mathbf{\Phi}'\mathbf{K}_{22,t+1}\mathbf{\Phi} \\ &-(\mathbf{C}'\mathbf{K}_{11,t+1} + \mathbf{\Phi}'\mathbf{K}_{21,t+1})\mathbf{B}(\mathbf{R}_t + \mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{B})^{-1}\mathbf{B}'(\mathbf{K}_{11,t+1}\mathbf{C} + \mathbf{K}_{12,t+1}\mathbf{\Phi}).\end{aligned} \tag{4.14c}$$

However it depends upon the vector $\mathbf{k}_t$ which can be partitioned as $\mathbf{k}_t = [\mathbf{k}'_{1,t} \quad \mathbf{k}'_{2,t}]'$ with

$$\mathbf{k}_{1,t} = \mathbf{q}_t + \mathbf{A}'\mathbf{k}_{1,t+1} - (\mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{A} + \mathbf{W}'_t)'(\mathbf{R}_t + \mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{B})^{-1}(\mathbf{B}'\mathbf{k}_{1,t+1} + \mathbf{r}_t) \tag{4.15a}$$

$$\begin{aligned}\mathbf{k}_{2,t} &= \mathbf{C}'\mathbf{k}_{1,t+1} + \mathbf{\Phi}'\mathbf{k}_{2,t+1} \\ &-(\mathbf{C}'\mathbf{K}_{11,t+1} + \mathbf{\Phi}'\mathbf{K}_{21,t+1})\mathbf{B}(\mathbf{R}_t + \mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{B})^{-1}(\mathbf{B}'\mathbf{k}_{1,t+1} + \mathbf{r}_t).\end{aligned} \tag{4.15b}$$

When $\boldsymbol{\nu}_{t+1} \equiv \boldsymbol{\omega}_{t+1}$, i.e. the same shock is used to determine both robust control and tvp-control, the latter is

$$\begin{aligned}&\mathbf{u}_t = \\ &-(\mathbf{R}_t + \mathbf{B}'\mathbf{K}^{+}_{11,t+1}\mathbf{B})^{-1}[(\mathbf{B}'\mathbf{K}^{+}_{11,t+1}\mathbf{A} + \mathbf{W}_t')\mathbf{y}_t + \mathbf{B}'\mathbf{K}^{+}_{11,t+1}\mathbf{K}^{-1}_{11,t+1}\mathbf{k}_{1,t+1} + \mathbf{r}_t]\end{aligned} \tag{4.16}$$

with

$$\mathbf{K}^{+}_{11,t+1} = [\mathbf{I}_n + (\mathbf{K}_{11,t+1}\mathbf{C} + \mathbf{K}_{12,t+1}\mathbf{\Phi})(\beta\theta\mathbf{I}_l - \mathbf{C}'\mathbf{P}_{t+1}\mathbf{C})^{-1}\mathbf{C}']\mathbf{K}_{11,t+1}. \tag{4.17}$$

The quantity $\mathbf{K}^{+}_{11,t+1}$ collapses to the 'robust' Riccati matrix $\mathbf{P}^{*}_{t+1}$ when $\mathbf{P}_{t+1} = \mathbf{K}_{11,t+1}$ and $\mathbf{K}_{12,t+1}$ is a null matrix. This means that robust is control is insensitive to the true value of $\mathbf{\Phi}$

appearing in the law of motion for the stochastic parameters. This is due to the fact that when the same objective function is optimized both in the robust and tvp-control problems, the only difference between the Bellman Eqs. (2.11) and (4.11) is that the former, implicitly, sets $\mathbf{P}_t = \mathbf{K}_{11,t}$, $\mathbf{p}_t = \mathbf{k}_{1,t}$ and $\mathbf{K}_{12,t}$, $\mathbf{K}_{21,t}$, $\mathbf{K}_{22,t}$ and $\mathbf{k}_{2,t}$ equal to null arrays. Therefore, by construction, the control applied by the decision maker who wants to be "robust to misspecifications of the approximating model" implicitly assumes that the $\boldsymbol{\omega}$'s in (2.2) are serially uncorrelated. This means that the vector process $\boldsymbol{\omega}_{t+1}$ included in the true model (2.2) can indeed describe only a very special kind of model misspecification. Alternatively put, given arbitrary desired paths for the states and controls, robust control is "robust" only when today's malevolent shock is linearly uncorrelated with tomorrow's malevolent shock.

The framework laid out in this section can be used also to study the case of robust control without commitment discussed in Sect. 3. Then, Equation (4.7) holds when

$$\boldsymbol{\alpha}_{t+1} = \begin{bmatrix} \boldsymbol{\alpha}_{1,t+1} \\ \boldsymbol{\alpha}_{2,t+1} \end{bmatrix} \text{ and } \quad \boldsymbol{\Phi} = \begin{bmatrix} \underset{l\times l}{\boldsymbol{\Phi}_{11}} & \mathbf{O} \\ \mathbf{O} & \underset{n_2\times n_2}{\boldsymbol{\Phi}_{22}} \end{bmatrix}$$

and the arrays $\mathbf{z}_t$ and $\mathbf{f}_{(\mathbf{z}_t,\mathbf{u}_t)}$ have dimension $n+(l+n_2)$. The objective function (3.8) is obtained defining the L's as in Eqt. (4.8) with $\mathbf{Q}_t^* = \beta^t \mathbf{Q}^*$, $\mathbf{R}_t = \beta^t \mathbf{R}$ and $\mathbf{W}_t^* = \beta^t \mathbf{W}^*$ where

$$\mathbf{Q}^* = \begin{bmatrix} \underset{n\times n}{\mathbf{Q}} & \underset{n\times(l+n_2)}{\mathbf{M}^*} \\ \mathbf{M}^{*\prime} & \underset{(l+n_2)\times(l+n_2)}{\Delta_{\boldsymbol{\omega}}(\Delta_t)} \end{bmatrix}, \ \mathbf{W}^* = \begin{bmatrix} \underset{n\times m}{\mathbf{W}} \\ \underset{(l+n_2)\times m}{\mathbf{E}^{*\prime}} \end{bmatrix} \text{ and } \underset{m\times(l+n_2)}{\mathbf{E}^*} = \begin{bmatrix} \underset{m\times l}{\mathbf{O}} & \underset{m\times n_2}{\mathbf{E}} \end{bmatrix}.$$

For this problem the tvp-control looks like

$$\begin{aligned} \mathbf{u}_t &= -\left(\mathbf{R}_t + \mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{B}\right)^{-1} \\ &\times\left[\left(\mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{A} + \mathbf{W}_t'\right)\mathbf{y}_t + \left(\mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{C} + \mathbf{B}'\mathbf{K}_{12,t+1}\boldsymbol{\Phi} + \mathbf{E}_t^*\right)\mathbf{v}_{t+1} + \left(\mathbf{B}'\mathbf{k}_{1,t+1} + \mathbf{r}_t\right)\right]. \end{aligned} \tag{4.18}$$

Then, when the same shock is used to determine both robust control and tvp-control, i.e. $\mathbf{v}_{t+1} \equiv [\boldsymbol{\omega}_{1,t}' \quad \boldsymbol{\omega}_{2,t}']'$ with $\boldsymbol{\omega}_{1,t}$ and $\boldsymbol{\omega}_{2,t}$ as in Sect. 3, Equation (4.18) takes the form

$\mathbf{u}_t =$ (4.19)

$$-\left\{\mathbf{R}_t + \mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{B} + \left(\mathbf{B}'\mathbf{K}_{11,t+1}\breve{\mathbf{C}}(\Delta_t) + \mathbf{B}'\mathbf{K}_{12,t+1}\mathbf{\Phi} + \mathbf{E}_t^*\right)\mathbf{\Theta}_t^{-1}\left(\mathbf{E}_t^{*\prime} + \breve{\mathbf{C}}(\Delta_t)'\mathbf{P}_{t+1}\mathbf{B}\right)\right\}^{-1}$$
$$\times\left\{\left[\mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{A} + \mathbf{W}_t' + \left(\mathbf{B}'\mathbf{K}_{11,t+1}\breve{\mathbf{C}}(\Delta_t) + \mathbf{B}'\mathbf{K}_{12,t+1}\mathbf{\Phi} + \mathbf{E}_t^*\right)\mathbf{\Theta}_t^{-1}\left(\mathbf{M}_t^{*\prime} + \breve{\mathbf{C}}(\Delta_t)'\mathbf{P}_{t+1}\mathbf{A}\right)\right]\mathbf{y}_t\right.$$
$$\left. + \left(\mathbf{B}'\mathbf{K}_{11,t+1}\breve{\mathbf{C}}(\Delta_t) + \mathbf{B}'\mathbf{K}_{12,t+1}\mathbf{\Phi} + \mathbf{E}_t^*\right)\mathbf{\Theta}_t^{-1}\breve{\mathbf{C}}(\Delta_t)'\mathbf{p}_{t+1} + \mathbf{B}'\mathbf{k}_{1,t+1} + \mathbf{r}_t\right\}.$$

The quantities

$$\mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{B} + \left(\mathbf{B}'\mathbf{K}_{11,t+1}\breve{\mathbf{C}}(\Delta_t) + \mathbf{B}'\mathbf{K}_{12,t+1}\mathbf{\Phi} + \mathbf{E}_t^*\right)\mathbf{\Theta}_t^{-1}\left(\mathbf{E}_t^{*\prime} + \breve{\mathbf{C}}(\Delta_t)'\mathbf{P}_{t+1}\mathbf{B}\right) \tag{4.20a}$$

$$\mathbf{B}'\mathbf{K}_{11,t+1}\mathbf{A} + \left(\mathbf{B}'\mathbf{K}_{11,t+1}\breve{\mathbf{C}}(\Delta_t) + \mathbf{B}'\mathbf{K}_{12,t+1}\mathbf{\Phi} + \mathbf{E}_t^*\right)\mathbf{\Theta}_t^{-1}\left(\mathbf{M}_t^{*\prime} + \breve{\mathbf{C}}(\Delta_t)'\mathbf{P}_{t+1}\mathbf{A}\right) \tag{4.20b}$$

$$\left(\mathbf{B}'\mathbf{K}_{11,t+1}\breve{\mathbf{C}}(\Delta_t) + \mathbf{B}'\mathbf{K}_{12,t+1}\mathbf{\Phi} + \mathbf{E}_t^*\right)\mathbf{\Theta}_t^{-1}\breve{\mathbf{C}}(\Delta_t)'\mathbf{p}_{t+1} + \mathbf{B}'\mathbf{k}_{1,t+1} \tag{4.20c}$$

in the denominator and numerator of (4.19) are identical to those on the left-hand side of (3.14a), (3.14b) and (3.14c), respectively, when $\mathbf{K}_{11,t+1} \equiv \mathbf{P}_{t+1}$ and $\mathbf{k}_{11,t+1} \equiv \mathbf{p}_{t+1}$ and $\mathbf{K}_{12,t+1}$ is a null matrix.[54] Therefore even this non "probabilistically sophisticated" decision maker implicitly assumes that $\boldsymbol{\omega}_{1,t}$ and $\boldsymbol{\omega}_{2,t}$, i.e. the perturbation to the distribution of $\boldsymbol{\varepsilon}_{t+1}$ and of the hidden state conditional on ($\mathbf{y}_{1,t}, \breve{\mathbf{y}}_{2,t}$), respectively, in (3.5) are both independent and serially uncorrelated. Again, given arbitrary desired paths for the states and controls, this robust control is "robust" only when today's malevolent shocks are linearly uncorrelated with tomorrow's malevolent shocks.

Before leaving this section it is worth it to emphasize two things. First of all the results (4.16)-(4.17) and (4.19)-(4.20) do not imply that robust control is implicitly based on a very specialized type of time-varying parameter model or that one of the two approaches is better than the other. Robust control and tvp-control represent two alternative ways of dealing with the problem of not knowing the true model 'we' want to control and are generally characterized by different solutions. In general, when the same objective function and terminal conditions are used, the main difference is due the fact that the former is determined assuming for $\boldsymbol{\omega}_{t+1}$ the worst-case value, whereas the latter is computed using the expected conditional mean of $\boldsymbol{\nu}_{t+1}$ and taking into account its relationship with next period conditional mean. As a side effect even the Riccati matrices common to the two procedures, named **P** and **p** in the robust control case and $\mathbf{K}_{11}$ and $\mathbf{k}_{11}$ in the tvp-case, are different. This is due to the fact that (2.18) and (2.19) are different from (4.14) and (4.15a), respectively.[55] The use of identical Riccati matrices and of an identical shock in the two alternative approaches, i.e. setting $\mathbf{K}_{11,t+1} \equiv \mathbf{P}_{t+1}$, $\mathbf{k}_{11,t+1} \equiv \mathbf{p}_{t+1}$ and $\boldsymbol{\nu}_{t+1} \equiv \boldsymbol{\omega}_{t+1}$ or $\boldsymbol{\nu}_{t+1} \equiv [\boldsymbol{\omega}'_{1,t} \quad \boldsymbol{\omega}'_{2,t}]'$, has the sole purpose of investigating some of the implicit assumptions of these procedures.

Secondly the results of this section do not claim that the 'malevolent shocks' are serially uncorrelated or that the perturbation to the distribution of $\boldsymbol{\varepsilon}_{t+1}$ and of the hidden state conditional on $(\mathbf{y}_{1,t}, \breve{\mathbf{y}}_{2,t})$ in (3.5) are both independent and serially uncorrelated. It simply shows that in all models where the agent is assumed to behave both in a "probabilistically sophisticated" and in a probabilistically 'unsophisticated' manner robust control implicitly assumes that these shocks are serially uncorrelated. This follows from the Bellman Equation associated with this type of problem.

5. Robust control in the presence of uncertain parameters in the structural model

The robust control problems discussed in the previous sections deal with unstructured uncertainty à la Hansen and Sargent.[56] However, sometimes robust control is applied to situations where uncertainty is related to unknown structural parameters. Giannoni (2002, 2007) considers an optimizing model for monetary policy. This is a structural forward-looking model where the constant structural parameters are unknown to the policymaker but are known to agents in the private sector. It "is composed of a monetary policy rules and two structural equations – an intertemporal IS equation and an aggregate supply equation - that are based on explicit microeconomic foundations ... (namely, they) can be derived as log-

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[54] By comparing the Bellman Eqs. (3.10) and (4.11), with the new definitions, it is apparent that the two are identical when $\mathbf{P}_t = \mathbf{K}_{11,t}$, $\mathbf{p}_t = \mathbf{k}_{1,t}$ and $\mathbf{K}_{12,t}$, $\mathbf{K}_{21,t}$, $\mathbf{K}_{22,t}$ and $\mathbf{k}_{2,t}$ are null arrays as for the "probabilistically sophisticated" case discussed above.

[55] The same consideration holds when (2.18) and (2.19) are replaced by (3.15) and (3.16), respectively.

[56] See Giannoni (2007, pp. 181−183) for a brief and updated overview of robust control literature.

linear approximations to equilibrium conditions of an underlying general equilibrium model with sticky prices" (Giannoni, 2002, pp. 112-114).[57]

In Giannoni (2007), the demand side of the economy is written as[58]

$$x_t = E_t x_{t+1} + \sigma^{-1} E_t \pi_{t+1} - \sigma^{-1} \hat{\imath}_t + \frac{\varpi}{(\varpi + \sigma)\sigma} \delta_t + \frac{1}{(\varpi + \sigma)} \eta_t \tag{5.1}$$

where E_t denotes the expectation formed at time *t*, x_t the output gap, π_t the rate of inflation, $\hat{\imath}_t$ the percentage deviation of the nominal interest rate from its constant steady state value, δ_t a demand shock and η_t an "adverse efficient supply shock". By output gap is meant the percentage deviation of actual output from its constant steady state value minus the percentage deviation of the efficient rate of output.[59] The aggregate supply curve takes the form[60]

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \frac{\kappa}{\varpi + \sigma} \mu_t \tag{5.2}$$

with μ_t the percent deviation of the desired markup from steady state,[61] κ a parameter greater than zero and β the discount factor.[62] As pointed out in Giannoni (2007, p. 187), the parameters σ and ϖ represent "the inverse of the intertemporal elasticity of substitution in private expenditure (and) … the elasticity of each firm's real marginal cost with respect to its own supply", respectively.[63] Finally, it is assumed that the exogenous shocks δ_t, η_t and

57 The model used to characterize the behavior of the private sector is a variant of the 'new keynesian' or 'new synthesis' model presented, e.g., in Clarida et al. (1999) and Woodford (2003). See also Giannoni (2007, pp. 186–188) for details.

58 See Giannoni (2002, pp. 113–115) for an intuitive description of a simplified version of this model.

59 Giannoni (2007, p. 187) defines the efficient rate of output as "the equilibrium rate of output that would obtain in the absence of price rigidities and market power".

60 Equations (5.1)–(5.2) correspond to (14)–(15) in Giannoni (2007, p. 188). The reader should be aware of the fact that the μ in this section bears no relationship with those appearing in the previous sections.

61 Giannoni (2007, p. 188) calls μ_t "the inefficient supply shock … since it represents a perturbation to the natural rate of output that is not efficient."

62 As noticed in Giannoni (2002, p. 114) "κ, which is the slope of the short run aggregate supply curve, can be interpreted as a measure of the speed of price adjustment. Finally β … (is) the discount factor of the price setters … (and) is supposed to be the same as the discount factor of the representative household."

63 In this model the opposite of σ is the slope of the intertemporal IS curve.

μ_t have zero (unconditional) mean, are independent of the parameters σ, κ and ϖ and follow an AR(1) process,[64] i.e.

$$\begin{bmatrix} \delta \\ \eta \\ \mu \end{bmatrix}_{t+1} = \begin{bmatrix} \rho_\delta & 0 & 0 \\ 0 & \rho_\eta & 0 \\ 0 & 0 & \rho_\mu \end{bmatrix} \begin{bmatrix} \delta \\ \eta \\ \mu \end{bmatrix}_t + \begin{bmatrix} \xi_\delta \\ \xi_\eta \\ \xi_\mu \end{bmatrix}. \tag{5.3}$$

In this model "monetary policy has real effects … because prices do not respond immediately to perturbations … only a fraction … of suppliers may change their prices at the end of each period" (Giannoni, 2007, p. 187). The controller determines the optimal monetary policy optimizing the following penalty function[65]

$$L_0 = E_0 \left\{ (1-\beta) \sum_{t=0}^{\infty} \beta^t \left[\pi_t^2 + \lambda_x \left(x_t - x^* \right)^2 + \lambda_i \hat{\iota}_t^2 \right] \right\} \tag{5.4}$$

where λ_x and λ_ι, both positive, "are weights placed on the stabilization of the output gap and the nominal interest rate and where $x^* \geq 0$ represents some optimal level of output gap" (Giannoni, 2007, p. 189).

Assuming rational expectations, the system (5.1)-(5.3) can be rewritten as (2.1) when $\mathbf{y}_t = (x_t \;\; \pi_t \;\; \delta_t \;\; \eta_t \;\; \mu_t)'$ and $\mathbf{u}_t = \hat{\iota}_t$. Then matrix **A** looks like

$$\mathbf{A} = \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{D} \\ \mathbf{O} & \mathbf{T} \end{bmatrix}, \tag{5.5}$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1+(\kappa/\beta\sigma) & -1/\beta\sigma \\ -\kappa/\beta & 1/\beta \end{bmatrix}, \; \mathbf{D} = \begin{bmatrix} -\varpi/(\varpi+\sigma)\sigma & -1/(\varpi+\sigma) & \kappa/(\varpi+\sigma)\beta\sigma \\ 0 & 0 & -\kappa/(\varpi+\sigma)\beta \end{bmatrix}, \tag{5.6}$$

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[64] See also footnote 10 in Giannoni (2007, p. 187).

[65] This corresponds to Eqt. (18) in Giannoni (2007, p. 189). As explained on page 199 of the same reference, it is assumed "that the preference parameters of the policymaker … (in Eqt. (5.4)) are known to the policymaker and are kept fixed regardless of the values of the structural parameters". For a discussion of the relationship between the parameters in the objective function and those in the underlying structural model see also footnote 27 on that page.

$\mathbf{T}$ is the 3×3 diagonal matrix on the right hand side of Eqt. (5.3) and $\mathbf{O}$ is a null 3×2 array, and $\mathbf{B}$ is defined as $\mathbf{B} = (\sigma^{-1} \ 0 \ 0 \ 0 \ 0)'$. The vector of disturbances $\boldsymbol{\varepsilon}_t = (\varepsilon_x \ \ \varepsilon_\pi \ \ \varepsilon_\delta \ \ \varepsilon_\eta \ \ \varepsilon_\mu)'$ has mean zero and identity covariance matrix and $\mathbf{C}$ is appropriately defined. Namely, it is such that $\mathbf{CC}' = E(\boldsymbol{\xi}_t \boldsymbol{\xi}_t')$ where $\boldsymbol{\xi}_t' = (\xi_x \ \ \xi_\pi \ \ \xi_\delta \ \ \xi_\eta \ \ \xi_\mu)$, with ξ_x and ξ_π the errors associated with the output gap and inflation, respectively, and $E(\boldsymbol{\xi}_t) = \mathbf{0}$, $E(\boldsymbol{\xi}_t \boldsymbol{\xi}_t') = \boldsymbol{\Sigma}$. Similarly, the one-period loss function implicit in (5.4) can be put in the format (2.6) when $\mathbf{W}$ is a null matrix, $\mathbf{Q} = (1-\beta)\ diag\ (\lambda_x \quad 1 \quad 0 \quad 0 \quad 0\ )$ and $\mathbf{R} = (1-\beta)\lambda_t$.

In the presence of uncertain parameters, the worst-case parameter vector results in worst-case matrices which can be viewed as the algebraic sum of the 'baseline case matrices', $\mathbf{A}$ and $\mathbf{B}$, and the 'worst-case discrepancies', $\mathbf{A}_\omega$ and $\mathbf{B}_\omega$.[66] It follows that the model in the worst case scenario can be written as[67]

$$\mathbf{y}_{t+1} = \left(\mathbf{A} + \mathbf{A}_\omega\right)\mathbf{y}_t + \left(\mathbf{B} + \mathbf{B}_\omega\right)\mathbf{u}_t + \mathbf{C}\boldsymbol{\varepsilon}_{\tau+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{u}_t + \mathbf{C}\boldsymbol{\varepsilon}_{\tau+1} + \left(\mathbf{A}_\omega \mathbf{y}_t + \mathbf{B}_\omega \mathbf{u}_t\right) \tag{5.7}$$

where the term in parenthesis on the right-hand side of the second equality sign plays the role of $\mathbf{C}\boldsymbol{\omega}_{t+1}$ in Eqt. (2.2). More precisely, the quantity $\mathbf{A}_\omega \mathbf{y}_t + \mathbf{B}_\omega \mathbf{u}_t$ replaces the malevolent vector defined in Eqt. (2.13) premultiplied by the volatility matrix $\mathbf{C}$ in a robust control model where uncertainty is à la Hansen and Sargent. Then, robust control is obtained by replacing this quantity into (2.12) to yield

$$\mathbf{u}_t = -(\mathbf{R}_t + \mathbf{B}'\mathbf{P}_{t+1}\mathbf{B}_\mathrm{W})^{-1}[(\mathbf{B}'\mathbf{P}_{t+1}\mathbf{A}_\mathrm{W} + \mathbf{W}_t')\mathbf{y}_t + \mathbf{B}'\mathbf{p}_{t+1} + \mathbf{r}_t] \tag{5.8}$$

where $\mathbf{A}_\mathrm{w} = \mathbf{A} + \mathbf{A}_\omega$ and $\mathbf{B}_\mathrm{w} = \mathbf{B} + \mathbf{B}_\omega$. The same 'malevolent shock' can be used in (4.12) to compute the associated tvp-control.

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66 As observed in Giannoni (2007, p. 205) when uncertainty is unstructured à la Hansen and Sargent the worst case scenario is always on the boundary of the set of relevant models. This is not necessarily true when uncertainty is associated to uncertain parameters. See also Giannoni (2002).

67 From (3.7) follows that the worst case in Giannoni's case corresponds to the worst case in Hansen and Sargent's approach when nature ignores the desired paths and $\mathbf{A}_\omega = \mathbf{C}(\beta\theta\mathbf{I}_l - \mathbf{C}'\mathbf{P}_{t+1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}_{t+1}\mathbf{A}$, $\mathbf{B}_\omega = \mathbf{C}(\beta\theta\mathbf{I}_l - \mathbf{C}'\mathbf{P}_{t+1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}_{t+1}\mathbf{B}$. It is then clear the relationship between the robustness parameter θ and the size of the confidence interval underlying $\mathbf{A}_\omega$ and $\mathbf{B}_\omega$.

6. Some numerical results

The permanent income model is a popular model in the robust control literature (see, e.g., Hansen and Sargent, 2001, 2003, 2008; Hansen et al. 1999, 2002). It is a linear quadratic stochastic growth model with a habit where a "probabilistically sophisticated" planner values a scalar process *s* of consumption services according to[68]

$$E_0\left[-\sum_{t=0}^{\infty}\beta^t\left(s_t-\mu_b\right)^2\right] \tag{6.1}$$

with μ_b a preference parameter governing the curvature of the utility function.[69] The service *s* is produced by the scalar consumption process c_t via the household technology

$$s_t = (1+\lambda)c_t - \lambda h_{t-1} \tag{6.2a}$$

$$h_t = \delta_h h_{t-1} + (1-\delta_h)c_t \tag{6.2b}$$

where $\lambda \geq 0$, $0 < \delta_h < 1$ and h_t is a stock of households habits given by a geometric weighted average of present and past consumption. Then a linear technology converts an exogenous (scalar) stochastic endowment d_t into consumption or capital, i.e.

$$k_t = \delta_k k_{t+1} + i_t \tag{6.3a}$$

$$c_t + i_t = \gamma k_{t+1} + d_t \tag{6.3b}$$

where k_t and i_t represent the capital stock and gross investment, respectively, at time *t*, γ the constant marginal product of capital and δ_k the depreciation factor for capital. The endowment is specified as the sum of two orthogonal AR(2) components, namely

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[68] This discussion draws heavily on Hansen et al. (2002, Sect. 4). The notation used in the presentation of this model is kept as close as possible to that used in the cited reference.

[69] See also pp. 47–53, 320–321 and Ch. 10 in Hansen and Sargent (2008) for a clear discussion of the main features of this model. Hall (1978), Campbell (1987), Heaton (1993) and Hansen et al. (1991) have applied versions of this model to aggregate U.S. time series data on consumption and investment. Aiyagari et al. (2002) discuss the connection between the permanent income consumer and Barro's (1979) model of tax smoothing.

$$d_{t+1} = \mu_d + d_{1,t+1} + d_{2,t+1} \tag{6.4}$$

where $d_{1,t+1}$ and $d_{2,t+1}$ are the permanent and transitory component, respectively, and

$$d_{1,t+1} = g_1 d_{1,t} + g_2 d_{1,t-1} + c_1 \varepsilon_{1,t+1}$$
$$d_{2,t+1} = a_1 d_{2,t} + a_2 d_{2,t-1} + c_2 \varepsilon_{2,t+1}$$

with $\varepsilon_{1,t+1}$ and $\varepsilon_{2,t+1}$ as in Sect. 2.[70]

Rewriting (6.3b) in terms of c_t and substituting it into (6.2a) yields

$$s_t = (1+\lambda)[\gamma k_{t+1} - i_t + d_t] - \lambda h_{t+1}. \tag{6.5}$$

Then the one-period loss function $(s_t - \mu_b)^2$ in (6.1) can be expressed as in (2.6) when $\mathbf{u}_t = i_t$, $\mathbf{u}_t^d = 0$, $\mathbf{y}_t = (h_{t\text{-}1} \quad k_{t\text{-}1} \quad d_{t\text{-}1} \quad 1 \quad d_t \quad d_{1,t} \quad d_{1,t\text{-}1})'$, $\mathbf{y}_t^d = (\mu_b \quad 0 \quad 0 \quad 0 \quad \mu_b \quad 0 \quad 0)'$,[71] $\mathbf{Q}$ = $diag(\mathbf{Q}^\dagger, \mathbf{O}_2)$ where

$$\mathbf{Q}^\dagger = \begin{bmatrix} \lambda^2 & \bullet & \bullet & \bullet & \bullet \\ -(1+\lambda)\lambda\gamma & (1+\lambda)^2\gamma^2 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet \\ -(1+\lambda)\lambda & (1+\lambda)^2\gamma & 0 & 0 & (1+\lambda)^2 \end{bmatrix} \tag{6.6}$$

and only the lower portion is reported because the matrix is symmetric, $\mathbf{O}_2$ is a square null matrix of dimension 2, $\mathbf{R} = (1+\lambda)^2$ and $\mathbf{W}$ = ($w_1 \quad w_2 \quad 0 \quad 0 \quad w_5 \quad 0 \quad 0$)$'$ with $w_1 = (1+\lambda)\lambda$, $w_2 = -(1+\lambda)^2\gamma$ and $w_5 = -(1+\lambda)^2$.

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[70] Solving (6.3a) for i_t and substituting it into (6.3b) yields $c_t + k_t = Rk_{t-1} + d_t$ with $R \equiv \delta_k + \gamma$ the "physical gross return on capital, taking into account that capital depreciates over time" (Hansen and Sargent, 2008, p. 226). This quantity coincides with the gross return on a one-period risk-free asset in the Hansen et al. (1999) model as noticed in fn. 9 of the cited reference.

[71] See also Hansen et al. (2002) and Hansen and Sargent (2008, Ch. 10).

When model misspecification is not ruled out, the equations for the permanent and transitory components of the endowment process are rewritten adding the quantities $c_1\omega_{1,t+1}$ and $c_2\omega_{2,t+1}$, respectively. Then problem (2.5) is solved subject to Eqt. (2.2) with the initial condition $\mathbf{y}_0$ given and the matrices of coefficients defined as

$$\mathbf{A}=\begin{bmatrix}\delta_h & (1-\delta_h)\gamma & 0 & 0 & (1-\delta_h) & 0 & 0\\ 0 & \delta_k & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & a_2 & \mu_d^* & a_1 & g_1-a_1 & g_2-a_2\\ 0 & 0 & 0 & 0 & 0 & g_1 & g_2\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ c_1 & c_2\\ c_1 & 0\\ 0 & 0\end{bmatrix} \qquad (6.7)$$

and $[1]$ As pointed out in Hansen et al. $(2002,\ \text{Sect. }4)$, although

Using the parameter estimates in Hansen et al. (2002), robust control for the permanent income model is computed for μ_b = 32 and different values of θ's.[72] The initial condition is set at $\mathbf{y}_0$ = (100 100 13.7099 1.0 13.7099 0 0)′, and it is assumed a time horizon of 2 periods.[73] As observed in Hansen and Sargent (2008, p. 47), a preference for robustness "leads the consumer to engage in a form of precautionary savings that … tilts his consumption profile toward the future relative to what it would be without a concern about misspecification of (the endowment) process." This is confirmed by the results reported in Tab. 6.1 where gross investment, the control variable, increases as θ gets smaller.[74]

Tab. 6.1 - Linear quadratic control (QLP) vs. robust control at time 0.*

[72] As reported in Hansen et al. (2002, Table 1) β = 9971, λ = 2.4433, δ_h = .6817, μ_d = 13.7099, α_1 = .8131, α_2 = .1888, ϕ_1 = .9978, ϕ_2 = .7044, c_1 = .1084 and c_2 = .1551. In addition the condition $\beta = (\delta_k + \gamma)^{-1}$ is imposed for stability reasons, with δ_k equal to .975, as in Hansen and Sargent (2008, p. 247). It should be emphasized that the derivations reported in Tucci (2006) are not general enough to handle this case. For this reason in the numerical example carried out in Section 7 of that work is set $\mu_b = 0$.

[73] The goal is to compare the first period control in the various cases. Given the recursive nature of this 'control game' considering a 3, 4 or 100 periods time horizon does not change the qualitative results.

[74] In this example, the admissible region for θ is approximately between .7 and infinity. When θ = .7, robust control is above 81 and nature controls are around -1935 and -1648. Robust control is −51.269, with nature controls equal to −.006 and −.01, for θ = 10000.

QLP Control	θ	Robust control	Nature controls	
−51.269756	100.0	−51.189376	−1.170727	−0.997060
	95.0	−51.185117	−1.232762	−1.049892
	85.0	−51.175084	−1.378892	−1.174346
	75.0	−51.162353	−1.564327	−1.332273
	65.0	−51.145665	−1.807385	−1.539276
	55.0	−51.122837	−2.139869	−1.822438
	45.0	−51.089718	−2.622255	−2.233267
	35.0	−51.037320	−3.385424	−2.883226
	25.0	−50.941904	−4.775164	−4.066811
	15.0	−50.713597	−8.100463	−6.898831
	5.0	−49.438034	−26.679050	−22.721448

* The QLP control is independent of θ and is reported only for the case $\theta = 100$.

When the observationally equivalent model of Sect. 4 is used, the endowment process and its permanent component have time-varying intercepts following a 'Return to Normality' model, namely

$$d_{t+1} = \mu^*_{d,t+1} + a_1 d_t + a_2 d_{t-1} + (g_1 - a_1) d_{1,t} + (g_2 - a_2) d_{1,t-1} \tag{6.8a}$$

$$d_{1,t+1} = \mu^*_{d1,t+1} + g_1 d_{1,t} + g_2 d_{1,t-1} \tag{6.8b}$$

with $\mu^*_{d,t+1} = \mu^*_d + c_1 \nu_{1,t+1} + c_2 \nu_{2,t+1}$ and $\mu^*_{d1,t+1} = c_1 \nu_{1,t+1}$ with $\nu_{i,t+1} \equiv \varepsilon_{i,t+1} + \omega_{i,t+1}$ for i = 1, 2. Then the time-invariant portion of the intercepts can be interpreted as[75]

$$\begin{bmatrix} \mu^*_d \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{6.9a}$$

and the stochastic component takes the form

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}_{t+1} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}_t + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}_{t+1} . \tag{6.9b}$$

[75] The implicit values of a_1 and a_2 are 0 and (μ_d^*/c_2), respectively.

In this case, setting $\varepsilon_{i,t+1} = 0$ and $\nu_{i,t+1} \equiv \omega_{i,t+1}$ for i = 1, 2, the results reported in Section 4 hold.

The relationship between the value of $\mathbf{\Phi}$ and the optimal control at various θ's is shown in Fig. 1 where $\mathbf{\Phi} = \phi\mathbf{I}$ and several values of ϕ are used. As shown in Sect. 4, the tvp-control derived assuming that the intercept follows a 'Return to Normality' model *and* $\phi = 0$ is identical to robust control when the same malevolent shocks are used. On the other hand knowing that tomorrow's shocks are negatively correlated with today's ones would make the household, facing a negative 'malevolent nature' shock, to save less for a given β. For

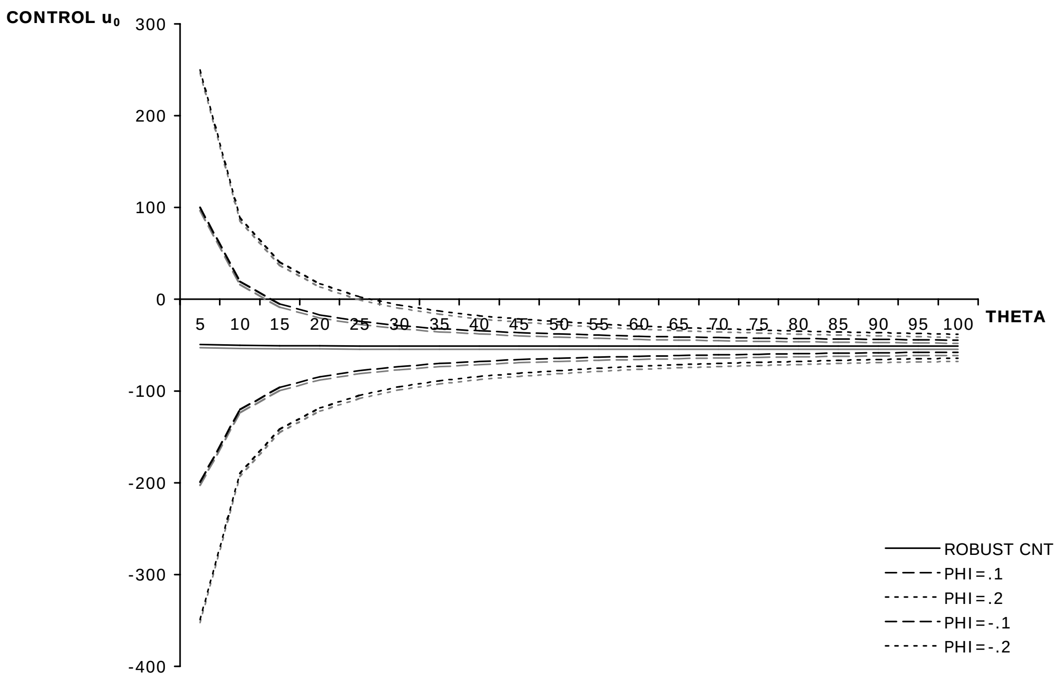

FIG. 1. First period robust control and tvp-control, for various values of Φ, at different levels of θ for the permanent income model.

instance, when θ = 100, savings decrease from −51.1894 at ϕ = 0 to −57.7563 at ϕ = −.1. Then the controls at the various θ's associated with negative ϕ's are always below the corresponding robust controls and they go farther and farther from them as the absolute value of ϕ increases. The opposite occurs for positive values of ϕ. Again, the line farther from the 'robust control line' is that associated with a higher absolute value of ϕ.

A meaningful example of robust control applied to situations where uncertainty is related to unknown structural parameters of the model has been discussed in Sect. 5. When the parameter values are as in Giannoni (2007, pp. 189-191 and 200) both for the baseline case and for the worst case, the matrices in (5.7) look like[76]

$$\mathbf{A}=\begin{bmatrix} 1.1530 & -6.4297 & -4.7781 & -1.5873 & .2429 \\ -.0240 & 1.0101 & 0 & 0 & -.0382 \\ 0 & 0 & .35 & 0 & 0 \\ 0 & 0 & 0 & .35 & 0 \\ 0 & 0 & 0 & 0 & .35 \end{bmatrix},\quad \mathbf{B}=\begin{bmatrix} 6.3654 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathbf{A}_{\omega}=\begin{bmatrix} .1870 & -4.6097 & -3.4856 & -1.0779 & .6633 \\ -.0071 & 0 & 0 & 0 & -.0448 \\ 0 & 0 & .45 & 0 & 0 \\ 0 & 0 & 0 & .45 & 0 \\ 0 & 0 & 0 & 0 & .45 \end{bmatrix},\quad \mathbf{B}_{\omega}=\begin{bmatrix} 4.5636 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

with[77]

$$\mathbf{C}=\begin{bmatrix} 2.1886 & 0 & 0 & 0 & 0 \\ 0.0060 & 1.51 & 0 & 0 & 0 \\ 0 & 0 & 1.7364 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16.2558 & 0 & 15.0793 \end{bmatrix}.$$

[76] The vector of uncertain parameters $\mathbf{p} = [\sigma\ \kappa\ \varpi\ \rho_\delta\ \rho_\eta\ \rho_\mu\ \nu]'$ is $\mathbf{p} = [.1571, .0238, .4729, .35, .35, .35, .5]'$ in the baseline case and $\mathbf{p}_w = [.0915, .0308, .2837, .8, -, .8, 1]'$ in the worst case, when it is assumed that "uncertainty about the critical structural parameters is given by the approximate 95% intervals" (Giannoni, 2007, p. 190). As explained on page 200 of the same reference, in the worst-case ρ_η. "may take any value in the allowed interval [0, 1] since the loss is maximized when … there are no efficient supply shocks."

[77] It is assumed that the $\mathbf{\Sigma}$ matrix is block diagonal. The 2×2 North-East block has been constructed using the variances of output and inflation reported in Tab. 2 of Rotemberg and Woodford (1998) on the main diagonal and their covariance, reflecting a correlation of .004 as in Fig. 2 of the cited work, as the off-diagonal element. The 3×3 South-West block is identical to that in Giannoni (2007, Tab. 1 on p. 192) for the case ν=1. The relationship between the models presented in Giannoni and Rotemberg and Woodford is discussed in footnote 13 on pages 189-190 of the same reference.

Setting the initial condition $\mathbf{y}_0 = (.03 \;\; .05 \;\; 1.0 \;\; 0 \;\; .01)'$, [78] the desired path for the output gap equal to .01 and the parameters in the penalty matrices equal to $\lambda_x = .0483$, $\lambda_\iota = .2364$ and $\beta = .99$ yields the results in Tab. 6.2, for a time horizon of 2 periods.[79] In this example, robust control is more active than that associated with the familiar linear regulator problem (or quadratic linear problem) and it is identical to the tvp-control when the transition matrix $\mathbf{\Phi}$ is equal to zero. For this problem specification, the tvp-control is higher than robust control when $\mathbf{\Phi}$ is positive. The opposite is true for negative values of $\mathbf{\Phi}$. As already noticed the difference between the two controls gets larger and larger as $\mathbf{\Phi}$ gets farther from the null matrix.

Tab. 6.2 - Linear quadratic control (QLP), robust control and tvp-control at time 0.*

QLP Control	Robust control	$\mathbf{\Phi}$	TVP-control
.75411	.79657	.0	.79657
		.1	.79667
		.2	.79677
		.5	.79707
		−.1	.79647
		−.2	.79637
		−.5	.79607

* The QLP control and robust control are independent of $\mathbf{\Phi}$.

[78] The initial values for the unexpected demand shock and inefficient supply shock are similar to those used in Giannoni (2007, pp. 194–195).

[79] When the demand and supply shocks are set to 0, i.e. at their unconditional mean level, robust control is .04975 and tvp-control is lower for positive values of $\mathbf{\Phi}$. It is equal to .04972 when $\mathbf{\Phi}=diag(.5, .5)$ and .04977 for $\mathbf{\Phi}=diag(-.5, -.5)$.

7. Conclusion

Tucci (2006) argues that, unless some prior information is available, the true model in a robust control setting à la Hansen and Sargent is observationally equivalent to a model with a time-varying intercept. Then he shows that, when the same "malevolent shock" is used in both procedures, the robust control for a linear system with an objective function having desired paths for the states and controls set to zero applied by a "probabilistically sophisticated" decision maker is identical to the optimal control for a linear system with an intercept following a 'Return to Normality' model and the same objective function *only when* the transition matrix in the law of motion of the parameters is zero. The goal of this paper has been to generalize this result

First, a robust control problem with unstructured uncertainty à la Hansen and Sargent and a "probabilistically sophisticated" decision maker has been introduced (Sect. 2). Then, in Sect. 3, an example of a non-"probabilistically sophisticated" decision maker has been discussed. At this point both problems have been reformulated as linear quadratic tracking control problems where the system equations have a time-varying intercept following a 'Return to Normality' model (Sect. 4). By comparing the solutions for the tvp models with those of the previous sections it has been confirmed that the latter imply strong assumptions. Both a "probabilistically sophisticated" and a non-"probabilistically sophisticated" decision maker who want to be robust against "misspecification of the approximating model" implicitly assume that the ω's are linearly uncorrelated. Alternatively put, given arbitrary desired paths for the states and controls, robust control is correct only when today's "malevolent shock" is linearly uncorrelated with tomorrow's "malevolent shock". This means that the vector process ω can describe only a very special kind of "misspecification" in robust control applications in the time domain. Section 5 shows that the same conclusion holds when uncertainty is associated with unknown structural parameters of the model.

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