A COMPLETE MARKOVIAN STOCHASTIC VOLATILITY MODEL IN THE HJM FRAMEWORK

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ABSTRACT. This paper considers a stochastic volatility version of the Heath, Jarrow and Morton (1992) term structure model. Market completeness is obtained by adapting the Hobson and Rogers (1998) complete stochastic volatility stock market model to the interest rate setting. Numerical simulation for a special case is used to compare the stochastic volatility model against the traditional Vasicek (1977) model.

1. INTRODUCTION

The Heath, Jarrow and Morton (HJM) term structure model provides a consistent framework for the pricing of interest rate derivatives. The model is automatically calibrated to the currently observed yield curve, and is complete in the sense that it does not involve the market price of interest rate risk, something which was a feature of the early generation of interest rate models, such as Vasicek (1977) and Cox, Ingersoll and Ross (1985). The fundamental quantity driving the dynamics of the HJM model is the forward rate volatility process, which, together with the initial yield curve, is the main input to the model.

A great deal of research in this area focused on the different classes of interest rate models that arise from different assumptions about the form of the forward rate volatility process. Originally the focus was on forward rate volatility processes that depended on some function of time to maturity and the instantaneous spot rate of interest rate, as in Cheyette (1992), Carverhill (1994), Ritchken and Sankarasubramanian (1995), Bhar and Chiarella (1997) and Inui and Kijima (1998). Subsequently Chiarella and Kwon (1998*c*) and de Jong and Santa-Clara (1999) considered forward rate volatility processes dependent on time to maturity and a vector of fixed tenor forward rates. The essential

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characteristic of all of these models is that the form of the forward rate volatility processes allows them to be transformed to Markovian form, at the expense, however, of increasing the dimension of the underlying state space.

Chiarella and Kwon (1998*b*) have considered forms of the forward rate volatility process that yield some of the popular interest rate models, such as the Hull and White one and two factor models. All of the forward rate volatility processes referred to above could be described as level dependent volatility processes. It is also of interest to consider volatility processes that are themselves diffusion processes. Chiarella and Kwon (1998*a*) have investigated such a class of models where the Wiener processes driving the diffusion process for the volatility process are independent of the Wiener processes driving the forward rate process. This class of models also turns out to be Markovian, and so it is possible to generate, in the HJM framework, a class of stochastic volatility models that are in some sense the counterpart of the Hull and White (1987), Heston (1993) and Scott (1997) stochastic volatility models. In common with these models, they are incomplete as they involve the market price of risk that arises from the independent Wiener processes that drive the stochastic volatility process.

Hobson and Rogers (1998) introduced a special class of complete stochastic volatility models in the standard Black and Scholes stock option framework. They obtained market completeness by setting up a class of diffusion processes which, ultimately, are driven by the same Wiener process that drives the underlying asset price.

The aim in this paper is to obtain the counterpart to the Hobson and Rogers (1998) complete stochastic volatility model in the HJM framework. This is achieved by suitably adapting the offset processes, which ultimately depend on the Wiener processes driving the forward rate process, and feed into the forward rate volatility process.

The plan of the paper is as follows. Section 2 outlines the model and obtains a formula for the bond price in terms of the state variables of the model. Section 3 considers a special case, analogous to the special case considered by Hobson and Rogers, which is the basis of the numerical calculations undertaken in Section 4, and Section 5 concludes.

2. The Model

The stochastic volatility term structure model introduced in this paper is based on Chiarella and Kwon (1998*a*), but with the volatility process modeled along the lines of the Hobson and Rogers (1998) stochastic volatility stock price model.

Let $\mathcal{M}(n)$ be an n-dimensional risk-neutral HJM model on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\mathcal{F}_t$ is generated by an n-dimensional standard $\mathbb{P}$-Wiener process $(W_t^i)_{1\leq i\leq n}$. Let $P_{t,T}$ be the price of a T-maturity *zero coupon bond* at time t. Then the instantaneous *forward rate* $f_{t,T}$ is defined as

$$f_{t,T} = -\frac{\partial \ln P_{t,T}}{\partial T}, \tag{2.1}$$

and the instantaneous *spot rate* is defined as $r_t = f_{t,t}$. The *money market account* representing the value of initial unit investment at time t is defined by $B_t = e^{\int_0^t r_s\,ds}$, and it was shown in Heath, Jarrow and Morton (1992) that the *discounted bond price*, $P_{t,T}/B_t$, is a $\mathbb{P}$-martingale. It was shown in Brace, Gatarek and Musiela (1997) that if a different parametrisation, $T = t+\tau$, is used for maturity, then the corresponding $\mathbb{P}$-martingale is $e^{-\int_0^t (r_s - f_{s,s+\tau})\,ds} P_{t,t+\tau}$.

Although most of the ideas in this paper remain valid for the general $\mathcal{M}(n)$, for simplicity of notation, computations will be carried out only for the special case $\mathcal{M} = \mathcal{M}(1)$, and the 1-dimensional Wiener process W_t^1 will be written W_t.

For $0 < \tau \in \mathbb{R}$ and $i \in \mathbb{N}$, define the *offset*[1] process $S_{t,\tau}^{(i)}$ by

$$S_{t,\tau}^{(i)} = \int_0^\infty \lambda e^{-\lambda u} \left(Z_{t,t+\tau} - Z_{t-u,t-u+\tau}\right)^i \, du, \tag{2.2}$$

where $Z_{t,t+\tau}$ is the logarithm of the $\mathbb{P}$-martingale process $e^{-\int_0^t (r_s - f_{s,s+\tau})\,ds} P_{t,t+\tau}$, so that

$$Z_{t,t+\tau} = \ln\left(e^{-\int_0^t (r_s - f_{s,s+\tau})\,ds} P_{t,t+\tau}\right). \tag{2.3}$$

For each i, the offset process $S_{t,\tau}^{(i)}$ is best regarded as an exponentially weighted historical moment of returns on a τ-maturity bond.

Fix $n_1, n_2 \in \mathbb{N}$ and $0 \le \tau_1 < \tau_2 < \cdots < \tau_{n_2} \in \mathbb{R}$, and define

$$\mathcal{S}_t = \{S_{t,\tau_j}^{(i)} : 1 \le i \le n_1,\, 1 \le j \le n_2\}. \tag{2.4}$$

Assume now that the forward rate volatility, $\sigma_{t,T}$, becomes stochastic through its dependence on the offset processes. Thus, in $\mathcal{M}$, $\sigma_{t,T}$ has the form

$$\sigma_{t,T} = \varsigma(\mathcal{S}_t)\, e^{-\int_t^T \kappa(u)\,du}, \tag{2.5}$$

where ς and κ are deterministic functions. Models similar to $\mathcal{M}$, but with $\sigma_{t,T}$ dependent at most on t, T and $r(t)$, were considered by Ritchken and Sankarasubramanian (1995), Inui and Kijima (1998) and Chiarella and Kwon (1998*d*), who showed that the respective HJM models transform to Markovian systems, and derived formulae for bond prices in terms of the Markovian state variables. It is possible to show that a similar reduction to a Markovian form is possible for $\sigma_{t,T}$ having the form (2.5).

[1] This terminology is borrowed from Hobson and Rogers (1998).

The reduction to a Markovian form requires the introduction of a number of subsidiary quantities. For $0 \leq \tau \in \mathbb{R}$, define

$$\begin{aligned} \alpha_{t,\tau} &= e^{-\int_t^{t+\tau} \kappa(u)\, du}, & \beta_{t,\tau} &= \alpha_{t,\tau} \int_t^{t+\tau} \alpha_{t,u-t}\, du \\ \hat{\alpha}_{t,\tau} &= \int_t^{t+\tau} \alpha_{t,u-t}\, du, & \hat{\beta}_{t,\tau} &= \int_t^{t+\tau} \beta_{t,u-t}\, du, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \xi_{t,\tau} &= \int_0^t \sigma_{s,t+\tau}^2\, ds, \\ \zeta_{t,\tau} &= \int_0^t \sigma_{s,t+\tau} \int_s^{t+\tau} \sigma_{s,u}\, du\, ds + \int_0^t \sigma_{s,t+\tau}\, dW_s. \end{aligned} \tag{2.7}$$

The Lemmas 2.1 and 2.2 show that the dynamics of the stochastic quantities $\xi_{t,\tau}$ and $\zeta_{t,\tau}$ can be expressed just in terms of $\xi_{t,0} \equiv \xi_t$ and $\zeta_{t,0} \equiv \zeta_t$.

Lemma 2.1. *Define $\xi_t = \xi_{t,0}$ and $\zeta_t = \zeta_{t,0}$. Then for any $0 \leq \tau \in \mathbb{R}$,*

$$\xi_{t,\tau} = \alpha_{t,\tau}^2\, \xi_t, \tag{2.8}$$

$$\zeta_{t,\tau} = \alpha_{t,\tau}\, \zeta_t + \beta_{t,\tau}\, \xi_t. \tag{2.9}$$

Proof. Note that $\sigma_{s,t+\tau} = \alpha_{t,\tau}\sigma_{s,t}$. So

$$\xi_{t,\tau} = \int_0^t \sigma_{s,t+\tau}^2\, ds = \int_0^t \alpha_{t,\tau}^2 \sigma_{s,t}^2\, ds = \alpha_{t,\tau}^2 \int_0^t \sigma_{s,t}^2\, ds = \alpha_{t,\tau}^2\, \xi_t,$$

which is (2.8). Similarly,

$$\begin{aligned} \zeta_{t,\tau} &= \int_0^t \sigma_{s,t+\tau} \int_s^{t+\tau} \sigma_{s,u}\, du\, ds + \int_0^t \sigma_{s,t+\tau}\, dW_s \\ &= \int_0^t \alpha_{t,\tau}\, \sigma_{s,t} \left(\int_s^t \sigma_{s,u}\, du\, ds + \int_t^{t+\tau} \sigma_{s,u}\, du\, ds \right) + \int_0^t \alpha_{t,\tau}\, \sigma_{s,t}\, dW_s \\ &= \alpha_{t,\tau}\zeta_t + \alpha_{t,\tau} \int_0^t \sigma_{s,t} \int_t^{t+\tau} \alpha_{t,u-t}\, \sigma_{s,t}\, du\, ds \\ &= \alpha_{t,\tau}\, \zeta_t + \beta_{t,\tau}\, \xi_t, \end{aligned}$$

which is (2.9). ☐

Lemma 2.2. *The variables ξ_t and ζ_t satisfy the sde*

$$d\xi_t = \left[\varsigma^2(\mathcal{S}_t) - 2\kappa(t)\xi_t \right]\, dt, \tag{2.10}$$

$$d\zeta_t = \left[\xi_t - \kappa(t)\zeta_t \right]\, dt + \varsigma(\mathcal{S}_t)\, dW_t. \tag{2.11}$$

Proof. First consider (2.10). Since $\sigma_{s,t} = \varsigma(\mathcal{S}_s)\, e^{-\int_s^t \kappa(u)\, du}$,

$$d\xi_t = d \int_0^t \sigma_{s,t}^2\, ds = \left[\sigma_{t,t}^2 + \int_0^t \frac{\partial \sigma_{s,t}^2}{\partial t}\, ds \right]\, dt = \left[\varsigma^2(\mathcal{S}_t) - 2\kappa(t)\xi_t \right]\, dt,$$

which is (2.10). Similarly,

$$\begin{aligned} d\zeta_t &= d\left[\int_0^t \sigma_{s,t}\int_s^t \sigma_{s,u}\,du\,ds + \int_0^t \sigma_{s,t}\,dW_s\right] \\ &= \left[\int_0^t \frac{\partial}{\partial t}\left(\sigma_{s,t}\int_s^t \sigma_{s,u}\,du\right) + \int_0^t \frac{\partial \sigma_{s,t}}{\partial t}\,dW_s\right]\,dt + \sigma_{t,t}\,dW_t \\ &= \left[\xi_t - \kappa(t)\zeta_t\right]\,dt + \varsigma(\mathcal{S}_t)\,dW_t, \end{aligned}$$

which is (2.11). □

The following two propositions show that, under the stated assumptions, the forward rate curve and the bond price can be completely characterised in terms of ξ_t and ζ_t.

Proposition 2.3. *The forward rate curve satisfies the equation*

$$f_{t,t+\tau} = f_{0,t+\tau} + \alpha_{t,\tau}\zeta_t + \beta_{t,\tau}\xi_t. \tag{2.12}$$

In particular, since $\alpha_{t,\tau}$ and $\beta_{t,\tau}$ are deterministic, the entire forward rate curve is Markovian with respect to the state variables ξ_t and ζ_t.

Proof. From definitions of $f_{t,t+\tau}$ and $\zeta_{t,t+\tau}$,

$$f_{t,t+\tau} = f_{0,t+\tau} + \zeta_{t,t+\tau} = f_{0,t+\tau} + \alpha_{t,\tau}\zeta_t + \beta_{t,\tau}\xi_t,$$

where the last equality follows from (2.9). □

Proposition 2.4. *The bond price $P_{t,t+\tau}$ is given by the formula*

$$P_{t,t+\tau} = \frac{P_{0,t+\tau}}{P_{0,t}}\, e^{-\hat{\alpha}_{t,\tau}\zeta_t - \frac{1}{2}\hat{\alpha}^2_{t,\tau}\xi_t}. \tag{2.13}$$

Proof. Only a sketch proof is given. For the details, see Ritchken and Sankarasubramanian (1995) p60, Inui and Kijima (1998) p48, or Chiarella and Kwon (1998*c*) Theorem 4.5. Now, since $P_{t,t+\tau} = e^{-\int_t^{t+\tau} f_{t,u}\,du}$, the integral $\int_t^{t+\tau} f_{t,u}\,du$ must be computed. But from (2.12), $f_{t,u} = f_{0,t+u} + \alpha_{t,u-t}\zeta_t + \beta_{t,u-t}\xi_t$, and so the integral reduces to integrating deterministic functions $\alpha_{t,u-t}$ and $\beta_{t,u-t}$ with respect to the variable u. The result of performing these integrals is (2.13). □

The dynamics of the bond price follows as an immediate corollary of the above results.

Corollary 2.5. *The bond price $P_{t,t+\tau}$ satisfies the sde*

$$dP_{t,t+\tau} = \mu_\tau(t,\xi_t,\zeta_t,\mathcal{S}_t)\,dt - \sigma_\tau(t,\xi_t,\zeta_t,\mathcal{S}_t)\,dW_t, \tag{2.14}$$

where μ_τ and σ_τ are given by

$$\mu_\tau(t,\xi_t,\zeta_t,\mathcal{S}_t) = P_{t,t+\tau}\left[(f_{0,t} - f_{0,t+\tau}) + (1-\alpha_{t,\tau})\zeta_t - \beta_{t,\tau}\xi_t\right] \tag{2.15}$$

$$\sigma_\tau(t,\xi_t,\zeta_t,\mathcal{S}_t) = P_{t,t+\tau}\,\hat{\alpha}_{t,\tau}\varsigma(\mathcal{S}_t). \tag{2.16}$$

Proof. Consequence of Lemma 2.2, Proposition 2.4, Proposition 2.3 and Itô's Lemma. □

Corollary 2.6. *The state variable $Z_{t,t+\tau}$ satisfies the sde*

$$dZ_{t,t+\tau} = -\frac{1}{2}\hat{\alpha}^2_{t,\tau}\varsigma^2(\mathcal{S}_t)\,dt - \hat{\alpha}_{t,\tau}\varsigma(\mathcal{S}_t)\,dW_t, \tag{2.17}$$

where $\hat{\alpha}_{t,\tau}$ is given by (2.6), and $\varsigma(\mathcal{S}_t)$ is as defined in (2.5).

Proof. From (2.3), $Z_{t,t+\tau} = -\int_0^t (r_s - f_{s,s+\tau})\,ds + \ln P_{t,t+\tau}$, and so

$$dZ_{t,t+\tau} = -r_t + f_{t,t+\tau} + d\ln P_{t,t+\tau}. \tag{2.18}$$

Application of Itô's Lemma to $\ln P_{t,t+\tau}$, together with Corollary 2.5, yields the desired result. □

The dependence of the volatility function, ξ_t and ζ_t on the offset process means that a stochastic differential equation for the offset process is required for the overall dynamics to be specified. The next lemma provides the required result.

Lemma 2.7. *The offset process $S^{(i)}_{t,\tau_j}$ satisfies the sde*[2]

$$dS^{(i)}_{t,\tau_j} = iS^{(i-1)}_{t,\tau_j}\,dZ_{t,t+\tau_j} + \frac{i(i-1)}{2}\,S^{(i-2)}_{t,\tau_j}\,d\langle Z\rangle_{t,t+\tau_j} - \lambda S^{(i)}_{t,\tau_j}\,dt. \tag{2.19}$$

Proof. The details of the proof are given in Lemma 3.1 of Hobson and Rogers (1998), and only a sketch is given here. Following Hobson and Rogers (1998), consider

$$\begin{aligned} e^{\lambda t}S^{(i)}_{t,\tau_j} &= \int_{-\infty}^t \lambda e^{\lambda u}\left(Z_{t,t+\tau_j} - Z_{t-u,t-u+\tau_j}\right)^i\,du \\ &= \sum_{k=0}^i \binom{i}{k} Z^k_{t,t+\tau_j}\int_{-\infty}^t \lambda e^{\lambda u}\left(-Z_{t-u,t-t+\tau_j}\right)^{i-k}\,du, \end{aligned}$$

and take the 'differential' of both sides. This results in the equation

$$\lambda e^{\lambda t}\,S^{(i)}_{t,\tau_j} + e^{\lambda t}\,dS^{(i)}_{t,\tau_j} = e^{\lambda t}\left[i\,S^{(i-1)}_{t,\tau_j}\,dZ_{t,t+\tau_j} + \frac{i(i-1)}{2}\,S^{(i-2)}_{t,\tau_j}\,d\langle Z\rangle_{t,t+\tau_j}\right],$$

which essentially is (2.19). □

Proposition 2.8. *The set $\{\xi_t, \zeta_t, Z_{t,t+\tau_j}, S^{(i)}_{t,\tau_j} : 1 \le i \le n_1,\ 1 \le j \le n_2\}$ forms a Markovian system.*

Proof. Follows from Lemma 2.2, Corollary 2.5, Corollary 2.6, and Lemma 2.7. □

[2]The reader may need to recall that the quadratic variation of the process for $Z_{t,t+\tau_j}$, in the current context, is given by

$$\langle Z\rangle_{t,t+\tau_j} = \int_0^t \sigma^2_\tau(s,\xi_s,\zeta_s,\mathcal{S}_s)\,ds.$$

3. A Special Case

In this section, consider $\mathcal{M}$ corresponding to the case in which $n_1 = n_2 = 1$. This is the direct analogue of the special case considered by Hobson and Rogers (1998) for the stock price model.

For notational convenience, write $\tau = \tau_1$ and $S_t = S^{(1)}_{t,\tau}$. Then (2.19) simplifies to

$$dS_t = dZ_{t,t+\tau} - \lambda S_t \, dt. \tag{3.1}$$

Substituting for $dZ_{t,t+\tau}$ from (2.17) yields

$$dS_t = -\left[\frac{1}{2}\hat{\alpha}^2_{t,\tau}\varsigma^2(S_t) + \lambda S_t\right] dt - \hat{\alpha}_{t,\tau}\varsigma(S_t)\, dW_t, \tag{3.2}$$

where $\hat{\alpha}_{t,\tau}$ is given by (2.6), and $\varsigma(\mathcal{S}_t)$ is as defined in (2.5). Note that, as in Hobson and Rogers (1998), S_t is adapted to the filtration $\mathcal{F}_t$ of W_t.

Lemma 3.1. *The set $\{\xi_t, \zeta_t, S_t\}$ forms a Markovian system.*

Proof. Follows from Proposition 2.8, and equations (2.10), (2.11) and (3.2), since the dynamics of the system $\{\xi_t, \zeta_t, S_t\}$ do not depend on $Z_{t,t+\tau}$. □

Since the discounted bond price process $P_{t,T}/B_t$ is a $\mathbb{P}$-martingale, the bond price is given by the expectation

$$P_{t,T} = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r(u)\,du} P_{T,T} \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r(u)\,du} \middle| \mathcal{F}_t \right], \tag{3.3}$$

and, using the Feynman-Kac Theorem, the bond price must satisfy the pde

$$\mathcal{K}P_{t,T} - (f_{0,t} + \zeta_t)P_{t,T} + \frac{\partial P_{t,T}}{\partial t} = 0, \tag{3.4}$$

subject to the terminal condition $P_{T,T} = 1$, where

$$\begin{aligned}\mathcal{K} = {} & \frac{1}{2}\varsigma^2(S_t)\frac{\partial^2}{\partial \zeta_t^2} + \frac{1}{2}\hat{\alpha}^2_{t,\tau}\varsigma^2(S_t)\frac{\partial^2}{\partial S_t^2} - \hat{\alpha}_{t,\tau}\varsigma^2(S_t)\frac{\partial^2}{\partial \zeta_t \partial S_t} \\ & + \left[\varsigma^2(S_t) - 2\kappa(t)\xi_t\right]\frac{\partial}{\partial \xi_t} + \left[\xi_t - \kappa(t)\zeta_t\right]\frac{\partial}{\partial \zeta_t} - \left[\frac{1}{2}\hat{\alpha}^2_{t,\tau}\varsigma^2(S_t) + \lambda S_t\right]\frac{\partial}{\partial S_t}.\end{aligned}$$

The system of sdes underlying the pde are (2.10), (2.11) and (3.2). More generally, if C_{t,T_C} is a T_C-maturity European option on $P_{t,T}$ with payoff $h(T_C)$, where $T_C < T$, then

$$C_{t,T_C} = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^{T_C} r(u)\,du} h(T_C) \middle| \mathcal{F}_t \right], \tag{3.5}$$

and C_{t,T_C} must satisfy the pde

$$\mathcal{K}C_{t,T_C} - (f_{0,t} + \zeta_t)C_{t,T_C} + \frac{\partial C_{t,T_C}}{\partial t} = 0, \tag{3.6}$$

subject to the terminal condition $C_{T_C,T_C} = h(T_C)$.

The pdes (3.4) and (3.6) have three spatial variables, as well as the time variable, and are further complicated by the absence of the second order term in ξ_t. These pdes are rather difficult to tackle using the standard pde solution techniques, and so Monte Carlo simulation is used in the numerical results that follow.

4. Numerical Example

Let $\mathcal{M}$ be the special case considered in Section 3, and for this section, assume further that κ is constant and

$$\varsigma(s) = \eta\sqrt{1+\epsilon s^2} \wedge N \tag{4.1}$$

where η, ϵ and N are constants. This form of the volatility ς was introduced in Hobson and Rogers (1998) for the stock price model.

Now, constant κ implies

$$\alpha_{t,\tau} = e^{-\kappa\tau}, \quad \hat{\alpha}_{t,\tau} = \frac{1}{\kappa}\left(1-e^{-\kappa\tau}\right), \quad \beta_{t,\tau} = \frac{1}{\kappa}e^{-\kappa\tau}\left(1-e^{-\kappa\tau}\right), \tag{4.2}$$

and the state variables ξ_t, ζ_t, and S_t satisfy the sdes

$$d\xi_t = \left[\eta^2(1+\epsilon S_t^2) \wedge N^2 - 2\kappa\xi_t\right]\, dt, \tag{4.3}$$

$$d\zeta_t = \left[\xi_t - \kappa\zeta_t\right]\, dt + \left[\eta\sqrt{1+\epsilon S_t^2} \wedge N\right]\, dW_t, \tag{4.4}$$

$$dS_t = -\left[\frac{1}{2\kappa^2}(1-e^{-\kappa\tau})^2\left[\eta^2(1+\epsilon S_t^2)\wedge N^2\right] + \lambda S_t\right]\, dt \tag{4.5}$$

$$-\frac{1}{\kappa}\left(1-e^{-\kappa\tau}\right)\left[\eta\sqrt{1+\epsilon S_t^2}\wedge N\right]\, dW_t. \tag{4.6}$$

This system of sdes was solved numerically using Monte Carlo simulation with antithetic variables. A flat initial term structure of 5% was assumed for the simulation, with parameter values

$$\eta = 0.025, \quad N = 0.2, \quad \kappa = 2, \quad \lambda = 5, \quad \tau = 3.$$

From the definition of ξ_t and ζ_t in (2.7), it is clear that the initial values for these state variables are $\xi_0 = 0$ and $\zeta_0 = 0$. The initial value of S_t was varied within the range -1 and 1.

The effect of varying the value of ϵ on the distribution of the bond price is shown in Figure 4.1. The figure shows an increasing variance in the bond price with increasing ϵ, which is expected. The figure also shows that as ϵ increases, one is less likely to observe bond prices about the mean value. Since $\epsilon = 0$ corresponds to the Vasicek model, this implies that the Vasicek model overvalues the deep in-the-money calls while undervaluing the deep out-of-the-money calls when compared with the stochastic volatility model of this paper.

For the remaining simulations, ϵ was set equal to 2. The price of a 3-year bond was computed using (2.13), and the price of a 3-month call option on the bond was computed. To compare the price against the traditional Vasicek model, the Vasicek volatility implied

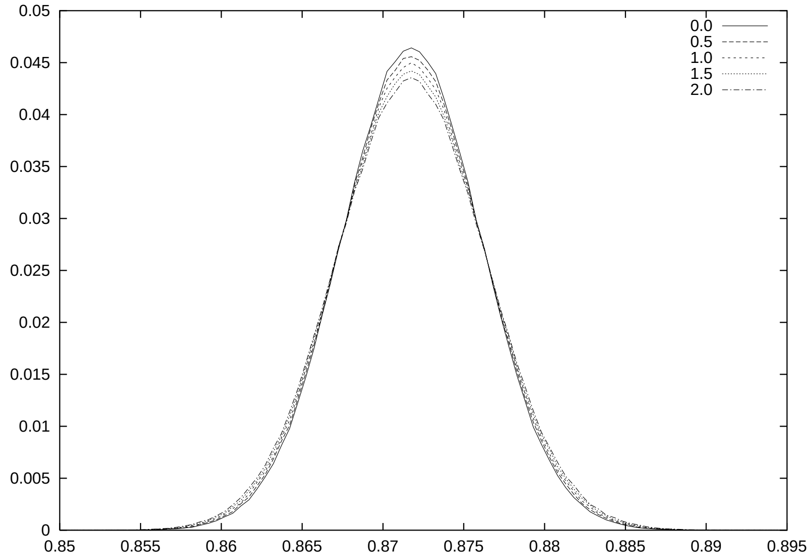

FIGURE 4.1. Distribution of Bond Price with Varying ϵ

by the call price was computed for various values of S_0 and the strike K. Under the Vasicek model, the forward rate volatility has the form

$$\sigma(s,t) = \sigma_0 e^{-\kappa(t-s)}, \tag{4.7}$$

and the price C_{t,T_C} of a T_C-maturity call option on a T_P maturity bond is

$$C_{t,T_C} = P_{t,T_P}\mathcal{N}\left[h_1(P_{t,T_P},t,T_C)\right] - KP_{t,T_C}\mathcal{N}\left[h_2(P_{t,T_P},t,T_C)\right], \tag{4.8}$$

where

$$h_1(P_{t,T_P},t,T_C) = \frac{\log P_{t,T_P} - \log P_{t,T_C} - \log K + \frac{1}{2}v^2(t,T_C)}{v(t,T_C)} \tag{4.9}$$

$$h_2(P_{t,T_P},t,T_C) = h_1(P_{t,T_P},t,T_C) - v(t,T_C), \tag{4.10}$$

$$v^2(t,T_C) = \frac{\sigma_0^2}{2\kappa^3}\left(e^{2\lambda(T_C-t)} - 1\right)\left(e^{-\kappa T_P} - e^{-\kappa T_C}\right)^2. \tag{4.11}$$

The implied volatility surface is shown in Figure 4.2. The figure shows that for each fixed S_0 there is a skew in the implied volatility curve, as was the case in the stock option case of Hobson and Rogers (1998).

In Figure 4.3, two implied volatility curves, for $S_0 = 0.2$ and $S_0 = -0.2$ are plotted to better illustrate their shape, and to illustrate the change in the direction of the skew with the change in the sign of S_0.

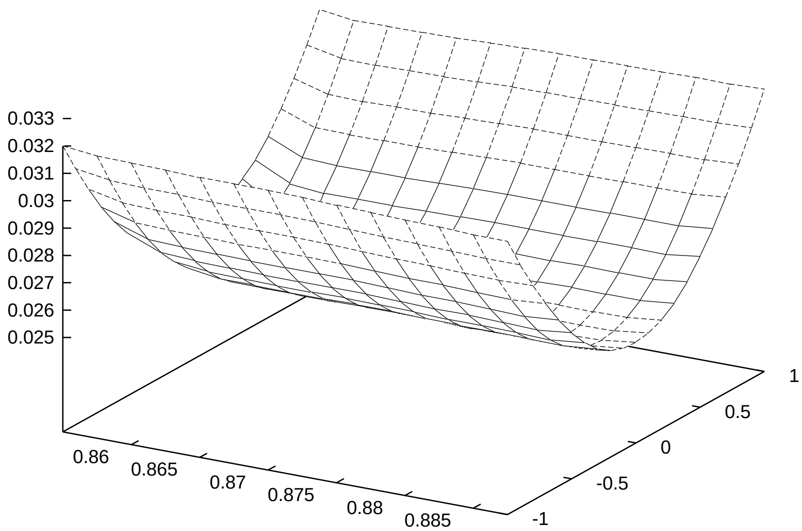

FIGURE 4.2. Implied Vasicek Volatility Surface. The strike increases from 0.855 to 0.8875 from left to right, and S_0 increases from -1 to 1 from front to back. The implied volatility ranges from 0.025 to 0.032.

5. CONCLUSION

This paper has considered the Hobson and Rogers (1998) technique for obtaining complete stochastic volatility models. In particular, the technique is used to obtain a complete stochastic model within the Heath, Jarrow and Morton (1992) interest rate framework. One of the main contributions of the paper has been to show how the stochastic dynamics can be reduced to a Markovian form. This allows the bond price to be expressed in terms of the underlying state variables, thus considerably reducing the computational burden required for the calculation of interest rate derivative prices. The model has been simulated in the simplest case, and an implied volatility surface based on the Vasicek (1977) model has been generated. These results indicate that the model is able to capture important features, such as skewness, of the implied volatility surfaces.

Future research should take further the numerical simulations reported here, perhaps experimenting with a wider specification of the volatility function. Empirical research on an appropriate form for the volatility functions also needs to be undertaken.

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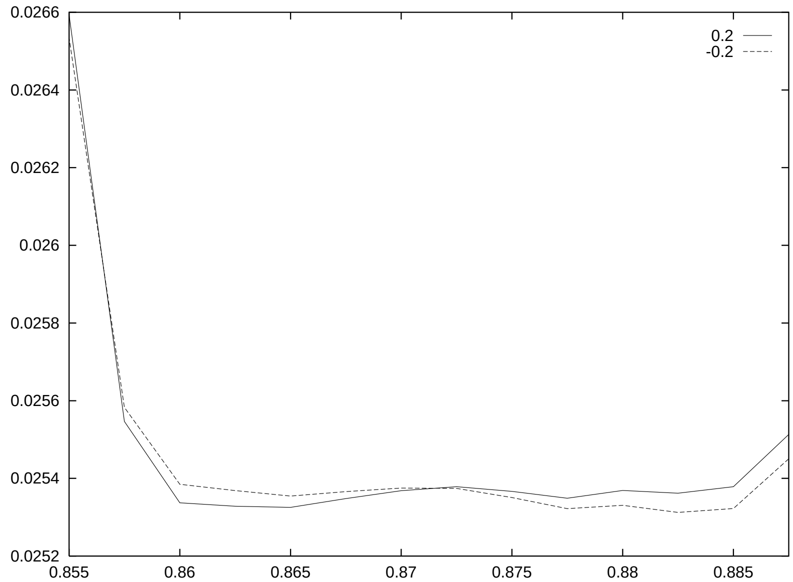

FIGURE 4.3. Implied volatility curves corresponding to $S_0 = 0.2$ and $S_0 = -0.2$.

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