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Liquidity as an Insurance Problem.

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Liquidity as an Insurance Problem*

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Abstract

Risk-averse individuals wish that assets concentrate their payoffs in states of high marginal value (that is, highly likely or low endowment states). An asset or portfolio may fail to do so, by having payoffs uncorrelated to its owner needs or, even worse, by having them inversely related. The latter, which we call tier 1 illiquidity, is shown to occur in non-Walrasian markets (where a trade involves bargaining) and in incomplete Walrasian markets where optimal trading strategies are non trivial. In both cases, the high valuation of the trader biases the equilibrium price against him.

The former, which we call tier 2 illiquidity, is shown to arise when individual shocks are privately observed, because moral hazard prevents contracting on them. Diamond and Dybvig (1983) and Holmström and Tirole (1998) present prominent examples of tier 2 illiquidity. However, a self-insurance model is offered to argue that the importance of this type of illiquidity is limited from a welfare perspective, provided individuals are patient enough and can trade in a perfectly competitive, complete–except for individual-level uncertainty– set of asset markets.

This article characterizes an asset's liquidity as the degree of insurance it provides, thereby identifying the basic economic problem behind liquidity as one of the familiar risk-sharing kind. It also shows, by means of examples, that the problem arises when asset markets are imperfectly competitive, incomplete, or both.

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1 Introduction

Although there is a large variety of meanings attributed to the word "liquidity" in the finance and economics literatures, (almost) all of them can be understood as the (partial) expression in particular contexts of the notion of insurability, that is, the ability of an asset (or portfolio) of providing his owner access to consumption goods precisely in those states in which he most needs them.

It is clarifying to decompose the insurability problem in two parts, or to think of it as a two-level problem, the separating line between them being the benchmark in which the asset's payoff (price) is insensitive to his owner's needs.

At the first or most basic level (tier 1 illiquidity), the illiquidity problem refers to a condition in which the asset's trading value depends adversely on his owner's needs, that is, a needy consumer finds that he is able to sell at a lower price the hurrier he is. This idea is certainly present in the usual intuition of illiquidity as the "loss due to rush selling," and is central to more precise definitions we encounter in the literature. Two important examples are:

- Lippman and McCall's (1986) operational definition, "the expected time until the asset is sold when following the optimal policy." They show in a search environment that the optimal selling strategy has the reservation price property, and that the reservation price depends inversely on the seller's discount factor. The degree of patience is then a key determinant of the expected price the seller gets.
- The wide-spread definition that relates price changes to the size of the transaction. For instance, in words of Garbade and Silver (1979): "A financial instrument is commonly considered liquid if [...] the instrument may be traded with a sufficient number of participants to make feasible purchases and sales on short notice at prices near the contemporaneous equilibrium value of the instrument. [...]" Part of the problem is then suffering from adverse price changes as a consequence of one's desire to sell; the sufficient number of participants is somehow viewed as a condition for avoiding that. In a similar fashion, Economides and Siow define "[...] a market as having *high liquidity* when the volume of trade is high and the corresponding variance of the price is low." A high number of participants, or a large trading volume, are commonly seen as conditions for more competitive markets[1]; more competitive markets are then

[1] Although this association is usually correct, Makowski and Ostroy (1995) have shown that large numbers are neither necessary nor sufficient conditions for perfect competition.

more liquid.

It should be clear that an asset that trades in a perfectly competitive market is, almost by definition, completely liquid in this first level sense: the trading price is common to all participants, and is not affected either by the inclusion of a new trader nor by the strength of his gains from trade. In view of this fact, it is not surprising how difficult it has been to accommodate liquidity considerations in existing asset pricing models, which are overwhelmingly based either on perfectly-competitive general equilibrium theory, or in price-taking no arbitrage theory.

At a second level, a liquid asset can be thought as one whose payoffs are higher precisely when his owner is in the highest need –an insurance contract being a prominent example–. This is in fact the notion of liquidity creation in the seminal work of Diamond and Dybvig (1983), notion which we also encounter in the recent work by Holmström and Tirole (1998).

Diamond and Dybvig (1983) study a model of state-dependent utility in which each individual can either be indifferent between consuming at date 1 or 2, or only value consumption at date 1. Which state is going to prevail at date 1 is unknown at the previous date, when the investment decision must be made. Real assets yield consumption flows at an exogenous pace, and are completely insensitive to the occurrence of liquidity shocks, which might make consuming earlier than planned –before investment matures– optimal. Similarly, Holmström and Tirole (1998a, b) characterize liquidity in the context of a firm, as a protection from unexpectedly high reinvestment needs to continue a project, case in which a restriction to borrowing imposed by moral hazard considerations would bind. Thus, firms would be averse to the risk of facing high investment (consumption) needs.

In both cases, perfectly competitive asset markets are assumed. Yet, it is the unobservability of the private shocks (to time preference in one case, and to reinvestment needs in the other) which prevents the use of direct insurance contracts–or contingent claims, for that matter–to transfer the risk, resulting in a tier-2 liquidity problem.

Diamond and Dybvig use that fact to argue that deposit contracts can be made to serve this function, whereby banks pool those individual risks and adjust the early-withdrawal payment to the optimal intermediate consumption level. A drawback in the argument is that this recomposition of consumption flows can not be made without creating arbitrage opportunities in the presence of other financial assets (see Freixas and Rochet (1997) and references therein). Hence, it would appear that the unobservability of the private shock would preclude an efficient allocation of risk.

Holmström and Tirole, instead of deducing an optimal financial instrument, obtain an optimal portfolio choice for firms. The optimal choice would in general leave a residual risk undistributed, despite of the fact that individual shocks are completely diversifiable.

The purpose of this paper is two-fold. On the one hand, it aims to persuade the reader that level-1 illiquidity refers to the obstacles that arise to insuring a desired expenditure flow in imperfectly competitive or incomplete asset markets. The point is made by a series of examples, whereby the illiquidity problem is shown to arise because of either market incompleteness or lack of perfect competition, or both. On the other hand, it illustrates with a particular example of the self-insurance kind, that type-2 illiquidity may not be a serious problem from a welfare viewpoint provided that individuals plan in long horizons.

In particular, section 2 extends Diamond and Dybvig's shock-to-time-preference setup to long horizons, where individuals preferences are subject to permanent, privately observed, and unanticipated shocks and have access to a complete set (except for individual shocks) of perfectly competitive asset markets. In the particular case of logarithmic preferences, it is shown that the utility loss derived from the unobservability of preference shocks is negligible if agents are infinitely lived and time discount factors are close enough to 1. The remaining of the paper is devoted to tier-1 liquidity.

Section 3 intends to understand the scope of Lippman and McCall's definition. The search world is one in which there is a unique asset, which can be sold at the current price or kept until a better offer arrives. If besides that Walrasian market there was a complete set of perfectly competitive asset markets –so that the first one is in fact redundant– then the expected time to sell it is always zero, for, as it is shown, any gains from waiting correspond to arbitrage opportunities. This fact could be interpreted as meaning that market completeness allows for increased competition from different date consumers.

Section 4 sketches a bargaining model as an example of a non-Walrasian trading environment. Being commonly known that the seller could be in a rush, it is optimal for the buyer to structure offers that will sort out patient and impatient sellers. Price offers will in equilibrium increase at a pace that impatient players cannot afford. Hence, a liquidity shock translates into a lower price or loss relative to patient individuals. We refer to this as "adverse bargaining" since its origin is the adverse conditions in which the liquidity-needed person bargains. It is well known however that outside opportunities will alleviate this problem, for instance through increased access to a larger pool of potential buyers. In the extreme, when all potential buyers are contacted and these outside opportunities are such that there is no room left for bargaining, we are back in a perfectly competitive world. The overall conclusion is that liquidity can only be properly studied outside the complete-market, perfectly competitive environment.

2 Liquidity and the investor's horizon

This section considers the decision problem of an investor with logarithmic preferences who, in each of the (possibly infinite) periods of his like makes consumption and investment decisions. In particular, it compares two situations. In the first one, there are complete markets in the standard sense, that is, there are assets whose payoffs span the whole consumption set, that is, any risk can be traded. In the second one, this spanning condition is only met for a subset of all states, which excludes the changes in time preference the individual experiences permanently over his lifetime. This particular form of market incompleteness, possibly due to the unobservability of personal shocks, is harmful from a welfare perspective because it impedes efficient risk-sharing.

However, this form of market incompleteness forces the individual to consume the same, regardless of whether he experienced a shock or not, only in the last period of his life, because at any other period he could have made his portfolio holdings sensitive to his personal shocks. This is not to say that only matters at the last period, because the last period constraint has an effect on all previous decisions. Nevertheless, it is intuitive that as the investor's horizon becomes farther away, this form of incompleteness becomes less and less important. In fact, the example that is presented below shows that the utility loss due to the incompleteness becomes arbitrarily small as $T \to \infty$ and the individual's time discount factor $\beta \to 1$.

The model is standard, with a unique consumption good and many periods indexed by $t \in \{0, 1, ..., T\}$ (where possibly $T \to \infty$). The economy is subject to production-related shocks. At any point in time, there is a finite number of events $\omega \in \Omega$, that describe current real-asset endowments and payoffs. That is, at every period, every individual chooses a production plan (real asset r) from a set $Y^i(\omega)$ which is affected by the state.

A production plan (or real asset r) is a sequence of contingent consumption flows $y_r = \{y_0(\omega_0), y_1(\omega_0\omega_1), ...y_M(\omega_0\omega_1...\omega_M)\}$ of size Ω^M, indicating the number of consumption goods that the real asset delivers in each event and time elapsed from the moment construction initiated $\tau+1$. Each flow $y_r(\omega_0\omega_1...\omega_\tau,\tau)$ is assumed to be bounded from above and below, could be 0, is negative if net investment is required, and it is assumed that no asset has a life-span larger than $M << T$, that is, $y_r(\cdot,\tau)=0$ for $\tau > M$, and for all r. Finally, not producing is always an alternative: $0 \in Y^i(\omega)$ for all i and ω.

Individuals are also subject to personal, unobservable shocks. In particular, the value of consumption at any period is not known but until the period starts. At the beginning of every period, one factor δ^i, $\delta \in \{\underline{\delta},...,\overline{\delta}\}$ is privately revealed to each individual to be his valuation, chosen by nature from the distribution function $\pi(\delta)$. Hence, at any point T of his life, the consumer $i \in [0,1]$ will evaluate the realized consumption path $(c_0^i,...,c_T^i)$ according to

$$U(c_0^i,...,c_T^i;\delta_0^i,...,\delta_T^i) = (1-\beta)\left\{\delta_0^i \ln(c_0^i) + \delta_1^i\beta\ln(c_1^i) + ... + \delta_T^i\beta^T\ln(c_T^i)\right\} \quad (1)$$

where β is a constant discount factor, common across individuals, and $(\delta_0^i,\delta_1^i,...,\delta_T^i)$ is the actual history of personal shocks. Ex-ante evaluation corresponds to the expectation of (1).

Hence, every person's utility will generally depend on both, the realized value of his personal shock δ^i and the production-related shock ω. Individual i's history up to time t is a sequence of the form $s_0^i s_1^i s_2^i \cdots s_t^i$ where each s^i is a pair $s^i = (\omega,\delta^i) \in S \equiv \Omega \times \Delta$, that is

$$h_t^i \in H_t^i = \underbrace{S \times S \times \cdots \times S}_{t+1 \text{ times}} = S^{t+1}$$

In the spirit of Aumann (1964), it is assumed that individuals are non atomistic and hence personal histories have no effect on the aggregate (that is, personal risks are perfectly diversifiable). The common part of history, in contrast, contains only the production-related shock, for the average personal shock is constant:

$$h_t \in H_t = \underbrace{\Omega \times \Omega \times \cdots \times \Omega}_{t+1 \text{ times}} = \Omega^{t+1}$$

As a notational matter, we will used interchangeably the following: $h_t^i = h_{t-1}^i s_t^i = s_0^i s_1^i s_2^i \cdots s_t^i$ and $h_t = h_{t-1}\omega_t = \omega_0\omega_1\omega_2\cdots\omega_t$.

Hence, each individual has to choose a production plan from $Y^i(\omega)$ every period. If individual i chooses the real asset $y_{h_t}^*$ in the event h_t, she will receive the flow $y_{h_t}^*(h_{t+\tau})$ at event $h_{t+\tau}$ and time $\tau = 0,1,...,M$. Each individual chooses also a consumption level, so as to

$$\max \sum_{t\geq 0} \beta^t \sum_{h_t^i \in H_t^i} \pi^i(h_t^i)\,\delta_t^i \ln\left(c\left(h_t^i\right)\right)$$

Remark 1 *Diamond and Dybvig's (1983) setting corresponds to a situation where individuals live for three periods, in the second period* $\delta_1 \in \{0,1\}$*, and investment projects pay off at the last period. Since there is a continuum of individuals, the aggregate state is unique, characterized by a fraction* $(1-\pi)$ *having discount factors of* 0*, and a fraction* π *of* 1*.*

The economy is assumed to be essentially complete in the sense that all production-related states are insurable, that is, there is at every node h_t a portfolio of real (and/or financial) assets $\widehat{a}(\omega\,|h_t)$ that pays off at time $t+1$ one unit of the consumption good in state ω and zero otherwise. We will refer to this concept as $\omega-$completeness, to distinguish it from full or s^i-completeness. Let $\widehat{q}(\omega\,|h_t)$ denote the time t price of such a portfolio, and $\widehat{q}(h_t\omega)$ its time 0 price. Individuals have to make consumption, production and portfolio decisions

$$\max_{\left\{c_{h_t},y_{h_t}\in Y^i(h_t),\left\{\widehat{a}_{h_t}(\omega)\right\}_{\omega\in\Omega}\right\}_{t=0}^{\infty}} \sum_{t\geq 0}\beta^t \sum_{h_t^i\in H_t^i} \pi^i\left(h_t^i\right)\delta_t^i \ln\left(c\left(h_t^i\right)\right) \tag{2}$$

$$s/t \qquad \sum_{t\geq 0}\sum_{h_t\in H_t} c(h_t)\,\widehat{q}(h_t) = \sum_{t\geq 0}\sum_{h_t\in H_t}\sum_{\tau\in\{1,\ldots,M\}} y_{h_t}(h_{t+\tau})\,\widehat{q}(h_t)$$

This lifetime budget constraint is obtained from recursive substitution of (3) as follows:

$$c\left(h_{t-1}^i\omega\delta\right) - \sum_{\tau=0}^{M} y_{h_{t-\tau}}(h_{t-1}\omega) = \widehat{a}\left(h_{t-1}^i\omega\right) - \sum_{\omega'\in\Omega}\widehat{a}\left(h_t^i\omega'\right)\widehat{q}(\omega'\,|h_t)$$

$$\widehat{q}(h_{t-1}\omega)\,c\left(h_{t-1}^i\omega\delta\right) - \sum_{\tau=0}^{M}\widehat{q}(h_t)\,y_{h_{t-\tau}}(h_{t-1}\omega) = \widehat{q}(h_t)\,\widehat{a}\left(h_{t-1}^i\omega\right) - \sum_{\omega'\in\Omega}\widehat{a}\left(h_t^i\omega'\right)\widehat{q}(h_t\omega')$$

$$\sum_{h_t^i\in H_t^i}\widehat{q}(h_{t-1}\omega)\,c\left(h_{t-1}^i\omega\delta\right) - \sum_{h_t^i\in H_t^i}\sum_{\tau=0}^{M}\widehat{q}(h_t)\,y_{h_{t-\tau}}(h_{t-1}\omega) = \sum_{h_t^i\in H_t^i}\widehat{q}(h_t)\,\widehat{a}\left(h_{t-1}^i\omega\right) - \sum_{h_t^i\in H_t^i}\sum_{\omega'\in\Omega}\widehat{a}\left(h_t^i\omega'\right)\widehat{q}(h_t\omega')$$

$$\sum_{t\geq 0}\sum_{h_t^i\in H_t^i}\widehat{q}(h_{t-1}\omega)\,c\left(h_{t-1}^i\omega\delta\right) - \sum_{t\geq 0}\sum_{h_t^i\in H_t^i}\sum_{\tau=0}^{M}\widehat{q}(h_t)\,y_{h_{t-\tau}}(h_{t-1}\omega) = \sum_{t\geq 0}\sum_{h_t^i\in H_t^i}\widehat{q}(h_t)\,\widehat{a}\left(h_{t-1}^i\omega\right) - \sum_{t\geq 0}\sum_{h_t^i\in H_t^i}\sum_{\omega'\in\Omega}\widehat{a}\left(h_t^i\omega'\right)\widehat{q}(h_t\omega') = 0$$

$$\Rightarrow \sum_{t\geq 0}\sum_{h_t^i\in H_t^i}\widehat{q}(h_t)\,c\left(h_t^i\right) = \sum_{t\geq 0}\sum_{h_t^i\in H_t^i}\widehat{q}(h_t)\left(\sum_{\tau=0}^{M} y_{h_{t-\tau}}(h_t)\right)$$

From (4), it follows directly that, despite the incompleteness of markers, production and consumption decisions are separable:

Proposition 1 *Production decisions are objective:* $\max \sum_{t\geq 0} \sum_{h_t^i \in H_t^i} \widehat{q}(h_t) \left(\sum_{\tau=0}^{M} y_{h_{t-\tau}}(h_t) \right)$ *requires* $y_{h_t}^* \in \arg\max_{y_{h_t} \in Y(h_t)} \sum_{\tau=0}^{M} \widehat{q}(h_{t+\tau}) y_{h_t}(h_{t+\tau})$.

Hence, we can analyze the consumer's problem separately by treating the $y_{h_t}^*(h_{t+\tau})$ as endowments:

$$\max_{\left\{c_{h_t}, \{\widehat{a}_{h_t}(\omega)\}_{\omega\in\Omega}\right\}_{t=0}^{\infty}} \sum_{t\geq 0} \beta^t \sum_{h_t^i \in H_t^i} \pi^i\left(h_t^i\right) \delta_t^i \ln\left(c\left(h_t^i\right)\right)$$

$$s/t \qquad \sum_{t\geq 0} \sum_{h_t \in H_t} c(h_t) \widehat{q}(h_t) = W$$

where $W \equiv \sum_{t\geq 0} \sum_{h_t \in H_t} \sum_{\tau \in \{1,\ldots,M\}} y_{h_t}^*(h_{t+\tau}) \widehat{q}(h_t)$. In recursive form it corresponds to

$$\begin{aligned} v(\widehat{a}(h_t), h_t^i) &= \max_{\left\{c\left(h_t^i\right), \left(\widehat{a}(h_{t+1})\right)_{\omega\in\Omega}\right\}} \left\{ \delta_t \ln\left(c\left(h_t^i\right)\right) + \beta \sum_{\omega'\in\Omega} \sum_{\delta\in\triangle} \pi\left(\omega' | h_t\right) \pi(\delta) v(\widehat{a}(h_{t+1}), h_{t+1}^i) \right\} \\ s/t \qquad c\left(h_t^i\right) &= \sum_{\tau=0}^{M} y_{h_{t-\tau}}^*(h_{t-1}\omega) + \widehat{a}\left(h_{t-1}^i \omega\right) - \sum_{\omega'\in\Omega} \widehat{a}\left(h_t^i \omega'\right) \widehat{q}\left(\omega' | h_t\right) \end{aligned}$$

with first order condition:

$$-\widehat{q}\left(\omega' | h_t\right) \delta_t \frac{1}{c\left(h_t^i\right)} + \beta \sum_{\delta\in\triangle} \pi\left(\omega' | h_t\right) \pi(\delta) \frac{\partial v(\widehat{a}(h_{t+1}), h_{t+1}^i)}{\partial \widehat{a}\left(h_t^i \omega'\right)} = 0$$

$$\frac{\partial v(\widehat{a}(h_t), h_t^i)}{\partial \widehat{a}\left(h_{t-1}^i \omega\right)} = \delta_t \frac{1}{c\left(h_t^i\right)}$$

$$\Rightarrow \delta_t \frac{1}{c\left(h_t^i\right)} = \frac{\beta \pi\left(\omega' | h_t\right)}{\widehat{q}\left(\omega' | h_t\right)} \sum_{\delta\in\triangle} \pi\left(\delta_{t+1}\right) \delta_{t+1} \frac{1}{c\left(h_{t+1}^i\right)} \tag{3}$$

In contrast, with full information we would have had

$$\begin{aligned} v(\widehat{a}\left(h_t^i\right), h_t^i) &= \max_{\left\{c\left(h_t^i\right), \left(\widehat{a}\left(h_{t+1}^i\right)\right)_{\omega\in\Omega}\right\}} \left\{ \delta_t \ln\left(c\left(h_t^i\right)\right) + \beta \sum_{\omega'\in\Omega} \sum_{\delta\in\triangle} \pi\left(\omega' | h_t\right) \pi(\delta) v(\widehat{a}\left(h_{t+1}^i\right), h_{t+1}^i) \right\} \\ s/t \qquad c\left(h_t^i\right) &= \sum_{\tau=0}^{M} y_{h_{t-\tau}}^*(h_{t-1}\omega) + \widehat{a}\left(h_t^i\right) - \sum_{\omega'\in\Omega} \sum_{\delta\in\triangle} \widehat{a}\left(h_{t+1}^i\right) \widehat{q}\left(h_{t+1}^i\right) \end{aligned}$$

with first order condition:

$$-\widehat{q}\left(h_{t+1}^{i}\right)\delta_{t}\frac{1}{c\left(h_{t}^{i}\right)}+\beta\pi\left(\omega'\left|h_{t}\right.\right)\pi\left(\delta\right)\frac{\partial v(\widehat{a}\left(h_{t+1}^{i}\right),h_{t+1}^{i})}{\partial\widehat{a}\left(h_{t}^{i}\omega'\right)}=0$$

$$\frac{\partial v(\widehat{a}\left(h_{t}^{i}\right),h_{t}^{i})}{\partial\widehat{a}\left(h_{t}\right)}=\delta_{t}\frac{1}{c\left(h_{t}^{i}\right)}$$

$$\Rightarrow\delta_{t}\frac{1}{c\left(h_{t}^{i}\right)}=\frac{\beta\pi\left(\omega'\left|h_{t}\right.\right)\pi\left(\delta_{t+1}\right)}{\widehat{q}\left(h_{t+1}^{i}\right)}\delta_{t+1}\frac{1}{c\left(h_{t+1}^{i}\right)} \tag{4}$$

Let us denote by c^{*} the solution to (4) (the optimal plan with fully complete markets) and by $\overline{c}$ the solution to (3) (the corresponding plan with $\omega-$complete markets). Hence,

$$\begin{aligned}\overline{c}\left(h_{t+1}^{i}\right)&=\frac{\beta\pi\left(\omega'\left|h_{t}\right.\right)}{\widehat{q}\left(\omega'\left|h_{t}\right.\right)}\frac{\sum_{\delta\in\triangle}\pi\left(\delta_{t+1}\right)\delta_{t+1}}{\delta_{t}}\overline{c}\left(h_{t}^{i}\right)\\ c^{*}\left(h_{t+1}^{i}\right)&=\frac{\beta\pi\left(\omega'\left|h_{t}\right.\right)\pi\left(\delta_{t+1}\right)}{\widehat{q}\left(h_{t+1}^{i}\right)}\frac{\delta_{t+1}}{\delta_{t}}c^{*}\left(h_{t}^{i}\right)\end{aligned}$$

In this particular case of logarithmic utility, we can actually solve for the optimal plan: The problem with T periods and complete markets has the solution

$$c_{t}^{*}(\delta_{t})=\frac{\beta^{t}\delta_{t}}{\delta_{0}+E\left[\delta\right]\left(\beta+...+\beta^{T}\right)}\rho^{t}W_{0}\quad\forall t=0,...,T$$

that is, each period consumption will be a fraction of wealth, in proportion to the current shock, while with incomplete markets the solution is

$$\begin{aligned}\overline{c}_{t}(\delta_{t})=&\frac{\left(1-\beta\right)\beta^{t}\delta_{t}}{\delta_{0}\left(1-\beta\right)+E\left[\delta\right]\beta\left(1-\beta^{T}\right)}\rho^{t}W_{0}\\ &\left\{\Pi_{j=1}^{t}\left(\frac{E\left[\delta\right]\left(1-\beta^{T-j+2}\right)}{\delta_{j}\left(1-\beta\right)+\beta E\left[\delta\right]\left(1-\beta^{T-j}\right)}\right)\right\}\qquad t=1,...,T\\ &=c_{t}^{*}(\delta_{t})\left\{\Pi_{j=1}^{t}\left(\frac{E\left[\delta\right]\left(1-\beta^{T-j+2}\right)}{\delta_{j}\left(1-\beta\right)+E\left[\delta\right]\beta\left(1-\beta^{T-j}\right)}\right)\right\}\end{aligned}$$

In the infinite horizon problem, the difference in utility levels attained under both situations becomes arbitrarily small:

Proposition 2

$$\lim_{\beta\to1}\left\{\left(1-\beta\right)\sum_{t=0}^{\infty}\beta^{t}\sum_{h_{t}^{i}\in H_{t}^{i}}\pi^{i}\left(h_{t}^{i}\right)\delta_{t}^{i}\ln c^{*}\left(h_{t}^{i}\right)-\left(1-\beta\right)\sum_{t=0}^{\infty}\beta^{t}\sum_{h_{t}^{i}\in H_{t}^{i}}\pi^{i}\left(h_{t}^{i}\right)\delta_{t}^{i}\ln\overline{c}\left(h_{t}^{i}\right)\right\}=0$$

Proof. In the appendix. ■

The reason for this result is that, even though it is not possible to buy a state-contingent consumption flow for the next period (and hence the marginal utility of consumption at t can only be made proportional to the expected marginal utility next period), once the shock is known consumption can be accommodated by trading in assets. The consumption path –and hence the resource allocation– becomes arbitrarily close to the one with fully complete markets when individuals are patient enough.

Hence, level-2 liquidity becomes unimportant –the ability of the asset to accommodate consumption needs can be substituted entirely by a carefully chosen portfolio management strategy–.

The following sections analyze two scenarios in which level-1 liquidity problems arise; in the former, it is because asset markets are incomplete, while in the latter because they are imperfectly competitive.

3 Liquidity and the expected time to sell

Lippman and McCall (1986) define liquidity as the expected selling time provided the optimal selling strategy is used. The more impatient the seller, the lower his reservation price and, equivalently, his expected selling price.

By its nature, this search-theoretic environment excludes the existence of other assets. In contrast, this section imagines that this particular asset is traded as in Lippman and McCall's environment, but that the individual has also access to an $\omega-$complete (in the sense of the previous section) set of perfectly competitive asset markets. In this new context it is shown that the expected time to sell is zero, that is, personal impatience does not lead to a lower reservation price.

Before the result can be stated, some notation needs to be introduced. So far we have dealt with pure securities. An ordinary security is a bundle of $f\left(h_t\left(\omega\right)\right)$ state claims in state $h_t\left(\omega\right)$, $\forall\omega\in\Omega, t=0,1,...$ Let $q_f(h_t\left(\omega\right))$ be the time t price of security f in state $h_t\left(\omega\right)$. Proposition 1 shows that under $\omega-$complete markets, production decisions are independent of preferences. Hence, the optimal policy for selling an asset is the solution to

$$v(h_t\left(\omega\right)) = \max\left\{q_f(h_t\left(\omega\right)), f\left(h_t\left(\omega\right)\right) + \sum_{\omega'\in\Omega}\widehat{q}(\omega'\left|h_t\left(\omega\right)\right)v(\omega'\left|h_t\left(\omega\right)\right)\right\} \qquad (5)$$

The following proposition says that the optimality of waiting is equivalent to the existence of an arbitrage opportunity.

Proposition 3 $v(h_t(\omega)) = q_f(h_t(\omega)) \qquad \forall h_t \in H_t, f \in F.$

Proof. Suppose not, that is, $q_f(h_t(\omega)) < f(h_t(\omega)) + \sum_{\omega' \in \Omega} \widehat{q}(\omega' | h_t(\omega)) v(\omega' | h_t(\omega))$. If $v(h_{t+1}(\omega)) = q_f(h_{t+1}(\omega))$, a portfolio of 1 unit of f and $-q_f(\omega' | h_t(\omega))$ units of state security in ω', $\forall \omega' \in \Omega$, costs today $q_f(h_t(\omega)) - f(h_t(\omega)) + \sum_{\omega' \in \Omega} \widehat{q}(\omega' | h_t(\omega)) [-q_f(\omega' | h_t(\omega))] < 0$ and pays off $q_f(h_{t+1}(\omega)) - q_f(h_{t+1}(\omega)) = 0$ in states $\omega \in \Omega$. If $v(h_{t+1}(\omega)) \neq q_f(h_{t+1}(\omega))$, the same argument can be reproduced with regard to the first time t^* in a history such that $v(h_{t^*}(\omega)) = q_f(h_{t^*}(\omega))$. ■

The key issue is that with $\omega-$complete markets, if waiting produces an expected gain, it must be so for all individuals, not just for patient persons. If a liquidity shocked person faces these misaligned prices, she can proceed as the proof suggests, buying a portfolio that not only doesn't force her to postpone consumption, but even gives her the possibility of consuming more immediately.

The role of $\omega-$completeness in the argument must be stressed. If the pure securities (or their equivalent) were not available, there would not be a way of transforming the future gain in present consumption; hence, a liquidity-shocked person may choose to pass it, depending on her time-preference. This is essentially the break-down of the separation theorem. Our point is that Lippmann and McCall's definition is meaningful only under $\omega-$incompleteness.

4 Liquidity and the bargaining problem

In contrast to the Walrasian environment discussed previously, this section addresses the problem of selling an asset to only one potential buyer. Bargaining games are very diverse and more so equilibrium outcomes. Yet, we develop a particularly simple example with the aim of illustrating the general point that one source of illiquidity is precisely the lack of competition among buyers at a particular moment in time, and more importantly, that in such situation the seller will face a loss if he is liquidity shocked at the time.

The model considers two players, a buyer whose valuation of 1 is common knowledge, and a seller with private valuation $v_s \in \{v_1, ..., v_S\}$, where $0 \leq v_1 < ... < v_S \leq 1$. Let π_s be the commonly known probability that the seller's valuation is v_s. There are $I > S$ periods $i = 1, 2, ..., I$, and in each of them the buyer makes an offer which the potential seller accepts or reject. Rejection leads to another offer, acceptance to a transaction in the proposed terms.

In order to stress the time pressure of the liquidity shocked person, we assume that if the transaction is made at time i, the utility of the seller is

$$\left(1-\frac{i}{I}\right)(q-v_s)$$

while the buyer's utility is $(1-q)$, independent of the date of the transaction.

This formulation can be made partially compatible with the model used in the previous two sections in the following way: imagine that between consumption dates t and $t+1$ there are many opportunities to bargain –made offers or respond to received offers–. Then, the impatience rate for the whole interval corresponds to the realized δ_t; the value of obtaining one unit of consumption in the logarithmic case would be $\frac{\delta}{c_t}$, which can be normalized to δ. Hence, the highest possible value of selling the asset would correspond to the highest δ, $\delta=\overline{\delta}=v_1$ while the lowest to $\underline{\delta}=v_S$.

Proposition 4 *The perfect bayesian equilibrium of this game is characterized by:*
The buyer making ascending offers that satisfy $q_{i+1}=\frac{I-i}{I-i-1}q_i-\frac{v_i}{I-i-1}$,
The seller of valuation i *accepting the price* i *at round* i, $i=1,2,...,S$,
where either $q_1=v_1$ *or* $q_S=v_S$*, depending on which leaves the buyer higher profit.*

The reason for this is that an impatient seller has higher gains from trade; that allows the buyer to screen the different types of sellers out, extracting their surplus up to the point that is allowed by the incentive compatibility constraints.

Admittedly, the illustration is very special in many respects. The property of separating ascending prices, however, arises in many bargaining models studied in the literature (see, for instance, Sobel and Takahishi (1983) and Fudenberg, Tirole and Levine (1985)).

The main point is that bargaining theory does support the idea that impatience (demand for immediacy) creates "losses," in the sense usually attributed to the notion of tier-1 liquidity: had the player been more patient, he would have gotten a better price. In a perfectly competitive economy, however, there is nothing to bargain over (Makowski and Ostroy (1995)).

5 Concluding remarks

The term liquidity was coined originally because it is evocative of the property of liquid elements to accommodate its shape to the recipient that contain them. Likewise, this article has characterized an asset's liquidity as its ability to accommodate its payoffs to its owner needs. Two levels of illiquidity were distinguished: tier-1, related to the property that the asset value does not depend adversely on its owner's needs, and tier-2, related to the property that the asset value does accommodate its value to its owner's needs.

It was argued that the general problem of liquidity could be modeled as a problem of protecting investors from personal shocks to time preference. If investors wish to accommodate consumption in states which they cannot foresee, and which they cannot credibly communicate, the unrestricted consumption plan cannot be implemented–that is, level 2 illiquidity arises–. It was shown, by means of an example, that the utility loss due to the referred unobservability of time-preference shocks is arbitrarily small if individuals are patient enough (this is to say, their horizon is long and their discount factor close to one) and have access to $\omega-$complete markets.

Market incompleteness and non-Walrasian markets worsen the problem in the same way: the investor is not only deprived of the possibility of receiving a higher payoff than if he was not shocked, but also his higher discount factor makes him willing to accept even lower prices than if unshocked.

The distinction of the two levels of liquidity seems to be useful because it facilitates the comparison of alternative definitions, which as we have seen, can usually be put into one of these categories. But the differences go beyond that. Coping with level-2 illiquidity is undoubtedly a portfolio problem, and hence it does not make complete sense to discuss the protection against personal shocks that particular assets may offer. In contrast, level-1 liquidity is a characteristic of the asset (although also affected by environmental characteristics). Perhaps this is why most definitions, and analysis of liquidity –especially in finance– only refer to this level.

This research suggests that the study of liquidity requires an understanding of the causes of market incompleteness and imperfect competition. In particular, whether there are trading environments and institutions that facilitate that the same assets be traded more competitively. The economic value of such structures goes beyond the static allocative efficiency usually attributed to perfectly competitive environments, because, as the ideas presented suggest, they also facilitate efficient risk-sharing.

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6 Appendix

Proof. [of Proposition 2.] We have that

$$\overline{c}_t(\delta_t) = \frac{(1-\beta)\beta^t\delta_t}{\delta_0(1-\beta)+E[\delta]\beta\left(1-\beta^T\right)}\rho^t W_0$$

$$\left\{\Pi_{j=1}^t\left(\frac{E[\delta]\left(1-\beta^{T-j+2}\right)}{\delta_j(1-\beta)+\beta E[\delta]\left(1-\beta^{T-j}\right)}\right)\right\} \quad t=1,...,T$$

$$\Rightarrow \lim_{T\to\infty}\overline{c}_t(\delta_t) = c_t^*(\delta_t)\left\{\Pi_{j=1}^t\left(\frac{E[\delta]}{\delta_j(1-\beta)+E[\delta]\beta}\right)\right\}$$

The expected lifetime utility differential is

$$\begin{aligned}
& (1-\beta)\left\{\sum_{t=0}^{\infty}\beta^t\sum_{h_t^i\in H_t^i}\pi^i\left(h_t^i\right)\delta_t^i\ln c^*\left(h_t^i\right)\right. \\
& \left. -\sum_{t=0}^{\infty}\beta^t\sum_{h_t^i\in H_t^i}\pi^i\left(h_t^i\right)\delta_t^i\ln c_t^*(\delta_t)\left\{\Pi_{j=1}^t\left(\frac{E[\delta]}{\delta_j(1-\beta)+E[\delta]\beta}\right)\right\}\right\} \\
= \;& (1-\beta)\left\{\sum_{t=0}^{\infty}\beta^t\sum_{h_t^i\in H_t^i}\pi^i\left(h_t^i\right)\delta_t^i\left[\ln c^*\left(h_t^i\right)-\ln c_t^*(h_t^i)\left\{\Pi_{j=1}^t\left(\frac{E[\delta]}{\delta_j(1-\beta)+E[\delta]\beta}\right)\right\}\right]\right\} \\
= \;& (1-\beta)\left\{\sum_{t=0}^{\infty}\beta^t\sum_{h_t^i\in H_t^i}\pi^i\left(h_t^i\right)\delta_t^i\ln\frac{c^*\left(h_t^i\right)}{c_t^*(h_t^i)\left\{\Pi_{j=1}^t\left(\frac{E[\delta]}{\delta_j(1-\beta)+E[\delta]\beta}\right)\right\}}\right\} \\
= \;& -(1-\beta)\left\{\sum_{t=0}^{\infty}\beta^t\sum_{h_t^i\in H_t^i}\pi^i\left(h_t^i\right)\delta_t^i\ln\Pi_{j=1}^t\left(\frac{E[\delta]}{\delta_j(1-\beta)+E[\delta]\beta}\right)\right\}
\end{aligned}$$

A history of personal shocks can be described by the number of times each obtained. If the possible shocks are N $(\delta_1,...,\delta_N)$, then let $(x_1,...,x_N)$ represent the history up to t, with $x_n\in\{0,1,...,t\}$ and $\sum_n x_n = t$. There are $\binom{t}{x_1}\binom{t-x_1}{x_2}...\binom{t-x_1-...-x_{N-1}}{x_N}$ different histories with the same consequence, and the chances of observing it are $p_1^{x_1} * ... * p_N^{x_N}$. Hence,

$$\pi^i(h_t = x_1,...,x_N) = \binom{t}{x_1}\binom{t-x_1}{x_2}...\binom{t-x_1-...-x_{N-1}}{x_N}p_1^{x_1}...p_N^{x_N}$$

and the utility differential can be written as

$$
\begin{aligned}
&= -(1-\beta)\left\{\sum_{t=0}^{\infty}\beta^t \sum_{x_1=0}^{t}\dots\sum_{x_N=0}^{t-x_1-\dots-x_{N-1}}\binom{t}{x_1}\dots\binom{t-x_1-\dots-x_{N-1}}{x_N}p_1^{x_1}\dots p_N^{x_N}\delta_t^i\right. \\
&\quad \left.\ln\left\{\left(\frac{E[\delta]}{\delta_1(1-\beta)+E[\delta]\beta}\right)^{x_1}\dots\left(\frac{E[\delta]}{\delta_N(1-\beta)+E[\delta]\beta}\right)^{x_N}\right\}\right\} \\
&= -(1-\beta)\left\{\sum_{t=0}^{\infty}\beta^t \sum_{x_1=0}^{t}\dots\sum_{x_N=0}^{t-x_1-\dots-x_{N-1}}\binom{t}{x_1}\dots\binom{t-x_1-\dots-x_{N-1}}{x_N}p_1^{x_1}\dots p_N^{x_N}\delta_t^i\right. \\
&\quad \left.\left\{x_1\ln\left(\frac{E[\delta]}{\delta_1(1-\beta)+E[\delta]\beta}\right)+\dots+x_N\ln\left(\frac{E[\delta]}{\delta_N(1-\beta)+E[\delta]\beta}\right)\right\}\right\} \\
\lim_{\beta\to 1}(\cdot) &= -(1-\beta)\sum_{t=0}^{\infty}\left\{\sum_{x_1=0}^{t}\dots\sum_{x_N=0}^{t-x_1-\dots-x_{N-1}}\binom{t}{x_1}\dots\binom{t-x_1-\dots-x_{N-1}}{x_N}p_1^{x_1}\dots p_N^{x_N}\delta_t^i\right. \\
&\quad \left.\lim_{\beta\to 1}\beta^t\left\{x_1\ln\left(\frac{E[\delta]}{\delta_1(1-\beta)+E[\delta]\beta}\right)+\dots+x_N\ln\left(\frac{E[\delta]}{\delta_N(1-\beta)+E[\delta]\beta}\right)\right\}\right\} \\
&= -(1-\beta)\sum_{t=0}^{\infty}\left\{\sum_{x_1=0}^{t}\dots\sum_{x_N=0}^{t-x_1-\dots-x_{N-1}}\binom{t}{x_1}\dots\binom{t-x_1-\dots-x_{N-1}}{x_N}p_1^{x_1}\dots p_N^{x_N}\delta_t^i\right. \\
&\quad \left.\{x_1\ln(1)+\dots+x_N\ln(1)\}\right\} \\
&= 0
\end{aligned}
$$

■