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ABSTRACT

The autoregressive conditional heteroskedasticity (ARCH) estimation procedure provides a specification of the error terms as well as estimates of the coefficients. A simple interest rate equation is estimated using least squares and also using ARCH. Then the stochastic simulation methodology is extended to the ARCH process and Treasury Bond call options are evaluated. Interestingly when ARCH is compared to least squares it is found that the difference in coefficients estimates has a small effect, while the different simulation procedures have a large effect on the value of Treasury Bond call options [1].

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1. Introduction

The autoregressive conditional heteroskedasticity estimator (ARCH) is particularly useful for the estimation of financial equations. This is a natural specification since asset prices are often assumed to come from a martingale, with changing variance. The simplest ARCH specification is from Engle (1982) and assumes that the error terms in an equation are unserially correlated, but that the variance of u_t is $\alpha_0 + \alpha_1 u_{t-1}^2$. Thus the variance is assumed to be a function of the previous period error term. For a survey of more complicated models of this form see Pagan and Schwert (1990).

In this paper we will consider the simple form of Engle (1982), where the conditional variance of u_t is given as $\alpha_0 + \alpha_1 u_{t-1}^2$. This model can be estimated using the asymptotically efficient estimator proposed by Engle (1982) as alternative to maximum likelihood (a description can be found, for example, in Judge et al., 1985, p.441). The result of estimating a single equation is a set of coefficients and a specification of the stochastic process generating the errors. In this paper we explore an example where this second feature is much more important than the first.

A relatively new asset is the interest rate option. For a full discussion of interest rate options see Hull (1989) [2]. Several different theoretical methods exist for pricing interest rate options such as Jamshidian (1989) and Hull and White (1990). These methods specify a relation between short and long term interest rates. They specify a stochastic process for interest rates and find an analytical interest rate formula based on this stochastic process. In this paper we will work directly from an estimated long term interest rate equation.

(2) The basic mechanics of an interest rate call option are that a striking price is set and a maturity date is set. At maturity of the option the holder of the option has the right to buy a specified bond at the striking price. We assume that the option can only be exercised

The price of the bond is found by taking the present discounted value of the future payments. This means that the value of the bond is a nonlinear function of the interest rate. Since the data we use for long term interest rates are derived from long term bond prices this is in fact an identity. The payoff of the call option is a nonlinear function of the value of the bond. The result is that the expected payoff of the call option is a function not only of the expected value of interest rates, but also of higher order moments of interest rates as well. Even if the interest rate is modelled as a simple linear equation with exogenous regressors, the expected payoff of the call option will depend upon the higher order moments of the interest rate, not just its expected value, thus it will depend upon the higher order moments of the stochastic error terms. In Ross (1978) and in Dybvig and Ross (1989) it is shown that in the absence of arbitrage possibilities a positive linear pricing rule exists and that for appropriately chosen probabilities the expected payoff is the price.

When an equation is estimated using ARCH, this not only changes the coefficients, but also provides a new specification for the stochastic error terms. In this paper we wish to explore the effects of the components of the ARCH process on the expected payoff to a call option.

on the final date. The holder of the option is not required to buy the bond, he simply has the option of doing so. The payoff to the option then is simply the amount by which the bond price exceeds the striking price if the bond price is above the striking price. If the bond price is below the striking price the payoff is zero. Bond options can be set up privately or publicly. The Chicago Board of Trade has its own interest rate options contract.

2. *The model*

A simple monthly model of long term interest rates is used. In this model the U.S. interest rate depends upon the money supply, the inflation rate as measured by the consumer price index, and the unemployment rate. The specification is given as

$$[1] \qquad R_t = b_1 + b_2 M_t + b_3 I_t + b_4 U_t + u_t$$

where R_t is the long term interest rate, M_t is the real money supply, I_t is the inflation rate, U_t is the unemployment rate, and u_t is the error term. The inflation rate is based on a moving average of the last two months rates of inflation. The money supply is the nonseasonally adjusted value of $M2$ divided by the price level to give real money supply. The interest rate and money supply data are from the Federal Reserve monthly bulletins, while the inflation rate data and unemployment rate data are from the Bureau of Labor Statistics (3). The equation is estimated with OLS and with ARCH, and the results are

$$[2] \qquad \underset{}{R_t \times 100} = \underset{(3.01)}{10.17} - \underset{(0.39)}{0.992}\, M_t + \underset{(0.58)}{0.737}\, I_t + \underset{(0.13)}{0.660}\, U_t + \hat{u}_t \qquad \begin{matrix} OLS \\ \hat{\sigma}^2 = 2.21 \end{matrix}$$

(3) We tried specifications of this equation using $M1$, but the fit was not as good as with $M2$. During the 1980's there was a widespread use of now accounts and $M2$ seems to have been a more reliable monetary policy instrument and naturally it works best in an interest rate equation.

[3]
$$R_t \times 100 = \underset{(1.64)}{9.12} - \underset{(0.21)}{0.810}\, M_t + \underset{(0.29)}{0.869}\, I_t$$

$$+ \underset{(0.07)}{0.673}\, U_t + \hat{u}_t \qquad \textit{ARCH} \quad \textit{uncnd.}\ \hat{\sigma}^2 = 1.90$$

where standard errors are given in parentheses. The ARCH estimates are done using the procedure described in Judge et al. (1985, pp. 441-444). The conditional variance estimator is $\hat{Var}(u_t | u_{t-1}) = \hat{\alpha}_0 + \hat{\alpha}_1 \hat{u}^2_{t-1}$, and $\hat{\alpha}_0$ is found to be 0.25, $\hat{\alpha}_1$ is found to be 0.87, with asymptotic standard errors 0.08 and 0.14, respectively. The unconditional variance in the ARCH equation is found as $\hat{\alpha}_0/(1 - \hat{\alpha}_1)$. The estimated ARCH coefficients are slightly different from the estimated ordinary least squares coefficients.

In both cases the coefficients are significant (except the inflation coefficient in equation 2) and the signs are reasonable. The real money supply has a negative effect on interest rates. Inflation has a positive effect on nominal interest rates. During recessions the real interest rate is high; this explains the positive coefficient on unemployment rates. The focus of this research is not on the specification itself, but rather how the ARCH process differs from the OLS process.

An interest rate is used to find the present discounted value of an 8% coupon bond. This value (price of the bond) is computed as

[4]
$$P_t = \frac{4}{r_t}\left[1 - \left(\frac{1}{1 + r_t}\right)^{60}\right] + \frac{100}{(1 + r_t)^{60}}$$

where r_t is the six months interest rate computed as

[5]
$$r_t = (1 + R_t)^{1/2} - 1$$

Then the payoff to a call option is found as

$$[6] \qquad C_t = Max\,[(P_t - S)\,,\ 0]$$

where S is the striking price. In our paper we used options with 3 months to mature. We also used two different values of the striking price. In one case the striking price was \$ 97 (note that when striking prices are listed in newspapers, they are given as price per \$ 100 face value, but bonds and calls are normally traded per \$ 100 000 value). The corresponding payoff to the call option will be indicated as $C1$. In the other case we used a striking price \$ 98 and the corresponding payoff to the call option will be indicated as $C2$.

3. *Simulating error terms with ARCH structure*

In this section we apply simulation techniques to examine the distribution of random error terms with ARCH structure. To begin with, we consider an ARCH structure with values of α_0 and α_1 equal to those obtained from the estimation of our model (3): $\alpha_0 = 0.25$ and $\alpha_1 = 0.87$. Since $\alpha_1 < 1$, the error process is variance stationary as the number of time periods increases, converging to a distribution with unconditional variance $\alpha_0/(1 - \alpha_1) = 1.90$. The unconditional standard deviation is $\sqrt{1.90} = 1.38$. If we take an initial value $u_t = 1.38$ and e is an independent standard normal deviate, then $u_{t+1} = (\alpha_0 + \alpha_1 u_t^2)^{1/2}e$ is a zero mean normal error with the same variance as the unconditional variance (1.90). Moving one period ahead, $u_{t+2} = (\alpha_0 + \alpha_1 u_{t+1}^2)^{1/2}e$ still has zero mean and the same variance (1.90), but a different distribution, since u_{t+1} is random. If we again move one period ahead, again mean and variance do not change, but the distribution does. We continue up to 20 periods

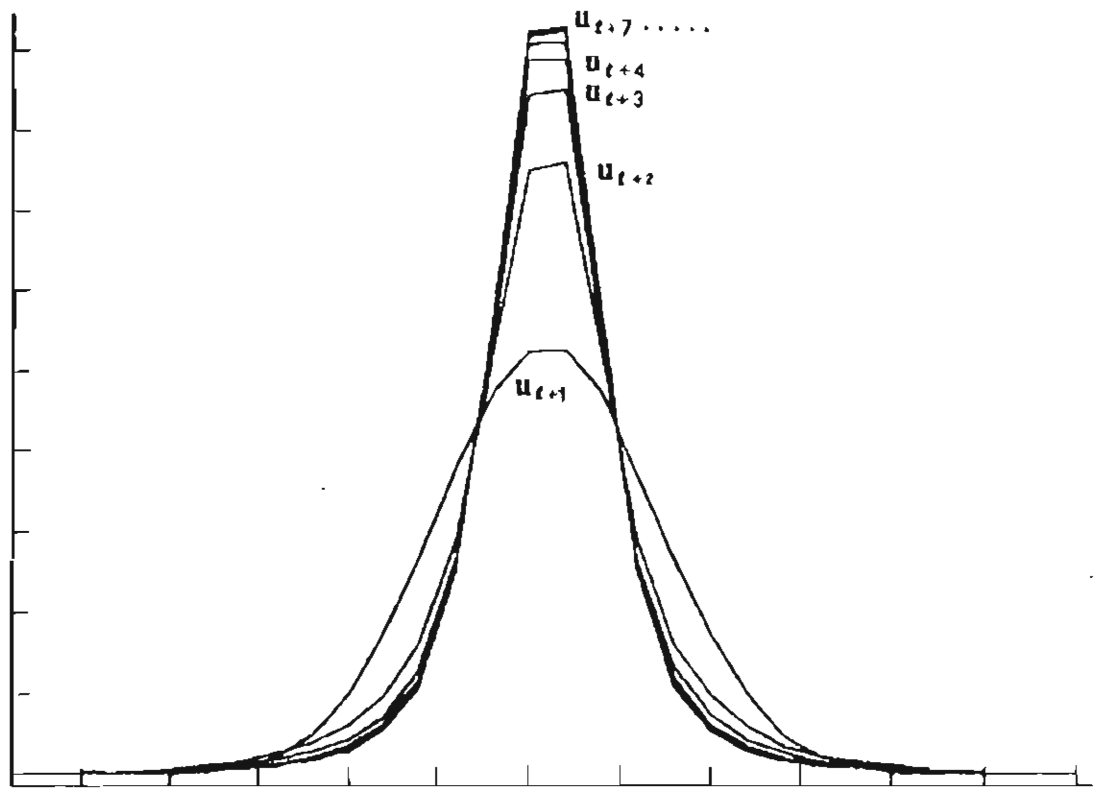

Fig. 1

Distributions of error terms with ARCH structure; $\alpha_0 = 0.25$; $\alpha_1 = 0.87$.

ahead, then we replicate the whole process a million times and plot the results.

A look at figure 1 gives a clear idea of how fast the ARCH generated errors converge in distribution. There is a big change from the distribution of the error terms in the first period, u_{t+1} (normal), to the distribution of u_{t+2} and u_{t+3} ; there are only small changes moving to u_{t+4} ,and u_{t+5} , u_{t+6} ; the changes are practically negligible after u_{t+7} , at least in the central region of the distribution; the tails of the distribution cannot be appreciated from the plot of figure 1, but a look at the numerical tables used for the plot shows only negligible changes after 8 or more time periods.

The converged distribution (practically the distribution of u_{t+8} and following) has a typical leptokurtic shape. Note however that the value of $\alpha_1 = 0.87$ does not satisfy the condition for the existence of a finite *4th* order moment ($3\alpha_1^2 < 1$, see Engle, 1982, p.992, or Harvey, 1990), so we cannot evaluate the kurtosis of the converged distribution. In practice, if we try to evaluate the kurtosis from the simulated values, we measure reliable values in the first period (about 3, as it should be), in the second period (about 7), in the third period (about 16), and not many more; already from the *6th* period the fluctuations in the sample kurtosis observed over replications indicate that a problem of non-finite fourth moment is likely to occur.

Compared to the normal, the tails of the distribution after 3 or more periods are quite fat. For example, in a million replications we have encountered less than 10 values of u_{t+1} over $\pm 5\sigma$. But we found about 1500 values of u_{t+2}, about 4000 values of u_{t+3} and about 4300 values of u_{t+7} over $\pm 5\sigma$. We found only one exceptional value of u_{t+1} over $\pm 7\sigma$, but about 1300 values of u_{t+3} and about 2000 values of u_{t+7}, u_{t+8} , etc. No values of u_{t+1} were generated over $\pm 10\sigma$, but about 300 values of u_{t+3} and about 800 values of u_{t+7}, as well as u_{t+8}, etc. Therefore we must be prepared for the presence of a non-negligible number of exceptionally large outliers in the simulation experiments on the model, not only when the ARCH process has come to a converged distribution, but already after 3 periods (starting from normal). This is an important point for our experiments. Large shocks are properly treated (downweighted) by the ARCH estimation method in the estimated equation (see, for example, the discussion in Hendry, 1986). But ARCH simulation generates these large shocks and introduces them into the rest of the model, where they are involved in nonlinear transformations.

We have repeated the same experiment with smaller values of α_1 . For example, figure 2 is related to an initial value of u_t = 1.38 , α_0 = 0.95 and α_1 = 0.50, which give the same value of the variance and of the unconditional variance as in the previous experiment. The difference between the normal and the ARCH distributions is obviously smaller than before. Convergence to the final distribution is even faster than in the previous case (we do not observe significant changes after about 4 periods). The tails in the distributions after 3 or more time periods are less fat than in the previous case. The existence of a finite moment of the fourth order clearly appears from the simulated values of the sample kurtosis that quickly converge to a value close to 7, without abnormal or unstable fluctuations.

4. Simulating the model

To simulate a model using the ordinary least squares approach, we first use the estimated coefficients of equation (2). We also use the variance estimate from equation (2). The errors in equation (2) are assumed to be from the normal distribution. A random error is drawn i.i.d. from the normal distribution with mean zero and variance $\hat{\sigma}^2$ for each of the next three periods. Then the interest rate path is found. Based on this path equations (4), (5) and (6) are solved, giving the present discounted value of the bond and the final payoff of the option. This process is repeated a large number of times and the results are averaged to give expected payoff. As usual (e.g. Hendry,

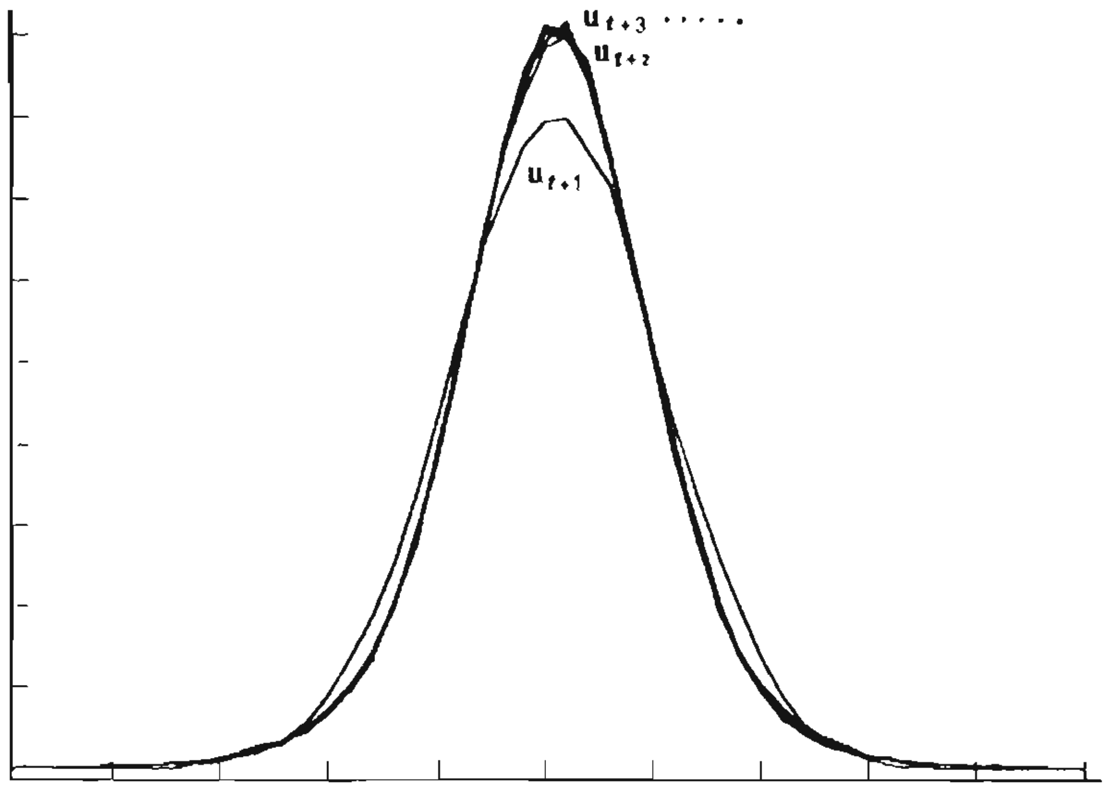

Fig. 2

Distributions of error terms with ARCH structure; $\alpha_0 = 0.95$; $\alpha_1 = 0.50$.

1984, Calzolari, 1979) strong computational benefits are achieved using simulation with the antithetic variates technique.

To simulate the model using the ARCH equation (3) we must use the ARCH coefficients and also the ARCH stochastic process. We start with the last residual $\hat{u}_t$. Based on this we find the conditional variance as $\hat{\alpha}_0 + \hat{\alpha}_1 \hat{u}_t^2$. We draw a random value of u_{t+1} from the normal distribution with this variance. We then solve for R_{t+1} and move to the next period. We use the random error drawn (u_{t+1}) to find the new conditional variance. The conditional variance of u_{t+2} is given by $\hat{\alpha}_0 + \hat{\alpha}_1 u_{t+1}^2$. We draw an independent random value e from a normal $N(0,1)$ and the random error term is obtained as $u_{t+2} = (\hat{\alpha}_0 + \hat{\alpha}_1 u_{t+1}^2)^{1/2} e$. Using this value of u_{t+2} the value of R_{t+2} is found and we move on to the next period. In the third period we use R_{t+3} to solve equations (4), (5) and (6) to give the payoff to the option. The process is repeated many times and the average is again computed (also in this case antithetic variates provide a large improvement of the computational efficiency). This gives the expected payoff to the call option. Although one period into the future the conditional distribution of u_{t+1} is normal, observe that when this process is repeated to give u_{t+3}, u_{t+2} is random, so that u_{t+3} comes from a more complicated distribution. The distribution of u_{t+3} is likely to exhibit substantial leptokurtosis. This is not surprising, as many financial time series exhibit leptokurtosis (see Ball, 1988). The distribution of u_{t+2} is a mixed normal distribution. As time progresses, the distribution of the ARCH errors becomes quite complicated.

We can work also in a different way. Assuming the ARCH process to be stationary, we generate u_{t+1}, u_{t+2} and u_{t+3} following the ARCH generation process after it comes to a converged distribution. In practice this means that we start our generating process a few periods earlier (t - 10, see section 3), then we use u_{t+1}, u_{t+2} and u_{t+3} in the simulation period. The starting value of the residual is no longer important, and the distribution of u_{t+1}, u_{t+2} and u_{t+3} is about the same, thus it is leptokurtic already for u_{t+1}.

The difference between these last two methods is the following. In both cases we make use of an ARCH structure for the error process, but with the last method we directly use the converged (unconditional) distribution. This looks like the standard simulation approach, with a distribution of the error terms that does not change over the forecast period, except that the distribution is chosen to be leptokurtic rather than normal in all periods. On the contrary, with the previous method the distribution of the simulated error terms is allowed to change over the forecast horizon, in the first period $(t + 1)$ being conditional upon the available information set at the forecast origin (t). As Engle and Bollerslev (1986, p.5) point out, this is analogous to the use of the conditional mean for forecasting with a time series model, rather than the unconditional mean.

5. *Simulation results*

In this section we consider several possibilities. Since the ARCH process has two components (one is the coefficients and the other is the stochastic process) we wish to see each of the components. We consider the following six possibilities.

1) Using OLS coefficients and standard simulation (with variance 2.21).

2) Using ARCH coefficients and standard simulation (with the same variance 2.21 as in the previous case).

3) Using OLS coefficients and ARCH simulation starting from a normal distribution in the first period.

4) Using ARCH coefficients and ARCH simulation starting from a normal distribution in the first period.

5) Using OLS coefficients and ARCH simulation with the converged distribution of the ARCH process already from the first period.

6) Using ARCH coefficients and ARCH simulation with the converged distribution of the ARCH process already from the first period.

In cases 3 and 4 the ARCH simulation is sensitive to the choice of the initial residual (at time t). The value we use in these first experiments is 1.38, which is $(\hat{\alpha}_0/(1 - \hat{\alpha}_1))^{1/2}$ (as in section 3; it gives a constant variance in each period, equal to the unconditional variance of the converged stationary ARCH process). Values for the exogenous variables are taken at the historical values of 1989/1 - 1989/3. The simulation starts in 1989/1 and the option is supposed to come to maturity at 1989/3. The results are displayed in table 1.

Beyond the deterministic solution values, the table displays expected values under the six different scenarios. The results are based on 100 000 couples of replications with antithetic variates. Below each expected value the Monte Carlo (experimental) standard deviation allows to appreciate the computational accuracy; it would be zero with infinitely many replications, if finite moments of the first two orders exist. In cases (5) and (6) use is done of the converged ARCH distribution of error terms already in the first period; in these two cases the simulation does not converge as the number of replications increases, suggesting that a finite mean does not exist either for P or for $C1$ and $C2$[4].

(4) Using the converged ARCH distribution from the first period (cases 5 and 6), but with the values of $\hat{a}_0$ and $\hat{a}_1$ adopted for figure 2, the simulation converges for P , $C1$ and $C2$, thus suggesting that finite means exist. The mean values are nearly the same as those obtained after three periods of ARCH simulation starting from normal (cases 3 and 4, obviously with the same values for the ARCH parameters). Results are not displayed here for brevity's sake.

Table 1

Bond price (P) and payoff to call option ($C1$, $C2$) corresponding to two different striking prices (97 and 98, respectively). ARCH simulation is done over three periods.

OLS coeff: deterministic solution	ARCH coeff: deterministic solution
P = 97.02	P = 95.06
$C1$ = 0.02	$C1$ = 0.0
$C2$ = 0.00	$C2$ = 0.0

Six sets of expected values

	(1)	(2)	(3)	(4)	(5)	(6)
P	99.17	97.11	99.24	97.17	-	-
	(0.03)	(0.03)	(0.17)	(0.16)	-	-
$C1$	7.37	6.18	6.04	4.92	-	-
	(0.06)	(0.06)	(0.19)	(0.18)	-	-
$C2$	6.89	5.74	5.56	4.53	-	-
	(0.06)	(0.06)	(0.19)	(0.18)	-	-

Increasing variance increases the expected payoff to the option. The variance is larger for the OLS simulations than for the ARCH simulations since $\hat{\alpha}_0/(1 - \hat{\alpha}_1)$ in this case is less than the variance of the OLS process.

In doing the ARCH simulation starting from a normal distribution in the first period (cases 3 and 4) the initial residual is quite important. To illustrate this point and to explore its importance we consider several different quantiles of the sample distribution of absolute residuals (10%, 25%, 50%, 75%, 90%); the value of the 50% quantile residual (median absolute residual) is 1.17, not too far from the value used for table 1. The other four residuals used are 0.15, 0.61, 1.78, and 2.49. To appreciate the importance of the choice of the initial residual (and therefore the conditional variance in the first period), table 2 displays the results related to a simulation in a single period (that is with normal error terms). The table shows the extreme sensitivity of the results to the starting point. The use of ARCH coefficients lowers price by about two. However the difference between an ARCH simulation beginning with the 10% residual and one beginning with the 90% residual is about five. This occurs whether the OLS coefficients are used or the ARCH coefficients. The results are even more noticeable for the payoff to the call option. For a given starting residual the choice of coefficients (ARCH or OLS) changes the expected payoff by less than 1.5. However the difference between the 10% residual and 90% residual as starting points is around 11. This is not surprising since the expected payoff to the call option is quite sensitive to the variance of the price of the bond.

We now consider again the results after three periods. This would be appropriate for finding the expected payoff of an interest rate option expiring in three months. Some interesting differences exist for the three periods as opposed to the one period result. First of all note that in the long run the variance should converge to $\hat{\alpha}_0/(1 - \hat{\alpha}_1)$. So for high initial starting variances we would expect on average a decline in variance. For low initial starting variance, the variance would on average rise

Table 2

Expected bond price (P) and payoff to call option ($C1$, $C2$) with OLS and ARCH coefficients, using standard simulation (normal errors). The variance of the bond price (P) is also displayed. The different variances of the normal random errors correspond to different quantiles in the sample distribution of residuals.

Init. resid.	0.15		0.61		1.17		1.78		2.49	
Percentile	10%		25%		50%		75%		90%	
Coefficients	OLS	ARCH	OLS	ARCH	OLS	ARCH	OLS	ARCH	OLS	ARCH
P	97.27	95.30	97.55	95.57	98.40	96.39	99.99	97.99	102.87	100.11
	(.004)	(.003)	(.008)	(.007)	(0.02)	(0.02)	(0.05)	(0.04)	(0.09)	(0.09)
$Var(P)$	29.12	27.12	63.05	58.70	166.6	155.0	383.1	355.8	857.8	794.5
$C1$	2.28	1.36	3.42	2.43	5.75	4.63	8.96	7.69	13.57	12.11
	(0.02)	(0.02)	(0.03)	(0.02)	(0.05)	(0.05)	(0.08)	(0.08)	(0.14)	(0.13)
$C2$	1.82	1.04	2.95	2.05	5.27	4.21	8.47	7.25	13.08	11.66
	(0.02)	(0.02)	(0.03)	(0.02)	(0.05)	(0.05)	(0.08)	(0.08)	(0.14)	(0.13)

toward $\hat{\alpha}_0/(1 - \hat{\alpha}_1)$. The distributions from different starting points would appear to be converging. Therefore one might expect less difference between methods as the number of periods into the future increases. At the same time the degree of leptokurtosis is higher for three periods than for one period. Thus there are rather wide tails in the ARCH simulations for three periods. The results are displayed in table 3. The results for the 90% case are particularly interesting. They show that if the variance becomes large, the results may be quite sensitive to a few replications.

Equation (4) is just the present discounted value of a bond in terms of r_t. It could be rewritten as

$$[7] \quad P_t = \frac{4}{(1+R_t)^{1/2}} + \frac{4}{(1+R_t)^{1}} + \frac{4}{(1+R_t)^{3/2}} + \dots + \frac{4}{(1+R_t)^{30}} + \frac{100}{(1+R_t)^{30}}$$

Inspecting the above expression shows that $\partial P/\partial R_t < 0$ and $\partial^2 P/\partial R_t^2 > 0$. Thus P is a convex function of R_t and therefore a convex function of u_t. Therefore the expected value of P is greater than the deterministic solution of P which occurs when $u_t = 0$. Also we can observe by looking at the equation above that the even numbered derivatives of P with respect to R_t are all positive. The value of R_t of course depends upon u_t. Define the deterministic value R_t^d equal to the value of R_t if $u_t = 0$. Then a Taylor series expansion of P may be found about the point R_t^d . This may be written as

$$[8] \quad P = f(R_t^d) + f'(R_t^d)u_t + \frac{1}{2}f''(R_t^d)u_t^2 + \frac{1}{6}f'''(R_t^d)u_t^3 + \frac{1}{24}f''''(R_t^d)u_t^4 + \dots$$

Table 3

Expected bond price (P) and payoff to call option ($C1$, $C2$) with OLS and ARCH coefficients, using ARCH simulation over 3 periods (starting from normal errors, cases 3 and 4). The variance of the bond price (P) is also displayed. Values of initial residual correspond to different quantiles in the sample distribution of residuals.

Init. resid.	0.15		0.61		1.17		1.78		2.49	
Percentile	10%		25%		50%		75%		90%	
Coefficients	OLS	ARCH	OLS	ARCH	OLS	ARCH	OLS	ARCH	OLS	ARCH
P	97.66	95.68	97.91	95.91	98.71	96.67	100.95	98.81	115.56	112.48
	(0.01)	(0.01)	(0.03)	(0.02)	(0.09)	(0.08)	(0.57)	(0.54)	(11.0)	(10.3)
$Var(P)$	77.96	72.52	115.9	107.6	346.5	318.2	3677.	3293.	255913	223850
$C1$	3.48	2.49	3.98	2.97	5.32	4.24	8.23	7.00	23.59	21.40
	(0.03)	(0.03)	(0.04)	(0.04)	(0.10)	(0.10)	(0.58)	(0.55)	(11.0)	(10.3)
$C2$	3.01	2.13	3.51	2.60	4.84	3.86	7.75	6.61	23.12	21.00
	(0.03)	(0.03)	(0.04)	(0.04)	(0.10)	(0.10)	(0.58)	(0.55)	(11.0)	(10.3)

Note that for the ARCH process which we are considering the odd numbered moments are zero and likewise for the normal. The ARCH process has high fourth moments and high sixth moments (also with $\alpha_1 = 0.87$ finite moments exist at time $t + 2$ and $t + 3$, starting from a normal distribution at time $t + 1$). The even numbered moments beyond the second moment are higher for the ARCH process than for the normal process. Since the even numbered derivatives of price with respect to R_t are positive, the even numbered terms such as $\frac{1}{24} f''''(R_t^e)u_t^4$ are larger for the ARCH process. Thus it is not surprising that the stochastic mean of P is larger for the ARCH process (table 3) than for the normal process (table 2).

We can now perform a group of experiments that can be considered intermediate between the pure simulation experiments of tables 2 and 3, and the analytical method of Black and Scholes (1973). The Black-Scholes formula gives the value of a call option on stock assuming that the stock price follows a Brownian motion diffusion process. Under the Brownian motion diffusion process the stock price is from the lognormal distribution. In our case we can simulate this as follows

$$[9] \qquad P = e^{(m+v)} \qquad v \sim N(0, V)$$

In this case, P has a lognormal distribution with mean and variance given by

$$[10] \qquad E(P) = e^{\left(m + \frac{V}{2}\right)}$$

$$[11] \qquad Var(P) = e^{(2m + 2V)} - e^{(2m + V)}$$

(see, for example, Mood, Graybill, and Boes, 1974, p.117). We now plug into equations (10) and (11) the values for $E(P)$ and

Var(*P*) displayed in tables 2 and 3, then invert the two equations solving for *m* and *V*. We can now use each of these pairs of *m* and *V* to generate *P* from the appropriate lognormal distribution. The generated value for the bond price *P* is then used to produce the payoff to the call options *C*1 and *C*2. 100 000 replications with control variates ensure a sufficient accuracy to the calculated expected values and variances, the only exception being the last two cases of table 5 (corresponding to the 90% initial residual), where the contribution of the control variates in improving the accuracy of results is negligible. Results are displayed in table 4 (to be compared with those of table 2) and in table 5 (to be compared with those of table 3). If enough replications were performed, the mean and variance of *P* would be the same in table 4 as in table 2. The main difference is that the simulations for table 4 assume that *P* is lognormally distributed.

The experiments with the lognormal distribution have produced slightly higher values of *C*1 and *C*2. The values in tables 4 and 5 should be close to the values in tables 2 and 3. With enough replications the mean and variance of *P* would be the same in table 4 as in table 2 and in table 5 as in table 3. The difference between these tables is the higher order moments of *P*. In other macroeconomic contexts higher order moments have smaller effects on stochastic means (see for example Sterbenz and Calzolari, 1990). The results in the tables show large differences in call values.

The results for call options are not very different between tables 2 and 4. This is not surprising since the lognormal distribution used in table 4 is close to the normal distribution in table 2. However there are large differences between tables 3 and 5. These results are somewhat surprising. In tables 3 and 5 the variances and means of *P* are constrained to be equal. The values in table 3 use an ARCH simulation and *P* is subject to greater kurtosis than for the lognormal. The result interestingly is that the expected payoff of the option is lower in table 3 than in table 5.

Table

Expected payoff to call option ($C1$, $C2$) with OLS and ARCH coefficients, simulating the bond price P with lognormal distribution: $P = e^{(m+v)}$, $v \sim N(0, V)$. Parameters m and V have been computed (numerically) to produce for P the same means and variances as in table 2 (here slightly different for simulation errors; experimental standard deviations, omitted for simplicity, are about the same as in table 2).

Init. resid.	0.15		0.61		1.17		1.78		2.49	
Percentile	10%		25%		50%		75%		90%	
Coefficients	OLS	ARCH	OLS	ARCH	OLS	ARCH	OLS	ARCH	OLS	ARCH
P	97.27	95.30	97.55	95.57	98.40	96.39	99.99	97.99	102.87	100.11
$Var(P)$	29.12	27.12	63.06	58.71	166.7	155.1	383.3	355.9	858.3	794.9
m	4.576	4.555	4.577	4.557	4.580	4.560	4.586	4.567	4.594	4.568
V	.00307	.00298	.00660	.00641	.01706	.01655	.03760	.03638	.07794	.07629
$C1$	2.28	1.36	3.43	2.42	5.80	4.67	9.17	7.94	14.24	12.49
$C2$	1.82	1.03	2.96	2.03	5.33	4.23	8.66	7.46	13.72	12.00

Table 5

Expected bond price (P) and payoff to call option ($C1$, $C2$) with OLS and ARCH coefficients, simulating the bond price P with lognormal distribution: $P = e^{(m+v)}$, $v \sim N(0, V)$. Parameters m and V have been computed (numerically) to produce for P the same means and variances as in table 3 (here slightly different for simulation errors; experimental standard deviations, omitted for simplicity, are about the same as in table 3).

Init. resid.	0.15		0.61		1.17		1.78		2.49	
Percentile	10%		25%		50%		75%		90%	
Coefficients	OLS	ARCH	OLS	ARCH	OLS	ARCH	OLS	ARCH	OLS	ARCH
P	97.66	95.68	97.91	95.91	98.71	96.67	100.94	98.80	115.11	112.07
$Var(P)$	77.98	72.53	115.9	107.6	346.6	318.3	3676.	3292.	196819	175075
m	4.577	4.557	4.578	4.558	4.575	4.555	4.461	4.448	3.248	3.259
V	.00814	.00789	.01202	.01163	.03494	.03348	.30808	.29064	3.0039	2.9282
$C1$	3.83	2.80	4.72	3.64	8.15	6.92	23.73	21,78	74.31	71.17
$C2$	3.36	2.40	4.25	3.22	7.71	6.47	23.31	21.37	74.09	70.95

6. Conclusions

We have explored the effect that the autoregressive conditional heteroskedasticity process has on the expected payoff of a call option on a Treasury bond. For a simple long term interest rate equation we have found that the ARCH coefficients produce different payoffs than OLS coefficients. We have found that the effect of using the ARCH stochastic process in simulating option payoffs is far greater than the effect of using ARCH coefficients.

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Contents