Counterparty Risk Subject To ATE

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Abstract

Rating trigger ATE (Additional Termination Event) is a counterparty risk mitigant that allows banks to terminate and close out bilateral derivative contracts if the credit rating of the counterparty falls below the trigger level. Since credit default is often preceded by rating downgrades, ATE clause effectively reduces the counterparty credit risk by early termination of exposure. However, there is still the risk that counterparty may default without going through severe downgrade. This article presents a practical model for valuating CVA in the presence of ATE.

Keywords: Counterparty Risk, Credit Valuation Adjustment, Rating Transition, Rating Trigger, Additional Termination Event.

1. Introduction

Counterparty credit risk refers to the risk that a counterparty to a bilateral financial derivative contract may fail to fulfill its contractual obligation causing financial loss to the non-defaulting party. Only over-the-counter (OTC) derivative contracts are subject to counterparty risk. Exchange traded derivatives have very little counterparty risk because the exchange or a clearing house is the central counterparty to the transaction. Exchanges/clearing houses are well protected by the financial industry.[1]

From the perspective of a bank, when the counterparty defaults, the portfolio of all OTC derivative contracts between the bank and the counterparty is marked-to-market (MTM) at the time of default.[2] If the value of the portfolio is negative to the bank, the bank is obliged to pay the full MTM value to the defaulting counterparty. If, however, the value is positive to the bank, the bank will recover only a percentage of that MTM value, usually after a lengthy bankruptcy proceeding.[3] If the recovered amount is less than 100%, ignoring the time value, the bank suffers a credit loss. This potential credit loss due to

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[1] Some clearing house uses haircut to set margin requirement. Clearing funds from participants provide additional protection in extreme market condition when margins are insufficient.

[2] Brigo and Morini (2010) raise the issue of settlement amount. They argue that if the residual deals are replaced by another counterparty, the settlement amount could be different than the risk-free amount. We do not consider such an issue.

[3] A lengthy bankruptcy proceeding can be costly to the bank as the money cannot earn interests or be invested.

the possibility that the counterparty may default must be factored into the deal price. In a similar way, the bank may also benefit from its own risk of bankruptcy.

An increasingly popular exposure control mechanism is to use some kind of break clause that allows the bank to terminate the portfolio with the counterparty in the event that some pre-agreed condition is breached by the counterparty.[4] ATE is such a break condition. ATE can take many forms of which most common is perhaps the credit ATE (Gregory 2010). In credit ATE, a credit trigger rating for a party is defined. If the party's credit rating crosses the trigger, the other party is entitled to terminate and close out the positions. Theoretically, ATE may effectively reduce the exposure if a default is preceded by an ATE event. One might view that ATE creates a sort of right-way exposure profile where the counterparty exposure is eliminated if the credit quality worsens significantly. From the modeling standpoint, ATE may also be considered as "lossless default" where the contracts are terminated with full recovery, as contrasted with default where the contracts are terminated with loss. However, as pointed out by Gregory (2010) that ATE might actually drive the counterparty into default if the positions are closed out and the counterparty is significantly out-of-the-money (OTM) on those positions.

While credit ATE can significantly reduce counterparty risk, it does not completely eliminate it because it is still possible that the counterparty might default before an ATE event. This is evident that firms may default before being significantly downgraded by the rating agencies. In other words, the residual counterparty risk in the presence of ATE comes from the possibility of counterparty default without ever crossing the credit trigger.

CVA modeling has attracted much attention recently. Alavian *et al* (2009), Gregory (2009, 2010), Pykhtin and Zhu (2007) provide excellent overviews of counterparty risk management practice and CVA pricing. Zhu and Lomibao (2005) propose a conditional valuation method to calculate exposures for instruments whose existence is path dependent. Gregory (2010) gives a general description of various forms of ATE. Recently, Yi (2010) proposed a model for CVA under rating trigger. In their model, the time of hitting the credit trigger and the time of jump-to-default are modeled by two Poisson processes. Both the mandatory and the optional settlements upon breaching the credit trigger are considered. However, calibration is a major issue. However, there appears a need for a practical model for bilateral CVA calculation subject to ATE.

In this paper, we present a practical model specifically for calculation of bilateral CVA (BCVA) of a portfolio subject to ATE rating trigger. The discretized formulation naturally leads to a rating-based Markov chain. We assume that portfolio termination and close-out is mandatory once the ATE trigger is breached. For exposition convenience, we assume that all trades in the portfolio have the same ATE trigger, so that once the party is downgraded to or below the trigger, the entire portfolio is terminated. We also suggest extension of the model to more complicated situations. The main theme of our paper is modeling CVA for a general derivative portfolio without regarding to the actual type of trade. We focus on how CVA should be calculated in the presence or credit ATE trigger.

[4] Reciprocally, the counterparty may terminate the deals with the bank if the bank breaches the condition.

Our model is rating based where obligors having the same rating would have the same rating transition probabilities. As such, the spread difference within a rating class is ignored.[5] We use a copula model for the joint rating transition and default. The risk neutral transition matrix is obtained by calibrating the historical transition matrix to rating-based generic CDS spread curves generated from the market CDS spreads.[6] To model the ATE trigger, we make the ratings that are equal to or below the ATE trigger rating absorbing states such that default paths cannot pass through the ATE trigger rating. This requires further modification of the calibrated risk neutral transition matrix. The resulting matrix is ATE rating specific.

We show that, in the presence of ATE rating trigger, the CVA is due to the possibility of the counterparty jumping to default without triggering the ATE event. For a given counterparty, the CVA is smaller the lower are the ATE trigger ratings.

Our contributions are that we extend the bilateral CVA calculation to including ATE trigger ratings through a rigorous mathematical model framework, and propose a practical model for implementation by financial institutions. A salient point is the introduction of the ATE transition matrix that enables to calculate the first passage time of the ATE trigger using the modified rating transition probabilities.

The rest of the article is organized as follows. Section 2 presents the mathematical formulations of the bilateral CVA in continuous time and their discretization. We point out how the base model can be extended to deal with margins and multiple ATE triggers. Section 3 outlines formulae for calculating the joint transition and conditional joint default probabilities. We define the ATE transition matrix that enables to calculate the probability of hitting ATE trigger in terms of transition probabilities. We also define the ATE factor profile can be used to estimate the effectiveness of ATE trigger for given exposure profile. Section 4 concludes the paper. Detailed formulation derivation is shown in appendices.

2. The Model

Throughout this article, we refer the two parties to the underlying derivative trades in the portfolio as the bank, denoted by B, and the counterparty, denoted by C. We use the term party to describe both B and C if it applies to both. We value the portfolio from the bank's perspective. As such, positive portfolio value or in-the-money (ITM) indicates the counterparty owes the bank money, and negative portfolio value or out-of-the-money (OTM) means the reverse.

2.1 Nomenclatures

Before describing the model, we define the notations that will be used throughout this article without further explanation.

- *UCVA*: Unilateral CVA.

[5] Spread difference within a rating class can vary substantially. Additionally, it is often observed in the CDS/bond markets that the credit spreads of a lower rated firm can be lower than those of a higher rated firm.

[6] For pricing credit risk of a firm without market CDS spread, banks create rating-based generic CDS curves that map credit ratings to market CDS spreads. Generic CDS curve is essentially some average market quotes of CDS spread grouped by credit rating.

- *BCVA*: Bilateral CVA.
- $\tilde{Q}(t,T)$: Risk-neutral transition matrix from *t* to *T*.
- $\tilde{Q}_{\omega}(t,T)$: Risk-neutral ATE transition matrix from *t* to *T*.
- $\Xi = \{1,2,\cdots,K-1,K\}$: Rating state space where 1 refers to the highest rating class (e.g. AAA/Aaa) and K-1 the lowest rating class. K is the default state. D is also used to refer to the default state.[7]
- $\beta_k(t), k = B, C$: The rating of party *k* at time *t*.
- $\omega_k,\ k = B, C$: The ATE trigger rating of party *k*. If $\beta_k(t)$ crosses ω_k, the portfolio is terminated.
- $\Omega_k = \{\omega_k, \omega_k + 1, \cdots, K-1\} = \{j | j \geq \omega_k, j \neq D\}, k = B, C$: Set of ratings that are equal to or worse than the ATE trigger rating of party k.
- $\Pi_k = \{1,2,\cdots,\omega_k - 1\} = \Xi \backslash \{\Omega_k \cup \{D\}\}, k = B, C$: Set of ratings that are better than ATE trigger of *k*.
- $\eta_k = Min\{t | \beta_k(t) \in \Omega_k\}, k = B, C$: The first time that the party *k* crosses its ATE trigger rating.
- $\eta = Min(\eta_B, \eta_C)$: The first-to-ATE time of both the bank B and the counterparty C.
- $\tau_k, k = B, C$: The default time of party *k* without ever crossing the trigger (ATE default time).
- $\tau = Min(\tau_B, \tau_C)$: The first-to-default time of both the bank and the counterparty.
- $W(s,q)$: Present (time *t*) value of the cashflow on the portfolio between *s* and *q* where *t* is the valuation time and the portfolio final maturity is *T*.
- $R_k, k = B, C$: Recovery rate of party *k*.
- $\delta_k = 1 - R_k, k = B, C$: Loss-Given-Default (LGD) of party *k*.
- $f^+ = Max(f, 0)$, $f^- = Max(-f, 0)$, and $f = f^+ - f^-$.

2.2 Credit Value Adjustment (CVA)

We consider a portfolio of derivative contracts that is uncorrelated with the credit quality of neither the counterparty nor the bank.[8] Hence, we assume that the portfolio value $W(t,T)$ is independent of either party.[9]

When a party defaults, one of the following scenarios will apply:

1) If $\tau > T$, no credit loss will incur to either the bank or the counterparty as the first default occurs after the final maturity of the portfolio.
2) If $\eta < \tau$, an ATE event occurs before the first-to-default time. Since the portfolio is terminated at the first-to-ATE time (mandatory termination),[10] the exposure to either party at default is zero and no credit loss to either party.
3) If $\tau \leq Min\ \{\eta, T\}$, a default event happens before the portfolio expiry date and before the first-to-ATE time. From the bank's perspective, the rule of default settlement is as follows:
    a. If the counterparty defaults first, $\tau_C < \tau_B$, then

[7] The default state *D* is technically not a valid credit rating class. However, transition matrix typically includes the default state for mathematical convenience.
[8] Examples of such trades are interest rate swaps, caps and swaptions, FX options and cross-currency swaps, equity options. However, CDS or CDO tranches where the reference entities are correlated with either party do not belong to this category.
[9] See Brigo and Chourdakis (2008) for a model of unilateral CVA of CDS when the counterparty and the CDS reference entity are correlated. In their case, the portfolio value strongly correlated with the counterparty.
[10] If termination is optional, additional criteria must be considered (Yi 2010).

   i. If the portfolio value at default $W(\tau_C, T) > 0$, i.e. the counterparty owes the bank money, the bank will receive from the counterparty the amount $R_C W(\tau_C, T)^+$.
   ii. Else, if $W(\tau_C, T) < 0$, the bank owes the counterparty money, the bank pays the counterparty the full portfolio value $W(\tau_C, T)^-$.

b. Conversely, if the bank defaults first, $\tau_B < \tau_C$, then
   i. The bank pays the counterparty $R_B W(\tau_B, T)^-$ if the bank owes the counterparty, or
   ii. The bank receives $W(\tau_B, T)^+$ if the counterparty owes the bank.

c. In the relatively rare situation where both the bank and the counterparty default at the same time, $\tau = \tau_C = \tau_B$, the bank would pay the counterparty $R_B W(\tau, T)^-$ if $W(\tau, T) < 0$, or receive from the counterparty $R_C W(\tau, T)^+$ if $W(\tau, T) > 0$. Since simultaneous defaults are usually much less frequent than single default, and loss to the bank due to counterparty default and the benefit to the bank from its own default cancel out to a significant extent, simultaneous default contribution to bilateral CVA is generally smaller than unilateral CVA.

The present value of the portfolio can be generally expressed as

$$\bar{V}(t,T) = E_t \left\{ \begin{matrix} 1(\eta \geq \tau)1(\tau > T)W(t,T) + 1(\eta < \tau)W(t,T) + \\ 1(\eta \geq \tau)1(\tau \leq T) \left\{ \begin{matrix} 1(\tau = \tau_C < \tau_B)[R_C W(\tau_C,T)^+ - W(\tau_C,T)^-] \\ +1(\tau = \tau_B < \tau_C)[W(\tau_B,T)^+ - R_B W(\tau_B,T)^-] \\ +W(t,\tau) - 1(\tau = \tau_C = \tau_B)W(\tau,T) \end{matrix} \right\} \end{matrix} \right\} \quad (1)$$

where $W(t, \tau)$ is the present value of cashflow of the portfolio from time *t* to the first-to-default time τ, and $W(\tau, T)$ is the present portfolio value at default time τ.

Eqn. (1) above extends the formulation of Gregory (2009) to the case of ATE. Specifically, if the portfolio $V(\tau, T)$ in Eqn. (5) of Gregory (2009) is replaced with $1(\eta \geq \tau)W(\tau, T)$, we obtain Eqn. (1) above. Put it another way, the exposure in the presence of ATE is contingent upon ATE event not occurring before default. However, the presence of $1(\eta \geq \tau)$ makes pricing CVA subject to ATE much more difficult to model because the first-to-ATE time η and the first-to-default time τ are generally correlated. Since function $1(\eta \geq \tau)$ decreases as the credit quality worsens, ATE creates a sort of "right-way" risk exposure, in the sense that the counterparty exposure is non-increasing as the counterparty's credit quality deteriorates.

Using the relation $W(t, T) = W(t, \tau) + W(\tau, T)$ and $W(\tau, T) = W(\tau, T)^+ - W(\tau, T)^-$, Eq. (1) can be written in a more convenient form

$$\bar{V}(t,T) = E_t\{W(t,T)\} - E_t \left\{ 1(\eta \geq \tau)1(\tau \leq T) \left\{ \begin{matrix} 1(\tau = \tau_C)\delta_C W(\tau_C,T)^+ \\ -1(\tau = \tau_B)\delta_B W(\tau_B,T)^- \end{matrix} \right\} \right\} \quad (2)$$

Eqn. (2) shows that the fair expected present value of the portfolio is equal to the counterparty risk-free value of the portfolio minus an adjustment due to default by either party or both. This adjustment is commonly referred to as credit value adjustment or CVA.

It is interesting to see that the risk-free value $E_t\{W(t,T)\}$ is independent of the ATE. This is expected because when the portfolio is terminated due to breach of ATE trigger, there is no credit loss. One might think of ATE event as a default with immediate full recovery.

The bilateral CVA, denoted by BCVA, is the net expected loss or gain due to default by the counterparty and/or by the bank itself,

$$BCVA = E_t\left\{1(\eta \geq \tau)1(\tau \leq T)\begin{Bmatrix} 1(\tau = \tau_C)\delta_C W(\tau_C, T)^+ \\ -1(\tau = \tau_B)\delta_B W(\tau_B, T)^- \end{Bmatrix}\right\} \quad (3)$$

The bilateral CVA contains two terms. The first term is the credit loss that the bank will suffer if the counterparty defaults first before the first-to-ATE time and the final maturity of the portfolio. This is the unilateral CVA, denoted by *UCVA*. The second term represents the gain by the bank should it default first before the first-to-ATE time and the final maturity of the portfolio. This second term is also called Debt Value Adjustment, or DVA, to reflect the benefit to the bank on its own debt. So the BCVA is equal to the difference between the unilateral CVA and DVA. The BCVA can be negative if DVA exceeds UCVA. An example of negative BCVA is a portfolio of short position of options. In this case, the bank is always OTM if it has collected all option premiums from the counterparty.

The unilateral CVA, UCVA, is the expected loss to the bank if the counterparty defaults, and can be obtained by setting $\eta_B = \infty$ and $\tau_B = \infty$ in eqn. (1), (also see Remark 2.3)

$$UCVA = E_t\{1(\eta_C \geq \tau_C)1(\tau_C \leq T)\delta_C W(\tau_C, T)^+\} \quad (4)$$

Remark 2.1: The CVA formulation of Gregory (2009) without ATE can be recovered by setting the ATE rating trigger to be equal to the default state ($\omega_B = \omega_C = D$), meaning that a default by either party is the only event that can terminate the portfolio prior to the final maturity. In this case, we have $\eta = \tau$, and hence $1(\eta \geq \tau) \equiv 1$.

Remark 2.2: In some cases, the deal contracts may require that ratings from both S&P and Moody's breach the ATE rating trigger. Ratings of these two rating agencies can occasionally differ (split ratings) although the difference is usually no more than one rating class. We do not consider split ratings and refer to the paper of Lando and Mortensen (2005).

Remark 2.3: Although the credit risk of the bank does not appear explicitly in the unilateral CVA formula (4), it does not necessarily imply that the bank has no influence on the unilateral CVA. One possibility is to model the default time τ_C using a correlated default model. Through the correlation, the bank's credit risk implicitly influences the default timing of the counterpart. An example is given by Gregory (2009).

Remark 2.4: In Eqn. (2), simultaneous default by B and C is implied by the condition $\tau = \tau_C = \tau_B$ and is modeled by the same joint transition/default model. This approach is appropriate under normal market condition when simultaneous default is infrequent. The likelihood of a simultaneous default tends to increase significantly when the credit market is under severe stress or in crisis mode. When the markets are in crisis, the systemic default risk is substantial.[11] Simultaneous default can be handled by specifically modeling a common default time τ (Gregory 2009).

2.3 Discretization

For ease of exposition of numerical implementation, we make the following assumptions:

(1) The portfolio value $W(t,T)$ is independent of the credit rating of either party. This assumption is mainly to simplify exposition. The model can easily be extended to rating dependent exposure, such as rating dependent margining.
(2) All obligors of a given rating are considered to have the same transition probabilities. Heterogeneity in credit spread between obligors of the same rating class is ignored. This is generally a restriction of rating based credit model. To model intra-rating spreads would greatly complicate the model.

We divide the time domain (t,T) into N sub-intervals, $t = t_0 < t_1 < \cdots < t_N = T$. We consider the loss due to default in time period $t_{k-1} < t \le t_k$ based on the exposure at t_k. In the discrete setting, we do not distinguish the time of default within the same time period, nor do we the ATE time. In other words, if party *C* defaults in the interval $(t_{k-1}, t_k]$ and if $\eta_C > t_{k-1}$, then we say $\eta_C \ge \tau_C$. Therefore, we have the following set relation

$$\{\eta_C \ge \tau_C, \tau_C \in (t_{k-1}, t_k]\} \cong \{\eta_C > t_{k-1}, \tau_C \in (t_{k-1}, t_k]\} \quad (5)$$

By definition, $\eta_C = Min\{t | \beta_C(t) \in \Omega_C\}$ is the first time that the rating of *C* crosses the trigger rating and migrates into set Ω_C. If we are sure that once in Ω_C the counterparty C has no chance to either default or come out of Ω_C, or equivalently, we force the non-defaulting rating class equal and below the ATE trigger ω_C absorbing state, we have[12]

$$\{\eta_C \le t\} = \{\beta_C(t) \in \Omega_C\} = \cup_{j\in\Omega_C} \{\beta_C(t) = j\} \quad (6)$$

Base on the similar reasoning, we can also derive

$$\{\eta_C > t\} = \{\beta_C(t) \in \Pi_C\} = \cup_{j\in\Pi_C} \{\beta_C(t) = j\} \quad (7)$$

The set relations (5-7) are important because they enable to express the probability of ATE hitting time $P(\eta_C \le t)$ or the survival probability $P(\eta_C > t)$ in terms of the transition probability $P\{\beta_C(t) = j \in \Pi_C\}$ which is the probability of counterparty C migrating from the current rating to rating j at a future time t under the restriction that the path of migration cannot cross the ATE trigger. It is easy to see that the transition probability $P\{\beta_C(t) = j\}$ is significantly easier to calculate than $P(\eta_C \le t)$.[13] In the following, we will use these set relations to calculate the probabilities of first-to-ATE and first-to-default when subject to ATE trigger.

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[11] An example is the financial crisis of 2008 brought upon by subprime mortgage and the overleveraged ABSCDO markets. See Hull (2009) for an excellent analysis.
[12] It may be helpful to consider Ω_C as a set of states similar to default state D in the sense that they are all absorbing states. Once C transits into a state in Ω_C, it will stay there with probability 1
[13] This is analogous to calculating the distribution of stock price hitting barrier vs the terminal stock price. The probability of hitting time or the first-passage-time is much harder to calculate.

2.3.1 Unilateral CVA Discretization

Utilizing relation (5), the Euler discretization of the *UCVA* of Eqn. (4) is

$$UCVA = \delta_C \sum_{k=1}^{N} P\{\tau_C \in (t_{k-1}, t_k], \eta_C > t_{k-1}\} E_t \left\{ W\big((t_k, T)\big)^+ \right\} \tag{8}$$

From the law of total probability and utilizing Eqn. (7), we can rewrite the probability in Eqn. (8) in terms of transition probability and one-period default probability

$$P\{\tau_C \in (t_{k-1}, t_k], \eta_C > t_{k-1}\} = \sum_{j \in \Pi_C} P(\beta_C(t_{k-1}) = j) P(\tau_C \leq t_k | \beta_C(t_{k-1}) = j) \tag{9}$$

where $P(\tau_C \leq t_k | \beta_C(t_{k-1}) = j)$ is the period k default probability with the starting rating j. These probabilities can be calculated from the martingale ATE rating transition matrix which we will describe later in this paper. We note that all default and rating transition when subject to ATE trigger is calculated using the ATE transition matrix.

Using Eqns. (9), we can rewrite Eqn. (8) as

$$UCVA = \delta_C \sum_{k=1}^{N} E_t \left\{ W\big((t_k, T)\big)^+ \right\} \sum_{j \in \Pi_C} P(\beta_C(t_{k-1}) = j) P(\tau_C \leq t_k | \beta_C(t_{k-1}) = j) \tag{10}$$

It is clear that the unilateral CVA of a portfolio subject to ATE rating trigger ω_C is caused by the possibility of counterparty jumping to default without going through any rating below or equal to the ATE trigger ω_C. The probability of this jump to default from rating j is $P(\tau_C \leq t_k | \beta_C(t_{k-1}) = j)$.

2.3.2 Bilateral CVA Discretization

The Euler discretization of BCVA of Eqn. (3) is

$$BCVA = \left\{ \begin{array}{c} \delta_C \sum_{k=1}^{N} P\{\tau_B > t_k, \tau_C \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} E_t\{W(t_k, T)^+\} \\ -\delta_B \sum_{k=1}^{N} P\{\tau_C > t_k, \tau_B \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} E_t\{W(t_k, T)^-\} \\ + \sum_{k=1}^{N} P\{\tau_C \in (t_{k-1}, t_k], \tau_B \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \\ \times \left\{ \delta_C E_t\{W(t_k, T)^+\} - \delta_B E_t\{W(t_k, T)^-\} \right\} \end{array} \right\} \tag{11}$$

In Eqn. (11), the first term represents the expected loss to the bank when the counterparty defaults first. The second term is the benefit the bank would gain if the bank defaults first. The third term is expected PnL when both B and C default simultaneously. It can be seen that the effect of simultaneous defaults is generally small compared with the first two terms due to the cancelation effect and the fact that the probability of simultaneous default is usually small.[14]

The joint default probabilities appearing in eqn. (11) are

$$P\{\tau_B > t_k, \tau_C \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} = P\{\tau_C > t_{k-1}, \tau_B > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}\}$$
$$-P\{\tau_C > t_k, \tau_B > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \tag{12}$$

[14] When the credit market is under stress and the systemic risk is high, simultaneous default risk can be high. In such a case, a term specifically modeling the simultaneous default should be added (see Gregory 2009).

$$P\{\tau_C > t_k, \tau_B \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} = P\{\tau_C > t_k, \tau_B > t_{k-1}, \eta_C > t_{k-1}, \eta_B > t_{k-1}\}$$

$$-P\{\tau_C > t_k, \tau_B > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \quad (13)$$

Using the set relation $\{\eta > t\} \cap \{\eta \geq \tau\} \supseteq \{\tau > t\} \cap \{\eta \geq \tau\}$ and from the law of total probability, we have

$$P(\tau_C > t_k, \tau_B > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}) =$$

$$\sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_C > t_k, \tau_B > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad (14)$$

$$P(\tau_C > t_{k-1}, \tau_B > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}) =$$

$$\sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_B > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad (15)$$

$$P(\tau_C > t_k, \tau_B > t_{k-1}, \eta_C > t_{k-1}, \eta_B > t_{k-1}) =$$

$$\sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_C > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad (16)$$

$$P(\tau_C > t_{k-1}, \tau_B > t_{k-1}, \eta_C > t_{k-1}, \eta_B > t_{k-1}) = \sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad (17)$$

Derivation of Eqns. (12-17) is provided in Appendix A. These formulas are based on the assumption of set Ω_k being absorbing, enabling expressing probability of first-passage-time of the trigger by the ATE rating transition probability.

From the law of total probability, we can express the one-period joint default probability as

$$P\{\tau_C \in (t_{k-1}, t_k], \tau_B \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} =$$

$$\sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_C \leq t_k, \tau_B \leq t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad (18)$$

We emphasize that rating transition and conditional default probabilities are based on the condition that no ATE trigger has been breached. To this end, we make sure that once a rating crosses its ATE trigger, it remains there. There is no possibility that a party can migrate to or below its ATE trigger and upgrade back. This can be easily achieved by making any rating class in the set $\Omega_k, k = B, C$ an absorbing state. As a result, the default probability is reduced as it is no longer possible to default from ratings in set Ω_k. As shown numerically in section 3.6, given a transition matrix and an ATE trigger rating, we can quantify the amount of reduction in default probability due to ATE.

2.4 Model Extensions

We have used the portfolio value $W(t, T)$ to derive exposure. This significantly simplifies notational exposition. In doing so, we implied that (a) all trades in the portfolio are nettable under a single netting agreement and (b) the portfolio is not collateralized or no margin requirement.[15] In this section, we suggest extension of the model to more practical and complicated portfolios. First, we note that extending the model to multiple netting agreements is straightforward (Pykhtin and Zhu 2007) since there is still only one ATE rating trigger.

2.4.1 Extension to Rating-Based Margin Threshold

Banks often impose margin threshold on their trading partners. The margin threshold is the maximum positive portfolio value that the counterparties do not need to pose margin. Furthermore, it makes economic sense for the margin threshold to decrease as the counterparty credit quality declines.

Let the margin threshold for the counterparty be denoted as $H\big(\beta_C(t)\big)$ where the time-dependency is through the counterparty's rating. The unilateral CVA is given by

$$UCVA = E_t\{1(\eta_C \geq \tau_C)1(\tau_C \leq T)\delta_C \times Min\big[W(\tau_C, T)^+, H\big(\beta_C(\tau_C -)\big)\big]\} =$$

$$\delta_C \sum_{k=1}^{N} E_t\left\{Min\left(W\big((t_k, T)\big)^+, H(j)\right)\right\} \sum_{j\in\Pi_C} P(\beta_C(t_{k-1}) = j) P(\tau_C \leq t_k | \beta_C(t_{k-1}) = j) \quad (19)$$

Note that we assume that the margin is determined at t_{k-1} for exposure time t_k. The corresponding BCVA can be derived similarly.

2.4.2 Extension to Multiple ATE Triggers

A recent trend is that banks are increasingly including a credit ATE clause in the trading agreement. This creates a counterparty portfolio consisting deals with heterogeneous ATE rating triggers. If the deal netting is based on ATE trigger where deals are nettable if they are subject to the same ATE trigger, the model outline above can be directly applied to each ATE trigger.

If, however, a single netting agreement covers multiple ATE rating triggers, extension is a little tricky. In such a case, the unilateral CVA is

$$UCVA = E_t\left\{1(\tau_C \leq T)\delta_C\big[\textstyle\sum_j 1\big(\eta^j \geq \tau_C\big)W_j(\tau_C, T)\big]^+\right\} \quad (20)$$

where η^j is the first hitting time of the *j*th ATE trigger denoted by ω^j, and W_j is the total value of all deals subject to the jth ATE trigger. Note that Eqn. (20) includes the case of no ATE clause where the ATE trigger rating is the default rating, $\omega^0 = K$. Because the default state is the only termination trigger, we have $\eta^0 = \tau_C$ and $1(\eta^0 \geq \tau_C) \equiv 1$.

Recognizing that all ATE rating triggers are associated with the same counterparty, breaching of a higher ATE trigger must be no later than a lower trigger. Therefore, if we index the ATE triggers $\omega^1 > \omega^2 > \cdots$, then the relationship $\eta^1 \geq \eta^2 \geq \cdots$ must hold, taking into consideration that multiple ATE's may be breached simultaneously.

[15] Collateralization and margin agreement are usually specified in CSA (Credit Support Annex). See Alavian *et al* (2009) and Gregory (2010) for an introduction of CSA.

A general discrete formulation for Eqn. (20) is rather cumbersome. Instead, we consider the case of two non-default ATE triggers plus the no ATE trigger. The possible ATE scenarios for this case are

$$\{\eta^1 \geq \eta^2 \geq \tau_C\},\ \{\eta^1 \geq \tau_C > \eta^2\}, \text{ and } \{\eta^1 < \tau_C, \eta^2 < \tau_C\} \tag{21}$$

By definition of ATE, $W_j(\tau_C, T) = 0$ if $\eta^j < \tau_C$. Consequently, we have

$$UCVA = \delta_C \sum_{k=1}^{N} \left\{ \begin{matrix} E_t(W_0 + W_1 + W_2)^+ P(\eta^1 \geq \eta^2 > t_{k-1}, t_{k-1} < \tau_C \leq t_k) \\ +E_t(W_0 + W_1)^+ P(\eta^1 > t_{k-1}, \eta^2 \leq t_{k-1}, t_{k-1} < \tau_C \leq t_k) \\ +E_t\left({W_0}^+\right) P(\eta^1 \leq t_{k-1}, \eta^2 \leq t_{k-1}, t_{k-1} < \tau_C \leq t_k) \end{matrix} \right\} \tag{22}$$

where $W_j = W_j(t_k, T)$.

A complication of calculating $P(\eta^1 \geq \eta^2 > t_{k-1}, t_{k-1} < \tau_C \leq t_k)$ is that we need to account for the possibility of $\omega^2 \leq \beta_C(t_{k-1}) < \omega^1$, i.e. there are two distinct ATEs, each violation only results in partial termination of the portfolio.

3. Transition and Conditional Default Probability

Eqns. (8-18) show that CVA calculation involves two components. One is the expected positive exposure (EPE) and the expected negative exposure (ENE) at time nodes t_k. The other involves the transition and default probability conditional on that the ATE trigger has not been breached. Since this paper focuses on modeling ATE trigger and assumes that the portfolio has a unique ATE trigger for each party, we will not consider exposure calculation. Interested readers may be referred to the paper of Zhu and Lomibao (2005) for some background and further references. In this section, we outline a model for calculating rating transition probability and default probability conditional on no prior violation of ATE trigger.

The presence of ATE necessitates modeling of rating transition in addition to default. Consequently, we use rating based models. As stated previously that, under the credit rating ATE clause, the portfolio between the bank and its counterparty is terminated and closed out once either party has crossed its ATE trigger. This implies that the paths of rating transition cannot cross the ATE trigger. This condition means rating class set Ω_k must be absorbing.

Our model contains the following steps:

1) Choose a historical transition matrix, say, Moody's one-year average default rate. Calculate the corresponding generator matrix using either JLT or IRW method.
2) Calibrate the time inhomogeneous generator matrix to the generic CDS spread term structure. This generates the martingale generator matrix.
3) For an ATE trigger rating ω_k, we modify the martingale generator matrix by setting rows from ω to $K-1$ to zero. The resulting ATE martingale generator matrix guarantees that the paths that lead to credit loss, or loss paths, cannot cross the trigger into Ω_k.
4) The ATE martingale generator matrix is used to calculate the required transition probabilities.
5) For BCVA, ATE martingale transition probabilities are calculated using the normal copula method.

3.1 The Historical Transition Matrix

Although its shortcomings for credit derivative pricing are well documented (Schonbucher 2003), all rating-based credit pricing models use a statistical transition matrix estimated from historical corporate default experience over a period of time. A major reason is that it is simply impractical to imply all entries of a rating transition matrix from market price data. A historical transition matrix provides a structure upon which (model based) adjustments can be made such that the model prices match the market prices. This adjustment transforms from the actuarial probability to risk-neutral probability, or equivalent martingale transition probability. This probability measure change is necessary because CVA is the price of counterparty default risk.

Given a historical transition matrix *P*, we calculate its generator matrix Λ.[16] Unfortunately, majority of the historical transition matrices does not admit a valid generator as they fail the test of Theorem 3.1(c) of the paper by Israel et al (2001) (referred to as IRW hereafter), thereby guaranteeing non-existence of a valid generator.[17]

One solution is to smooth the empirical transition matrix (Lando and Mortensen 2005) to avoid obvious violation of Theorem 3.1 of IRW, and then calculate a generator matrix. Even with smoothing, the existence of a valid generator is still not assured.[18] Another approach is to assume that a generator exists and proceed to search for one. Methods of generator matrix calculation can be found, for example, in IRW, Jarrow *et al* (1997) (JLT hereafter), and Kreinin *et al* (KS) (2001). The JLT method is the simplest and guarantees to give a valid generator. The IRW method has shown to be more accurate as it yields a smaller fitting error to the historical transition matrix, but is more computational involved.

In this paper, we also assume a generator matrix exists. We follow the suggestion of JLT (p. 504) that the martingale generator matrix be expressed as a product of a historical generator matrix and a time-dependent diagonal matrix which can be interpreted as risk premium. We apply the JLT method (p. 505) to estimate a generator matrix for the Moody's one-year average transition matrix (Moody's 2009) proportionally adjusted for WR. Next section describes the calibration of martingale generator matrix.

3.2 Risk-Neutral Generator Matrix Calibration

We assume that, for each rating class, there exists a generic term structure of CDS premium. Assuming a recovery rate, usually 40%, a rating specific term structure of PD, denoted by $PD(t,\beta)$, can be obtained by bootstrapping the generic CDS curves.[19] We adjust the one-step generator matrix by equating at each t_k and for each rating β the cumulative martingale default probability with $PD(t_k,\beta)$.

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[16] A generator matrix $\Lambda = (\lambda_{ij})$ of transition matrix *P* is a $K \times K$ matrix with the property (1) $P = Exp(\Lambda)$, and (2) $\lambda_{i,j\neq i} \geq 0, \lambda_{ii} = -\sum_{j\neq i} \lambda_{ij}$, where λ_{ij} is the intensity of jump to rating *j* from *i* in an infinitesimal time interval.

[17] KS examined 32 empirical matrices and found that all passed the test of Theorem 3.1 (a) and (b) of IRW, but most failed test of (c). If the smoothed matrix is without zero entries, it passes test of (c).

[18] Theorem 3.1 of IRW is a sufficient but not necessary condition for non-existence of a valid generator matrix.

[19] An alternative is to calibrate to the CDS spreads or bond prices directly.

Let $\tilde{Q}(t,T)$ be the martingale transition matrix from t to T, and recall that the martingale generator $\widetilde{\Lambda}(t)$ is the product of the historical generator Λ and a diagonal matrix[20]

$$U(t) = Diag(\mu_1(t), \mu_2(t), \cdots, \mu_{K-1}(t), 1),\ \widetilde{\Lambda}(t) = U(t)\Lambda \tag{23}$$

We assume that risk-premium matrix $U(t)$ is piecewise constant, and define the transition matrix over the period (t_{k-1}, t_k) as

$$\tilde{Q}(t_{k-1}, t_k) = \left(\tilde{q}_{ij}(t_{n-1}, t_n)\right)_{K\times K} = Exp\{U(t_{k-1})\Lambda\Delta t_k\} \tag{24}$$

Because the rating transition is a Markov chain, we can express the transition matrix at t_n recursively

$$\tilde{Q}(0, t_n) = \prod_{k=1}^{n} \tilde{Q}(t_{k-1}, t_k) = \tilde{Q}(0, t_{n-1})\tilde{Q}(t_{n-1}, t_n) \tag{25}$$

Since the rating-based default probability at t_n is given by the *K*th (last) column of $\tilde{Q}(0, t_n)$, and suppose we have calibrated $U(t)$ up to t_{n-1}, the risk premium matrix $U(t_{k-1})$ satisfies

$$\tilde{Q}(0, t_{n-1}) \begin{pmatrix} \tilde{q}_{1K}(t_{n-1}, t_n) \\ \tilde{q}_{2K}(t_{n-1}, t_n) \\ \vdots \\ \tilde{q}_{K-1,K}(t_{n-1}, t_n) \\ 1 \end{pmatrix} = \begin{pmatrix} PD(t_n, 1) \\ PD(t_n, 2) \\ \vdots \\ PD(t_n, K-1) \\ 1 \end{pmatrix} \tag{26}$$

Eqn. (24) is nonlinear and requires calculating matrix exponential which is expensive. For small time period Δt_k, we approximate Eqn. (24) by the first-order expansion

$$\tilde{Q}(t_{n-1}, t_n) = Exp\{U(t_{k-1})\Lambda\Delta t_k\} \approx I + U(t_{k-1})\Lambda\Delta t_k \tag{27}$$

From Eqn. (27), the last column of the one-period transition matrix $\tilde{Q}(t_{n-1}, t_n)$ is

$$\begin{pmatrix} \tilde{q}_{1K}(t_{n-1}, t_n) \\ \tilde{q}_{2K}(t_{n-1}, t_n) \\ \vdots \\ \tilde{q}_{K-1,K}(t_{n-1}, t_n) \\ 1 \end{pmatrix} = \Delta t_n \begin{pmatrix} \mu_1(t_{n-1})\lambda_{1K} \\ \mu_2(t_{n-1})\lambda_{2K} \\ \vdots \\ \mu_{K-1}(t_{n-1})\lambda_{K-1,K} \\ 1/\Delta t_n \end{pmatrix} \tag{28}$$

Thus, Eqn. (26) can be solved to yield

$$\begin{pmatrix} \mu_1(t_{n-1}) \\ \mu_2(t_{n-1}) \\ \vdots \\ \mu_{K-1}(t_{n-1}) \\ 1 \end{pmatrix} = \mathrm{A}\tilde{Q}(0, t_{n-1})^{-1} \begin{pmatrix} PD(t_n, 1) \\ PD(t_n, 2) \\ \vdots \\ PD(t_n, K-1) \\ 1 \end{pmatrix} / \Delta t_n \tag{29}$$

where

[20] The last element of $U(t)$ does not matter because the last row of the historical generator Λ is zero.

$$\mathrm{A} = Diag\left(1/\lambda_{1K}, 1/\lambda_{2K}, \cdots, 1/\lambda_{K-1,K}, \Delta t_n\right)_{K\times K} \tag{30}$$

Empirical evidence appears to indicate that $p_{jK} = 0$ leads to $\lambda_{jK} = 0$, but the reverse is not true. The JLT method guarantees positive (zero) λ_{jK} if the rating *j*'s historical default probability p_{jK} is positive (zero). Unfortunately, the one-year rating transition matrices published by Moody's and S&P have consistently shown that the one-year PD for Aaa/AAA rating is zero, although the five-year PD is not zero. The zero one-year PD for Aaa/AAA rating is an issue of data reliability which we do not address here.

Based on economic justification, Lando and Mortensen (LM) (2005) proposed a method to transform the original historical transition matrix into a smoothed on that satisfies the imposed conditions. Their results showed (Appendix C of LM) that the smoothed matrix has no zero PD. Another, more *ad hoc,* approach, used by JLT, is simply assign a reasonable non-zero value to p_{jK} whenever it is equal to zero. For example, we may equate the one-year PD with the scaled five-year historical PD for AAA, and adjust other entries in the same row accordingly. In this paper, we adopt the JLT approach. First, we assign one fifth of the five-year PD to the one-year PD. Then the historical transition matrix P has no zero entry in the last column. We then obtain the historical generator matrix Λ from P using the JLT method. The historical generator Λ guarantees no zero entry in the last column so the diagonal matrix A defined in Eqn. (30) is well defined.

Having calibrated the risk premium matrix $U(t_{n-1})$, we can calculate the one-period incremental risk-neutral transition matrix $\tilde{Q}(t_{n-1}, t_n)$ using Eqn. (27). From Eqn. (25), we obtain the time t_n martingale transition matrix $\tilde{Q}(0, t_n)$. However, the martingale transition matrix $\tilde{Q}(0, t_n)$ cannot be directly used for CVA calculation under ATE as it did not exclude the possibility that the obligor can breach the ATE trigger and recover. This brings us to the subject of adjusting the generator matrix to account for ATE.

Remark 3.1: Since the risk premium matrix $U(t)$is time-dependent but deterministic, the transition process is Markov but time inhomogeneous.

3.3 The ATE Generator Matrix and Transition Matrix

The main idea of using a credit ATE trigger as counterparty risk mitigant is that it eliminates the bank's exposure to counterparty by terminating the portfolio when the counterparty's credit quality reduces to or below a threshold. Consequently, there is no exposure to either party after the first crossing of ATE trigger.

Effective modeling of ATE rating trigger requires that the set Ω_k be absorbing, such that the probability of jump to default without going through ATE is not overstated. This can be achieved by modifying either the martingale transition matrix or the martingale generator matrix. The difference is that the generator matrix is for continuous time Markov chain, and the transition matrix is for discrete Markov chain. Under the continuous Markov chain setting, there is absolutely no possibility to migrate out of Π_k, whereas in the discrete Markov chain, migration into and then out of Π_k in the same period is still technically possible. As will be shown in section 3.5, the probability of default and the probability of not breaching ATE are smaller with the generator matrix approach than with the transition matrix approach.

Given the ATE trigger rating ω, we define the risk-neutral ATE generator matrix as

$$\widetilde{\Lambda}_{\omega}(t) = \Psi(\omega)\widetilde{\Lambda}(t) = \Psi(\omega)U(t)\Lambda \tag{31}$$

where

$$\Psi(\omega) = \begin{pmatrix} I_{(\omega-1)\times(\omega-1)} & 0 \\ 0 & 0 \end{pmatrix}_{K\times K} \tag{32}$$

is a matrix operator that sets the rows of the operand equal or below ω to zero.

We define the one-period risk-neutral ATE transition matrix by

$$\tilde{Q}_{\omega}(t_{k-1}, t_k) = Exp\{\widetilde{\Lambda}_{\omega}(\mathrm{t_{k-1}})\Delta t_k\} \cong I + \widetilde{\Lambda}_{\omega}(\mathrm{t_{k-1}})\Delta t_k \tag{33}$$

Clearly, for rows of $\tilde{Q}(t_{k-1}, t_k)$ that are equal to or greater than ω, the off-diagonal entries are equal to zero and the diagonal entries are equal to one. This guarantees that the set $\Omega = \{\omega, \omega + 1, \cdots, K - 1\}$ be absorbing.

The time t_n martingale ATE transition matrix is

$$\tilde{Q}_{\omega}(0, t_n) = \prod_{k=1}^{n} \tilde{Q}_{\omega}(t_{k-1}, t_k) \tag{34}$$

Given the current (t = 0) counterparty rating $\beta_C(0)$ and by definition, we can calculate the probability in Eqn. (10) as

$$P(\beta_C(t_{n-1}) = j) = \{\tilde{Q}_{\omega_C}(0, t_{n-1})\}_{\beta_C(0),j} \tag{35}$$

$$P(\tau_C \le t_n | \beta_C(t_{n-1}) = j) = \{\tilde{Q}_{\omega_C}(t_{n-1}, t_n)\}_{jK} \tag{36}$$

3.4 The Joint Rating Transition

If the rating transition of the bank is independent of the counterparty, the bilateral CVA is simply the difference of the two standalone CVAs, one for counterparty risk and the other (DVA) for bank's own default risk, each evaluated using the unilateral CVA model described above without regarding the other.

However, as shown by Gregory (2009), default correlation between the parties can significantly impact the CVA, even the unilateral CVA. For calculating bilateral CVA under the rating-based Markov chain model described above, it is convenient to adopt a simple normal copula model for correlated rating transition.[21] The details are given in Appendices B and C. Here we give the formulae for rating the joint transition and conditional joint default probabilities.[22]

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[21] Other copula models, such as t-copula, can also be used in place of the normal copula. The *t*-copula has tail dependency resulting in a larger joint default probability. The normal copula has no tail dependency.

[22] N and N_2 are respectively the standard and the bivariate normal distribution functions.

The time t_n transition probability $P(\beta_C(t_n)=j,\beta_B(t_n)=i)$ is given by Eqn. (B.5). For rating pair $(j,i)\in\Pi_C\times\Pi_B$ and from Eqn. (B.6), we obtain

$$P(\tau_C>t_n|\beta_C(t_{n-1})=j,\beta_B(t_{n-1})=i)=1-P(\tau_C\le t_n|\beta_C(t_{n-1})=j,\beta_B(t_{n-1})=i)$$
$$=1-\sum_{k\in\Xi}P(\beta_C(t_n)=D,\beta_B(t_n)=k|\beta_C(t_{n-1})=j,\beta_B(t_{n-1})=i) \quad (37)$$

Defining $\Gamma=\Xi\backslash D=\{1,2,\cdots,K-1\}$ as the set of non-default rating classes, using Eqn. (B.5) and the relationship

$$\{\tau_B\le t_n\}=\{\beta_B(t_n)=D\} \text{ and } \{\tau_B>t_n\}=\{\beta_B(t_n)\in\Gamma\} \quad (38)$$

we obtain

$$P(\beta_C(t_n)=D,\beta_B(t_n)=m|\beta_C(t_{n-1})=j,\beta_B(t_{n-1})=i)=N_2\left(\sigma_D^C,\sigma_m^B;\rho\right) \quad (39)$$

and

$$P(\tau_C>t_n,\tau_B>t_n|\beta_C(t_{n-1})=j,\beta_B(t_{n-1})=i)=$$
$$\sum_{(a,b)\in\Gamma\times\Gamma}P(\beta_C(t_n)=a,\beta_B(t_n)=b|\beta_C(t_{n-1})=j,\beta_B(t_{n-1})=i)=$$
$$\sum_{(a,b)\in\Gamma\times\Gamma}\left\{N_2\left(\sigma_a^C,\sigma_b^B;\rho\right)-N_2\left(\sigma_{a+1}^C,\sigma_b^B;\rho\right)-N_2\left(\sigma_a^C,\sigma_{b+1}^B;\rho\right)+N_2\left(\sigma_{a+1}^C,\sigma_{b+1}^B;\rho\right)\right\} \quad (40)$$

Where $(k=B,C)$, and

$$\sigma_m^k=N^{-1}\left(\sum_{q=m}^K\tilde{Q}_{\omega_k}(t_{n-1},t_n)_{\beta_k(t_{n-1}),q}\right) \quad (41)$$

3.5 The ATE Factor Profiles

Since the main purpose of ATE trigger is to mitigate counterparty credit risk, a natural question to ask is: how much CVA would ATE be able to reduce? The answer is clearly that it depends on the exposure profiles $EPE=E_t\{W(t_k,T)^+\}$ and $ENE=E_t\{W(t_k,T)^-\}$. It is useful to build a profile independent of the actual exposure that can be used to estimate the effectiveness of the ATE trigger.

Notice that the BCVA without ATE trigger is given by

$$BCVA_{No-ATE}=\begin{Bmatrix}\delta_C\sum_{k=1}^N P\{\tau_B^R>t_k,\tau_C^R\in(t_{k-1},t_k]\}E_t\{W(t_k,T)^+\}\\ -\delta_B\sum_{k=1}^N P\{\tau_C^R>t_k,\tau_B^R\in(t_{k-1},t_k]\}E_t\{W(t_k,T)^-\}\\ +\sum_{k=1}^N P\{\tau_B^R\in(t_{k-1},t_k],\tau_C^R\in(t_{k-1},t_k]\}\\ \times\left\{\delta_C E_t\{W(t_k,T)^+\}-\delta_B E_t\{W(t_k,T)^-\}\right\}\end{Bmatrix} \quad (42)$$

where τ_k^R is the default time without ATE for party $k=B,C$.

We define bilateral ATE factor profiles. One is for the counterparty, and the other for the bank.[23]

[23] Obviously, unilateral ATE factor profile can be similarly defined, and we do not repeat here.

$$R^{C,ATE}(t_{k-1},t_k) = \begin{Bmatrix} P\{\tau_B > t_k, \tau_C \in (t_{k-1},t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \\ +P\{\tau_B \in (t_{k-1},t_k], \tau_C \in (t_{k-1},t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \end{Bmatrix}$$

$$\div \left\{P\{\tau_B^R > t_k, \tau_C^R \in (t_{k-1},t_k]\} + P\{\tau_B^R \in (t_{k-1},t_k], \tau_C^R \in (t_{k-1},t_k]\}\right\} \tag{43}$$

$$R^{B,ATE}(t_{k-1},t_k) = \begin{Bmatrix} P\{\tau_B \in (t_{k-1},t_k], \tau_C > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \\ +P\{\tau_B \in (t_{k-1},t_k], \tau_C \in (t_{k-1},t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \end{Bmatrix}$$

$$\div \left\{P\{\tau_B^R \in (t_{k-1},t_k], \tau_C^R > t_k\} + P\{\tau_B^R \in (t_{k-1},t_k], \tau_C^R \in (t_{k-1},t_k]\}\right\} \tag{44}$$

As mentioned earlier that, while the paths of τ_k cannot go though Ω_k, there is no such restriction on τ_k^R, i.e. the possible paths of τ_k is a subset of those of τ_k^R. Hence, τ_k^R and τ_k follow different distributions with $P\left(\tau_k^R \in (t_{k-1},t_k]\right) \geq P(\tau_k \in (t_{k-1},t_k], \eta_k > t_{k-1})$. Therefore, we deduce the bounds

$$0 \leq R^{C,ATE}(t_{k-1},t_k) \leq 1 \text{ and } 0 \leq R^{B,ATE}(t_{k-1},t_k) \leq 1 \tag{45}$$

In terms of the ATE factor profiles, we can rewrite the BCVA formulation (11) as

$$BCVA = \begin{Bmatrix} \delta_C \sum_{k=1}^{N} R^{C,ATE}(t_{k-1},t_k) P\{\tau_B^R > t_k, \tau_C^R \in (t_{k-1},t_k]\} E_t\{W(t_k,T)^+\} \\ -\delta_B \sum_{k=1}^{N} R^{B,ATE}(t_{k-1},t_k) P\{\tau_C^R > t_k, \tau_B^R \in (t_{k-1},t_k]\} E_t\{W(t_k,T)^-\} \end{Bmatrix} \tag{46}$$

The bilateral ATE factor profiles may be used as guideline in setting ATE trigger by estimating how effective a potential ATE trigger might be in reducing CVA, both unilateral and bilateral.

3.6 Effect of ATE on Transition Matrix: An Example

We use an example to demonstrate the effects of ATE trigger on the transition and default probability. Generally, ATE trigger increases the probability of rating migrating into the trigger rating, but reduces the probabilities of transiting to all other ratings and default. This effect is more pronounced if the ATE matrix is calculated from the ATE generator since the generator does not allow any possibility that the rating can transit from the ATE trigger rating or below. This latter observation is the reason we use the risk-neutral ATE generator to calculate the risk-neutral transition matrix. The fact that ATE reduces the probability of default is the fundamental reason for CVA reduction.

Suppose we are given a transition matrix *P* of four rating classes A, B, C and D where D is the default state. From the matrix *P*, we calculate the generator Λ.

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -0.5507 & 0.3535 & 0.1294 & 0.0678 \\ 0.1531 & -0.8222 & 0.4719 & 0.1972 \\ 0.1767 & 0.4482 & -1.0463 & 0.4213 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Without ATE trigger, the two-period transition probability is the probability of transition from one rating to another or default over the two period time where any non-default rating can transit to another rating or default. For example, rating A may transit to rating C at the end of period one and from rating C back to rating A in the second period. Thus, over the two periods, rating A stays put actually consists of three possible transition paths, $A \to A \to A$, $A \to B \to A$ and $A \to C \to A$. The probability is the sum of probability over these three paths. The non-ATE two-period transition matrix is given by

$$P2 = P \times P = \begin{pmatrix} 0.39 & 0.24 & 0.14 & 0.23 \\ 0.13 & 0.31 & 0.19 & 0.37 \\ 0.12 & 0.2 & 0.21 & 0.47 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.6.1 Case 1: ATE Trigger Rating B

Setting the ATE trigger to rating *B*, the one- and two-period ATE transition matrices directly modified from *P* are

$$P^B = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad P2^B = P^B \times P^B = \begin{pmatrix} 0.36 & 0.32 & 0.16 & 0.16 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Comparing the two-period ATE transition matrix $P2^B$ with the non-ATE matrix *P2* shows that the presence of ATE, which makes it necessary to treat ratings B and C as absorbing,

1) Reduces the probability of remaining at rating A as well as the PD;
2) Increases the probability of being in ratings B and C.

The ATE generator matrix Λ^B and its associated two-period ATE transition matrix $\tilde{Q}^B$ are

$$\Lambda^B = \begin{pmatrix} -0.5507 & 0.3535 & 0.1294 & 0.0678 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\tilde{Q}^B(0,2) = Exp\left(2\Lambda^B\right) = \begin{pmatrix} 0.3324 & 0.4285 & 0.1569 & 0.0822 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Comparing with $P2^B$, we see that in $\tilde{Q}^B(0,2)$,

a) The probability of default and remaining at A is further reduced. The reduction in PD is significant;
b) The probability of being in the ATE trigger rating B is further increased.

Again, we emphasize that the underlying reason for this pattern is that $P2$ disallow only inter-period transition from ratings *B* and *C* back to rating *A*. But it implicitly permits intra-period rating transition from *B* and *C*. on the contrary, $\tilde{Q}^B(0,2)$ strictly prohibits migration from *B* and *C* in any time interval.

3.6.2 Case 2: ATE Trigger Rating *C*

We now move the trigger rating a notch lower from rating *B* to rating *C*. We want to illustrate that trigger *C* is less effective than trigger *B.*

Setting the ATE trigger to rating *C*, the one- and two-period ATE transition matrices directly modified from *P* are

$$P^C = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad P2^C = P^C \times P^C = \begin{pmatrix} 0.38 & 0.22 & 0.2 & 0.2 \\ 0.11 & 0.27 & 0.31 & 0.31 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The ATE generator matrix Λ^C and its associated two-period ATE transition matrix $\tilde{Q}^C(0,2)$ are

$$\Lambda^C = \begin{pmatrix} -0.5507 & 0.3535 & 0.1294 & 0.0678 \\ 0.1531 & -0.8222 & 0.4719 & 0.1972 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\tilde{Q}^C(0,2) = Exp\left(2\Lambda^C\right) = \begin{pmatrix} 0.3632 & 0.1879 & 0.3045 & 0.1443 \\ 0.0814 & 0.2189 & 0.4922 & 0.2075 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It can be easily verified that all conclusions from Case 1 still hold.

From comparing $\tilde{Q}^C(0,2)$ with $\tilde{Q}^B(0,2)$, we see that

1) The two-period default probability of rating A is 0.1443 when the trigger is *C* (Case 2) which is significantly higher than the PD of 0.0822 when the trigger is B (case 1).
2) The total ATE termination probability is 0.5854 in Case 1, and is 0.3045 in case 2. So there is a much greater likelihood that ATE occurs in case 1 than case 2.

Since CVA is a measure of loss upon default and the likelihood of default, and that ATE termination is equivalent to lossless default, CVA is smaller in case 1. This demonstrates that the ATE is more effective when it is placed near the current rating than distant.

4. Conclusions

We presented a rating-based Markov chain model for valuation of bilateral CVA of a derivative portfolio subject to ATE rating trigger. Starting from the first principle, we derive, in the continuous time setting, a fundamental formulation for the bilateral CVA as the adjustment to the portfolio value without accounting for the counterparty credit risk. The continuous time formulation is then discretized using the first-order Euler scheme. The discretized CVA formulation is convenient for using the Markov chain/rating transition model.

The rating-based CVA model consists of several key components. First, as most rating-based credit risk model, we take a (one-year) historical rating transition matrix. This matrix is readily available from the major rating agencies. From this historical transition matrix, we compute the historical generator matrix which can be considered as the starting point of the model. Second, we use the JLT method to calculate the risk-neutral generator matrix by calibrating to the market implied CDS spread curves for ratings. However, the martingale generator matrix needs further modification to account for ATE.

Third, we define the ATE transition matrix for those rating transition paths where either the portfolio survives or a party defaults. Assuming mandatory termination and close-out of the portfolio upon first breaching the ATE trigger, we model the ATE trigger by making the rating classes equal to or below the ATE trigger absorbing states. This is achieved by setting to zero the rows of the risk-neutral generator matrix that are equal to and below the trigger rating. This permits to transform the problem of calculating the probability of first crossing the ATE trigger into the problem of calculating transition probability under the constraints that any path, whether leading to default or to remaining above the ATE trigger rating, cannot cross the ATE trigger. The resulting ATE generator matrix is used to calculate the transition matrix.

The ATE transition matrix contains the probability of transiting from one rating to another without ever crossing the ATE trigger. As such, the default under ATE is the jump-to-default from above the ATE trigger rating. Jump means jumping over all ratings below the ATE trigger. The paths of jump-to-default are only a subset of all possible paths that lead to default without ATE. Hence, the PD under ATE is smaller than he PD of the party. It is this reduction in PD that mitigates the counterparty default risk.

Fourth, we use the normal copula model for joint rating transition where the marginal rating transition thresholds are mapped to the rating transition probability of each party viewed standalone.

We defined the ATE factor profile which is the ratio of ATE PD and PD without ATE. The ATE factor profile depends only on the party's credit rating and the ATE triggers, and does not involve the actual deal. It can be used to estimate the potential effectiveness of ATE trigger for given exposure profile. Given the counterparty, this is potentially useful in deciding where the ATE trigger should be.

Appendix A: Rating Transition and Conditional Default Probability

In this section, we provide detailed derivation of Eqns. (12-17) used in the bilateral CVA calculation. We stress that in the presence of ATE, the rating class equal to and below the ATE trigger are absorbing states. For party *k*, any rating transition path leading to either the default or another rating above the ATE trigger cannot pass through the set Ω_k at any time.

The bilateral CVA calculation requires joint transition and default probabilities. From the set relationship

$$\{\tau_B > t_k, \tau_C > t_k, \eta_B > t_{k-1}, \eta_C > t_{k-1}\} = \{\tau_B > t_k, \tau_C > t_k\} \cap \{\eta_B > t_{k-1}, \eta_C > t_{k-1}\}$$

$$= \cup_{\substack{j \in \Pi_C \\ i \in \Pi_B}} \{\{\tau_B > t_k, \tau_C > t_k\} \cap \{\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i\}\} \quad \text{(A.1)}$$

and the law of total probability, we can verify that

$$P(\tau_B > t_k, \tau_C > t_k, \eta_B > t_{k-1}, \eta_C > t_{k-1}) =$$

$$\sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_B > t_k, \tau_C > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad \text{(A.2)}$$

which is Eqn. (17).

To prove Eqn. (15), we notice that

$$\{\tau_B > t_k, \tau_C > t_{k-1}, \eta_B > t_{k-1}, \eta_C > t_{k-1}\} = \{\tau_B > t_k, \tau_C > t_{k-1}\} \cap \{\eta_B > t_{k-1}, \eta_C > t_{k-1}\}$$

$$= \cup_{\substack{j \in \Pi_C \\ i \in \Pi_B}} \{\{\tau_B > t_k\} \cap \{\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i\}\} \quad \text{(A.3)}$$

The term $\tau_C > t_{k-1}$ drops out by virtue of relationship (A.1).

Eqn. (A.3) implies that

$$P(\tau_B > t_k, \tau_C > t_{k-1}, \eta_B > t_{k-1}, \eta_C > t_{k-1})$$

$$= \sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_B > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad \text{(A.4)}$$

The remaining equations can be similarly proven.

Appendix B: A Copula Model for Joint Rating Transition

We describe a simple one-factor normal copula model for joint rating transition of the counterparty and the bank. This model is similar to that used in the credit VaR model (RiskMetrics 1997).

Let X_k denote the credit quality index of party *k*. further, we assume that X_k can be decomposed into a systemic component *Y* (market factor) that is common to both parties and an idiosyncratic component ϵ_k (idiosyncratic factor) that is specific to party *k*

$$X_k = \sqrt{\rho} Y + \sqrt{1-\rho}\epsilon_k, k = B, C \quad \text{(B.1)}$$

where *Y* and ϵ_k are assumed to follow the standard normal distribution. ρ is the correlation that measures the co-movement of the credit indices. Eqn. (B.1) is used for modeling the conditional joint default over the one-step (t_{n-1}, t_n) as well as the long step $(0, t_n)$ rating transition matrix. As such, the correlation ρ perhaps should be associated with a maturity tag and calibrated accordingly.

Suppose we know, for party k, the martingale ATE transition matrix $\tilde{Q}_{\omega_k}(0, t_n)$ and the current rating $\beta_k(0)$, we calculate $P(\beta_C(t_n) = j, \beta_B(t_n) = i)$ where $j \in \Pi_C, i \in \Pi_B$. The use of ATE transition matrices guarantees there would be no prior breach of ATE trigger by either party.

We map X_k onto a grid of rating change from the current rating $\beta_k(0)$ and the martingale ATE transition matrix $\tilde{Q}_{\omega_k}(0, t_n)$. To this end, we introduce the time t_n rating transition thresholds for party k

$$-\infty = \alpha_{K+1}^k < \alpha_K^k < \alpha_{K-1}^k < \cdots < \alpha_2^k < \alpha_1^k = \infty \tag{B.2}$$

Starting from the initial rating $\beta_k(0)$, the thresholds are defined that, if $X_k \in \left(\alpha_{j+1}^k, \alpha_j^k\right), \ \beta_k(t_n) = j$. Mathematically, this can be expressed as

$$P\left(\alpha_{j+1}^k < X_k \le \alpha_j^k\right) = \tilde{Q}_{\omega_k}(0, t_n)_{\beta_k(0),j} \tag{B.3}$$

where $\tilde{Q}_{\omega_k}(0, t_n)_{\beta_k(0),j}$ denotes the $(\beta_k(0), j)th$ entry of $\tilde{Q}_{\omega_k}(0, t_n)$.

Recall that X_k obeys the standard normal distribution, solving for α_j^k recursively for $j = K, \cdots, 2$ yields

$$\alpha_j^k = N^{-1}\left(\textstyle\sum_{i=j}^{K} \tilde{Q}_{\omega_k}(0, t_n)_{\beta_k(0),i}\right) \tag{B.4}$$

Having obtained the rating transition thresholds for both the bank and the counterparty, the joint transition probability from the initial rating state $\left(\beta_C(0), \beta_B(0)\right)$ to a state $\left(\beta_C(t_n), \beta_B(t_n)\right) = (j, i)$ is given by

$$\begin{aligned} P(\beta_C(t_n) = j, \beta_B(t_n) = i) &= P\left\{\alpha_{j+1}^C < X_C \le \alpha_j^C, \alpha_{i+1}^B < X_B \le \alpha_i^B\right\} \\ &= N_2\left(\alpha_j^C, \alpha_i^B; \rho\right) - N_2\left(\alpha_{j+1}^C, \alpha_i^B; \rho\right) - N_2\left(\alpha_j^C, \alpha_{i+1}^B; \rho\right) + N_2\left(\alpha_{j+1}^C, \alpha_{i+1}^B; \rho\right) \end{aligned} \tag{B.5}$$

Setting the current time to t_{n-1} and following the same procedure, we obtain the one-step conditional joint default probability

$$\begin{aligned} &P(\tau_C \le t_n, \tau_B \le t_n | \beta_C(t_{n-1}) = j, \beta_B(t_{n-1}) = i) \\ &\quad = N_2\left(N^{-1}\left(\tilde{Q}_{\omega_C}(t_{n-1}, t_n)_{jK}\right), N^{-1}\left(\tilde{Q}_{\omega_B}(t_{n-1}, t_n)_{iK}\right); \rho\right) \end{aligned} \tag{B.6}$$

Eqn. (B.6) is the base formula from which other conditional joint probabilities in section 3.4 are obtained.

Appendix C: Correlation Estimation

If the credit index X_k in Eqn. (B.1) is the log asset return of the party k, the parameter ρ is the pair-wise asset return correlation between the two parties. Asset correlation estimation is extremely difficult.

Here, we selectively mention several methods for estimating asset return correlation. These are by no means the only asset correlation estimation models.

Other than picking a fixed number, the simplest estimating/forecasting method is to use the equity correlation as proxy primarily due to the availability of equity data. CreditMetrcs uses a method that maps equity return to country/industry indices and assigns weights to the indices by the firm's industry participation (RiskMetrics 1997). Asset value is unobservable and is calculated from the firm's equity value, debt and capital structure. A major drawback of this approach is that it ignores the significant difference between equity and asset, especially for financial firms (Zeng and Zhang, 2001a).

Another approach is to infer the pair-wise asset correlation from the joint and the single name default probabilities of the two parties.

Zeng and Zhang (2001b) assessed the performance of three widely used asset correlation estimation models - historical models, average models and factor models. They concluded that the KMV's Global Correlation Model performed best.

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